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# A large scale consensus reaching process managing group hesitation

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ABSTRACT

Nowadays due to the social networks and the technological development, large-scale group decision making (LS-GDM) problems are fairly common and decisions that may affect to lots of people or even the society are better accepted and more appreciated if they agreed. For this reason, consensus reaching processes (CRPs) have attracted researchers attention. Although, CRPs have been usually applied to GDM problems with a few experts, they are even more important for LS-GDM, because differences among a big number of experts are higher and achieving agreed solutions is much more complex. Therefore, it is necessary to face some challenges in LS-GDM. This paper presents a new adaptive CRP model to deal with LS-GDM which includes: (i) a clustering process to weight experts' sub-groups taking into account their size and cohesion, (ii) it uses hesitant fuzzy sets to fuse expert's sub-group preferences to keep as much information as possible and (iii) it defines an adaptive feedback process that generates advice depending on the consensus level achieved to reduce the time and supervision costs of the CRP. Additionally, the proposed model is implemented and integrated in an intelligent CRP support system, so-called AFRYCA 2.0 to carry out this new CRP on a case study and compare it with existing models.

## 1. Introduction

A recent and challenging problem in the decision making field, driven by the current technological developments (social networks, P2P) and societal demands (e-group shopping, group marketing), is the engagement of a large number of people in different decision problems. Consequently, large-scale group decision making (LS-GDM) is becoming an important topic in the decision making field [26–28,47]. Unlike classical GDM problems in which a decision framework with a few number of experts is assumed, LS-GDM problems deal with a large number of experts (in [10] was pointed out more than 20 experts, but here we may assume several hundreds even thousands). This situation implies new challenges pointed out in previous researches in this topic [24,33,35], such as: (i) Scalability, (ii) Time cost, (iii) Constant preference supervision, iv) Stronger disagreement positions, v) Difficulties to understand/visualize current state of agreement, etc.

The study of LS-GDM has been mainly focused on four major topics:

- Clustering methods in LS-GDM [26,53].
- Consensus reaching processes in LS-GDM [34,48,49].
- LS-GDM methods [27,28].
- LS-GDM support systems [8,35].

Due to the fact that, consensual decisions for conflicting problems that may affect groups of people are better adopted and much more appreciated [13], the study and development of consensus reaching processes (CRPs) for GDM has been then a fruitful, interesting and necessary area of research in recent years [16,33,36]. However, most of results presented in this area are focused on GDM problems assuming just a few number of experts involved in the decision process. Notwithstanding, in LS-GDM this type of process seems to be even more important, because opinions among a larger number of people tend to be easily controversial and conflicting. Main shortcomings of classical CRPs when they are applied to LS-GDM problems have been identified [24] and initial CRP proposals for LS-GDM do not have overcame these shortcomings yet [34,49].

In light of the multiple challenges and shortcomings of classical CRPs for LS-GDM problems [24], this paper introduces a new *adaptive* CRP model for LS-GDM to overcome *scalability* problems and *experts' preference supervision* that is highly related to *time cost*. Therefore, to achieve these goals, our proposal incorporates to the CRP applied to LS-GDM the following novelties:

- *Clustering process for weighting experts' sub-groups*: the large number of experts in the LS-GDM problem are clustered into sub-groups according to their preferences and the *importance* of each sub-group

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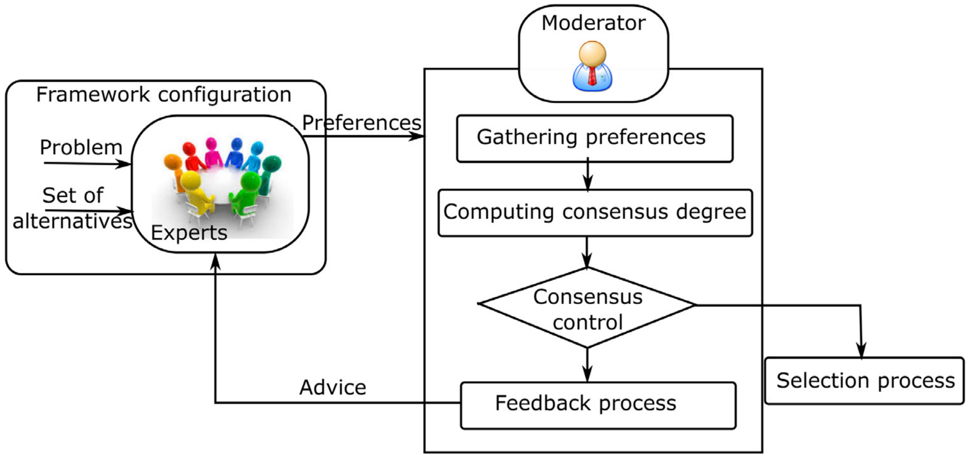


**Fig. 1.** General scheme of a consensus reaching process.

in the CRP that is computed considering two features such as its ***size*** and its ***cohesion***.

- *Grouping Opinions*: so far, most of CRPs aggregate experts' preferences from early stages of the process, the aggregation may result in a loss of important features of the information, such as distribution or shape [18]. In order to avoid such situations, our proposal will model experts' sub-group preferences by means of hesitant fuzzy sets (HFS), introduced by Torra [45] for representing the expert's hesitation to assign a degree of membership in a fuzzy set; so, it will be assumed that experts' preferences in a sub-group represent the group hesitation to express its fuzzy preference [9,39,52].
- *Adaptive feedback process*: last but not least, the negotiation process in a CRP is usually driven by a feedback mechanism [31] that is often time consuming even more in LS-GDM [33]; therefore, our proposal develops a new adaptive feedback mechanism process that guides the consensus process according to the level of agreement achieved by softening experts' preference supervision and reducing the time cost of the CRP.

Finally, the proposed CRP is implemented and integrated in the intelligent CRP support system so-called AFRYCA 2.0 [23,33] to compute the results of the case study, visualize the CRP and carry out a comparison with other consensus models.

The remainder of this paper is structured as follows: Section 2 revises some preliminary concepts about LS-GDM problems, CRPs and hesitant fuzzy information. Section 3 presents a novel adaptive consensus model based on clustering and hesitant fuzzy information to deal with LS-GDM problems. Section 4 introduces a case study to show the utility and applicability of the proposed model using an intelligent CRP support system and presents a comparison with other models. Finally, in Section 5 some concluding remarks are pointed out.

## 2. Preliminaries

This section revises different concepts about LS-GDM, CRPs and HFS that will be used in the proposed consensus model for LS-GDM.

### 2.1. Large-Scale group decision making

Even though the concept of GDM has been widely studied in decision theory [4,6,19,29], recently the concept of LS-GDM has risen because of the societal demand of involving crowds in important decision processes [12,13] that is facilitated by current technologies and tools [43]. Hence, the concept of LS-GDM is quite similar to GDM, but differs because in the former the number of experts eliciting their preferences on a set of alternatives, $X$, is much greater than in the latter. Formally, a LS-GDM problem consists of: (i) a set of alternatives $X=\{x_1, \ldots, x_n\}$, $(n \geq 2)$, which can be selected as possible solutions for the problem, and (ii) a set of experts $E=\{e_1, \ldots, e_m\}$, $(m > > n)$, who express their judgements on the set of alternatives $X$. Fuzzy preference relations [32], $P=(p_{ij})_{n\times n} \subset X \times X$, $p_{ij} \in [0, 1]$, are a common structure for eliciting preferences in both types of group decision problems [7].

Due to its similar structure, LS-GDM can be solved by a selection process similar to the one used in GDM [41] with an aggregation and exploitation phase. In such a case, this selection process does not always guarantee that the solution obtained would be accepted by all experts involved in the decision problem, because several of them might consider that their opinions were not taken sufficiently into account [42]. A usual solution to overcome this drawback and obtain agreed decisions accepted by the whole group is the application of a CRP [5,46]. In spite of the existence of different interpretations of consensus [30], in this paper it is understood as "*a state of mutual agreement among members of a group in which the decision made satisfies all of them*" [42]. Usually, achieving a consensus requires that experts modify their preferences bringing them closer to each other toward a collective opinion which is satisfactory for all of them [19,37].

A consensus process is an iterative and dynamic discussion process that can be carried out in different ways, Palomares et al. introduced in [33] a deep revision and a taxonomy of the different types of models for performing it, and a general scheme of a CRP sketched in Fig. 1 that is briefly described below:

- *Framework configuration*: it sets up the GDM problem determining the set of alternatives, the set of experts engaged in the decision making and fixing the consensus threshold to reach.
- *Gathering preferences*: the preferences provided by experts are gathered.
- *Computing the consensus degree*: by using a consensus measure [17] which is based on distance measures and aggregation operators [2,15]. This degree reflects the level of agreement in the group.
- *Consensus control*: if the obtained consensus degree is greater than the consensus threshold, a selection process is applied, otherwise more discussion rounds are required.
- *Feedback process*: the preferences causing disagreement are identified and advice is generated to guide experts how to modify their preferences and make them closer. Afterwards, another round starts by gathering preferences again.

In order to cope with the necessity of achieving agreed solutions in LS-GDM problems, several proposals have been introduced in the literature. Palomares et al. [34] proposed a consensus model to detect and manage non-cooperative behaviors and developed a visual tool based on self-organizing maps to facilitate the monitoring of the process performance [35]. Taking into account such a model, Xu et al. [49] proposed a consensus model for multi-criteria LS-GDM dealing with emergency problems that considers non-cooperative behaviors and minority opinions. Quesada et al. [38] introduced a weighting method for CRPs dealing with LS-GDM which includes the use of uninorm aggregation operators to compute experts weights taking into account their behaviors.

Previous proposals aggregate the experts' preferences in early stages of the decision process that may imply disregarding important information [18] and not considering the different levels of agreement across the CRP that can provoke a high time cost due to a greater experts' preference supervision during the feedback and discussion processes.

Therefore, to overcome these drawbacks our proposed model first, will include an approach to detect and weight sub-groups. Second, to keep as much information and avoid the loss of information, the sub-groups preferences will be fused by using HFS instead of aggregating them. Finally, a new adaptive feedback process based on previous inputs will be defined.

### 2.2. Hesitant information

The concepts of HFS and hesitant fuzzy preference relation have been widely applied to decision making [39], in this section these are briefly reviewed to facilitate the understanding of their use in our proposal for modelling experts sub-group preferences in order to keep as much information as possible during the proposed CRP.

HFSs [45] are an extension of fuzzy sets with the aim at modelling the uncertainty provoked by the doubt that an expert can have when she/he wants to assign the membership degree of an element in a fuzzy set. A HFS allows assigning several membership degrees of an element to a fuzzy set. Formally, a HFS is defined in terms of a function that obtains a set of membership degrees for each element in the domain.

**Definition 1** (*[45]*). Let X be a reference set, a HFS on X is a function $\mathfrak{h}$ that returns a subset of values in [0,1]:

$$\mathfrak{h}: X \to \wp([0,1]) \tag{1}$$

Previous definition was completed with the following mathematical representation of a HFS:

$$A = \{\langle x, h_A(x)\rangle : x \in X\},$$

where $h_A(x)$ is called Hesitant Fuzzy Element (HFE) that is a set of some values in [0,1], denoting the possible membership degrees of the element $x \in X$ to the set $A$. A HFS can also be seen as a mapping of HFEs, one for each element in the reference set. Therefore, if $h(x)$ is the HFE associated to $x$, $\cup_{x\in X} h(x)$ is then a HFS.

By using the concepts of fuzzy preference relation and HFS, the concept of Hesitant Fuzzy Preference Relation (HFPR) was proposed [56].

**Definition 2** (*[56]*). Let $X$ be a reference set, a HFPR on $X$ is represented by a matrix $H = (h_{ij})_{n\times n} \subset X \times X$, where $h_{ij} = \left\{ p_{ij}^s \middle| s = 1, 2, \ldots, \#h_{ij} \right\}$ ($\#h_{ij}$ is the number of elements in $h_{ij}$) is a HFE that indicates the membership degrees that denote to which extent $x_i$ is preferred to $x_j$. Additionally, $h_{ij}$ should satisfy the following conditions: $p_{ij}^{\sigma(s)} + p_{ji}^{\sigma'(s)} = 1$, $p_{ii} = \{0.5\}$, $\#h_{ij} = \#h_{ji}$, $i, j = \{1, 2, \ldots, n\}$ $p_{ij}^{\sigma(s)} < p_{ij}^{\sigma(s+1)}$, $p_{ji}^{\sigma'(s+1)} < p_{ji}^{\sigma'(s)}$, where $\{\sigma(1), \ldots, \sigma(\#h_{ij})\}$ is a permutation of $\{1, \ldots, \#h_{ij}\}$, i.e., $p_{ij}^{\sigma(s)}$ is the smallest element in $h_{ij}$, and $\{\sigma'(1), \ldots, \sigma'(\#h_{ji})\}$ is a permutation of $\{1, \ldots, \#h_{ji}\}$, i.e., $p_{ji}^{\sigma'(s)}$ is the largest element in $h_{ji}$.

During the CRP in LS-GDM with HFSs, it might happen that the cardinality of HFEs in HFPRs would be different, i.e, $h_{ij}^a \in H^a = (h_{ij}^a)_{n\times n}$ and $h_{ij}^b \in H^b = (h_{ij}^b)_{n\times n}$ with $\#h_{ij}^a \neq \#h_{ij}^b$ (e.g., $\#h_{ij}^a < \#h_{ij}^b$). In such a case, it is necessary to normalize the $h_{ij}^a$ with smaller cardinality until both have the same cardinality to operate correctly between them. Xu and Zhang [51] proposed the $\beta$-normalization, based on the optimization parameter, $\eta$.

**Definition 3** (*[51]*). Let $h_i$ be the HFE with the smaller cardinality and $h_i^- = min\{\gamma|\gamma \in h_i\}$ and $h_i^+ = max\{\gamma|\gamma \in h_i\}$, then the value $\gamma'$ to add in the HFE $h_i$, is computed as:

$$\gamma' = \eta h_i^+ + (1-\eta)h_i^-, \tag{2}$$

where $\eta(0 \le \eta \le 1)$.

The value of the optimization parameter, $\eta$, relies on experts' risk attitudes. If $\eta = 1$, the value added is $\gamma' = h_i^+$, which indicates an optimistic point of view; if $\eta = 0$, the value added is $\gamma' = h_i^-$, which indicates a pessimistic point of view; and if $\eta = 1/2$, then $\gamma' = 1/2(h_i^+ + h_i^-)$, which means that expert is neutral. Consequently, by using $\eta$, *if* $\#h_{ij}^a < \#h_{ij}^b$ the HFPR, $H^a$, is normalized as:

**Definition 4** (*[55]*). Let $H^a = (h_{ij}^a)_{n\times n} \subset X \times X$ be a HFPR and $\eta\,(0 \le \eta \le 1)$ an optimization parameter to add values into $h_{ij}^a (i<j)$, moreover $1-\eta$ is used to add values into $h_{ji}^a (i<j)$, a normalized HFPR $\overline{H}^a = (\overline{h}^a{}_{ij})_{n\times n}$, is obtained satisfying the following conditions, $\#\overline{h}^a{}_{ij} = max\{\#h_{ij}^a | i, j = 1, 2, \ldots, n;\ i \neq j\}$ $\overline{\gamma}_{ij}^{\sigma(s)} + \overline{\gamma}_{ji}^{\sigma'(s)} = 1$, $\overline{h}_{ii}^a = \{0.5\}$, $\overline{\gamma}_{ij}^{\sigma(s)} \le \overline{\gamma}_{ij}^{\sigma(s+1)}$, $\overline{\gamma}_{ji}^{\sigma'(s+1)} \le \overline{\gamma}_{ji}^{\sigma'(s)}$, where $\{\sigma(1), \ldots, \sigma(\#h_{ij}^a)\}$ is a permutation of $\{1, \ldots, \#h_{ij}^a\}$, i.e., $\overline{\gamma}_{ij}^{\sigma(s)}$ is the smallest element in $h_{ij}^a$, and $\{\sigma'(1), \ldots, \sigma'(\#h_{ji}^a)\}$ is a permutation of $\{1, \ldots, \#h_{ji}^a\}$, i.e., $\overline{\gamma}_{ji}^{\sigma'(s)}$ is the largest element in $h_{ji}^a$.

Even though, our proposal avoids aggregation operations in early stages, there are several procedures in the CRP that need to aggregate and compute distances with HFSs. Despite there exist multiple proposals to carry out such operations [40]. Here, the Hesitant Fuzzy Weighted Average (HFWA) operator and the Euclidean distance which are used for sake of clarity in the proposed consensus model for LS-GDM are just revised.

**Definition 5** (*[54]*). Let $H$ be a HFS and $h_i (i = 1, \ldots, n)$ be a collection of HFEs, $h_i \in H$, the Hesitant Fuzzy Weighted Average operator is a mapping $H^n \to H$ such that

$$HFWA(h_1, \ldots, h_n) = \oplus_{i=1}^n (w_i h_i) = \bigcup_{\gamma_1^{\sigma(s)} \in h_1, \ldots, \gamma_n^{\sigma(s)} \in h_n} \left\{ \sum_{i=1}^n w_i \gamma_i^{\sigma(s)} \right\}, \tag{3}$$

where $w = (w_1, w_2, \ldots, w_n)^T$ is the weighting vector of $h_i (i = 1, \ldots, n)$ with $w_i \in [0, 1]$ and $\sum_{i=1}^n w_i = 1$.

**Definition 6** (*[50]*). Let $H_1$ and $H_2$ be two HFSs on $X = \{x_1, \ldots, x_n\}$, the hesitant normalized Euclidean distance is defined as follows,

$$d_{hne}(H_1, H_2) = \left[ \frac{1}{n} \sum_{i=1}^n \left( \frac{1}{\#h} \sum_{s=1}^{\#h} |\gamma_2^{\sigma(s)}(x_i) - \gamma_1^{\sigma(s)}(x_i)|^2 \right) \right]^{1/2} \tag{4}$$

where $\#h$ is the cardinality of any HFE $h_i \in H_1, H_2$, considering that all of them are equal cardinality.

Additionally, during the CRP will be necessary to compare HFEs of the HFS. Therefore, one suitable function will be the below one:

**Definition 7** (*[14]*). Let $h$ be a HFE, the score function of $h$ is given by,

$$score(h) = \frac{\sum_{s=1}^{\#h} \gamma^s \tau(s)}{\sum_{s=1}^{\#h} \tau(s)} \tag{5}$$

where $\{\tau(s)\}_{s=1}^{\#h}$ is a positive-valued monotonic increasing sequence of index $s$.

## 3. An adaptive consensus model for large scale group decision making based on group hesitation

The goal of this paper is to introduce a novel CRP for LS-GDM problems able to tackle the *scalability* and *time cost* challenges of a CRP in this type of decision problems.

- To cope with the former one, a clustering process to detect experts sub-groups based on their preference similarity is done. And such sub-groups' preferences are modelled as the group's hesitation by means of HFSs; eventually the hesitant preferences are weighted according to the *size* and *cohesion* of the group.
- On the other hand, the latter challenge is managed by an *adaptive process* that varies the feedback procedure in the CRP between two levels according to the level of consensus achieved at each discussion round.

The proposed adaptive consensus model based on group hesitation for LS-GDM extends the general scheme shown in Fig. 1 by introducing two new phases:

- *Sub-groups management* that clusters similar experts' opinions, maintaining the maximum possible information by HFSs and computing the relevance of the sub-groups.
- A new *adaptive feedback process* that adapts the feedback to the current agreement among experts.

Besides, these new phases, other two of the general scheme are modified (dashed lines):

- *Framework configuration* in which a new parameter to deal with the adaptivity is introduced.
- *Computing the consensus degree* to deal with hesitant information.

So, the proposed model consists of six main phases (see Fig. 2), but only the new and modified ones (previously enumerated) will be further detailed below.

### 3.1. Framework configuration

In a LS-GDM problem there are two important elements (Section 2.1): a set of alternatives $X = \{x_1, \ldots, x_n\}$ and a large number of experts $E = \{e_1, \ldots, e_m\}$ who are involved in the problem, being $m > > n$.

Classically, two parameters are established, the consensus threshold and the maximum number of discussion rounds. However, in our proposal a new parameter is necessary to introduce the adaptivity during the consensus process. Therefore, three parameters are defined in our adaptive CRP:

- $\vartheta \in [0, 1]$: It is the consensus threshold established to achieve the consensus among experts.
- $\delta \in [0, 1], \delta < \vartheta$: It is a parameter used in the adaptive feedback process to determine the level of consensus reached (high or low), such that different rules for the advice generation can be applied.
- $Maxround$: This parameter controls the maximum allowed number of discussion rounds for the LS-GDM problem.

### 3.2. Sub-groups management: Managing scalability in LS-GDM

To tackle the scalability problem in LS-GDM, we consider that among a large number of experts there will be sub-groups of them with similar preferences. Therefore, with this idea in mind, this phase reduces the number of preferences to manage by means of a three-step process (further detailed in the coming subsections):

1. ***Detection***: A clustering process is applied to detect experts' groups with similar opinions.

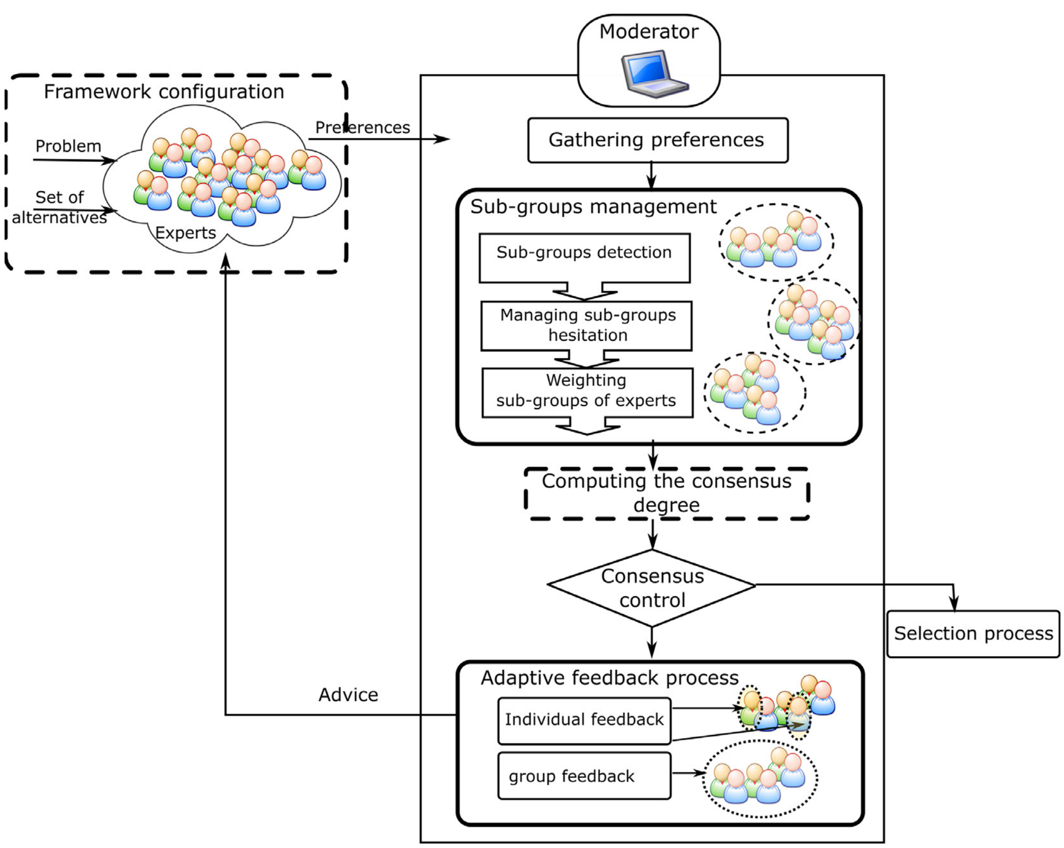


**Fig. 2.** Scheme of the proposed consensus model.

```
Require: Map preferences
Require: List alternatives
Require: List experts
Ensure: List clusters
n ← lenght(alternatives)
m ← lenght(experts)
for l=1 to n do
    clusters(l) ← generateCluster(l) //Including initialization of centroid
iteration ← 0
variation ← 1
repeat
    for r=1 to m do
        for l=1 to n do
            degree ← computeMembership(getPreferences(experts(r)), getCentroid(clusters(l))
            membershipDegrees(l) ← degree
        matrixMembershipDegrees(r) ← membershipDegrees
        idCluster ← maximumMembershipDegree(membershipDegrees)
        for l=1 to n do
            if (idCluster == getIdCluster(clusters(l))) then
                if (contain(clusters(l),experts(r)))==false then
                    assignPreferencesCluster(getPreferences(experts(r)), clusters(idCluster))
    allMembershipDegrees(iteration) ← matrixMembershipDegrees
    if (iteration > 0) then
        variation ← compareMembershipDegrees(allMembershipDegrees(iteration), allMembershipDegrees(iteration −
1))
    if (variation > ϵ) then
        for l=1 to n do
            updateCentroid(clusters(l))
    iteration ← iteration + 1
until (variation ≤ ϵ) return clusters
```

**Algorithm 1.** Fuzzy c-means algorithm applied to expert fuzzy preference relations.

2. ***Hesitation modelling***: The experts' opinions in each sub-group are modelled by means of a HFS that represents the hesitation of the group
3. ***Weighting***: The importance of the sub-group's opinion should reflect its features, in our case the size and cohesion of a subgroup is considered.

*3.2.1. Sub-groups detection*

To detect experts' groups with similar opinions, an adapted fuzzy c-means based algorithm [3] that assigns a membership degree to each data object for each cluster according to the distance between the data object and the corresponding centroid is presented. The nearer the data object is to the centroid, the higher its membership degree with respect to this centroid is. Both centroids and memberships degrees are iteratively updated until an optimal solution is found.

1. The number of clusters can be randomly selected, in this proposal, the initial number of clusters is the number of different alternatives, $C = \{C^1, \dots, C^n\}$, because we want to find the clusters of experts supporting each different alternative.
2. A centroid represents each cluster $c^l$, $l \in \{1, \dots, n\}$. Centroids can be either randomly initialized or assigned to a value from the dataset, but their initialization is very sensitive to converging [1,21]. In this case, as the problem is known, each centroid is initialized with a fuzzy preference relation that ideally prefers the corresponding alternative over all the others, i.e. for alternative $x_k$, the centroid $c^k$ contains $c^{kj} = 1$, $c^{jk} = 0$ $(j \in \{1, \dots, n\})$ and for the remaining ones the preference is 0.5 that representing indifference.

$$c^1 = \begin{pmatrix} - & 1 & 1 & \cdots & 1 \\ 0 & - & 0.5 & \cdots & 0.5 \\ 0 & 0.5 & - & \cdots & 0.5 \\ \vdots & \cdots & \cdots & \cdots & \vdots \\ 0 & 0.5 & 0.5 & \cdots & - \end{pmatrix}, \quad c^2 = \begin{pmatrix} - & 0 & 0.5 & \cdots & 0.5 \\ 1 & - & 1 & \cdots & 1 \\ 0.5 & 0 & - & \cdots & 0.5 \\ \vdots & \cdots & \cdots & \cdots & \vdots \\ 0.5 & 0 & 0.5 & \cdots & - \end{pmatrix} \cdots$$

$$c^n = \begin{pmatrix} - & 0.5 & 0.5 & \cdots & 0 \\ 0.5 & - & 0.5 & \cdots & 0 \\ 0.5 & 0.5 & - & \cdots & 0 \\ \vdots & \cdots & \cdots & \cdots & \vdots \\ 1 & 1 & 1 & \cdots & - \end{pmatrix}$$

3. Centroids are computed in each iteration $t$, and the membership degree of each experts' fuzzy preference relation $P^r$ to each centroid $c^{l,\,t}$, $\mu_{c^{l,t}}(P^r) \in [0, 1]$, is calculated by:

$$\mu_{c^{l,t}}(P^r) = \frac{(1/d(P^r, c^{l,t}))^{1/(b-1)}}{\sum_{u=1}^{n} (1/d(P^r, c^{u,t}))^{1/(b-1)}} \tag{6}$$

where $d(P^r, c^{l,\,t})$ is the Minkowski distance, $t$ is the current iteration, and $b$ indicates the fuzziness degree of the clusters. The larger $b$, the fuzzier the cluster [3]. A common value for this parameter is $b = 2$.

**Definition 8 (*[25]*).** Let $P^r$ be a fuzzy preference relation provided by the expert $e_r$, and $c^{l,\,t}$ be the centroid for the cluster $C^l$ at iteration $t$, the Minkowski distance is defined as follows,

$$d(P^r, c^{l,t}) = \left( \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j=1, i\neq j}^{n} \left| p_{ij}^r - c_{ij}^{l,t} \right|^{\lambda} \right)^{1/\lambda} \tag{7}$$

being $\lambda > 0$.In our proposal, $\lambda = 2$ that is the Euclidean distance.

4. The preference relation $P^r$ of expert $e_r$ is assigned to the cluster for which, the membership degree is maximum.

$$C^{l,t}(P^r) = \text{argmax}_l\, \mu_{c^{l,t}}(P^r) \tag{8}$$

5. New centroids are computed according to the experts preference relations included in each cluster.

$$c_{ij}^{l,t+1} = \frac{1}{|C^{l,t}|} \sum_{P^r \in C^{l,t}} p_{ij}^r,\ i, j \in \{1, \cdots, n\}, \tag{9}$$

where $|C^{l,\,t}|$ is the number of preference relations that belong to the cluster $C^l$ at iteration $t$.

6. The algorithm stops when all clusters stabilize. This happens when the variation of the membership degrees between two consecutive iterations approaches to zero. Formally, the iterative process stops when

$$\frac{\sum_{r=1}^{m} \sum_{l=1}^{n} |\mu_{c^{l,t}}(P^r) - \mu_{c^{l,t-1}}(P^r)|}{m \cdot n} \leq \epsilon \tag{10}$$

where $\epsilon$ is a threshold value that should be close to zero.

Algorithm 1 formalizes previous steps. The outcome provides clusters, $C^l$, containing a sub-group of experts, $G^l$, with similar opinions.

*3.2.2. Sub-groups hesitation modelling*

The classification into sub-groups according to preferences similitude aims at reducing scalability problems, however it is necessary to establish how to model the sub-group preferences. There exist several possibilities, from using the centroid that represents the sub-group's cluster to aggregate all the expert's preferences in the sub-group. But bearing in mind our goal of keeping as much information as possible in the CRP, unlike of oversimplifying the preferences modelling with aggregation procedures, our proposal considers that the different experts' preferences elicited in the sub-group despite of being similar, show a kind of hesitation in the group regarding such preferences.

Therefore, let $G^l = \{e_1^l, \dots, e_k^l\}$ be the sub-group of experts belonging to cluster, $C^l$, whose preference relations are, $P^{lk} = \left(p_{ij}^{lk}\right)_{n\times n}$. From such preference relations a HFPR, $HP^l = (h_{ij}^l)_{n\times n}$, $l \in \{1, \dots, n\}$ is built, that fuses all experts' preferences in $G^l$ such that, $h_{ij}^l = \{p_{ij}^{lk} | k = 1, 2, \dots, |G^l|\}$, $|G^l|$ is the cardinality of $G^l$ and will be the number of preferences in the HFE $\#h_{ij}^l$ which represents the sub-group's preference over the pair of alternatives $(x_i, x_j)$ provided by all experts in $G^l$.

At this moment, the large number of experts $E = \{e_1, \dots, e_m\}$ and their respective preference relations, $P^r$, have been replaced by a smaller number of sub-groups, $G^l$, and their respective HFPRs, $HP^l$, that will be the input for the CRP in the LS-GDM.

*3.2.3. Sub-groups weighting*

To conduct a fair CRP taking into account the previous elements, $G^l$ and $HP^l$, it is necessary to characterize the sub-groups by computing their importance. Our proposal takes into account their *size* and *cohesion* [44] to reflect their weight:

- *Size*: number of experts in the sub-group.
- *Cohesion*: level of togetherness among the experts' preferences in a sub-group.

Therefore, the importance of the sub-groups is based on the two following statements:

- The greater the group the more important.
- The more cohesive the more important.

Hence, the weight of a sub-group of experts will be based on the *size* and *cohesion*. The former is directly obtained from the *sub-group detection* process, and the latter needs further computation. So, to obtain the weights for all sub-groups of experts it is necessary to carry out three steps: a) to compute the cohesion of each group, b) to compute the size of each group and c) to obtain the group's weight. These steps are further detailed below:

*(a) Computing the cohesion of a sub-group*. For the sake of clarity, a geometric description of the *cohesion* of experts' preferences, $HP^l = (h_{ij}^l)_{n\times n}$, in a sub-group $G^l$ is introduced. First, the area

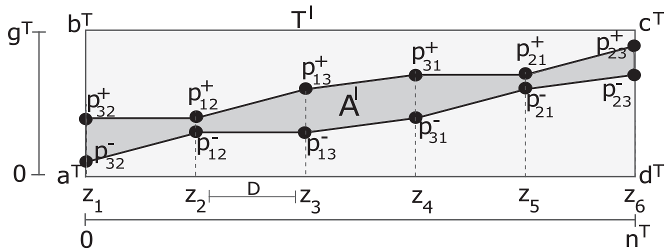


**Fig. 3.** Graphical representation for computing the cohesion of a sub-group.

delimited by the maximum and minimum assessments in $h^l_{ij}$, over the set of alternatives $X$ is computed. For instance, let $G^l = \{e_1, e_2\}$ be a sub-group of experts, $X = \{x_1, x_2, x_3\}$ a set of alternatives, and $HP^l$ the HFPR representing the preferences of the sub-group $G^l$.

$$HP^l = \begin{pmatrix} - & \{0.3, 0.4\} & \{0.3, 0.6\} \\ \{0.7, 0.6\} & - & \{0.7, 0.9\} \\ \{0.7, 0.4\} & \{0.3, 0.1\} & - \end{pmatrix}$$

The preference degrees in the HFE, $h^l_{ij}$, elicited by experts on $(x_i, x_j)$ are shown in Fig. 3. The X-axis represents a discrete set $Z$ formed by all pair of alternatives over $X$ where each pair $z_t = (x_i, x_j)$ $i, j \in \{1, 2, 3\}$, $i \neq j$ is positioned equidistantly on the $X$-axis. In order to compute the area, the maximum, $p^+_{ij}$, and minimum, $p^-_{ij}$, assessments, for each pair of alternatives are obtained. To do so, it is necessary to establish the order in which the pairs of alternatives are located across the $X$-axis. In this approach, we have considered the minimum assessments in increasing order.

The *cohesion* of, $G^l$, is related to the dark shadowed area, $A$ (the larger, $A$, the lower the cohesion), that is computed as follows:

(i) Let $T^l$ be the total area of the rectangle formed by the points $a^T$, $b^T$, $c^T$ and $d^T$ (see Fig. 3), i.e., $T^l = g^T \times n^T$ in which $g^T$ corresponds to the height of the rectangle and $n^T = (n^2 - n) - 1$, corresponds to the number of pairs of alternatives (considering $p_{ii}$ is not assessed) minus 1, because an area needs at least two pairs in $A$.

(ii) Let $I = \bigcup_{i,j \in n, i \neq j} \{(i, j)\}$ be the $n^T$ pairs over the set of alternatives $X = \{x_1, \ldots, x_n\}$. The $p^+_{ij}$, and $p^-_{ij}$, assessments for each $p_{ij}$ taking into account all the preferences in $G^l$ are obtained as:

$$p^-_{ij} = min\left\{p^1_{ij}, p^2_{ij}, \ldots, p^s_{ij}\right\}, \ \forall\ (i, j) \in I \tag{11}$$

$$p^+_{ij} = max\left\{p^1_{ij}, p^2_{ij}, \ldots, p^s_{ij}\right\}, \ \forall\ (i, j) \in I \tag{12}$$

The first and last pair of alternatives considered in the $X$-axis are obtained by,

$$p^-_{ab} = min_{i,j \in I}\left\{p^-_{ij}\right\}, \ (a, b) \in I \tag{13}$$

$$p^+_{cd} = max_{i,j \in I}\left\{p^-_{ij}\right\}, \ (c, d) \in I \tag{14}$$

A function $f$ is defined to obtain the indexes of the pairs of alternatives.

**Definition 9.** Let $f$ be a function that returns the indexes of a pair of alternatives,

$$f: \{z_1, z_2, \ldots, z_{n(n-1)}\} \to I \tag{15}$$

being $f(z_1) = (a, b) \in I$ such that, $p^-_{ab} = \min_{i,j \in I}\left\{p^-_{ij}\right\}$,

$$f(z_t) = (e, f) \in I \text{ where } p^-_{ef} = \min_{i,j \in I/\{f(z_1), f(z_2), \ldots, f(z_{t-1})\}}\left\{p^-_{ij}\right\},$$

$$f(z_{n(n-1)}) = (e, f) \in I \quad \text{with} \quad p^-_{ef} = \min_{i,j \in I/\left\{f(z_1), f(z_2), \ldots, f\left(z_{n(n-1)-1}\right)\right\}}\left\{p^-_{ij}\right\} = \max_{i,j \in I}\left\{p^-_{ij}\right\},$$

therefore, $f(z_{n(n-1)}) = (c, d) \in I$.The area $A^l$, between the maximum and minimum assessments ordered in the $X$-axis by the minimum is computed by,

$$A^l = \left[\sum_{i,j \in I}\left(p^+_{ij} - p^-_{ij}\right) - \frac{(p^+_{ab} - p^-_{ab}) + (p^+_{cd} - p^-_{cd})}{2}\right] \cdot D \tag{16}$$

where $D$ is the distance between $z_i$ and $z_{i+1}$, that in our case it is 1.

(iii) Finally, the cohesion of a sub-group of experts $G^l$ is given by,

$$cohesion(G^l) = 1 - \frac{A^l}{T^l} \in [0, 1], \tag{17}$$

*(b) Computing the size of a sub-group*. The value of the *size* of the group, $G^l$, is directly obtained from the *sub-group detection* process, but its representation should be adjusted and adapted to the number of experts involved in the LS-GDM problem. Therefore, a adaptation process based on computing with words [38] is proposed in which, the *size* is modelled by a fuzzy membership function $\mu_{\text{size}}$ shown in Fig. 4, such that the universe of discourse is the number of experts in a sub-group and the membership degree reflects group's influence regarding all the experts involved in the LS-GDM.

The points $a$ and $b$ of this membership function depend on the number of alternatives and experts in the LS-GDM problem, where the highest membership degree is for values above $b$ and the lowest membership degree is for values below $a$ and different importance is assigned in between.

*(c) Computing the relevance of a sub-group*. Eventually, for weighting the sub-groups, the values of their *size* and *cohesion* are aggregated, our proposal defines a function to fuse both values making such a computation more flexible according to the specific LS-GDM.

**Definition 10.** Let $Y_{G^l} = \{y_1, y_2\}$ be the values obtained for cohesion and size, respectively, $y_1, y_2 \in [0, 1]$, of the sub-group $G^l$ which are aggregated as follows,

$$\varphi(Y_{G^l}) = (1 + y_2)^{y_1 \beta} \tag{18}$$

being $\beta > 0$ a parameter to increase/decrease the impact of the cohesion in the computation of the sub-group's weight.

The aggregated values, $\varphi(Y_{G^l})$, reflects the relevance of the sub-group, $G^l$. Finally, such values are normalized.

$$w_l = \frac{\varphi(Y_{G^l})}{\sum_{z=1}^{n} \varphi(Y_{G^z})}, \forall\ l \in \{1, \ldots, n\}. \tag{19}$$

Below, an example shows how the aggregation function performs and the influence of parameter $\beta$ on the computation of the sub-groups' weight.

Let suppose a LS-GDM problem with 80 experts distributed into four sub-groups, $G = \{G^1, G^2, G^3, G^4\}$ whose size, membership degree and

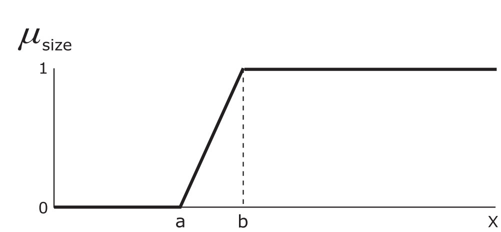


**Fig. 4.** Membership function for the sub-group *size*.

**Table 1**
Weights for different values of $\beta$.

| | | | | Weigths | | | |
|---|---|---|---|---|---|---|---|
| | Size | Memb. Degree | Coh. | $\beta = 1$ | $\beta = 1.5$ | $\beta = 2$ | $\beta = 3$ |
| $G^1$ | 11 | 0.25 | 0.55 | 0.219 | 0.205 | 0.191 | 0.166 |
| $G^2$ | 14 | 0.5 | 0.55 | 0.243 | 0.238 | 0.234 | 0.224 |
| $G^3$ | 23 | 1 | 0.49 | 0.273 | 0.284 | 0.295 | 0.318 |
| $G^4$ | 32 | 1 | 0.45 | 0.265 | 0.272 | 0.279 | 0.292 |

cohesion are depicted in Table 1. Different values for the parameter $\beta$ have been used to solve Eq. (18).

Note that the weights in Table 1 are already normalized. We can observe that the sub-groups $\{G^1, G^2\}$ have different size but equal cohesion, therefore, when the value of $\beta$ increases, the sub-group's weight with higher size, $G^2$, decreases slower than $G^1$. On the other hand, sub-groups $\{G^3, G^4\}$ have the same membership degree, therefore, when the value of $\beta$ increases, the sub-group's weight $G^3$ increases more than the sub-group $G^4$, because its cohesion is higher. Thus, the parameter $\beta$ allows to increase/decrease the impact of the cohesion in the computing weights.

### 3.3. Computing the consensus degree

Our CRP model modifies the way of computing the level of agreement among experts shown in Fig. 1 by adapting the three-step process introduced in [31], to deal with the HFSs obtained in the previous phase.

1. *Pairwise similarity matrix*: For each pair of sub-groups $G^l$ and $G^k$, a similarity matrix $SM^{lk} = (sm_{ij}^{lk})_{n\times n}$ is obtained, being $sm_{ij}^{lk} \in [0, 1]$ the similarity between $h_{ij}^l$ and $h_{ij}^k$:

$$sm_{ij}^{lk} = 1 - d(h_{ij}^l, h_{ij}^k) \tag{20}$$

being $d$ a distance measure for HFEs [40] (see Remark 1). In this proposal $d$ is the Euclidean distance (see Def. 6).

**Remark 1.** The number of values in the HFEs of each HFPR, $HP^l$, might be different. In such a case, based on Definition 4 and using the optimization parameter $\eta^l$, all HFPRs, $HP^l(h_{ij}^l)_{n\times n}$, are normalized, $\overline{HP}^l = (\bar{h}_{ij}^l)_{n\times n}$, before carrying out the computations.

2. *Consensus matrix*: The similarity matrices are aggregated to obtain a consensus matrix $CM = (cm_{ij})_{n\times n}$. Though, different aggregation operators may be used, without loss of generality in this proposal the arithmetic mean is applied:

$$cm_{ij} = \frac{\sum_{u=1}^{l-1}\sum_{k=u+1}^{l} sm_{ij}^{lk}}{l(l-1)/2} \tag{21}$$

with $l(l-1)/2$ the number of pairwise sub-group comparisons.

3. The *consensus degree* is calculated at two different levels using the consensus matrix $CM$:

- Level of alternatives ($ca_i$): the consensus degree of each alternative $x_i \in X$ is computed as,

$$ca_i = \frac{1}{n-1}\sum_{j=1, i\neq j}^{n} cm_{ij} \tag{22}$$

- Level of preference relation ($cr$): the consensus degree among all experts participating in the LS-GDM problem is computed by,

$$cr = \frac{1}{n}\sum_{i=1}^{n} ca_i \tag{23}$$

```
Require: cr
Require: δ
Require: ϑ
if cr > ϑ then
    Consensus achieved
else
    if cr ≥ δ then
        Consensus level high
        Individual feedback process
    else
        Consensus level low
        Group feedback process
```

**Algorithm 2.** Adaptive feedback process.

### 3.4. Adaptive feedback process: Time cost supervision

When the consensus degree, $cr$, achieved in a consensus round is not high enough, i.e. $cr < \vartheta$, another discussion round is necessary to increase the agreement among experts. This new discussion round is usually guided by a feedback process [33]. So far, CRPs introduced for dealing with LS-GDM [34,49] do not consider the agreement achieved in each round to adapt the feedback process making the expert's preference supervision harder and the consensus process longer. Due to the fact that, one goal of our proposal is to reduce the time cost and soften the preference supervision, this CRP model for LS-GDM proposes an *adaptive* procedure that adapts the feedback process according to the rules for the advice generation based on the consensus level achieved (see Algorithm 2). According to such a level, the generated feedback is intended for the whole *group* or for several *individuals*. The adaptivity of the feedback is based on the consensus threshold, $\vartheta$, fixed to achieve the consensus, and the parameter $\delta$ that distinguishes between the two feedback processes. From this definition, the adaptive feedback process consists of three steps:

1. A *collective matrix* that represents the collective opinion of the experts involved in the LS-GDM problem is computed by aggregating the normalized HFPRs $\{\overline{HP}^1, \ldots, \overline{HP}^n\}$. Different hesitant fuzzy aggregation operators [40,54] can be used, here the Hesitant Fuzzy Weighted Average operator is used. This operator is revised in Definition 5, and adapted for our proposal.

**Definition 11.** Let $\overline{HP}^l = (\bar{h}_{ij}^l)_{n\times n}$, $(l = 1, \ldots, n)$, be the normalized HFPRs of the $n$ sub-groups $G^l$, and $w = (w_1, w_2, \ldots, w_n)^T$ the weighting vector for those sub-groups (see Section 3.2.3), the collective HFPR, $HP^C = (h_{ij}^C)_{n\times n}$, is computed as,

$$h_{ij}^C = \oplus_{l=1}^{n}(w^l \bar{h}_{ij}^l) = \bigcup_{\bar{\gamma}_{ij}^{l,s} \in \bar{h}_{ij}^l} \left\{ \sum_{l=1}^{n} w^l \bar{\gamma}_{ij}^{l,s} \right\}, \quad \forall\, i, j \in \{1, \ldots, n\} \tag{24}$$

being $HP^C$ a normalized HFPR.

2. The *proximity* between each sub-group represented by a normalized HFPR $\{\overline{HP}^1, \ldots, \overline{HP}^n\}$, and the collective matrix $HP^C$, is calculated by using a similarity measure like in Eq. (20).

$$pr^l = sim(HP^C, \overline{HP}^l) = 1 - d_{hne}(HP^C, \overline{HP}^l) \tag{25}$$

Proximity values, $pr^l$, are used to identify the sub-groups that are furthest from the collective opinion.

3. *Adapting the feedback*: depending on the consensus level reached $cr$, the feedback process will be aimed at all experts of the furthest sub-groups or just for several further experts. Both processes are explained in more detail:

**i) Group feedback process. Low consensus level**

In this case $cr < \delta$, that means the consensus level is "low" and consensus is still far away, therefore quite a lot more changes are necessary, consequently all experts of the furthest sub-groups will obtain suggestions for modifying their preferences over the pair of alternatives identified in disagreement. To identify the furthest sub-groups, the proximity value of each sub-group $pr^l$ is compared with the average of the proximity values $\overline{pr}$, such that,

$$\overline{pr} = \frac{1}{n}\sum_{l=1}^{n} pr^l \tag{26}$$

and to select the pair of alternatives to be changed, the proximity value of each pair of alternatives, $pr^l_{ij}$, is compared with the average of the proximity value for such an alternative $\overline{pr_i}$, such that,

$$\overline{pr_i} = \frac{1}{n}\sum_{j=1}^{n} pr^l_{ij}, \;\; with \; pr^l_{ij} = 1 - d(h^C_{ij}, h^l_{ij}) \tag{27}$$

where $h^C_{ij} \in HP^C$ and $h^l_{ij} \in \overline{HP}^l$.

Therefore,

1. If $pr^l \le \overline{pr}$ then the sub-group $G^l$ is selected.
2. If $ca_i \le \vartheta$ then the alternative $x_i$ is selected and it is necessary to look for the pair of alternatives,
   (a) if $pr^l_{ij} \le \overline{pr_i}$, then the pair of alternatives $(x_i, x_j)$ is selected.

Once the sub-groups and pair of alternatives have been identified, a suggestion indicating the right direction of the preference changes (increase or decrease) to improve the agreement among experts is provided, according to the following direction rules:

- If $score(h^l_{ij}) < score(h^C_{ij})$, then all experts who belong to the sub-group $G^l$ should increase their preferences degrees for the pair of alternatives $(x_i, x_j)$.
- If $score(h^l_{ij}) > score(h^C_{ij})$, then all experts who belong to the sub-group $G^l$ should decrease their preferences degrees for the pair of alternatives $(x_i, x_j)$.

Being $score(h^l_{ij})$ and $score(h^C_{ij})$ the score function for the HFEs $h^l_{ij} \in \overline{HP}^l$ and $h^C_{ij} \in HP^C$, respectively (see Eq. (5)).

**ii) Individual feedback process. High consensus level**

In this case $\delta \le cr < \vartheta$, that means the consensus level is "high" but not enough yet. Therefore not many changes should be necessary, hence those experts whose opinion differs most from the collective opinion will obtain advice to modify their opinions. Thus, it would be necessary to identify the sub-group, $G^l$, the pair of alternatives $(x_i, x_j)$ and experts $e_r$ who should modify their preferences in disagreement:

1. If $pr^l \le \overline{pr}$ then the sub-group $G^l$ is selected.
2. If $ca_i \le \vartheta$ then the alternative $x_i$ is selected and,
   (a) If $pr^l_{ij} \le \overline{pr_i}$, then the pair of alternatives $(x_i, x_j)$ is selected.
3. If $\left(1 - d\left(h^C_{ij}, p^{lr}_{ij}\right) \le \overline{pr_i}\right)$, then the expert $e_r$ is selected to change his/her preference.

The direction in which the selected expert should change his/her preferences is determined as follows:

- If $\left(p^{lr}_{ij}\right) < score(h^C_{ij})$, then expert $e_r \in G^l$ should increase his/her preference degree for the pair of alternatives $(x_i, x_j)$.
- If $\left(p^{lr}_{ij}\right) > score(h^C_{ij})$, then the expert $e_r \in G^l$ should decrease his/her preference degree for the pair of alternatives $(x_i, x_j)$.
- If $\left(p^{lr}_{ij}\right) = score(h^C_{ij})$, then it is not necessary to make changes.

Being $score(h^C_{ij})$ the score function of the HFE $h^C_{ij}$, in the collective matrix $HP^C$ calculated by Eq. (5). After this process, the CRP will go to the *sub-groups management* phase again.

## 4. Case study

This section presents a case study to show the usefulness of the proposed CRP for LS-GDM. To do so, firstly the LS-GDM problem is described. Afterwards, the problem is solved by means of the proposed model which has been implemented and integrated into the intelligent CRP support system, AFRYCA 2.0 [23,33]. A comparison with some existing models is then shown, and finally an analysis to display distinctive characteristics regarding existing approaches is introduced.

### 4.1. Definition of the LS-GDM problem

The GDM problem is formulated as follows: let $E = \{e_1, e_2, \ldots, e_{50}\}$, be the students of the course of basic programming of Computer Science degree. The professor asks them which programming language they would like to use for the practices in the laboratory and he provides four options, $X = \{x_1$: *C*, $x_2$: *C* + +,$x_3$: *Java*, $x_4$: *Python*$\}$. The professor wants an agreed solution because once the language is selected, they cannot change it for another one. Students provide their preferences by fuzzy preference relations over the four options. For the sake of space, the preferences have been included as a supplementary material document which is available at http://sinbad2.ujaen.es/afryca/sites/default/files/app/computerScienceDegree-programmingLanguage.pdf.

Additionally to the experts and alternatives, it is necessary to establish the following parameters:

- Consensus threshold: $\vartheta = 0.85$
- Level of consensus for the advice generation: $\delta = 0.7$
- Maximum number of rounds allowed: max_round = 15

### 4.2. Resolution of the LS-GDM problem

In order to solve the problem and achieve the consensus, the new adaptive CRP is applied and the intelligent CRP support system is used to carry out the computations and visualize the CRP.

1. *Framework configuration*: all the parameters necessary in this phase have been already defined previously.

2. *Sub-groups management*: The fuzzy c-means based algorithm explained in Section 3.2.1 is applied to obtain the clusters containing the sub-groups of experts with similar opinions. Table 2 shows the sub-groups of experts $G = \{G^1, G^2, G^3, G^4\}$ in the first round.

Afterwards, a HFPR for each sub-group of experts is built and they are the input for the proposed CRP for LS-GDM.

The points $a$ and $b$ to define the membership function for the sub-group size are computed according to the number of experts $m$ involved in the LS-GDM problem and the number of alternatives $n$. In this case study, we have considered 10% of experts to define the point $a$ and the number of experts divided by the number of alternatives to define the point $b$, i.e. experts are equally distributed in the clusters obtained, but any other technique can be used.

$$a = Round(m \cdot 10/100), \quad b = Round(m/n)$$

Therefore, the points are $a = Round(50 \cdot 10/100) = 5$ and $b = Round(50/4) = 13$, (see Fig. 5), where $Round(\cdot)$ is the round function.

The weights of each sub-group of experts considering its size and

**Table 2**
Sub-group of experts in the first round.

| | |
|---|---|
| $G^1$ | $e_1, e_9, e_{24}, e_{27}, e_{30}, e_{32}, e_{33}, e_{37}, e_{48}$ |
| $G^2$ | $e_6, e_8, e_{11}, e_{21}, e_{25}, e_{28}, e_{35}, e_{47}, e_{50}, e_{39}, e_{36}, e_4, e_{19}$ |
| $G^3$ | $e_2, e_{12}, e_{13}, e_{15}, e_{34}, e_{40}, e_{45}, e_5, e_{43}, e_{46}, e_{44}, e_{29}, e_{41}, e_3, e_{20}, e_{26}, e_{38}$ |
| $G^4$ | $e_7, e_{10}, e_{14}, e_{16}, e_{18}, e_{22}, e_{31}, e_{42}, e_{49}, e_{23}, e_{17}$ |

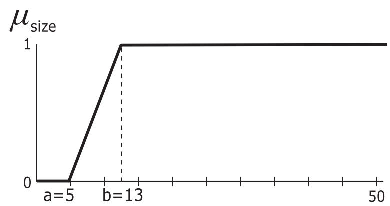


**Fig. 5.** Membership function for the sub-group *size*.

**Table 3**
Weights of the sub-groups of experts in the first round.

| Sub-groups | $G^1$ | $G^2$ | $G^3$ | $G^4$ |
|---|---|---|---|---|
| Size | 9 | 13 | 17 | 11 |
| Membership degree sub-group size | 0.5 | 1 | 1 | 0.75 |
| Cohesion | 0.51 | 0.49 | 0.44 | 0.59 |
| Weights | 0.19 | 0.29 | 0.25 | 0.27 |

**Table 4**
Consensus degree and consensus level for each round.

| round | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| cr | 0.64 | 0.66 | 0.69 | 0.72 | 0.74 | 0.75 | 0.79 | 0.83 | **0.85** |
| level | low | low | low | high | high | high | high | high | high |

cohesion are computed by using Eq. (18), in which the parameter $\beta$ has been established after several experiments as $\beta = 3.0$ to increase the impact of the cohesion. Table 3 shows the size, cohesion and weight for each sub-group of experts in the first round.

3. Computing the consensus degree: The consensus degree obtained in the first round is $cr = 0.64$.

4. Adaptive feedback process: As the consensus degree achieved is not enough, another discussion round is necessary. Applying the Algorithm 2, it is easy to see that the consensus level is low, because $0.64 < \delta = 0.7$, thus a group feedback process is carried out to identify the furthest sub-groups and suggest them to modify their preferences and increase the consensus degree in the next round.

This adaptive CRP is repeated until the consensus threshold is achieved. Table 4 shows the consensus degrees obtained for each round and indicates the consensus level reached in such rounds. Fig. 6 shows the visualization of the CRP obtained by using the statistics tool implemented in AFRYCA 2.0 that is able to carry out Multi-Dimensional Scaling (MDS) [22] of preferences.

### 4.3. Comparison with previous CRP models

Even though, previous results provide a good performance according to our goals. It should seem convenient to compare such results with other previous proposals for CRP. First, we compare our model with two well-known and widespread CRP proposals, Chiclana's approach [11] and Kacprzyk's approach [20]. Our hypothesis was that our proposal should reduce the cost to achieve the consensus (rounds, supervisions,...) and in both cases it is necessary to carry out more rounds to achieve the consensus (see Table 5). Fig. 7 shows the visualization of the CRP for each approach.

**Table 5**
Classical approaches.

| | Chiclana's approach | Kacprzyk's approach |
|---|---|---|
| Initial consensus degree | 0.63 | 0.53 |
| consensus degree achieved | 0.85 | 0.86 |
| rounds | 12 | 15 |

Second, a fairer comparison would consist of comparing our proposal with other CRPs for LS-GDM [34,38,48,49], but most of them [34,38,49] are focused on managing non-cooperative behaviours, therefore the comparison with our proposal is not fair, because their main feature is useless in our case study. And the CRP proposal in [48] for LS-GDM is incomparable, because the constrains imposed in it (see remark 2).

**Remark 2.** This approach represents the group preferences by using a possibility distribution based on hesitant fuzzy elements. The use of this type of information limits the elicitation of preferences, because experts have to use a discrete scale such as $\{0.1,0.2,0.3,0.4,0.5,0.6,0.7,0.8,0.9\}$ and in the feedback process, a set of values from the same scale is computed as possible suggestions for experts to change their preferences. This implies another limitation in the feedback process because experts cannot change their preferences as they want, and the minimum change is $\pm 0.1$. It is remarkable that in the feedback process the decision problem shows changes of 0.3 regarding the original preference in just one round which is not realistic, because usually experts do not want to make big changes in their preferences. Additionally, we have found some errors in the resolution process of the emergency decision making problem presented in the paper which makes difficult a comparison.

### 4.4. Analyzing the results

As result of the previous sections the following points must be highlighted:

- The proposed model performs effectively the CRP in LS-GDM as can be seen in Fig. 7 by adapting the process to the consensus degree in each round and reducing the preferences by a clustering process.
- Classical models compared, Chiclana's approach and Kacprzyk's approach, obtain from the initial round a consensus degree lower than the proposed model.
- The necessary number of rounds to achieve the required consensus degree with classical approaches is greater than in our proposal, therefore the latter reduces the time cost.
- The use of cohesion in the proposed model facilitates that experts are close to each other in the solution achieved unlike Kacprzyk's

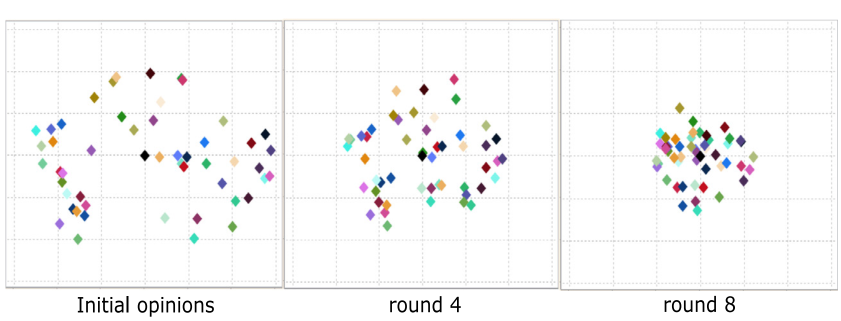


**Fig. 6.** MDS visualization of proposed CRP.

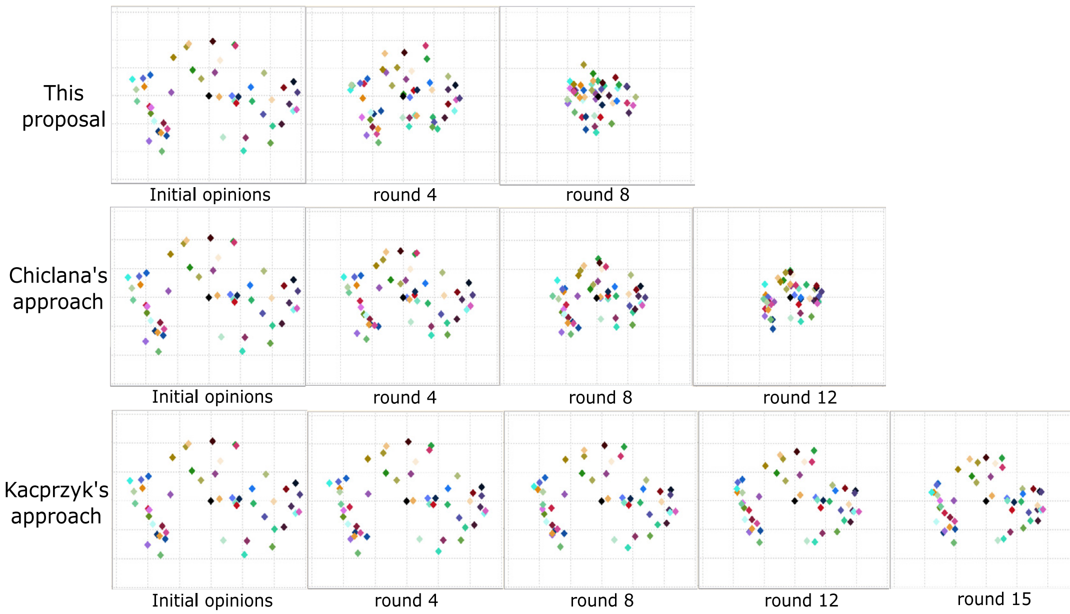


**Fig. 7.** MDS visualization of the CRPs.

approach and quicker than in Chiclana's one.

- The proposed CRP does not need to impose any limitation regarding the elicitation of preferences to achieve its goal, meanwhile others in the literature as Wu and Xu's approach limits the values of preferences elicited.

## 5. Conclusions and future research

Consensual decisions is a growing societal demand nowadays that becomes harder and more challenging in those decision making problems that involve a large number of experts. Despite its importance, most of current proposals in specialized literature are still focused on group decision situations with a few number of experts that present scalability and time cost limitations.

A novel CRP for LS-GDM based on a clustering process for weighting experts' preferences by using the size and cohesion of the clusters together a preference modelling with HFS and an adaptive feedback process has been introduced and compared with previous CRPs models. The results obtained show that the new CRP model for LS-GDM can effectively deal with these types of problems overcoming challenges proper of LS-GDM. This model has been implemented and integrated in an intelligent CRP support system.

As future research, we will study how the minimum cost can be used in the CRP to decrease the number of rounds to achieve the consensus and how to manage experts' behaviour that can make difficult to reach the consensus.

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