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Decision Support

Relationship between the distance consensus and the consensus degree in comprehensive minimum cost consensus models: A polytope-based analysis

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ABSTRACT

Agreement in Group Decision-Making problems has recently been tackled through the use of Minimum Cost Consensus (MCC) models, which are associated with solving convex optimization problems. Such models minimize the cost of changing experts' preferences towards reaching a mutual consensus, and establish that the distance between the modified individual preferences and the collective opinion must be bounded by the threshold $\varepsilon > 0$. A recent MCC-based model, called the Comprehensive Minimum Cost Consensus (CMCC) model, adds another constraint related to a parameter $\gamma \in [0, 1]$ to the above constraint related to the parameter ε to enforce modified expert preferences in order to achieve a minimum level of agreement dictated by the consensus threshold $1 - \gamma \in [0, 1]$. This paper attempts to analyze the relationship between the aforementioned constraints in the CMCC models from two different perspectives. The first is based on inequalities and allows simple bounds to be determined to relate the parameters ε and γ. The second one is based on Convex Polytope Theory and provides algorithms that compute more precise bounds to relate these parameters, and could also be applied to other similar optimization problems. Finally, several examples are provided to illustrate the proposal.

1. Introduction

Group Decision-Making (GDM) problems are those situations in which a group of individuals or experts should decide, from a collective point of view, which alternative is the most suitable to solve a problem. Even though different rules such as majority, unanimity, or Borda count, among others, have been proposed in the classic literature to model these situations (Butler & Rothstein, 2006), the use of these rules in the formation of group opinions could leave some Decision Makers (DMs) feeling dissatisfied by not taking their opinions sufficiently into account in group opinion formation (Palomares, Estrella, Martinez, & Herrera, 2014). Therefore, it is of utmost interest to resolve conflicts among decision makers before forming a collective opinion to ensure that everyone is satisfied with unanimous acceptance.

Consensus Reaching Processes (CRPs) have been proposed (Labella, Liu, Rodríguez, & Martínez, 2018; Zhang, Dong, Chiclana, & Yu, 2019) to deal with conflicts between DMs' opinions in GDM. They consist of iterative discussion processes, usually coordinated by a human figure, called a moderator, which aim to smooth out conflicts in a GDM situation (Palomares et al., 2014). This iterative process is controlled by the measure of the level of agreement among decision makers, which we refer to as *consensus measure*. If the value obtained from this measure in a given round exceeds the consensus threshold set $\mu \in [0, 1]$, or the number of iterations exceeds the maximum number of rounds allowed $MaxRounds \in \mathbb{N}$ (the set of natural numbers), the CRP ends (Palomares et al., 2014). Often, these CRPs (we also refer to them as *consensus models*) are time-consuming as they require several discussion rounds among experts. Consequently, other consensus models have been proposed that aim to achieve agreement among experts quickly and automatically (Gong, Zhang, Forrest, Li, & Xu, 2015; Zhang, Dong, & Xu, 2012).

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Minimum Cost Consensus (MCC) models (Ben-Arieh & Easton, 2007; Zhang, Dong, Xu, & Li, 2011) stand out from other automatic CRPs because they express GDM problems in terms of an optimization model based on minimizing a cost function in the space of preferences (Ben-Arieh & Easton, 2007; Zhang et al., 2011). These models consider that the constraints defining the feasible region of the minimization problem are given by a maximum distance $\varepsilon > 0$ between the DMs and the collective opinion (Gong et al., 2015; Zhang, Dong, & Xu, 2013), but neglect the minimum level of agreement that characterizes classical CRPs (Zhang et al., 2012; Zhang, Gong, & Chiclana, 2017). To overcome this drawback, Comprehensive MCC (CMCC) models were introduced (Labella, Liu, Rodríguez, & Martínez, 2020; Rodríguez, Labella, Dutta, & Martínez, 2021) to generalize previous MCC approaches by including a constraint involving a consensus measure that enforces adjusted preferences to ensure a consensus degree $1-\gamma \in [0,1]$. However, the inclusion of such an additional inequality presents a major drawback in terms of redundancy, in some situations, of both types of constraints.

Rodríguez et al. (2021) observed that for a given fixed value of ε some constraints of the γ values become redundant and vice versa. Furthermore, the calculations shown by Labella et al. (2020) indicate that the parameters ε and γ could be related to one another. Concretely, one of the proposed examples shows that, for a fixed value of the parameter γ, the value of the cost of modifying the original preferences remains the same for several values of ε. Similarly, for certain fixed values of ε, the value of the cost function also remains invariant for some specific values of γ.

Although there are some initial observations on the dynamics of the relationship of the parameters ε and γ and their corresponding inequalities, a proper understanding of the behavior of the parameters with respect to the optimal solutions and specific theoretical results are yet to be established. From a practical application point of view, it is of utmost interest to understand the proper relationship between these parameters, since if the redundant constraints can be identified a priori, they can be suppressed in the computational resolution of the CRP. This simplification can be especially relevant when the number of DMs involved in the GDM is high because CMCC models are based on solving mathematical programming problems that can be especially slow in these situations.

Therefore, this paper is devoted to analyzing the relationship between the two parameters of the CMCC models. In particular, we attempt to explore the structure of such a relationship in light of the following research questions.

- RQ1: For a fixed value of γ (resp. ε), which values of ε (resp. γ) imply that the ε (resp. γ) constraint is redundant in CMCC?
- RQ2: For a fixed value of γ (resp. ε), which values of ε (resp. γ) imply that the γ (resp. ε) constraint is redundant in CMCC?

To answer these questions, this proposal studies the influence of the parameters γ and ε on the CMCC model using two different approaches. We will start with an inequality-based approach, which allows us to derive approximate bounds to relate the parameters that are very simple to calculate. The second approach relies on Convex Polytope Theory (Henk, Richter-Gebert, & Ziegler, 2018; Ziegler, 1995) to determine more precise bounds for these parameters. In addition, this polytope-based approach also provides a generic solution to the abstract problem of establishing a relationship between any linear constraints that define the feasible region of a convex minimization problem. In summary, the main novelties of this contribution are:

- The interactions between the consensus measure and the maximum distance between experts and the group in CMCC are formally analyzed from two different perspectives, one based on inequalities and the other on Polytope Theory, to explore the full potential of CMCC models in practice by providing a comprehensive view of parameter dynamics.
- A generic polytope-based algorithm is proposed to analyze the relationship between linear constraints in convex minimization problems, i.e., whether some of them are redundant and, consequently, determine the same feasible region.

A deeper understanding of the relationship between parameters in CMCC has the following main implications:

- It provides an explanation of certain peculiarities of the behavior of the cost function in the CMCC models that have been pointed out in the literature (Labella et al., 2020; Rodríguez et al., 2021).
- It simplifies CMCC models by eliminating redundant constraints.
- It helps the moderator to conduct the CRP more efficiently by explicitly detecting unnecessary consensus conditions and understanding possible changes in cost and solutions for different parameters configurations.
- It can significantly reduce the computational cost of generating experts' modified preferences in CMCC, which implies an immediate improvement in the total time required for the CRP.

The remainder of this contribution is set out as follows. Section 2 provides the necessary background on GDM, CRPs, MCC models, CMCC models, and Polytope Theory to easily understand this proposal. In Section 3, the minimization problem is reformulated by using a novel notation to simplify this proposal. Section 4 provides several relationships between the parameters γ and ε which have been obtained by chaining inequalities, and in Section 5 a novel approach based on Polytope Theory is developed to obtain more precise relationships between these parameters. In Section 6 several examples are proposed to illustrate this research. Finally, Section 7 concludes the paper.

2. Preliminaries

This section provides the background required to fully understand this proposal and introduces basic concepts about GDM and CRPs. In addition, a brief review of the historical evolution of MCC models is developed to emphasize the link between the various formulations. Finally, we provide a brief introduction to the fundamental concepts of convex polytopes.

2.1. Group decision-making

GDM problems are those situations in which several individuals or experts have to decide which alternative, out of a given set of possible solutions to a given problem, is the most appropriate (Butler & Rothstein, 2006; Kacprzyk, 1986). Formally, a GDM problem consists of:

- A set $X = \{x_1, x_2, \ldots, x_n\}$ of possible solutions to the problem.
- A set $E = \{e_1, e_2, \ldots, e_m\}$ of experts who express their preferences about the alternatives in X through a certain preference structure.

For the sake of simplicity, in this work, we restrict our investigation to two preference elicitation approaches, namely, numerical scale and Fuzzy Preference Relation (FPR). In numerical scale settings, experts evaluate the alternatives by using a number from $[0,1]$. On the other hand, in the FPR setting, it is assumed that preferences are elicited from experts by using Fuzzy Preference Relations (FPRs), a widely used structure

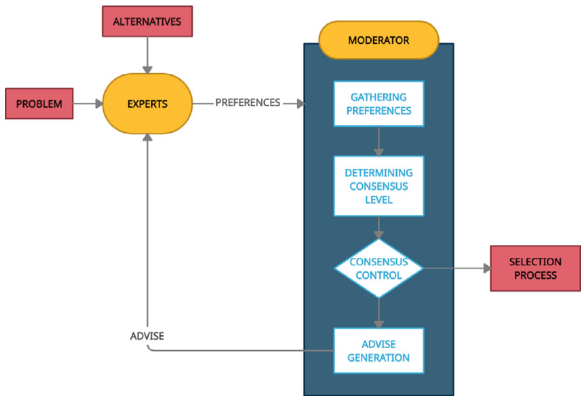

Fig. 1. Scheme of a CRP.

that has been shown to be effective in dealing with uncertainty (Bryson, 1996; Herrera-Viedma, Herrera, & Chiclana, 2002). These FPRs are obtained by asking the expert e_k to assess how much they prefer the alternative x_i to the alternative x_j using a value $p^k_{i,j}$ in the interval $[0, 1]$. The FPR associated with the expert e_k is the matrix $P_k = (p^k_{i,j}) \in \mathcal{M}_{n\times n}([0, 1])$, whose elements must satisfy the symmetry condition $p^k_{i,j} + p^k_{j,i} = 1$.

The classical scheme for a GDM problem consists of two phases:

- Aggregation. An aggregation operator is used to fuse the preferences elicited from the experts.
- Exploitation. The best alternative is selected taking into account the results of the previous phase.

Based on this scenario for a GDM problem, we will illustrate the key elements for carrying out the GDM process in the following subsection.

2.2. Consensus reaching processes and consensus measures

Classically, several rules have been used to select the best alternative in a GDM problem, such as the majority rule, the minority rule, or unanimity (Butler & Rothstein, 2006) but, when using these classic rules in the GDM solving process, some experts may disagree with the solution chosen by the group (Labella et al., 2018; Palomares et al., 2014).

A Consensus Reaching Process (CRP) is an iterative discussion process in which experts must modify their initial opinions to reach a collective agreement. CRPs have been developed to avoid disagreements and reach a collective opinion that satisfies all individuals who participate in the GDM problem.

Several consensus models have been proposed in the literature (Palomares et al., 2014). The general scheme for these models (see Fig. 1) is:

- *Consensus Measurement*. The preferences elicited from the experts are gathered and the level of agreement is computed using consensus measures (Beliakov, Calvo, & James, 2014).
- *Consensus control*. The obtained level of agreement is compared with a fixed consensus threshold $\mu \in [0, 1]$. If the level of agreement is greater than this threshold, a selection process is applied. Otherwise, another round of discussions is conducted. In order to avoid an endless process, a maximum number of rounds, $MaxRounds \in \mathbb{N}$, must be established beforehand.
- *Consensus Progress*. A moderator identifies experts' preferences that are difficult for the agreement process and generates recommendations for the experts to consider.

According to the taxonomy developed in Palomares et al. (2014), the consensus measures can be classified into two groups:

- Consensus measures based on the distance between each expert and the collective opinion,
- Consensus measures based on distances between experts.

Based on this background on the elements of the GDM process, in the following we will describe the automatic cost-based consensus models, which are the type of models for which we will study the relationships between parameters in this proposal.

2.3. Minimum cost consensus models

To study the cost of modifying experts' preferences, Ben-Arieh & Easton (2007) proposed the notion of MCC and introduced a model that considers consensus as being the minimum distance between each expert and the collective opinion, which is calculated using a weighted mean. This model aims to minimize the cost of moving preferences using a linear function. Specifically, for a set of experts $E = \{e_1, e_2, \ldots, e_m\}$ who express the preferences $o = (o_1, o_2, \ldots, o_m)$ over a certain alternative, the proposed optimization model is as follows:

$$\min_{(x_1,\ldots,x_m)} \sum_{i=1}^{m} c_i|x_i - o_i| \\ s.t. \begin{cases} \bar{x} = \sum_{i=1}^{m} w_i x_i \\ |x_i - \bar{x}| \le \varepsilon, i = 1, 2, \ldots, m \end{cases} \tag{M-1}$$

where the parameter $c_i \in \mathbb{R}_+$ models the cost of moving the opinion of the expert e_i one unit and $w_i \in [0, 1]$, $\sum_{i=1}^m w_i = 1$, is the importance of the expert e_i when aggregating the preferences.

By solving the non-linear programming problem defined in (M-1), a vector of optimal preferences $\hat{o} = (\hat{o}_1, \hat{o}_2, \ldots, \hat{o}_m)$ is obtained that satisfies that the distance between its coordinates and the collective opinion $\bar{\hat{o}} = \sum_{i=1}^m w_i \hat{o}_i$ is bounded by ε.

Zhang et al. (2011) improved this previous proposal by considering that collective opinion could be calculated using different aggregation operators. To do so, the previous model was modified as follows:

$$\min_{(x_1,\ldots,x_m)} \sum_{i=1}^{m} c_i|x_i - o_i| \\ s.t. \begin{cases} \bar{x} = F(x_1, x_2, \ldots, x_m) \\ |x_i - \bar{x}| \le \varepsilon, i = 1, 2, \ldots, m \end{cases} \tag{M-2}$$

where F is an aggregation operator.

2.4. Comprehensive minimum cost consensus

Recent studies (Gong et al., 2015; Zhang et al., 2013; 2012; Zhang et al., 2017) have introduced new MCC approaches based on the original model proposed in Ben-Arieh & Easton (2007), but they all consider the distance of each expert from the collective opinion, ignoring a minimum level of agreement among experts, which is a milestone for CRPs (Chiclana, Mata, Martinez, Herrera-Viedma, & Alonso, 2008; Kacprzyk & Zadrożny, 2010). In order to deal with this shortcoming, Comprehensive MCC (CMCC) models were developed (Labella et al., 2020).

A CMCC model is a modification of the model (M-2) which includes a new constraint related to preferences holding a minimum consensus level:

$$\min_{(x_1,\ldots,x_m)\in[0,1]^m} \sum_{i=1}^{m} c_i|x_i - o_i| \qquad \text{(M-3)}$$
$$s.t. \begin{cases} \bar{x} = F(x_1,\ldots,x_m) \\ |x_i - \bar{x}| \le \varepsilon, i = 1,2,\ldots,m \\ consensus(x_1,\ldots,x_m) \ge \mu, \end{cases}$$

where the function *consensus* : $[0,1]^m \to [0,1]$ measures the level of consensus reached by experts, $\mu \in [0,1]$ is a consensus threshold that is fixed a priori, $F : [0,1]^m \to [0,1]$ is an averaging aggregation operator, and ε is a parameter that measures the distance between each expert's adjusted opinion and the collective opinion.

2.4.1. MCC models dealing with numerical values

The model (M-3) was first adapted to two possible types of consensus measures (Palomares et al., 2014): those based on the distance between experts and collective opinion are modeled using (M-4) and those based on the distance between experts are modeled using (M-5), both of which are detailed below:

$$\min_{(x_1,\ldots,x_m)\in[0,1]^m} \sum_{i=1}^{m} c_i|x_i - o_i| \qquad \text{(M-4)}$$
$$s.t. \begin{cases} \bar{x} = \sum_{i=1}^{m} w_i x_i \\ |x_i - \bar{x}| \le \varepsilon, i = 1,2,\ldots,m \\ \sum_{i=1}^{m} w_i |x_i - \bar{x}| \le \gamma, \end{cases}$$

$$\min_{(x_1,\ldots,x_m)\in[0,1]^m} \sum_{i=1}^{m} c_i|x_i - o_i| \qquad \text{(M-5)}$$
$$s.t. \begin{cases} \bar{x} = \sum_{i=1}^{m} w_i x_i \\ |x_i - \bar{x}| \le \varepsilon, i = 1,2,\ldots,m \\ \sum_{i=1}^{m-1} \sum_{j=i+1}^{m-1} \frac{w_i + w_j}{m-1} |x_i - x_j| \le \gamma. \end{cases}$$

where $\gamma = 1 - \mu$ and $w_i \in [0,1]$ ($\sum_{i=1}^{m} w_i = 1$) are the importance values of the expert e_i.

2.4.2. MCC models dealing with FPRs

The models provided in the previous section were also adapted for FPRs. Model (M-4) was rewritten as (M-6), while model (M-5) becomes (M-7). Given the fuzzy preference relations $P_k = (p_{ij}^k) \in \mathcal{M}([0,1])_{n\times n}$, $(k = 1,\ldots,m)$, the model (M-6) is defined as follows:

$$\min_{(x_{ij}^k)\in\mathcal{M}_{n\times n}([0,1])} \sum_{k=1}^{m} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} c_k |x_{ij}^k - p_{ij}^k|$$
$$s.t. \begin{cases} \bar{x}_{ij} = \sum_{k=1}^{m} w_k x_{ij}^k \\ |x_{ij}^k - \bar{x}_{ij}| \le \varepsilon, k = 1,\ldots,m, i = 1,\ldots,n-1, j = i+1,\ldots,n \\ \frac{2}{n(n-1)} \sum_{k=1}^{m} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} w_k |x_{ij}^k - \bar{x}_{ij}| \le \gamma, \end{cases} \qquad \text{(M-6)}$$

where p_{ij}^k is the original preference of the expert e_k for the pair of alternatives x_i and x_j. Following the same scheme, the model (M-7) was defined as:

$$\min_{(x_{ij}^k)\in\mathcal{M}_{n\times n}([0,1])} \sum_{k=1}^{m} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} c_k |x_{ij}^k - p_{ij}^k|$$
$$s.t. \begin{cases} \bar{x}_{ij} = \sum_{k=1}^{m} w_k x_{ij}^k \\ |x_{ij}^k - \bar{x}_{ij}| \le \varepsilon, k = 1,\ldots,m, i = 1,\ldots,n-1, j = i+1,\ldots,n \\ \frac{2}{n(n-1)} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \sum_{k=1}^{m-1} \sum_{l=k+1}^{m} \frac{w_k + w_l}{m-1} |x_{ij}^k - x_{ij}^l| \le \gamma. \end{cases} \qquad \text{(M-7)}$$

2.5. Polytopes

In this subsection, some basic notions about the Convex Polytope Theory are introduced. Convex Polytopes are the generalization of the 2-dimensional notion of convex polygon or the 3-dimensional concept of polyhedron. After introducing two different definitions for the concept of Polytope that are present in the literature (Henk et al., 2018; Ziegler, 1995), the Main Theorem of Polytope Theory, which unifies these two definitions, is stated.

Definition 1 (V-Polytope)**.** $\mathcal{R} \subset \mathbb{R}^m$ is said to be a V-polytope if $\mathcal{R}$ can be expressed as the convex hull of a finite set $V = \{v_1, v_2, \ldots, v_n\} \subset \mathbb{R}^m$, i.e.:

$$\mathcal{R} = \left\{ \sum_{k=1}^{n} \lambda_k v_k \ : \ \lambda_k \ge 0 \ \forall\ k = 1,2,\ldots,n \text{ and } \sum_{k=1}^{n} \lambda_k = 1 \right\}$$

The set V is called the set of vertices of $\mathcal{R}$.

Definition 2 (H-Polytope)**.** $\mathcal{R} \subset \mathbb{R}^m$ is said to be a H-polytope if $\mathcal{R}$ can be expressed as the bounded solution set of a finite system of q linear inequalities, i.e., we can find $A \in \mathcal{M}_{q\times m}(\mathbb{R})$ and $B \in \mathcal{M}_{q\times 1}(\mathbb{R})$:

$$\mathcal{R} = \{x \in \mathbb{R}^m \ : \ Ax \le B\} \subset B(0,r) \text{ for a certain } r > 0,$$

where $B(0,r)$ denotes the ball of center $0 \in \mathbb{R}^m$ and radius $r > 0$.

The following fundamental theorem on the representation of polytopes (Henk et al., 2018; Ziegler, 1995), which relates the notions of V-polytope and H-polytope representations, forms the basis of our polytope-based analysis of the relationship between the inequalities involving the parameters μ and γ and the finding of tight bounds.

Theorem 1 (Main Theorem of Polytope Theory)**.** *The definitions of V-polytopes and of H-polytopes are equivalent. That is, every V-polytope has a description by a finite system of inequalities, and every H-polytope can be obtained as the convex hull of a finite set of points (its vertices).*

Definition 3 (Convex Polytope)**.** $\mathcal{R} \subset \mathbb{R}^m$ is said to be a convex polytope if $\mathcal{R} \subset \mathbb{R}^m$ is an H-polytope or a V-polytope.

From the computation point of view, it is important to understand how to switch between polytope representations. It was pointed out in Henk et al. (2018), Ziegler (1995) that there are three types of algorithms that allow us to transform one representation of a convex polytope into the other, namely, inductive algorithms (inserting vertices), projection algorithms and reverse search methods. Irrespective of the merits and demerits of the algorithms for numeric computations, this work will use the popular package `vertexenum` Robere (2018) developed for the software R (R Core Team, 2017), which allows us to transform the inequality-based representation of a convex polytope (H-polytope) into the vertex-based representation (V-polytope).

3. Problem description

This section is devoted to establishing a common notation to simplify the study of the relationships between the constraint induced by the consensus measure and the constraints involving the distances between preferences and collective opinion in CMCC models.

First, note that all models proposed in Section 2.4 can be described as follows.

$$\min \sum_{k=1}^{m} c_k|x_k - o_k| \quad s.t.\begin{cases}(x_1, x_2, \ldots, x_m) \in \mathcal{R}_\varepsilon,\\ (x_1, x_2, \ldots, x_m) \in \mathcal{R}_\gamma,\end{cases} \tag{M-G}$$

where

- $\mathcal{R}_\varepsilon$ and $\mathcal{R}_\gamma$ are the sets of points in $[0,1]^m$ that satisfy certain constraints related to the parameters ε and γ, respectively, that is,

  $$\mathcal{R}_\varepsilon := \{x \in [0,1]^m \ : \ g_k(x) \le \varepsilon \ \forall \ k = 1, 2, ..., m\},$$
  $$\mathcal{R}_\gamma := \{x \in [0,1]^m \ : \ g_0(x) \le \gamma\},$$

  where $g_0, g_1, \ldots, g_m : [0,1]^m \to \mathbb{R}_+$ are defined as the composition of a linear function with the absolute values of some other linear combination of variables, that is,

  $$g_k(x_1, x_2, \ldots, x_m) = \sum_{i=1}^{m} w_i^k |L_i(x_1, \ldots, x_m)|, \quad \forall (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m, \tag{1}$$

  for all $k = 0, 1, \ldots, m$, where $L_i : \mathbb{R}^m \to \mathbb{R}$ is a linear function for all $i = 1, 2, ..., m$ and $w_i^k \ge 0$.
- $(c_1, c_2, \ldots, c_m) \in \mathbb{R}_+^m$ are the constant values for the cost of moving the opinion of each expert one unit,
- $(o_1, o_2, \ldots, o_m) \in [0,1]^m$ are the initial values for experts' preferences.

As mentioned earlier, the experimental results conducted in Labella et al. (2020) suggest that the parameters γ and ε appearing in the models (M-4), (M-5), (M-6) and (M-7) could be related. Here, we are interested in exploring the relationships of these parameters by analyzing the containment relationship between the regions $\mathcal{R}_\gamma$ and $\mathcal{R}_\varepsilon$ that involve the inequalities corresponding to the parameters γ and ε, which attempt to restrict the adjusted preferences.

In light of the inclusion relationships of these regions, we first attempt to study if, given a value for $\gamma \in [0,1]$, certain values of ε exist for which the region $\mathcal{R}_\gamma$ is encapsulated in $\mathcal{R}_\varepsilon$. In other words, we want to find a value $\varepsilon_1 \in [0,1]$ such that $\mathcal{R}_\gamma \subseteq \mathcal{R}_\varepsilon \ \forall \ \varepsilon \ge \varepsilon_1$. This setup leads to some interesting consequences in the geometry of the regions that we describe in the following.

Denote $\mathcal{R}_{\gamma,\varepsilon} := \mathcal{R}_\gamma \cap \mathcal{R}_\varepsilon \ne \emptyset$ ($\overline{0} \in \mathcal{R}_{\gamma,\varepsilon}$). Note that for a fixed value of γ, the following statements are equivalent:

- $\mathcal{R}_\gamma \subseteq \mathcal{R}_\varepsilon \ \forall \ \varepsilon \ge \varepsilon_1$.
- $\mathcal{R}_{\gamma,\varepsilon} = \mathcal{R}_\gamma \ \forall \ \varepsilon \ge \varepsilon_1$.
- The constraints associated with ε, that is, $|x_k - \overline{x}| \le \varepsilon, k = 1, 2, ..., m$, do not affect the shape of $\mathcal{R}_{\gamma,\varepsilon} \ \forall \ \varepsilon \ge \varepsilon_1$.
- The values of the preferences that satisfy the constraint of $\mathcal{R}_\gamma$ also satisfy the constraints of $\mathcal{R}_\varepsilon$ for all $\varepsilon \ge \varepsilon_1$.

Remark 1. The notation $\mathcal{R}_\gamma^i, \mathcal{R}_\varepsilon^i$ and $\mathcal{R}_{\gamma,\varepsilon}^i$, $i = 4, 5, 6, 7$ will be used to relate these regions with the models (M-4), (M-5), (M-6) and (M-7). When referring to the generic model (M-G) the notation $\mathcal{R}_\gamma, \mathcal{R}_\varepsilon$ and $\mathcal{R}_{\gamma,\varepsilon}$ is kept for the sake of simplicity.

Keeping these consequences in mind, one can observe that for a fixed γ, studying the shape of the region $\mathcal{R}_{\gamma,\varepsilon}$ is equivalent to finding if a maximum value for the distance between the experts' opinions and the respective collective opinion ε is guaranteed. This fact leads to the following definitions:

Definition 4. ($\varepsilon_1(\gamma)$) For a fixed $\gamma \in [0,1]$ we will denote by $\varepsilon_1(\gamma)$, or simply ε_1 if no confusion is possible, the infimum value of ε such that $\mathcal{R}_\gamma \subseteq \mathcal{R}_\varepsilon$.

Definition 5. ($\gamma_1(\varepsilon)$) For a fixed $\varepsilon \in [0,1]$, we will denote by $\gamma_1(\varepsilon)$, or simply γ_1 if no confusion is possible, the infimum value of γ such that $\mathcal{R}_\varepsilon \subseteq \mathcal{R}_\gamma$.

The purpose of $\varepsilon_1(\gamma)$ and $\gamma_1(\varepsilon)$ is to detect when the region $\mathcal{R}_{\gamma,\varepsilon}$ starts to differ from the region $\mathcal{R}_\gamma$ or $\mathcal{R}_\varepsilon$, respectively. Note that finding the values $\varepsilon_1(\gamma)$ and $\gamma_1(\varepsilon)$ is equivalent to providing an answer to RQ1. However, a change in the region does not necessarily imply a change in the solution of the minimization problem. In order to control this change, we introduce the following alternative definition for the notion of boundary, which is more suitable for the problem that we are dealing with in this study:

Definition 6. Let $\mathcal{R} := \{x \in [0,1]^m \ : \ g(x) \le r\}$, for some $g : \mathbb{R}^m \to \mathbb{R}$ and $r > 0$, be a region in the unit hypercube. Then, the modified boundary of $\mathcal{R}$, $Bound^*(\mathcal{R})$, is defined as follows:

$$Bound^*(\mathcal{R}) := ((\mathcal{R}')^c \setminus (\mathcal{R}')^\circ) \cap [0,1]^m$$

where $\mathcal{R}' := \{x \in \mathbb{R}^m \ : \ g(x) \le r\}$ and c and $\circ$ denote, respectively, the standard closure and interior of a set.

The point of this definition is to avoid the interference of values such as $(0,0,\ldots,0)$ or $(1,1,\ldots,1)$ in the computation of the boundaries of the regions $\mathcal{R}_\gamma$ and $\mathcal{R}_\varepsilon$.

Based on this notion of a modified boundary, we propose the following definitions in order to find the values of the parameters ε and γ for which a change in the solution is ensured when the other parameter is fixed.

Definition 7. ($\varepsilon_2(\gamma)$) For a fixed $\gamma \in [0,1]$, we will denote by $\varepsilon_2(\gamma)$, or simply ε_2 if no confusion is possible, the supremum of the values of ε such that $Bound^*(\mathcal{R}_\gamma) \cap Bound^*(\mathcal{R}_{\gamma,\varepsilon}) = \emptyset$, or equivalently $\mathcal{R}_\varepsilon \subset \mathcal{R}_\gamma$.

Definition 8. ($\gamma_2(\varepsilon)$) For a fixed $\varepsilon \in [0,1]$, we will denote by $\gamma_2(\varepsilon)$, or simply γ_2 if no confusion is possible, the supremum of the values of γ such that $Bound^*(\mathcal{R}_\varepsilon) \cap Bound^*(\mathcal{R}_{\gamma,\varepsilon}) = \emptyset$, or equivalently $\mathcal{R}_\gamma \subset \mathcal{R}_\varepsilon$.

Note that if these boundaries have no points in common, the solution will inevitably change. In addition, by finding these values, an answer to RQ2 would be provided. To clarify this, consider a fixed value $\gamma \in [0,1]$ and the sequence $\{\varepsilon_n\} = \{\frac{1}{n}\}, n \in \mathbb{N}$. Note that for $\varepsilon_1 = 1$, the constraints related to ε_1 in the region $\mathcal{R}_{\gamma,\varepsilon_1}$ are always satisfied and consequently $\mathcal{R}_{\gamma,\varepsilon_1} = \mathcal{R}_\gamma$. If we increase the value of n, we will find some n_1 such that $\varepsilon_{n_1} < \varepsilon_1(\gamma)$, which means that for $n \ge n_1$, there is at least one point in $\mathcal{R}_\gamma$ which does not belong to $\mathcal{R}_{\varepsilon_n}$. However, the solution will not necessarily change until a value $n_2 \ge n_1$ such that $\varepsilon_{n_2} < \varepsilon_2(\gamma)$ is found and consequently $Bound^*(\mathcal{R}_\gamma) \cap Bound^*(\mathcal{R}_{\gamma,\varepsilon_n}) = \emptyset$ for any value $n \ge n_2$ (see Fig. 2).

Based on this formalization of our research questions in the form of the definitions mentioned above, we attempt to find rough bounds for the parameters in the next section.

4. An approach based on inequalities

In this section, we provide rough bounds for the values of $\varepsilon_1(\gamma)$ and $\gamma_1(\varepsilon)$ for each CMCC model. Specifically, we obtain the values ε_1' and γ_1' such that $\varepsilon_1' \ge \varepsilon_1(\gamma)$ and $\gamma_1' \ge \gamma_1(\varepsilon)$ for every model.

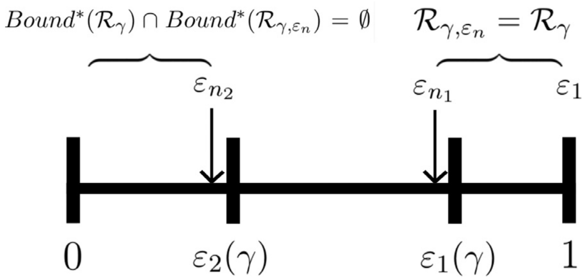

Fig. 2. A sketch of $\varepsilon_1(\gamma)$ and $\varepsilon_2(\gamma)$ for a fixed value of $\gamma \in [0,1]$.

4.1. The model (M-4)

Adopting our earlier introduced notations, the regions associated with the consensus model (M-4) can be written as follows:

$$\mathcal{R}^4_\varepsilon := \{x \in [0,1]^m \ : \ |x_i - \bar{x}| \le \varepsilon, \ \forall \ i = 1,2,\dots,m\},$$
$$\mathcal{R}^4_\gamma := \left\{x \in [0,1]^m \ : \ \sum_{i=1}^m w_i|x_i - \bar{x}| \le \gamma\right\},$$
$$\mathcal{R}^4_{\gamma,\varepsilon} := \mathcal{R}^4_\varepsilon \cap \mathcal{R}^4_\gamma.$$

The rough bound of γ for a given $\varepsilon > 0$ in terms of Definition 5 is illustrated in the following proposition.

Proposition 1. *For a given $\varepsilon > 0$ in the consensus model (M-4), the value of γ that ensures the satisfaction of the constraints of $\mathcal{R}^4_\gamma$ is $\gamma'_1(\varepsilon) = \varepsilon$, that is, $\mathcal{R}^4_\varepsilon = \mathcal{R}^4_{\gamma,\varepsilon} \ \forall \ \gamma \ge \varepsilon$.*

Proof. Suppose that $x \in \mathcal{R}^4_\varepsilon$. Then $|x_i - \bar{x}| \le \varepsilon$ and $\sum_{k=1}^m w_k|x_i - \bar{x}| \le \varepsilon$. Therefore, if $\gamma \ge \varepsilon$, $\mathcal{R}^4_\varepsilon \subseteq \mathcal{R}^4_{\gamma,\varepsilon}$ and consequently $\mathcal{R}^4_\varepsilon = \mathcal{R}^4_{\gamma,\varepsilon}$. □

The opposite case is described in the following proposition.

Proposition 2. *For a given $\gamma > 0$ in the consensus model (M-4), the values of ε that ensure the satisfaction of the constraints in $\mathcal{R}^4_\varepsilon$ are $\varepsilon'_1(\gamma) = \frac{\gamma}{\min_{k=1,2,\dots,m}\{w_k\}}$, that is,*

$$\mathcal{R}^4_\gamma = \mathcal{R}^4_{\gamma,\varepsilon} \ \forall \ \varepsilon \ge \frac{\gamma}{\min_{k=1,2,\dots,m}\{w_k\}}$$

Proof. Let us suppose that $x \in \mathcal{R}^4_\gamma$. Then $\sum_{i=1}^m w_i|x_i - \bar{x}| \le \gamma$ and $w_i|x_i - \bar{x}| \le \gamma \ \forall \ i = 1,2,\dots,m$. Therefore,

$$|x_i - \bar{x}| \le \frac{\gamma}{\min_{k=1,2,\dots,m}\{w_k\}} \ \forall \ i = 1,2,\dots,m.$$

So, if $\varepsilon \ge \varepsilon'_1 := \frac{\gamma}{\min_{k=1,2,\dots,m}\{w_k\}}$, $x \in \mathcal{R}^4_{\gamma,\varepsilon}$ and consequently, $\mathcal{R}^4_\gamma = \mathcal{R}^4_{\gamma,\varepsilon}$. □

4.2. The model (M-5)

The regions associated with the consensus model (M-5) can be cast as follows:

$$\mathcal{R}^5_\varepsilon := \{x \in [0,1]^m \ : \ |x_i - \bar{x}| \le \varepsilon, \ \forall \ i = 1,2,\dots,m\},$$
$$\mathcal{R}^5_\gamma := \left\{x \in [0,1]^m \ : \ \sum_{i=1}^{m-1}\sum_{j=i+1}^m \frac{w_i + w_j}{m-1}|x_i - x_j| \le \gamma\right\},$$
$$\mathcal{R}^5_{\gamma,\varepsilon} := \mathcal{R}^5_\varepsilon \cap \mathcal{R}^5_\gamma.$$

Similarly, rough bounds for the parameters ε and γ when one of them is given can be obtained for (M-5) and are given in the following propositions.

Proposition 3. *For a given $\varepsilon > 0$ in the consensus model (M-5), the values of γ that ensure the satisfaction of the constraint of $\mathcal{R}^5_\gamma$ is $\gamma'_1(\varepsilon) = 2\varepsilon$, that is, $\mathcal{R}^5_\varepsilon = \mathcal{R}^5_{\gamma,\varepsilon} \ \forall \ \gamma \ge 2\varepsilon$.*

Proof. First, note that

$$\begin{aligned}\sum_{i=1}^{m-1}\sum_{j=i+1}^m \frac{w_i + w_j}{m-1} &= \frac{1}{m-1}\left(\sum_{i=1}^{m-1}(m-i)w_i + \sum_{i=2}^m (i-1)w_i\right)\\ &= \frac{1}{m-1}\Bigg((m-1)w_1 + \sum_{i=2}^{m-1}(m-i)w_i\\ &\quad + (m-1)w_m + \sum_{i=2}^m (i-1)w_i\Bigg)\\ &= \frac{1}{m-1}\left((m-1)(w_1 + w_m) + \sum_{i=2}^{m-1}(m-1)w_i\right) = 1.\end{aligned}$$

For an arbitrary $x \in \mathcal{R}^5_\varepsilon$, we have

$$\begin{aligned}\sum_{i=1}^{m-1}\sum_{j=i+1}^m \frac{w_i + w_j}{m-1}|x_i - x_j| &\le \sum_{i=1}^{m-1}\sum_{j=i+1}^m \frac{w_i + w_j}{m-1}\big(|x_i - \bar{x}| + |\bar{x} - x_j|\big)\\ &\le 2\varepsilon \sum_{i=1}^{m-1}\sum_{j=i+1}^m \frac{w_i + w_j}{m-1} = 2\varepsilon\end{aligned}$$

Hence, $x \in \mathcal{R}^5_{\gamma,\varepsilon} \ \forall \ \gamma > 2\varepsilon$. □

Proposition 4. *For a given $\gamma > 0$ in the consensus model (M-5), the values of ε that ensure the satisfaction of the constraints in $\mathcal{R}^5_\varepsilon$ are $\varepsilon'_1(\gamma) = (m-1)\gamma$, that is,*

$$\mathcal{R}^5_\gamma = \mathcal{R}^5_{\gamma,\varepsilon} \ \forall \ \varepsilon \ge (m-1)\gamma$$

Proof. Let $x \in \mathcal{R}^5_\gamma$. Then

$$\begin{aligned}|x_i - \bar{x}| &= |x_i\sum_{j=1}^m w_j - \sum_{j=1}^m w_j x_j| \le \sum_{j=1, j\ne i}^m w_j|x_i - x_j|\\ &\le \sum_{i=1}^{m-1}\sum_{i=j+1}^m (w_i + w_j)|x_i - x_j| \le \gamma(m-1).\end{aligned}$$

So, if $\varepsilon \ge (m-1)\gamma$, $\mathcal{R}^5_\gamma = \mathcal{R}^5_{\gamma,\varepsilon}$. □

4.3. The model (M-6)

Let us define $\mathcal{M}_{n\times n}([0,1])^m := \mathcal{M}_{n\times n}([0,1]) \times \overset{m \text{ times}}{\cdots} \times \mathcal{M}_{n\times n}([0,1])$, whose elements are vectors of n-dimensional square matrices. With this notation, the regions associated with (M-6) can be represented as follows:

$$\begin{aligned}\mathcal{R}^6_\varepsilon : &= \Big\{X \in \mathcal{M}_{n\times n}([0,1])^m \ : \ |x^k_{ij} - \bar{x}_{ij}|\\ &\le \varepsilon, \ k = 1,\dots,m, i = 1,\dots,n-1,\\ &j = i+1,\dots,n\},\\ \mathcal{R}^6_\gamma : &= \{X \in \mathcal{M}_{n\times n}([0,1])^m \ :\\ &\frac{2}{n(n-1)}\sum_{k=1}^m\sum_{i=1}^{n-1}\sum_{j=i+1}^n w_k|x^k_{ij} - \bar{x}_{ij}| \le \gamma\Bigg\},\\ \mathcal{R}^6_{\gamma,\varepsilon} : &= \mathcal{R}^6_\varepsilon \cap \mathcal{R}^6_\gamma.\end{aligned}$$

Similar results for the rough bounds for the parameters can also be obtained for (M-6) and are given in the following propositions.

Proposition 5. *For a given $\varepsilon > 0$ in the consensus model (M-6), the values of γ that ensure the satisfaction of the constraint of $\mathcal{R}^6_\gamma$ is $\gamma'_1(\varepsilon) = \varepsilon$, that is, $\mathcal{R}^6_\varepsilon = \mathcal{R}^6_{\gamma,\varepsilon}$ $\forall\ \gamma \geq \varepsilon$.*

Proposition 6. *For a given $\gamma > 0$ in the consensus model (M-6), the values of ε that ensure the satisfaction of the constraints in $\mathcal{R}^6_\varepsilon$ are $\varepsilon'_1(\gamma) = \frac{\gamma(n-1)n}{2\min_{k=1,2,\ldots,m}\{w_k\}}$, that is,*

$$\mathcal{R}^6_\gamma = \mathcal{R}^6_{\gamma,\varepsilon}\ \forall\ \varepsilon \geq \frac{\gamma(n-1)n}{2\min_{k=1,2,\ldots,m}\{w_k\}}.$$

4.4. The model (M-7)

Analogously, we can cast the attached regions to the model (M-7) and find the results of the rough bounds for the parameters as follows:

$$\mathcal{R}^7_\varepsilon := \Big\{X \in \mathcal{M}_{n\times n}([0,1])^m\ :\ |x^k_{ij} - \bar{x}_{ij}| \leq \varepsilon,\ k = 1,\ldots,m, i = 1,\ldots,n-1, j = i+1,\ldots,n\Big\},$$

$$\mathcal{R}^7_\gamma := \left\{X \in \mathcal{M}_{n\times n}([0,1])^m\ :\ \frac{2}{n(n-1)}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{k=1}^{m-1}\sum_{l=k+1}^{m}\frac{w_k+w_l}{m-1}|x^k_{ij} - x^l_{ij}| \leq \gamma\right\},$$

$$\mathcal{R}^7_{\gamma,\varepsilon} := \mathcal{R}^7_\varepsilon \cap \mathcal{R}^7_\gamma.$$

The proof for the following results is analogous to some of those given previously, and is therefore omitted.

Proposition 7. *For a given $\varepsilon > 0$ in the consensus model (M-7), the values of γ that ensure the satisfaction of the constraint of $\mathcal{R}^7_\gamma$ is $\gamma'_1(\varepsilon) = 2\varepsilon$, that is, $\mathcal{R}^7_\varepsilon = \mathcal{R}^7_{\gamma,\varepsilon}$ $\forall\ \gamma \geq 2\varepsilon$.*

Proposition 8. *For a given $\gamma > 0$ in the consensus model (M-7), the values of ε that ensure the satisfaction of the constraints in $\mathcal{R}^7_\varepsilon$ are $\varepsilon'_1(\gamma) = (m-1)n(n-1)\frac{\gamma}{2}$, that is,*

$$\mathcal{R}^7_\gamma = \mathcal{R}^7_{\gamma,\varepsilon}\ \forall\ \varepsilon \geq (m-1)n(n-1)\frac{\gamma}{2}.$$

Therefore, we have obtained rough bounds for the parameters ε and γ that correspond to the different consensus models by treating regions $\mathcal{R}_\gamma$ and $\mathcal{R}_\varepsilon$ as abstract spaces. Although these rough bounds are very easy to obtain and provide an immediate idea of the variations and a partial answer to RQ1 and RQ2, they are not very precise. In the following, we attempt to find more precise bounds for these parameters.

5. Approach based on polytopes

In this section, we further explore the geometry of the regions to obtain more precise bounds of the parameters ε and γ. We start by characterizing the regions $\mathcal{R}_\gamma$ and $\mathcal{R}_\varepsilon$ as convex polytopes. Subsequently, the properties of convex polytopes are used to derive the numerical bounds of the parameters along with the connection to the optimal solution of the models.

5.1. Properties of the regions $\mathcal{R}_\gamma$ and $\mathcal{R}_\varepsilon$

In this subsection, we show that both regions $\mathcal{R}_\gamma$ and $\mathcal{R}_\varepsilon$ in the generic model (M-G) are convex polytopes.

The first result proves that any system of inequalities involving compositions of linear functions with absolute values can be reduced to a system of linear inequalities.

Proposition 9. *Let $w = (w_1, w_2, \ldots w_m) \in \mathbb{R}^m$ and $\mu \in \mathbb{R}$, and consider the function $g : \mathbb{R}^m \to \mathbb{R}$ defined as*

$$g(x_1, x_2, \ldots, x_m) = \sum_{k=1}^{m} w_k|x_k|\ \forall\ (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m.$$

Then, for any $x = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m$, the following relations are equivalent:

- $g(x) \leq \mu$,
- $< w_\sigma, x > \leq \mu$ *for every* $\sigma = (\sigma_1, \ldots, \sigma_m) \in \{-1, 1\}^m$.

where $< \cdot, \cdot >$ denotes the standard dot product and $w_\sigma = (w_1\sigma_1, w_2\sigma_2, \ldots, w_m\sigma_m)$.

Proof. For $m = 1$, the statement is an immediate consequence of the properties of the absolute value function. Consider the case $m = 2$. Thus, $w_1, w_2 \geq 0$ and $g : \mathbb{R}^2 \to \mathbb{R}$ is defined as $g(x_1, x_2) = w_1|x_1| + w_2|x_2|$, for all $(x_1, x_2) \in \mathbb{R}^2$. Note that

$$w_1|x_1| + w_2|x_2| \leq \mu \iff w_1|x_1| \leq \mu - w_2|x_2| \iff \begin{array}{c} w_1x_1 \leq \mu - w_2|x_2| \\ -w_1x_1 \leq \mu - w_2|x_2| \end{array}$$

$$\iff \begin{array}{c} w_2|x_2| \leq \mu - w_1x_1 \\ w_2|x_2| \leq \mu + w_1x_1 \end{array} \iff \begin{array}{c} w_2x_2 \leq \mu - w_1x_1 \\ -w_2x_2 \leq \mu - w_1x_1 \\ w_2x_2 \leq \mu + w_1x_1 \\ -w_2x_2 \leq \mu + w_1x_1 \end{array} \iff \begin{array}{c} w_1x_1 + w_2x_2 \leq \mu \\ w_1x_1 - w_2x_2 \leq \mu \\ -w_1x_1 + w_2x_2 \leq \mu \\ -w_1x_1 - w_2x_2 \leq \mu \end{array}.$$

Therefore, this is also true for $m = 2$. The rest of the proof is an obvious induction. □

Note that the resulting linear inequalities are those obtained by considering all possible combinations of the different signs for the weights $w_1, \ldots, w_m$, as shown in the proof of Proposition 9.

This representation of inequalities involving absolute values allows us to characterize regions $\mathcal{R}_\gamma$ and $\mathcal{R}_\varepsilon$ as polytopes as stated in the following corollary.

Corollary 1. *Let $n \in \mathbb{N}$ and consider a set of m-dimensional weighting vectors $\{w^1, w^2, \ldots, w^n\}$, i.e., $w^k = (w^k_1, w^k_2, \ldots w^k_m) \in [0,1]^m$ such that $\sum_{i=1}^m w^k_i = 1$, $\forall\ k = 1, 2, ..., m$, and $\mu_1, \ldots, \mu_m \in \mathbb{R}_+$. Consider the functions $g_k : \mathbb{R}^m \to \mathbb{R}$ defined as*

$$g_k(x_1, x_2, \ldots, x_m) = \sum_{i=1}^{m} w^k_i|x_i|,\ \forall\ (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m, k = 1, 2, ..., m.$$

Then, the region $\mathcal{R} = \{x \in [0,1]^m\ :\ g_k(x) \leq \mu_k\ \forall\ k = 1, 2, ..., m\}$ is a non-empty polytope.

Remark 2. Note that the point $(0, 0, \ldots, 0) \in \mathbb{R}^m$ always satisfies the conditions that define $\mathcal{R}$.

Furthermore, we can assure that the obtained polytope is convex, even if the arguments of the absolute values are replaced with linear combinations of the variables.

Corollary 2. *Let $n \in \mathbb{N}$ and consider a set of m-dimensional weighting vectors $\{w^1, w^2, \ldots, w^n\}$, i.e., $w^k = (w^k_1, w^k_2, \ldots w^k_m) \in [0,1]^m$, such that $\sum_{i=1}^m w^k_i = 1$ $\forall\ k = 1, 2, \ldots, n$, and $\mu_1, \ldots \mu_m \in \mathbb{R}_+$. Consider the functions $g_k : \mathbb{R}^m \to \mathbb{R}$ defined as*

$$g_k(x_1, x_2, \ldots, x_m) = \sum_{i=1}^{m} w^k_i|L_i(x_1, \ldots, x_m)|,\ \forall\ (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m, k = 1, 2, \ldots, n.$$

where $L_i : \mathbb{R}^m \to \mathbb{R}$ is a linear function for every $i = 1, 2, ..., m$. Then, the region

$$\mathcal{R} = \{x \in [0,1]^m\ :\ g_k(x) \leq \mu_k\ \forall\ k = 1, 2, \ldots, n\}$$

is a convex non-empty polytope.

Proof. Since the linear function L_i does not alter the linearity of the equations obtained by using Corollary 1, it is clear that the region $\mathcal{R}$ is a nonempty polytope. To show that $\mathcal{R}$ is convex, let us pick any $x, y \in \mathcal{R}$. By definition of $\mathcal{R}$, we have $g_k(x) \le \mu_k$ and $g_k(y) \le \mu_k$ for all $k = 1, 2, ..., n$. Now, we observe that for any $t \in [0, 1]$

$$\begin{aligned} g_k((1-t)x+ty) &= \sum_{i=1}^{m} w_i^k |L_i((1-t)x+ty)| \\ &= \sum_{i=1}^{m} w_i^k |(1-t)L_i(x)+tL_i(y)| \\ &\le \sum_{i=1}^{m} w_i^k ((1-t)|L_i(x)|+t|L_i(y)| \\ &= (1-t)g_k(x)+tg_k(y) \le \mu_k. \end{aligned}$$

Therefore, $(1-t)x+ty \in \mathcal{R}$ and, consequently, $\mathcal{R}$ is convex. □

The following result provides the sufficient and necessary conditions for a convex polytope to be contained in a fixed convex set.

Proposition 10. *Let $\mathcal{S} \subset \mathbb{R}^m$ be a convex set, and consider a convex polytope $\mathcal{R} \subset \mathbb{R}^m$. Then $\mathcal{R} \subseteq \mathcal{S}$ if and only if all the vertices of $\mathcal{R}$ belong to $\mathcal{S}$.*

Proof. The proof of the sufficient condition is straightforward. To prove the necessary condition, suppose that all vertices, say $\{v_1, v_2, \dots, v_n\}$, of $\mathcal{R}$ are contained in $\mathcal{S}$. As $\mathcal{R}$ is a convex polytope, any point $x \in \mathcal{R}$ can be expressed as a convex combination of its vertices $x = \sum_{k=1}^{n} a_k v_k$ for certain scalars $a_1, a_2, \dots, a_n$, such that $a_k \ge 0 \ \forall\ k = 1, 2, ..., n$ and $\sum_{k=1}^{n} a_k = 1$. Since $\mathcal{S}$ is convex and $\{v_1, v_2, \dots, v_n\} \in \mathcal{S}$, then $x \in \mathcal{S}$. Hence, $\mathcal{R} \subseteq \mathcal{S}$. □

Note that the constraints which define the feasible region in the models (M-4), (M-5), (M-6), and (M-7) are similar to the hypothesis constraints in Corollary 2. Based on Corollary 2 and Proposition 10, we can establish the containment relationship between regions $\mathcal{R}_\gamma$ and $\mathcal{R}_\varepsilon$ in terms of vertex representation, and that result is outlined in the following theorem.

Theorem 2. *The regions $\mathcal{R}_\gamma^i$ and $\mathcal{R}_\varepsilon^i$ $(i = 4, 5, 6, 7)$ which appear in the models (M-4), (M-5), (M-6) and (M-7) are convex nonempty polytopes. Furthermore, for any of these models, the region $\mathcal{R}_\gamma^i$ is contained in $\mathcal{R}_\varepsilon^i$ if and only if all the vertices of $\mathcal{R}_\gamma^i$ belong to the region $\mathcal{R}_\varepsilon^i$. In the same way, the region $\mathcal{R}_\varepsilon^i$ is contained in $\mathcal{R}_\gamma^i$ if and only if all the vertices in $\mathcal{R}_\varepsilon^i$ belong to the region $\mathcal{R}_\gamma^i$.*

Although Theorem 2 sketches the condition for the containment between the regions $\mathcal{R}_\gamma$ and $\mathcal{R}_\varepsilon$, it does not make any comment on the behavior of the optimal solution of the consensus model. Below, we take a closer look at this issue.

5.2. Existence of solution to (M-G)

In this section, we investigate the existence of a solution to the general optimization model (M-G). To do so, the link between this optimal solution and the region $\mathcal{R}_{\gamma,\varepsilon}$ is analyzed from a theoretical point of view.

Proposition 11. *The regions $\mathcal{R}_{\gamma,\varepsilon}, \mathcal{R}_\gamma$ and $\mathcal{R}_\varepsilon$ are compact subsets of the Euclidean spaces in which they are defined for any values of $\gamma, \varepsilon \in [0, 1]$.*

Proof. Note that all of these regions are contained in the unit hypercube of their respective Euclidean spaces, and therefore they are bounded. In addition, due to the fact that all of them are convex polytopes, they are also closed subsets in their respective Euclidean spaces. □

Proposition 12. *The function $c : [0, 1]^m \to [0, 1]$ defined as $c(x_1, x_2, \dots, x_m) := \sum_{i=1}^{m} c_i |x_i - o_i|$, for all $(x_1, x_2, \dots, x_m) \in [0, 1]^m$, where $o = (o_1, o_2, \dots, o_m) \in [0, 1]^m$ are the original preferences, is convex.*

Proof. Let us consider $x, y \in [0, 1]^m$. Then, for any $\alpha \in [0, 1]$

$$\begin{aligned} c(\alpha x+(1-\alpha)y) &= \sum_{i=1}^{m} c_i |\alpha x_i+(1-\alpha)y_i-o_i| \\ &= \sum_{i=1}^{m} c_i |\alpha x_i+(1-\alpha)y_i-(\alpha o_i+(1-\alpha)o_i)| \\ &\le \sum_{i=1}^{m} c_i |\alpha(x_i-o_i)+(1-\alpha)(y_i-o_i)| \\ &\le \alpha c(x)+(1-\alpha)c(y) \end{aligned}$$

which proves the convexity of c. □

Proposition 13. *(Corollary 2.8.1 in Giorgi, Guerraggio, & (Eds) (2004)) Let $A \subset \mathbb{R}^m$ be a nonempty convex subset and consider a convex function $f : A \to \mathbb{R}$. Then, if f reaches a local minimum at the point $x_0 \in A$, x_0 is also a global minimum for f.*

Based on the above results, we establish a link between the optimal solution and the region $\mathcal{R}_{\gamma,\varepsilon}$ in the following theorem.

Theorem 3. *The model (M-G) always has a solution. If the original preferences $o = (o_1, \dots, o_m)$ satisfy $o \in \mathcal{R}_{\gamma,\varepsilon}$, the solution to the minimization problem is the original preference vector o. Otherwise, the solution for the minimization problem is always obtained in the boundary of the region $\mathcal{R}_{\gamma,\varepsilon}$.*

Proof. Since c is a continuous function defined in the compact subset $\mathcal{R}_{\gamma,\varepsilon}$, it will always reach its maximum and minimum value within the region $\mathcal{R}_{\gamma,\varepsilon}$, so a solution for the (M-G) model always exist.

Suppose that the solution x_0 for the (M-G) model is an interior point of the region $\mathcal{R}_{\gamma,\varepsilon}$. Then, x_0 is a local minimum for the function c and according to Proposition 13 it is a global minimum. But it is obvious that the global minimum of c is the original preferences $o = (o_1, o_2, \dots, o_m)$, so it must be $x_0 = o$. Otherwise, if x_0 is not an interior point, then it must belong to the boundary of $\mathcal{R}_{\gamma,\varepsilon}$. □

So far, we have theoretically established the proper geometry of the regions associated with consensus models and the link of the optimal solution to these regions in light of the convex polytope-based analysis. We will further explore this theoretical foundation to compute more precise bounds for the parameters ε and γ.

5.3. Algorithms used to establish the relationship between γ and ε

In this section, we develop several generic algorithms to find numeric approximations for the bounds of the parameters when one is determined based on the key results of Theorems 2 and 3.

5.3.1. Obtaining ε_1 and γ_1

In this section, we take advantage of the fact that the containment relationship between the $\mathcal{R}_\varepsilon$ and $\mathcal{R}_\gamma$ regions is based on the vertex representation of convex polytopes in order to decisively find the value of the parameters. The idea is to first generate the vertex representation of a region for which the parameter value is given via successive applications of Proposition 9 and the `vertexenum` algorithm. We then successively reduce the value of the parameter we are looking for from the initialized level with a constant step-size until the containment relationship between regions holds. This principle of finding the parameters has been

Algorithm 1: Find $\gamma_1(\varepsilon)$, when ε is given.

Input : ε-related constrains $\mathcal{R}_\gamma$, γ-related constraint $\mathcal{R}_\varepsilon$, threshold $\varepsilon > 0$, step-size $\delta > 0$.
find the linear inequality-based representation of $\mathcal{R}_\varepsilon$ of the form $Az \le \varepsilon$ using Proposition 9;
obtain vertices of $\mathcal{R}_\varepsilon$ as $V = \texttt{vertexenum}(Az \le \varepsilon)$;
initialize: $\gamma_1 = 1$;
while $V \subseteq \mathcal{R}_\gamma$ **do**
  $\gamma := \gamma - \delta$;
end
Return $\gamma_1(\varepsilon) := \gamma$;
Output: $\gamma_1(\varepsilon)$ up to a precision equal to the step-size δ.

Algorithm 2: Find $\varepsilon_1(\gamma)$, when γ is given.

Input : ε-related constrains $\mathcal{R}_\gamma$, γ-related constraint $\mathcal{R}_\varepsilon$, threshold $\gamma > 0$, step-size $\delta > 0$.
find the linear inequality-based representation of $\mathcal{R}_\gamma$ of the form $Az \le \varepsilon$ using Proposition 9;
obtain vertices of $\mathcal{R}_\gamma$ as $V = \texttt{vertexenum}(Az \le \gamma)$;
initialize: $\varepsilon_1 = 1$;
while $V \subseteq \mathcal{R}_\varepsilon$ **do**
  $\varepsilon := \varepsilon - \delta$;
end
Return $\varepsilon_1(\gamma) := \varepsilon$;
Output: $\varepsilon_1(\gamma)$ up to a precision equal to the step-size δ.

summarized in Algorithms 1–2. Theorem 2 guarantees the proper functioning of the algorithms.

If these algorithms are applied, for each parameter ε and γ the guaranteed value of the other parameter, i.e., γ_1 and ε_1, is obtained. Note that the algorithms estimate ε_1 and γ_1 with the maximum error bounded by $\delta > 0$. Depending on the precision requirement, the value of $\delta > 0$ may be adjusted. This algorithm provides a clear answer to RQ1. As long as the containment relationship is maintained, the region associated with one parameter becomes redundant and there is no change to the optimal solution. Now the question is: What parameter values would make changes to the optimal solution? We attempt to answer this question below by developing algorithms to find such parameter values using Theorem 3.

5.3.2. Obtaining ε_2 and γ_2

To develop the algorithms of this subsection, we have taken into account that Theorem 3 guarantees that the solution of the minimization problem is always reached in the boundary of $\mathcal{R}_{\gamma,\varepsilon}$ whenever the solution is not trivial. For example, if γ is fixed, when ε is decreased so that none of the vertices of $\mathcal{R}_\varepsilon$ belong to the boundary of $\mathcal{R}_\gamma$ (with the exception of those vertices like $(0, 0, \ldots, 0)$, which always satisfy the constraints), we can ensure that the solution of the minimization problem will change. Based on this idea, we summarize the steps for computing ε_2 and γ_2 in Algorithms 3–4.

When using these algorithms, for each parameter ε or γ the guaranteed value of the other parameter, resp. γ_2 and ε_2, is obtained. Again, the error is given by the value $\delta > 0$. These algorithms give the answer to RQ2. We propose a generalization of these algorithms below for the more generic case of the (M-G) optimization model.

Remark 3. It must be highlighted that the output of these algorithms does not depend on the values of the preferences given by the experts, but on the weights that have been assigned to them. In other words, if such weights are fixed, we can determine the relationship between γ and ε and, consequently, whether it is possible to simplify the CMCC model for any of their preference values by only running the algorithm once.

Algorithm 3: Find $\gamma_2(\varepsilon)$, when ε is given.

Input : ε-related constrains $\mathcal{R}_\gamma$, γ-related constraint $\mathcal{R}_\varepsilon$, threshold $\varepsilon > 0$, step-size $\delta > 0$.
find the linear inequality-based representation of $\mathcal{R}_\varepsilon$ of the form $Az \le \varepsilon$ using Proposition 9;
obtain vertices of $\mathcal{R}_\varepsilon$ as $V = \texttt{vertexenum}(Az \le \varepsilon)$
construct refinement of V, say, V' by removing vertices of V which automatically satisfy γ constraints;
initialize: $\gamma = 1$;
while $V' \cap \mathcal{R}_\gamma \neq \emptyset$ **do**
  $\gamma := \gamma - \delta$
end
Return $\gamma_2(\varepsilon) := \gamma$;
Output: $\gamma_2(\varepsilon)$ up to a precision equal to the step-size δ.

Algorithm 4: Find $\varepsilon_2(\gamma)$, when γ is given.

Input : ε-related constrains $\mathcal{R}_\gamma$, γ-related constraint $\mathcal{R}_\varepsilon$, threshold $\gamma > 0$, step-size $\delta > 0$.
find the linear inequality-based representation of $\mathcal{R}_\gamma$ of the form $Az \le \gamma$ using Proposition 9;
obtain vertices of $\mathcal{R}_\gamma$ as $V = \texttt{vertexenum}(Az \le \gamma)$
construct refinement of V, say, V' by removing vertices of V which automatically satisfy ε constraints;
initialize: $\varepsilon = 1$;
while $V' \cap \mathcal{R}_\varepsilon \neq \emptyset$ **do**
  $\varepsilon := \varepsilon - \delta$;
end
Return $\varepsilon_2(\gamma) := \varepsilon$;
Output: $\varepsilon_2(\gamma)$ up to a precision equal to the step-size δ.

5.3.3. Generalizing the algorithm

To end this section, a generalization of this algorithm is provided to compare the constraints that define the feasible region of a minimization problem.

Theorem 4. *Let $\mathcal{P} \subset \mathbb{R}^m$ be a convex polytope in $\mathbb{R}^m$ and consider a continuous convex function $c : \mathcal{P} \to \mathbb{R}$. Consider two compact intervals I_1 and I_2 in $\mathbb{R}$ and the linear functions $g^1_k : \mathbb{R}^m \to \mathbb{R}$, for $k = 1, 2, \ldots, n_1$ and $g^2_k : \mathbb{R}^m \to \mathbb{R}$ for $k = 1, 2, \ldots, n_2$. For $\alpha \in I_1$ and $\beta \in I_2$ define the polytopes*

$$\mathcal{P}_\alpha : = \left\{ x \in \mathcal{P} \;:\; g^1_k(x) \le \alpha \;\forall\; k = 1, 2, \ldots, n_1 \right\},$$
$$\mathcal{P}_\beta : = \left\{ x \in \mathcal{P} \;:\; g^2_k(x) \le \beta \;\forall\; k = 1, 2, \ldots, n_2 \right\}.$$

Then:

- *The optimization problem*

$$\min_{x \in \mathcal{P}} \{c(x)\}$$
$$\text{s.t.} \begin{cases} x \in \mathcal{P}_\alpha \\ x \in \mathcal{P}_\beta \end{cases}$$

  always has a solution.
- *For any values $\alpha \in I_1$ and $\beta \in I_2$, $\mathcal{P}_\alpha \subseteq \mathcal{P}_\beta$ if and only if the vertices set of the polytope $\mathcal{P}_\alpha$ is contained in the polytope $\mathcal{P}_\beta$*

Given the hypotheses of the previous theorem, Algorithms 5 and 6 determine respectively when the feasible region changes and when the solution changes.

Algorithm 5: Algorithm for obtaining $\beta_1(\alpha)$, the infimum value of β such that $\mathcal{P}_\alpha \subseteq \mathcal{P}_\beta$.

Input: Problem defined in Theorem 4, a fixed $\alpha \in I_1$ and step-size $\delta > 0$.
find the linear inequality-based representation of $\mathcal{P}_\alpha$ of the form $Az \leq \alpha$ using Proposition 9;
obtain vertices of $\mathcal{P}_\alpha$ as $V = \texttt{vertexenum}(Az \leq \alpha)$;
Initialize $\beta := \max I_2$;
while $V \subset \mathcal{P}_\beta$ **do**
  $\beta := \beta - \delta$
end
Return $\beta_1(\alpha) := \beta$;
Output: $\beta_1(\alpha)$ up to a precision equal to the step-size δ.

Algorithm 6: Algorithm for obtaining $\beta_2(\alpha)$, the supremum value of β such that the solution of the optimization problem has changed.

Input: Problem defined in Theorem 4, a fixed $\alpha \in I_1$ and step size $\delta > 0$.
find the linear of the form representation of $\mathcal{P}_\alpha$ of the form $Az \leq \alpha$ using Proposition 9;
obtain vertices of $\mathcal{P}_\alpha$ as $V = \texttt{vertexenum}(Az \leq \alpha)$
construct refinement of V, say, V' by removing vertices of V which automatically satisfy β constraints;
initialize: $\beta = 1$;
while $V' \cap Bound^*(\mathcal{P}_\beta) \neq \emptyset$ **do**
  $\beta := \beta - \delta$;
end
Return $\beta_2(\alpha) := \beta$;
Output: $\beta_2(\alpha)$ up to a precision equal to the step-size δ.

6. Illustrative examples

In this section, a couple of examples are proposed to show the performance of the polytope-based algorithms developed in the previous section.

6.1. Example 1: Labella et al. (2020) GDM problem

The first illustrative example is related to one of the GDM problems with five experts provided by Labella et al. (2020) when they defined CMCC models for the first time, which has motivated this study.

Labella et al. considered a set of experts $E = \{e_1, e_2, e_3, e_4, e_5\}$ whose weights were

$$W = (0.375, 0.1875, 0.25, 0.0625, 0.125),$$

and their preferences were $(o_1, o_2, o_3, o_4, o_5) = (0, 0.09, 0.36, 0.45, 1)$. The values of the costs of moving experts' opinions were established as $(c_1, c_2, c_3, c_4, c_5) = (6, 3, 4, 1, 2)$. The values obtained by Labella et al. (2020) for the cost function for different values of ε and γ when using the model (M-5) are summarized in Table 1.

Note that Table 1 clearly suggests that the relationship between the parameters γ and ε exists. For instance, if we compare the values of the cost function for $\gamma = 0.1$ and the different values of ε, we can see that all of them, except the one obtained when $\varepsilon = 0.05$, are the same.

When CMCC models were first proposed, the authors were unable to provide any explanation for this phenomenon. In fact, existing studies do not offer any insight into the relationship of these parameters, how it influences the minimum consensus cost, and how one parameter could be computed when the other is given. Without such knowledge of the proper dynamics of CMCC models, the moderator, in practice, is forced to choose the other parameter on a hit and trial basis and cannot take advantage of computational cost reduction, which results in inefficiency when conducting the CRP. However, using the results of our study and the developed algorithms, it is not only possible to explain the rationale behind the behavior of the cost function, but we are also able to get a complete picture of how different parameter configurations will impact the cost of reaching consensus, which can help the moderator manage the CRP more efficiently. To explore this fact, we assume here that the parameter γ is given at the beginning for the problem described above, and the moderator wants to know how the values of the parameter ε impact the cost of solving the CMCC model. Therefore, we are going to use the results of our proposal to obtain the different bounds of the parameter ε for different values of γ (see Table 2). Specifically, the following values are collected:

Table 1
The costs with different values of ε and γ of (M-5) in Example 1.

	$\gamma = 0.3$	$\gamma = 0.25$	$\gamma = 0.2$	$\gamma = 0.15$	$\gamma = 0.1$	$\gamma = 0.05$
$\varepsilon = 0.30$	1.01	1.20	1.60	2.14	2.69	3.24
$\varepsilon = 0.25$	1.13	1.20	1.60	2.14	2.69	3.24
$\varepsilon = 0.20$	1.32	1.32	1.60	2.14	2.69	3.24
$\varepsilon = 0.15$	1.81	1.81	1.81	2.14	2.69	3.24
$\varepsilon = 0.10$	2.47	2.47	2.47	2.47	2.69	3.24
$\varepsilon = 0.05$	3.13	3.13	3.13	3.13	3.13	3.24

Table 2
Different bounds for ε when γ is fixed in (M-5) in Example 1.

	$\gamma = 0.3$	$\gamma = 0.25$	$\gamma = 0.2$	$\gamma = 0.15$	$\gamma = 0.1$	$\gamma = 0.05$
$\varepsilon_1'(\gamma)$	1.2	1	0.8	0.6	0.4	0.2
$\varepsilon_1(\gamma)$	0.94	0.79	0.64	0.48	0.32	0.16
$\varepsilon_2(\gamma)$	0.23	0.19	0.15	0.11	0.07	0.03

- $\varepsilon_1'(\gamma)$: The upper bound for ε_1 provided by the results presented in Section 4, which are based on inequalities (Proposition 4). This value represents an easy-to-compute rough bound for which the constraints provided by ε in the optimization problem are guaranteed by the γ condition.
- $\varepsilon_1(\gamma)$: An estimate for ε_1 (up to a precision $\delta = 0.01$) calculated using the algorithms in Section 5 (Algorithm 2). Although this threshold also stands for a bound for the values of ε whose constraints are redundant with the one related to γ, this value is more accurate than the previous one.
- $\varepsilon_2(\gamma)$: An estimation for ε_2 (up to a precision $\delta = 0.01$) computed using the algorithms in Section 5 (Algorithm 4). This threshold for ε indicates when the feasible region related to ε is strictly contained in the one related to γ. In other words, if the moderator chooses a value of ε lower than this bound, the γ constraint is always guaranteed and the DMs will move their opinions just to satisfy the ε conditions.

If we look at the column $\gamma = 0.1$, we can deduce that the feasible region will not change until $\varepsilon < \varepsilon_1(0.1) = 0.32$ and we cannot ensure that the solution of the optimization problem changes until $\varepsilon < \varepsilon_2(0.1) = 0.07$. In fact, Table 1 shows that the values of the cost function are the same for all $\varepsilon > 0.07$.

Furthermore, the inequality-based approach reveals that for $\varepsilon > \varepsilon_1'(0.1) = 0.4$ the constraints related to the distance between experts' opinions and collective opinions are guaranteed by the constraint corresponding to the parameter γ and therefore they could be omitted in the resolution of the mathematical programming model, resulting in an immediate reduction of the computational costs. In fact, if more precision was necessary, the polytope-based

Table 3
The costs with different values of ε and γ of (M-5) in Example 2.

	$\gamma = 0.3$	$\gamma = 0.25$	$\gamma = 0.2$	$\gamma = 0.15$	$\gamma = 0.1$	$\gamma = 0.05$
$\varepsilon = 0.30$	0.19	0.26	0.32	0.39	0.46	0.53
$\varepsilon = 0.25$	0.19	0.26	0.32	0.39	0.46	0.53
$\varepsilon = 0.20$	0.21	0.26	0.32	0.39	0.46	0.53
$\varepsilon = 0.15$	0.30	0.30	0.33	0.39	0.46	0.53
$\varepsilon = 0.10$	0.40	0.40	0.40	0.40	0.46	0.53
$\varepsilon = 0.05$	0.50	0.50	0.50	0.50	0.50	0.53

Table 4
Different bounds for ε when γ is fixed in (M-5) in Example 2.

	$\gamma = 0.3$	$\gamma = 0.25$	$\gamma = 0.2$	$\gamma = 0.15$	$\gamma = 0.1$	$\gamma = 0.05$
$\varepsilon_1'(\gamma)$	0.60	0.50	0.40	0.30	0.20	0.10
$\varepsilon_1(\gamma)$	0.55	0.46	0.37	0.28	0.19	0.10
$\varepsilon_2(\gamma)$	0.19	0.16	0.13	0.09	0.06	0.03

algorithm guarantees that it is possible to ignore all the constraints obtained when $\varepsilon > \varepsilon_1(0.1) = 0.32$. In other words, if the moderator considers that a maximum distance between DMs and the collective equal to 0.35 is sufficient when $\gamma = 0.1$, all the constraints related to ε could be erased from the optimization model when implementing it in a numerical solver because they are already granted by the consensus constraint. In fact, we have used the Clp solver with the Julia programming language (Bezanson, Edelman, Karpinski, & Shah, 2017) to implement the linearized version of (M-5) with and without ε constraints when $\varepsilon = 0.3$ and considering the threshold $\gamma = 0.1$. While the first model takes around 300 milliseconds to be solved, the second one only needs around 200 milliseconds, yet both provide the same solution as guaranteed by our algorithms.

Similarly, since for $\varepsilon < 0.07 = \varepsilon_2'(0.1)$ the feasible region related to ε is strictly contained in the one determined by $\gamma = 0.1$, the constraints associated with the latter parameter could be omitted in the resolution of the model. In this case, we have also solved the optimization model in both scenarios: considering both $\gamma = 0.1$ and $\varepsilon = 0.05$ constraints and only taking into account the $\varepsilon = 0.05$ constraints. As before, deleting the $\gamma = 0.1$ constraints has also implied accelerating the model by about 50%.

6.2. Example 2: The 3-dimensional GDM problem

This second GDM situation aims at explaining the relationship between the parameters ε and γ from a geometrical point of view. To do so, a consensus process involving three experts has been studied. Note that if three experts are considered, their preferences can be represented as a point in a 3-dimensional space (o_1, o_2, o_3). This makes it possible to provide a graphical visualization of the relationship between the parameters γ and ε.

In this case, we consider a set of experts $E = \{e_1, e_2, e_3\}$ whose weights are given by $W = (0.5, 0.05, 0.45)$. Their initial preferences are $(o_1, o_2, o_3) = (0.3, 0.6, 0.9)$ and the cost vector is $(c_1, c_2, c_3) = (1, 1, 1)$. The optimization problem (M-5) has been solved for different values of ε and γ, and the values obtained for the cost function have been compiled in Table 3.

Let us compare these results with the bounds obtained by the algorithms proposed in this paper. To do so, we have compiled in Table 4 the different bounds for ε obtained when γ is fixed and the step-size $\delta = 0.01$ is considered.

We provided a graphical analysis of these values by taking advantage of the 3-dimensional nature of this decision problem below. Let us fix $\gamma = 0.2$. According to Table 4, the region will not change for $\varepsilon \in [0.4, 1]$, and we are sure that it changes when $\varepsilon < 0.37$. Fig. 3, which has been plotted using the Mathematica software, shows the evolution of the corresponding feasible region

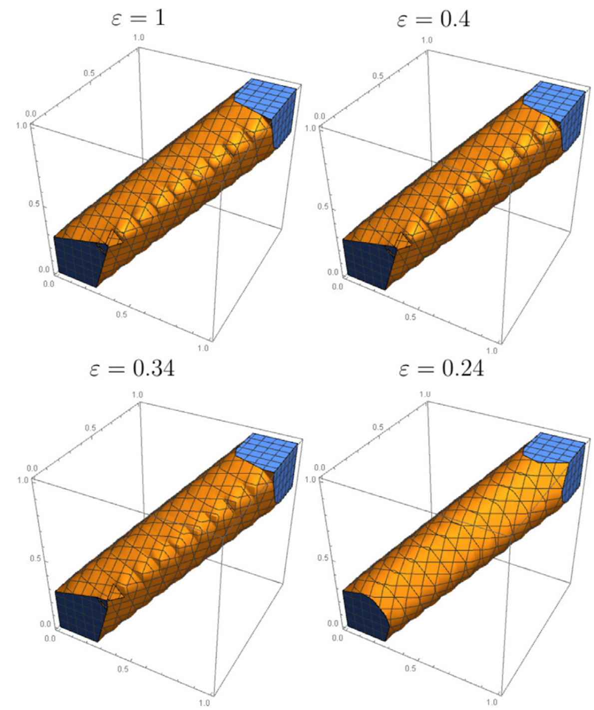

Fig. 3. The change of the region $\mathcal{R}_{\gamma,\varepsilon}$ for $\gamma = 0.2$ in Example 2.

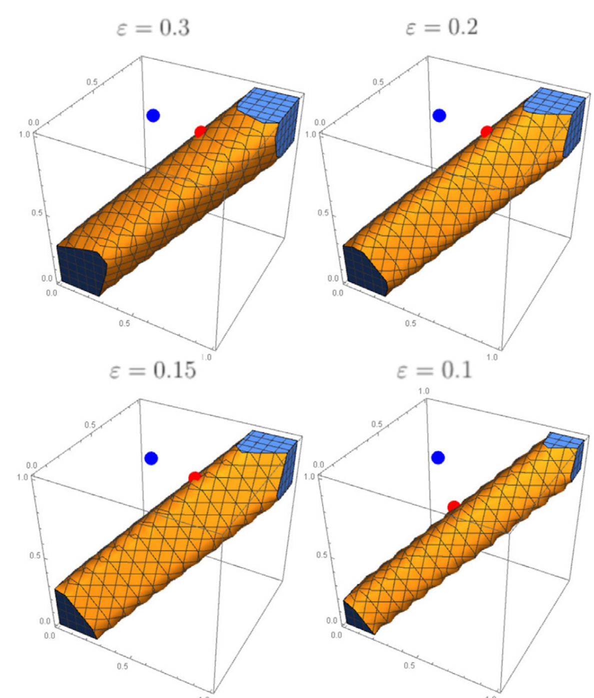

Fig. 4. The change of the region $\mathcal{R}_{\gamma,\varepsilon}$ and its solutions for $\gamma = 0.2$ in Example 2. The blue dot represents the original preferences and the red dot the solution for the given value of ε. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

$\mathcal{R}_{0.2,\varepsilon}$ for some values of ε. Note that the plots obtained for $\varepsilon = 1$ and $\varepsilon = 0.4$ are the same. However, the one obtained for $\varepsilon = 0.34$ is slightly different from the previous plots, and the plot corresponding to $\varepsilon = 0.24$ is completely different.

However, a change in the region does not necessarily imply a change in the solution. Fig. 4 shows the feasible region $\mathcal{R}_{0.2,\varepsilon}$, the position of the original preferences (blue dot), and the correspond-

Table 5
Computational time required for each GDM problem.

Decision-makers	5	10	50	100
$M-7$	10.0 ms	46.0 ms	16947.0 ms	480451.0 ms
$M-7'$	3.0 ms	7.0 ms	132.0 ms	531.0 ms

ing modified preferences obtained by solving the CMCC model (red dot). According to Table 4, we cannot guarantee that the solution will change until $\varepsilon < 0.13$, and this is exactly what the figure shows. Note that even though the values of ε are lower than $\varepsilon_1(0.2) = 0.37$ and we know that the feasible region is changing, the solution of the optimization problem is the same for $\varepsilon = 0.3$, $\varepsilon = 0.2$ and $\varepsilon = 0.15$. In contrast, the solution obtained for $\varepsilon = 0.1 < \varepsilon_2(\gamma) = 0.13$ changes.

6.3. Example 3: Simplifying CMCC in large-scale GDM

The following example is intended to show how the appropriate use of the relationship between the parameters ε and γ can be applied when solving a GDM problem to considerably improve computational time costs.

Let us consider a GDM problem which is going to be resolved by using the consensus model M-7. For a value $\varepsilon > 0$, Proposition 7 guarantees that, for any $\gamma \geq 2\varepsilon$, the model $M-7$ is equivalent to the following simplified version, which does not depend on the γ constraint:

$$\min_{(x_{ij}^k)\in\mathcal{M}_{n\times n}([0,1])} \sum_{k=1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} c_k|x_{ij}^k - p_{ij}^k|$$
$$s.t. \begin{cases} \overline{x}_{ij} = \sum_{k=1}^{m} w_k x_{ij}^k \\ |x_{ij}^k - \overline{x}_{ij}| \leq \varepsilon, k = 1,\ldots,m, i = 1,\ldots,n-1, j = i+1,\ldots,n \end{cases} \tag{M-7$'$}$$

Note that, even though this model does not require the γ constraint, any solution for M-7′ ($\varepsilon > 0$) is also a solution for M-7 ($\varepsilon > 0$, $\gamma \geq 2\varepsilon$) and vice-versa.

To analyze the computational cost of using M-7 or M-7′, we implemented their respective linearized versions (Rodríguez et al., 2021) in the Julia 1.6 programming language (Bezanson et al., 2017) on the Google Colaboratory cloud service (Bisong, 2019) (2.20GHz Intel(R) Xeon(R) CPU and 13 GB RAM).

Four GDM problems were solved, each considering a different number of experts. For these scenarios, the experts' preferences were randomly generated, and the consensus parameters were set to $\varepsilon = 0.1, \gamma = 0.2$. We obtained the consensus solution to the different GDM situations for both M-7 and M-7′ by using the solver *GLPK* (GNU Linear Programming Kit) and we then measured the time required for the solver to compute the optimal solution in milliseconds (ms) (see Table 5).

The results show that the resolution of the M-7 model is much more time-consuming than the resolution of its modified version M-7′. While the former considerably increases the time cost when the number of experts also increases, the latter requires less than a second to provide a solution to optimization problems with the same characteristics as the former. In this case, understanding of the relationship between the parameters γ and ε has been key to shortening the computational time when dealing with many decision-makers.

7. Conclusions

CRPs, specifically MCC models, have been widely applied to GDM problems to achieve consensus solutions. Classical CRPs and MCC models fix different constraints using parameters specifically associated with the consensus level ($\gamma = 1 - \mu$) and the absolute distance (ε), to obtain a more general consensus solution. CMCC models include both types of constraints to obtain the agreed solution for the GDM problems. Nevertheless, the values of these parameters in CMCC have been traditionally fixed a priori by the moderator of the decision process according to the desired level of agreement among the stakeholders. However, our study shows that some configurations of such parameters may imply redundant constraints in the mathematical programming model, which, in practice, leads to higher computational costs. In this regard, our proposal defines a method to identify for which parameter pairs the optimization model can be simplified by removing such redundant inequalities. Understanding the proper dynamics of these parameters and corresponding constraints over the optimal consensus solution is key to applying these models in real-life large-scale group decision-making scenarios.

This paper has analyzed the relationship between both parameters γ and ε, and the associated constraints that determine the feasible region of the CMCC models. The use of inequalities for this analysis provides simple and straightforward relationships that are not very precise, but the analysis based on polytopes has provided a novel algorithm that relates the parameters of the linear constraints of any convex optimization problem, providing an accurate relationship between them. Based on the point of view of decision support in CMCC-based CRPs, the proposed algorithms could help the moderator set these parameters when one is given, and provide information on the impact of different configurations of these parameters in optimal solutions. Furthermore, understanding the relationship between these parameters also implies an immediate computational improvement because it allows redundant inequalities to be neglected and thus a better performance is achieved.

In the future, it would be interesting to analyze CMCC models that not only consider the minimization of the cost of changing opinions, but also attempt to minimize the number of opinion changes in light of the machinery developed in this work. Furthermore, investigations could be oriented to apply this algorithm to other optimization problems. For example, it would be interesting to develop a deeper analysis of the computational impact of removing the corresponding constraints when dealing with large-scale GDM problems where hundreds or thousands of DMs are considered. In this regard, such research could also highlight any differences when applying our algorithms to different preferences structures, such as utility vectors or FPRs.

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Supplementary material

Supplementary material associated with this article can be found, in the online version, at doi:10.1016/j.ejor.2022.08.015.

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