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Improving decision making approaches based on fuzzy soft sets and rough soft sets

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ABSTRACT

Hybrid soft sets, such as fuzzy soft sets and rough soft sets, have been extensively applied to decision making. However in both cases, there is still a necessity of providing improvements on approaches to obtain better decision results in different situations. In this paper several proposals for decision making are provided based on both hybrid soft sets. For fuzzy soft sets, a computational tool called D-score table is introduced to improve the decision process of a classical approach and its convenience has been proved when attributes change across the decision process. In addition, a novel adjustable approach based on decision rules is introduced. Regarding rough soft sets, several new decision algorithms to meet different decision makers' requirements are introduced together a multi-criteria group decision making approach. Several practical examples are developed to show the validity of such proposals.

1. Introduction

Classical mathematical tools, which require all notions to be exact, usually fail to handle the uncertainty, imprecision and vagueness in a wide variety of practical fields. Although theories such as fuzzy set theory [1], rough set theory [2], intuitionistic fuzzy set theory [3] and vague set theory [4] have been proved useful mathematical approaches in modeling these uncertainties, all of them have a common limitation—the inadequacy of the parameterization tool. In 1999, soft set theory was put forward by Molodstsov [5] as a new mathematic tool for dealing with uncertainty, which is free from the above mentioned limitation. Afterwards, the generalized models of soft sets (hybrid soft sets) come forth rapidly and there has been an increasing interest in the practical applications of hybrid soft set theories, especially with regard to their applications in decision making [6–17].

The popular hybrid soft set models contain two main categories: (1) The combination of soft set theory with fuzzy set theory and the generalized models of fuzzy set theory [18–24]; (2) The combination of soft set theory with rough set theory and the generalized models of rough set theory [25–28]. As two representative hybrid soft set models in these two different categories, fuzzy soft sets [18] and rough soft sets [26] are interconnected [29]. All decision making methods based on fuzzy soft sets or rough soft sets have the potential to be extended to deal with more complex hybrid soft set models situations. For instance, Jiang et al. [30] and Zhang et al. [31] extended Feng et al.'s decision making approach based on fuzzy soft sets in [32] to come up with an intuitionistic fuzzy soft sets based decision making approach and an interval-valued intuitionistic fuzzy soft set based decision making approach, respectively.

In terms of fuzzy soft set based decision making methods, Roy and Maji [33] provided a novel method (the score based method) for decision making based on fuzzy soft sets, which builds upon concepts such as the comparison table and the scores of objects. However, no researchers have paid attention to the improvement of the score based method in order to overcome its own limitations and make it fit for more practical situations until now, although its reasonability has already been verified [32]. With the development of information technology in modern society, the practical information updates rapidly as time goes by, adding new data and removing old data. In this paper, we will improve the score based method by introducing a new mathematic tool called D-Score table and then successfully make it more convenient to obtain the decision result when parameters should be added/deleted in decision making problems, this improvement will be useful for practical problems solving which contains updating information. Furthermore, we propose a new approach to fuzzy soft set based decision making by introducing comparison thresholds when comparing two membership values to obtain different kinds of scores for

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objects. After choosing different comparison thresholds, we will construct different level D-Score tables and then obtain different optimal decision sets, which makes the new approach adjustable. In this way, the new approach can be successfully used to deal with some problems which cannot be solved by the initial score based method.

In terms of rough soft sets based decision making approaches, the researches are so far few. Recently, Ma et al. [34] introduced some initial algorithms for decision making based on rough soft sets and an algorithm for group decision making based on MSR-set [35]. However, the algorithms proposed by Ma et al. [34] are far from enough to meet various practical demands. Furthermore, the group decision making method based on rough soft sets has not been studied yet. In the present work, with the construction of rough soft sets and a fuzzy soft set, we determine the weights of experts by using the similarity measure of soft sets and then provide a new approach based on rough soft sets to solve the group decision making problem. As an important hybrid soft set model generated with rough set theory, rough soft sets has huge potential to be used in dealing with practical problems that contain uncertainty, and it is a promising topic to find out more decision making approaches based on rough soft sets to meet the different demand of decision.

Although some researchers have systematically discussed the decision making approaches based on fuzzy soft sets and rough soft sets recently [34,36], they concentrated on proposals review or revision, rather than improving existing approaches or providing new approaches to meet various application demands. There still exist arguments on the fuzzy soft sets based decision making approaches [32,37], and it can be said that the research of rough soft sets based decision making approaches is still in an initial stage, that is the reason why it necessary to carry out a research focuses on the improvement of decision making approaches based on fuzzy soft sets and rough soft sets. In the current work, the limitations of some popular existing proposals will be systematically discussed and afterwards several solutions will be provided. All the improved proposals or new approaches provided in the current research have the potential to be extended to more complex hybrid models situations.

The present paper is organized as follows: Some basic notions on soft sets, fuzzy soft sets and rough soft sets are reviewed in Section 2. In Section 3, we recall an existing argument of fuzzy soft sets based decision making approaches, provide our opinion on this argument, afterwards present an improvement of the score based method. On the basis of this improved score based method, a new adjustable decision making approach based on fuzzy soft sets is proposed. In Section 4, we discuss the limitation of existing rough soft set based decision making methods and the necessary to enrich the approaches, afterwards two algorithms are provided to conquer these limitations and to meet various practical demands. It is worth noticing that we originally apply rough soft set as a tool to deal with group decision making problems, which successfully solve the problems according to assessments on alternatives provided by decision makers, rather than according to specific decision results made by separate decision makers which have been adopted in some other existing approaches. Finally, conclusions are given in Section 5.

2. Preliminaries

In this section we briefly recall some concepts that will be useful in subsequent discussions.

Let U be the initial universe of objects and E be the set of attributes related to objects in U. Both U and E are assumed to be nonempty finite sets. Let $P(U)$ be the power set of U and $A \subseteq E$.

Definition 1. [5]: A pair (F, A) is called a soft set over U, where F is a mapping given by $F: A \longrightarrow P(U)$.

For any attribute $e \in A$, $F(e) \subseteq U$ may be considered as the set of e-approximate elements of the soft set (F, A). In other words, the soft set is not a kind of set in the ordinary sense, but a attributeized family of subsets of U. We denote by (U, E) the set of all soft sets over U.

For two soft sets (F, A) and (G, B) over a common universe U, we say that (F, A) is a soft subset of (G, B) if

(1) $A \subseteq B$;
(2) $\forall e \in A, F(e) \subseteq G(e)$.

This relationship is denoted by $(F, A) \subseteq (G, B)$. (G, B) is said to be a soft super set of (F, A), if (F, A) is a soft subset of (G, B).

Definition 2. [38]: A mapping $S: (U, E) \times (U, E) \rightarrow [0, 1]$ is said to be a similarity measure if the following axioms hold for arbitrary $(F, A), (G, B) \in (U, E)$:

(1) $0 \leq S((F, A), (G, B)) \leq 1$;
(2) $S((F, A), (F, A)) = 1$;
(3) $S((F, A), (G, B)) = S((G, B), (F, A))$;
(4) If $(F, A) \subseteq (G, B) \subseteq (H, C)$, then $S((F, A), (H, C)) \leq S((F, A), (G, B))$, $S((F, A), (H, C)) \leq S((G, B), (H, C))$.

The theory of fuzzy sets, first introduced by Zadeh [1] in 1965, provides an appropriate framework for representing and processing vague concepts by allowing partial memberships. A fuzzy set F in the universe U is defined as $F = \{(x, \mu_F(x))/x \in U, \mu_F(x) \in [0, 1]\}$. μ_F is called the membership function of F and $\mu_F(x)$ indicates the membership degree of x to F. The family of all fuzzy sets on U is denoted by $F(U)$.

In 2001, Maji et al. [18] initiated the study on hybrid structures involving both fuzzy sets and soft sets. They introduced the notion of fuzzy soft sets, which can be seen as a fuzzy generalization of crisp soft sets.

Definition 3. [18]: A pair (F, A) is called a fuzzy soft set over U, where $A \subseteq E$ and F is a mapping given by $F: A \longrightarrow F(U)$.

For any attribute $e \in A$, $F(e)$ is a fuzzy subset of U and it is called fuzzy value set of attribute e. If for any attribute $e \in A$, $F(e)$ is a crisp subset of U, then the fuzzy soft set (F, A) degenerated to the standard soft set. Let us denote $\mu_{F(e)}(x)$ the membership degree that object x holds attribute e where $x \in U$ and $e \in A$. Then $F(e)$ can be written as $F(e) = \{ < x, \mu_{F(e)}(x) > | x \in U\}$.

For two fuzzy soft sets (F, A) and (G, B) over a common universe U, we say that (F, A) is a fuzzy soft subset of (G, B) if

(1) $A \subseteq B$;
(2) $\forall e \in A$, $F(e)$ is a fuzzy subset of $G(e)$, that is $\mu_{F(e)}(x) \leq \mu_{G(e)}(x)$ for all $x \in U$.

This relationship is denoted by $(F, A) \subseteq (G, B)$. (F, A) and (G, B) are said to be fuzzy soft equal if and only if $(F, A) \subseteq (G, B)$ and $(F, A) \supseteq (G, B)$, We write $(F, A) = (G, B)$.

A fuzzy soft set (F, A) over U is said to be null fuzzy soft set, if for $\forall e \in A$, we have $F(e) = \emptyset$; A fuzzy soft set (F, A) over U is said to be absolute fuzzy soft set, if for $\forall e \in A$, we have $F(e) = U$.

The rough set theory proposed by Pawlak [2] provides a systematic approach for dealing with vague concepts caused by indiscernibility in situation with insufficient and incomplete information.

Definition 4. [2]: Let R be an equivalence relation on the universe U. (U, R) is called a Pawlak approximation space. For any $X \subseteq U$, the lower approximation $\underline{Apr}_R(X)$ and the upper $\bar{Apr}_R(X)$ of X are defined as:

$\underline{Apr}_R(X) = \{x \in U : [x]_R \subseteq X\}$,

$\bar{Apr}_R(X) = \{x \in U : [x]_R \cap X \neq \emptyset\}$.

A subset $X \subseteq U$ is called definable if $\underline{Apr}_R(X) = \bar{Apr}_R(X)$; otherwise, X is said to be a rough set.

Considering the lower and upper approximations of a soft set in a Pawlak approximation space, Feng et al. [26] introduced the concept of rough soft sets.

Definition 5. [26]: Let (U, R) be a Pawlak approximation space and $\mathfrak{S} = (F, A)$ be a soft set over U. The lower and upper rough approximations of $\mathfrak{S} = (F, A)$ with respect to (U, R) are denoted by $\underline{Apr}_R(\mathfrak{S}) = (\underline{F}_R, A)$ and $\bar{Apr}_R(\mathfrak{S}) = (\bar{F}_R, A)$, which are soft sets over U with the set-valued mappings given by

$\underline{F}_R(e) = \underline{Apr}_R(F(e))$,

$\bar{F}_R(e) = \bar{Apr}_R(F(e))$, where $e \in A$. The operators $\underline{Apr}_R$ and $\bar{Apr}_R$ are called the lower and upper rough approximation operators on soft sets. If $\underline{Apr}_R(\mathfrak{S}) = \bar{Apr}_R(\mathfrak{S})$ the soft set $\mathfrak{S}$ is said to be definable; otherwise $\mathfrak{S}$ is called a rough soft set.

3. Improving decision making approaches based on fuzzy soft sets

In this section, first we will state our opinion on an existing argument upon two popular decision making approaches based on fuzzy soft sets. Afterwards, we summarize the limitations of these two approaches and furthermore provide solutions. Especially, some improvement approaches are proposed to overcome the limitations of "the score based method" (Roy-Maji method) [33].

3.1. An argument on fuzzy soft sets based decision making

Which fuzzy soft sets based decision making method is more reasonable: "the score based method" or "the fuzzy choice value based method"? There has been a strong argument on this question. As follows we will present our opinion based on a brief list of these arguments and summarize the main limitations of both approaches.

By introducing the concept of comparison table and a measure called the score of object, Roy and Maji [33] introduced an original decision making method as below (see Algorithm 1).

Algorithm 1.

[Step 1.] Input the fuzzy soft sets (F, A), (G, B) and (H, C).
[Step 2.] Input the attribute set P as observed by the observer.
[Step 3.] Compute the corresponding resultant fuzzy soft set (S, P) from the fuzzy soft sets (F, A), (G, B), (H, C) and place it in tabular form.
[Step 4.] Construct the comparison table of the fuzzy soft set (S, P) and compute r_i and t_i for o_i, $\forall i$.
[Step 5.] Compute the score $s_i = r_i - t_i$ of o_i, $\forall i$.
[Step 6.] The decision is o_k if $s_k = max_i s_i$.
[Step 7.] If k has more than one value then any one of o_k may be chosen.

The comparison table is a square table in which both rows and columns are labelled by the objects $o_1, o_2, \ldots, o_n$, and the entry c_{ij} indicates the number of attributes for which the membership value of o_i exceeds or equals to the membership value of o_j. Clearly, $0 \le c_{ij} \le m$ and $c_{ii} = m (\forall i, j)$, where m is the number of attributes.

The row-sum r_i of object o_i is computed by

$$r_i = \sum_{j=1}^{n} c_{ij} \quad (1)$$

Table 1
Tabular representation of fuzzy soft set (S, P).

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	Choice value(c_i)
o_1	0.3	0.1	0.4	0.4	0.1	0.1	0.5	$c_1 = 1.9$
o_2	0.3	0.3	0.5	0.1	0.3	0.1	0.5	$c_2 = 2.1$
o_3	0.4	0.3	0.5	0.1	0.3	0.1	0.6	$c_3 = 2.3$
o_4	0.7	0.4	0.2	0.1	0.2	0.1	0.3	$c_4 = 2.0$
o_5	0.2	0.5	0.2	0.3	0.5	0.5	0.4	$c_5 = 2.6$
o_6	0.3	0.5	0.2	0.2	0.4	0.3	0.3	$c_6 = 2.2$

Table 2
Comparison table of the fuzzy soft set (S, P).

	o_1	o_2	o_3	o_4	o_5	o_6
o_1	7	4	2	4	4	4
o_2	6	7	5	5	3	3
o_3	6	7	7	5	3	3
o_4	4	4	4	7	2	3
o_5	3	4	4	6	7	6
o_6	4	5	4	6	3	7

Table 3
Score table of the fuzzy soft set (S, P).

	Row-sum(r_i)	Column-sum(t_i)	Score(s_i)
o_1	25	30	−5
o_2	29	31	−2
o_3	31	26	5
o_4	24	33	−9
o_5	30	22	8
o_6	29	26	3

The column-sum t_j of object o_j is computed by

$$t_j = \sum_{i=1}^{n} c_{ij} \quad (2)$$

Finally, the score s_i of object o_i is defined as

$$s_i = r_i - t_i \quad (3)$$

When dealing with decision making problems by Algorithm 1, the objects with the maximum score computed from the comparison table will be regarded as the optimal decision, so this method is called "the score based method" in this paper. Here is an example to illustrate it:

Example 1. [33]: Let $U = \{o_1, o_2, \ldots, o_6\}$ be the universe of objects. The tabular representation of the fuzzy soft set (S, P) (with choice values) is given by Table 1. The comparison table of (S, P) is shown by Table 2. Then we obtain Table 3, namely the score table of (S, P), by computing the row-sum r_i, column-sum t_i and the score s_i for each object o_i. From Table 3, it is clear that the optimal decision is o_5 since it has the maximum score $s_5 = 8$.

In [37], Kong et al. argued that "the score based method" was incorrect since the decision result obtained by using "the score based method" is not always the object with the maximum choice value. Besides, they revised Algorithm 1 from Step 4 by redesigning c_{ij} and r_i as follows.

$$c_{ij} = \sum_{k=1}^{m} (f_{ik} - f_{jk}) \quad (4)$$

$$r_i = \sum_{j=1}^{m} c_{ij} \quad (5)$$

where f_{ik} is the membership value of object o_i for the kth attribute, m is the number of attributes. The decision set obtained by Kong et al.'s revised algorithm is o_k if $r_k = max_i r_i$.

In [32], Feng et al. deduced Kong's c_{ij} as follows

$$c_{ij} = \sum_{k=1}^{m} f_{ik} - \sum_{k=1}^{m} f_{jk} = c_i - c_j \quad (6)$$

where c_i is the sum total of all membership values of object o_i with respect to different attributes, which is called the fuzzy choice value of object o_i.

The object with the maximum fuzzy choice value, instead of the object with the maximum score, will be selected as the optimal decision by Kong's redesigned algorithm [32]. Hence, Kong et al.'s algorithm [37] can be called "the fuzzy choice value based method". However, Feng et al. [32] argued that the direct addition of all the membership values with respect to different attributes in a fuzzy soft set is not always reasonable, it no longer represents the number of (good) attributes possessed by an object in decision making. Furthermore, they provided an example without setting a real-life circumstance to argue that the score based method is more suitable than the fuzzy choice value based method.

Actually, it is hard to determine whether "the score based method" or "the fuzzy choice value based method" is more reasonable without setting a certain circumstance, since their decision criteria are different. "the score based method" is to used to select the objects which cooperate with more attributes in quantity, whereas "the fuzzy choice value based method" is used to select the objects which cooperate with more attributes in quality. Both approaches have some limitations, and there should be proposed new methods to overcome these limitations.

1. For fuzzy choice value based method, the main limitation is that the direct addition of all membership values with respect to attributes is not always reasonable. To overcome this limitation:
   - Method 1: Firstly, if the direct addition of all membership values is not reasonable in some cases, we can consider other synthesized measures to construct the fuzzy choice value by the membership values with respect to every attribute. For instance, weight the attributes with the help of relevant experts and then use the OWA operator to compute the fuzzy choice value of each object.
   - Method 2: When it is hard to determine the weights for different attributes in some specific cases, Feng et al.'s adjustable approach in [32], i.e. translating a fuzzy soft set into a soft set by using threshold values is another choice to make the decision result more reasonable. By selecting certain thresholds and using corresponding decision rules, a fuzzy soft set will be translated into a crisp soft set, then choice value of objects in a soft set, instead of fuzzy choice value of objects in a fuzzy soft set, will be used to measure objects. If this approach is taken, selecting the most suitable threshold values according to practical circumstances becomes the most important task.
2. For the score based method, two main limitations of Algorithm 1 can be listed as below. To overcome these limitations, we will provide an improved algorithm and some new algorithms in Sections 3.2 and 3.3, respectively.
   - Limitation 1: During the process of decision making, sometimes new attributes need be added if the existing attributes are not enough to embody the character of objects. On the contrary, some attribute need to be deleted if these attributes are proven to be ineffective to the decision result. According to the calculation mechanism for scores of objects in Algorithm 1, a new comparison table has to be conducted when a set of attributes need to be added/deleted, which indicates a large amount of recalculations should be involved in order to obtain a new solution set.
   - Limitation 2 [32]: There exist some fuzzy soft set based decision problems in which Algorithm 1 cannot be successfully used to reach an optimal decision.

Table 4
Tabular representation of the fuzzy soft set (F_1, E_1) with choice values and scores.

	e_1	e_2	e_3	e_4	Choice value(c_i)	Score(s_i)
o_1	0.92	0.88	0.08	0.12	$c_1 = 2.0$	$s_1 = 0$
o_2	0.82	0.60	0.18	0.40	$c_2 = 2.0$	$s_2 = 0$
o_3	0.24	0.46	0.83	0.47	$c_3 = 2.0$	$s_3 = 0$
o_4	0.12	0.40	0.96	0.52	$c_4 = 2.0$	$s_4 = 0$

Example 2. [32]: Let (F_1, E_1) be a fuzzy soft set and Table 4 be its tabular representation. From Table 4, it is clear that all these objects have the same score (i.e., $s_1 = s_2 = s_3 = s_4 = 0$) and the same fuzzy choice value (i.e., $c_1 = c_2 = c_3 = c_4 = 2.0$). By using both the score based method and the fuzzy choice value based method we could hardly arrive at the final optimal decision, since any one of them could be selected as the optimal candidate.

3.2. An improved method of "the score based method"

Based on the limitations analysis of "the score based method" in Section 3.1, in this subsection, by introducing a tool called D-Score table, we will provide an equivalence approach of Algorithm 1 which successfully overcomes Limitation 1 of "the score based method".

Definition 6. Let $U = \{o_1, o_2, \ldots, o_n\}$ be the universe and $A = \{e_1, e_2, \ldots, e_m\}$ be the attribute set. The D-Score of object o_i on e_l is denoted by $S(o_i)(e_l)$ and defined by

$$S(o_i)(e_l) = R(o_i)(e_l) - T(o_i)(e_l) \quad (7)$$

where $R(o_i)(e_l) = |\{o_j \in U | \mu_{F(e_l)}(o_i) \geq \mu_{F(e_l)}(o_j)\}|$, $T(o_i)(e_l) = |\{o_j \in U | \mu_{F(e_l)}(o_j) \geq \mu_{F(e_l)}(o_i)\}|$.

The D-Score of object o_i is denoted by S_i and defined as

$$S_i = \sum_{l=1}^{m} S(o_i)(e_l). \quad (8)$$

The D-Score table is a table in which rows are labelled by the attributes $e_1, e_2, \ldots, e_m$ and columns are labelled by the objects $o_1, o_2, \ldots, o_n$. The entry corresponding to attribute e_l and object o_i is denoted by $S(o_i)(e_l)$.

An algorithm based on the D-Score table of a fuzzy soft set is given (see Algorithm 2).

Algorithm 2.

[Step 1.] Input a fuzzy soft set (F, A).
[Step 2.] Present the D-Score table for (F, A) and compute the D-Score S_i of o_i, $\forall i$.
[Step 3.] The optimal decision is to select o_j if $S_j = max_i S_i$.
[Step 4.] If j has more than one value, then any o_j can be chosen as the decision result.

Theorem 1. *Let (F, A) be a fuzzy soft set on U. For any $o_i \in U$, calculate its score s_i by Algorithm 1 and its D-Score S_i by Algorithm 2, then we have $s_i = S_i$.*

Proof. Since $\sum_{j=1}^{n} c_{ij} = \sum_{l=1}^{m} R(o_i)(e_l)$ and $\sum_{j=1}^{n} c_{ji} = \sum_{l=1}^{m} T(o_i)(e_l)$, we obtain $s_i = r_i - t_i = \sum_{j=1}^{n} c_{ij} - \sum_{j=1}^{n} c_{ji} = \sum_{l=1}^{m} R(o_i)(e_l) - \sum_{l=1}^{m} T(o_i)(e_l) = \sum_{l=1}^{m} (R(o_i)(e_l) - T(o_i)(e_l)) = \sum_{l=1}^{m} S(o_i)(e_l) = S_i$.

Example 3. Consider the fuzzy soft set (S, P) in Example 1, the D-Score table of (S, P) is presented as Table 5. For any $o_i \in \{o_1, o_2, \ldots, o_6\}$, its score s_i in Table 3 is equal to its D-Score S_i in Table 5.

By Theorem 1, we know that for any object $o_i \in U$, its D-Score S_i obtained by Algorithm 2 is always the same as its score s_i obtained

Table 5
The D-Score table of (S, P) with D-Scores.

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	D-Score (S_i)
o_1	−1	−5	1	5	−5	−2	2	S_1 = −5
o_2	−1	−2	4	−3	0	−2	2	S_2 = −2
o_3	3	−2	4	−3	0	−2	5	S_3 = 5
o_4	5	1	−3	−3	−3	−2	−4	S_4 = −9
o_5	−5	4	−3	3	5	5	−1	S_5 = 8
o_6	−1	4	−3	1	3	3	−4	S_6 = 3

Table 6
The tabular representation of (G, P').

	e'_1	e'_2	e'_3
o_1	0.3	0.3	0.4
o_2	0.4	0.7	0.5
o_3	0.4	0.3	0.6
o_4	0.5	0.2	0.7
o_5	0.6	0.1	0.8
o_6	0.3	0.5	0.2

Table 7
D-Score table of $(H, P \cup P')$.

	S_i	e'_1	e'_2	e'_3	S'_i	$S_i + S'_i$
o_1	−5	−4	0	−3	−7	−12
o_2	−2	0	5	−1	4	2
o_3	5	0	0	1	1	6
o_4	−9	3	−3	3	3	−6
o_5	8	5	−5	5	5	13
o_6	3	−4	3	−5	−6	−3

by Algorithm 1, which indicates that the optimal decision sets obtained by Algorithms 1 and 2 are always the same, and thus Algorithms 1 and 2 are equivalent. The "score" and "D-Score" of an object will not be distinguished in the following discussion since they are always equal.

Suppose that (F, E_1) is an original fuzzy soft set and a new attribute set $E_2 = \{e'_1, e'_2, \ldots, e'_r\}$ should be added to E_1. If we use Algorithm 1, to obtain the scores of objects, we have to compute the new comparison table for the new fuzzy soft set $(H, E_1 \cup E_2)$. If we use Algorithm 2, after adding attributes, we only need to calculate the D-Score table of (G, E_2) to obtain the D-Score table for $(H, E_1 \cup E_2)$. In this way, we say although Algorithms 1 and 2 are equivalent, Algorithm 2 has the advantage that reduce the time consumption by avoid redundant computations of Algorithm 1 when attributes are added/deleted in a decision making problem. Adding attributes and deleting attributes are similar cases, so we only discuss the cases when attributes should be added.

Here is an example to illustrate the convenience of Algorithm 2 in avoiding redundant computations:

Example 4. Let (S, P) be the fuzzy soft set on U given by Table 1 in Example 1. Suppose that some new attributes $P' = \{e'_1, e'_2, e'_3\}$ should be added to P, let (G, P') be the corresponding fuzzy soft set which is shown by Table 6. If we use Algorithm 2, then we only need calculate the D-Score table for (G, P'). For an object o_i, its D-Score in $(H, P \cup P')$ is the sum of its D-Score in (S, P) and its D-Score in (G, P'), i.e., $S_i + S'_i$. The D-Score table of $(H, P \cup P')$ is shown by Table 7. In contrast, if we use Algorithm 1, we need recalculate all issues in the new comparison table of $(H, P \cup P')$, which is shown by Table 8.

Here is an example carried on data of moderate size to illustrate the advantage of Algorithm 2 in reducing time consumption:

Example 5. Suppose that there are n objects that are related with m attributes in the fuzzy soft set which we will apply to make the decision. By writing codes in C++, an experiment is performed on a PC Intel Core i-5 with 4 GB RAM and Windows 7 as operating system. This experiment is divide into two stages. In the first stage, the time consumption will be tested when the number of objects (n) keeps constant as 200 while the number of parameters "m" changes among 50, 100, 150, 200, 250, 300, 350, 400, 450, as is shown in Table 9. In the second stage, 10 new parameters and corresponds new data will be added to the initial data set, and the time consumption are shown in Table 10 for obtaining the final decision results on the basis of the median decision results obtained in stage 1. From Table 9 we can easily observe that based on the same scale of initial fuzzy soft sets, the time consumption for achieving decision results by using Algorithms 1 and 2 are very similar. However, Table 10 show that in the second stage when 10 new parameters needed to be considered, the second-stage time consumption increases with the number of the whole parameter sets increases if Algorithm 1 is applied, whereas the time consumption stay unchanged if Algorithm 2 is adopted. This example serves as a strong evidence for that algorithm 2 effectively decreases the time consumption when parameters are requested to be added to a decision making problem during the decision process.

Table 8
Comparison table of the fuzzy soft set $(H, P \cup P')$.

	o_1	o_2	o_3	o_4	o_5	o_6
o_1	10	4	3	5	5	6
o_2	9	10	7	6	4	6
o_3	9	9	10	6	4	5
o_4	6	6	6	10	3	5
o_5	5	6	6	8	10	8
o_6	6	5	5	7	4	10

Table 9
Time consumption in the first stage.

m	50	100	150	200	250	300	350	400	450
A1[1] (s)	0.020	0.040	0.060	0.081	0.100	0.120	0.140	0.160	0.180
A2 (s)	0.020	0.040	0.060	0.081	0.101	0.121	0.143	0.164	0.184

[1] In Tables 9 and 10, Algorithms 1 and 2 are denoted by A1 and A2, respectively. The time consumption is measured in seconds.

Table 10
Time consumption in the second stage.

m	60	110	160	210	260	310	360	410	460
A1 (s)	0.024	0.044	0.064	0.084	0.104	0.124	0.145	0.164	0.184
A2 (s)	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004

Lemma 1. *Let $x_1, x_2, x_3, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ be two number sequences, if $x_i \le x_j \Leftrightarrow y_i \le y_j$ (for $\forall i, j \in \{1, 2, 3, \ldots, n\}$), the two sequences are called the same ordered and this relationship is denoted by $q(x_1, x_2, x_3, \ldots, x_n) = q(y_1, y_2, \ldots, y_n)$.*

Theorem 2. *Let (F, A) and (F', A) be two fuzzy soft sets on the universe U. Suppose that o_i is an object in the universe U. In the fuzzy soft set (F, A), denote the D-Score of o_i on e_l by $S(o_i)(e_l)$ and calculate the D-Score of o_i by $S_i = \sum_{l=1}^{m} S(o_i)(e_l)$. In the fuzzy soft set (F', A), denote the D-Score of o_i on e_l by $S'(o_i)(e_l)$ and calculate the D-Score of o_i by $S'_i = \sum_{l=1}^{m} S'(o_i)(e_l)$. If $q(S(o_1)(e_l), S(o_2)(e_l), \ldots, S(o_n)(e_l)) = q(S'(o_1)(e_l), S'(o_2)(e_l), \ldots, S'(o_n)(e_l))$ for all $e_l \in A$, then we have $S_i = S'_i$.*

When dealing with a decision making problem by Algorithms 1 or 2, if the membership value of one object is larger than another object with respect to one attribute, then it is supposed that the former object relatively possesses that attribute. Under this supposition, how much the membership value of one object is larger than another has not been taken into consideration. That is, no matter the membership value of one object is larger than another by a little or by a lot, we all conclude the same when we computing D-

Table 11
Tabular representation of a fuzzy soft set (F, A) with D-Scores.

	e_1	e_2	e_3	e_4	D-Score(S_i)
o_1	0.8	h_{12}	0.1	0.1	$S_1 = -6$
o_2	0.5	0.7	0.5	0.3	$S_2 = 0$
o_3	0.3	0.6	0.8	0.9	$S_3 = 2$
o_4	0.2	0.9	h_{43}	0.8	$S_4 = 4$

Scores of objects. With respect to an attribute, as long as the order of the membership values of objects stay unchanged, the D-Scores of objects with respect to this attribute are determined. Hence, as long as the order of the membership values of objects with respect to every attribute stay unchanged, then the D-Scores of objects are determined, and thus the optimal decision set will stay unchanged.

Example 6. Let (F, A) be a fuzzy soft set and Table 11 be the tabular representation of it. When the value of h_{12} varies between [0, 0.6), the value of h_{43} varies between (0.8, 1], the order of membership values of objects with respect to every attribute stay unchanged, hence the D-Scores of objects are determined, i.e., $S_1 = -6$, $S_2 = 0$, $S_3 = 2$, $S_4 = 4$. By Algorithm 2, we obtain that the optimal candidate is o_4.

3.3. An adjustable decision making approach based on fuzzy soft sets

Benefitting from Feng's idea of introducing thresholds in [32], a comparison threshold will be taken into consideration when comparing the membership values of two objects with respect to a common attribute. In this way, a new approach will be provided on this basis of algorithm 2 (see Section 3.2). Only if the exceed degree of one membership value over another is not less than the comparison threshold, we say that the object relatively possesses that attribute. People can obtain different optimal decision set by setting different comparison threshold, which make this approach adjustable. This new adjustable approach can also be regarded as an improvement of the score based method in [33] since it follows the initial idea of "scores" of objects and successfully overcomes Limitation 2 of Algorithm 1.

3.3.1. t-Level D-Score table of fuzzy soft sets.

By introducing a measure called "t-level D-Score" of object and the new tool called t-level D-Score table, we present an adjustable approach to solve fuzzy soft set based decision making problems (see Algorithm 3).

Definition 7. Let $U = \{o_1, o_2, \ldots, o_n\}$ be the universe, E be the attribute set, $A \subseteq E$ and $A = \{e_1, e_2, \ldots, e_m\}$. Suppose that (F, A) is a fuzzy soft set over U. For $t \in [0, 1]$, the t-level D-Score of object o_i on e_l is denoted by $S(o_i)(e_l)_t$ and defined by

$$S(o_i)(e_l)_t = R(o_i)(e_l)_t - T(o_i)(e_l)_t, \tag{9}$$

where $R(o_i)(e_l)_t = |o_j \in U \backslash \{o_i\} : \mu_{F(e_l)}(o_i) - \mu_{F(e_l)}(o_j) \geq t|$ and $T(o_i)(e_l)_t = |o_j \in U \backslash \{o_i\} : \mu_{F(e_l)}(o_j) - \mu_{F(e_l)}(o_i) \geq t|$.

The t-level D-Score of object o_i is denoted by S_i^t and defined by

$$S_i^t = \sum_{l=1}^{m} S(o_i)(e_l)_t. \tag{10}$$

The t-level D-Score table is a square table in which rows are labelled by the attribute names $e_1, e_2, \ldots, e_m$, columns are labelled by the object names $o_1, o_2, \ldots, o_n$, and the entry corresponding to attribute e_l and object o_i is $S(o_i)(e_l)_t$. The t-level D-Score table can be regarded as an extension of the D-Score table for a fuzzy soft set, and $t \in [0, 1]$ can be viewed as a comparison threshold between membership values of two objects with regard to each attribute. For real-life application of fuzzy soft set based decision making, the threshold t can be chosen by decision makers according to their requirement.

3.3.2. Level D-Score table with respect to a comparison threshold fuzzy set

In the definition of t-level D-Score table, the level comparison threshold assigned to each attribute is always a constant value $t \in [0, 1]$. However, it may happen that the decision makers would like to impose different comparison thresholds on different attributes in some special decision making problems. To deal with such situations, we can use a function instead of a constant number as the comparison threshold.

Now we introduce a measure called the level D-Score with respect to λ, and the new tool called the level D-Score table with respect to a fuzzy set λ.

Definition 8. Let $U = \{o_1, o_2, \ldots, o_n\}$ be the universe, E be the attribute set, $A \subseteq E$ and $A = \{e_1, e_2, \ldots, e_m\}$. Let $\lambda : A \longrightarrow [0, 1]$ be a fuzzy set on A which is called a comparison threshold fuzzy set. The level D-Score of object o_i on e_l with respect to λ is denoted by $S(o_i)(e_l)_\lambda$ and defined by

$$S(o_i)(e_l)_\lambda = R(o_i)(e_l)_\lambda - T(o_i)(e_l)_\lambda \tag{11}$$

where $R(o_i)(e_l)_\lambda = |o_j \in U \backslash \{o_i\} : \mu_{F(e_l)}(o_i) - \mu_{F(e_l)}(o_j) \geq \lambda(e_l)|$ and $T(o_i)(e_l)_\lambda = |o_j \in U \backslash \{o_i\} : \mu_{F(e_l)}(o_j) - \mu_{F(e_l)}(o_i) \geq \lambda(e_l)|$. The level D-Score of object o_i with respect to λ is denoted by S_i^λ and defined by

$$S_i^\lambda = \sum_{l=1}^{m} S(o_i)(e_l)_\lambda. \tag{12}$$

The level D-Score table with respect to the fuzzy set λ is a square table in which rows are labelled by the attributes $e_1, e_2, \ldots, e_m$, columns are labelled by the objects $o_1, o_2, \ldots, o_n$, and the entry corresponding to attribute e_l and object o_i is $S(o_i)(e_l)_\lambda$.

The D-Score table with respect to a comparison threshold fuzzy set generalize the t-level D-Score table by substituting a fuzzy set $\lambda : A \longrightarrow [0, 1]$ for a constant $t \in [0, 1]$. Let $\hat{t}$ denote the constant fuzzy set on A given by $\hat{t}(e) = t$ for $e \in A$, then we immediately have $S(o_i)(e_l)_{\hat{t}} = S(o_i)(e_l)_t$, that is, the level D-Score table with respect to the constant fuzzy set $\hat{t}$ coincides with the t-level D-Score table.

Example 7 *(The mid-level-comparison threshold of a fuzzy soft set).* Let the universe $U = \{o_1, o_2, \ldots, o_n\}$, E be the attribute set, $A \subseteq E$ and $A = \{e_1, e_2, \ldots, e_m\}$. Based on the fuzzy soft set (F, A), we can define a fuzzy set $\lambda_F^{mid} : A \longrightarrow [0, 1]$ by

$$\lambda_F^{mid}(e_l) = \frac{1}{|U|}(\vee_{o_i \in U}\mu_{F(e_l)}(o_i) - \wedge_{o_i \in U}\mu_{F(e_l)}(o_i)),$$

for all $e_l \in A$. The fuzzy set λ_F^{mid} is called the mid-level-comparison threshold of fuzzy soft set (F, A). In addition, the level D-Score table with respect to the mid-level-comparison threshold fuzzy set λ_F^{mid} is called the mid-level D-Score table of fuzzy soft set (F, A). In what follows the mid-level-comparison rule will mean using the mid-level-comparison threshold and considering the mid-level D-Score table of a fuzzy soft set in decision making process.

For a concrete example of mid-level-comparison threshold fuzzy set and mid-level D-Score table, let us reconsider the fuzzy soft set (F_1, E_1) with its tabular representation given by Table 4. It is clear that the mid-level-comparison threshold of (F_1, E_1) is a fuzzy set

$$\lambda_{F_1}^{mid} = \{(e_1, 0.20), (e_2, 0.12), (e_3, 0.22), (e_4, 0.10)\}.$$

Example 8 *(The min-level-comparison threshold of a fuzzy soft set).* Let the universe $U = \{o_1, o_2, \ldots, o_n\}$, E be the attribute set, $A \subseteq E$ and

Table 12
The 0.15-level D-Score table of (F_1, E_1) with 0.15-level D-Scores.

	e_1	e_2	e_3	e_4	$S_i^{0.15}$
o_1	2	3	−2	−3	$S_1^{0.15}=0$
o_2	2	0	−2	1	$S_2^{0.15}=1$
o_3	−2	−1	2	1	$S_3^{0.15}=0$
o_4	−2	−2	2	1	$S_4^{0.15}=-1$

Table 13
The mid-level D-Score table of (F_1, E_1) with mid-level D-Scores.

	e_1	e_2	e_3	e_4	S_i^{mid}
o_1	2	3	−2	−3	$S_1^{mid}=0$
o_2	2	1	−2	0	$S_2^{mid}=1$
o_3	−2	−2	2	1	$S_3^{mid}=-1$
o_4	−2	−2	2	2	$S_4^{mid}=0$

$A=\{e_1, e_2, \ldots, e_m\}$. Based on the fuzzy soft set (F, A), we can define a fuzzy set $\lambda_F^{\min}: A \longrightarrow [0, 1]$ by

$$\lambda_F^{\min}(e_l) = \wedge_{\{o_i, o_j \in U\}} |\mu_{F(e_l)}(o_i) - \mu_{F(e_l)}(o_j)|,$$

for all $e_l \in A$. The fuzzy set $\lambda_F^{\min}$ is called the min-level-comparison threshold of the fuzzy soft set. In addition, the level D-Score table with respect to the min-level-comparison threshold fuzzy set $\lambda_F^{\min}$ is called the min-level D-Score table of fuzzy soft set (F, A). In what follows the min-level-comparison rule will mean using the min-level-comparison threshold and considering the min-level D-Score table of a fuzzy soft set in decision making process.

For a concrete example of min-level-comparison threshold fuzzy set and min-level D-Score table, let us reconsider the fuzzy soft set (F_1, E_1) with its tabular representation given by Table 4. It is clear that the min-level-comparison threshold of (F_1, E_1) is a fuzzy set

$$\lambda_{F_1}^{\min}(e_1) = |\mu_{F_1(e_1)}(o_1) - \mu_{F_1(e_1)}(o_2)| = 0.10,$$
$$\lambda_{F_1}^{\min}(e_2) = |\mu_{F_1(e_2)}(o_3) - \mu_{F_1(e_2)}(o_4)| = 0.06,$$
$$\lambda_{F_1}^{\min}(e_3) = |\mu_{F_1(e_3)}(o_1) - \mu_{F_1(e_3)}(o_2)| = 0.10,$$
$$\lambda_{F_1}^{\min}(e_4) = |\mu_{F_1(e_4)}(o_3) - \mu_{F_1(e_4)}(o_4)| = 0.05,$$
$$\lambda_{F_1}^{\min} = \{(e_1, 0.10), (e_2, 0.06), (e_3, 0.10), (e_4, 0.05)\}.$$

Example 9 *(The max-level-comparison threshold of a fuzzy soft set).* Let the universe $U=\{o_1, o_2, \ldots, o_n\}$, E be the attribute set, $A \subseteq E$ and $A=\{e_1, e_2, \ldots, e_m\}$. Based on the fuzzy soft set (F, A), we can define a fuzzy set $\lambda_F^{\max}: A \longrightarrow [0, 1]$ by

$$\lambda_F^{\max}(e_l) = \vee_{\{o_i, o_j \in U\}} (\mu_{F(e_l)}(o_i) - \mu_{F(e_l)}(o_j)),$$

for all $e_l \in A$. The fuzzy set $\lambda_F^{\max}$ is called the max-level-comparison threshold of the fuzzy soft set. In addition, the level D-Score table with respect to the max-level-comparison threshold fuzzy set $\lambda_F^{\max}$ is called the max-level D-Score table of fuzzy soft set (F, A). In what follows the max-level-comparison rule will mean using the max-level-comparison threshold and considering the max-level D-Score table of a fuzzy soft set in fuzzy soft set based decision making.

For a concrete example of max-level-comparison threshold fuzzy set and max-level D-Score table, let us reconsider the fuzzy soft set (F_1, E_1) with its tabular representation given by Table 4. It is clear that the max-level-comparison threshold of (F_1, E_1) is a fuzzy set

$$\lambda_{F_1}^{\max}(e_1) = \mu_{F_1(e_1)}(o_1) - \mu_{F_1(e_1)}(o_4) = 0.80,$$
$$\lambda_{F_1}^{\max}(e_2) = \mu_{F_1(e_2)}(o_1) - \mu_{F_1(e_2)}(o_4) = 0.48,$$
$$\lambda_{F_1}^{\max}(e_3) = \mu_{F_1(e_3)}(o_4) - \mu_{F_1(e_3)}(o_1) = 0.88,$$
$$\lambda_{F_1}^{\max}(e_4) = \mu_{F_1(e_4)}(o_4) - \mu_{F_1(e_4)}(o_1) = 0.40,$$
$$\lambda_{F_1}^{\max} = \{(e_1, 0.80), (e_2, 0.48), (e_3, 0.88), (e_4, 0.40)\}.$$

In the fuzzy soft set (F, A), the level D-Score of object o_i with respect to λ_F^{mid}, $\lambda_F^{\min}$ and $\lambda_F^{\max}$ are denoted by S_i^{mid}, $S_i^{\min}$ and $S_i^{\max}$, respectively. Now we present the level D-Scores based decision making approach as below (see Algorithm 3).

Algorithm 3.

[Step 1.] Input a fuzzy soft set (F, A).
[Step 2.] Input a comparison threshold fuzzy set $\lambda: A \longrightarrow [0, 1]$ (or give a comparison threshold value $t \in [0, 1]$; or choose the mid-level-comparison decision rule; or choose the min-level-comparison decision rule; or choose the max-level-comparison decision rule) for decision making.
[Step 3.] Present the level D-Score table with respect to fuzzy set λ of (F, A) and compute the level D-Score of o_i with respect to λ, i.e. S_i^λ, $\forall i$ (or present the t-level D-Score table of (F, A) and compute the t-level D-Score S_i^t of o_i, $\forall i$; or present the mid-level D-Score table of (F, A) and compute the mid-level D-Score S_i^{mid} of o_i, $\forall i$; or present the min-level D-Score table of (F, A) and compute the min-level D-Score $S_i^{\min}$ of o_i, $\forall i$; or present the max-level D-Score table of (F, A) and compute the max-level D-Score $S_i^{\max}$ of o_i, $\forall i$).
[Step 4.] The optimal decision, which is denoted by $D((F, A), \lambda)$, is to select o_j if $S_j^\lambda = \max_i S_i^\lambda$ (or denoted by $D((F, A), t)$ and select o_j if $S_j^t = \max_i S_i^t$; or denoted by $D((F, A), \lambda_F^{mid})$ and select o_j if $S_j^{mid} = \max_i S_i^{mid}$; or denoted by $D((F, A), \lambda_F^{\min})$ and select o_j if $S_j^{\min} = \max_i S_i^{\min}$; or denoted by $D((F, A), \lambda_F^{\max})$ and select o_j if $S_j^{\max} = \max_i S_i^{\max}$).
[Step 5.] If j has more than one value then any one of o_j may be chosen.

In the last step of Algorithm 3, one may go back to the second step and change the comparison threshold that he/she once used so as to adjust the final optimal decision, especially when there are too many "optimal choices" to be chosen.

When comparing membership values of objects with respect to different attributes to evaluate the level D-Scores of objects, by introducing the comparison thresholds, Algorithm 3 takes into account both the quality and the quantity of attributes each object occupies. In this way, some decision making problems which cannot be dealt with by using Algorithm 1 can be solved by using Algorithm 3. In other words, Algorithm 3 overcomes Limitation 2 of Algorithm 1. Here is an example to illustrate.

Example 10. It is clear the 0.15-level D-Score table of (F_1, E_1) in Example 2 is given by Table 12. From Table 12, the 0.15-level D-Scores of objects are: $S_1^{0.15}=0$, $S_2^{0.15}=1$, $S_3^{0.15}=0$, $S_4^{0.15}=-1$. It indicates that when using a comparison threshold value $t=0.15$, we can obtain that o_2 is the optimal candidate by Algorithm 3.

The mid-level D-Score table of (F_1, E_1) is given by Table 13. From Table 13, we obtain $S_1^{mid}=0, S_2^{mid}=1, S_3^{mid}=-1, S_4^{mid}=0$. It follows that if the mid-level-comparison decision rule is chosen, we also obtain o_2 as the optimal candidate by Algorithm 3.

Theorem 3. *Assuming that an actual decision making context is reduced to a fuzzy soft set (F, A) on the universe U. Let $D((F, A), \lambda)$ be the optimal decision set got by Algorithm 3, where λ is a comparison threshold fuzzy set of (F, A). If $\lambda(e_l) > \lambda_F^{\max}(e_l)$ for $\forall e_l \in A$, then we have $D((F, A), \lambda)=U$.*

3.3.3. *Weighted D-Score based decision making*

Now we introduce the concepts of weighted D-Scores, weighted t-level D-Scores and weighted level D-Scores with respect to a fuzzy set, and pay attention to their applications in decision making problems based on weighted fuzzy soft set.

Definition 9. [32] Let E be a set of attributes and $A \subseteq E$. A weighted fuzzy soft set is a triple $\mathfrak{T} = \{F, A, w\}$ where (F, A) is a fuzzy soft set over U, and $w: A \longrightarrow [0, 1]$ is a weight function specifying the weight $w_l = w(e_l)$ for each attribute $e_l \in A$.

Definition 10. Let $U=\{o_1, o_2, \ldots, o_n\}$ be the universe, E be the attribute set, $A \subseteq E$ and $A=\{e_1, e_2, \ldots, e_m\}$. Let $\mathfrak{T} = \{F, A, w\}$ be a weighted fuzzy soft set where (F, A) is a fuzzy soft set over U and $w : A \longrightarrow [0, 1]$ is a weight function specifying the weight $w_l = w(e_l)$ for each attribute $e_l \in A$. The weighted D-Score of $o_i \in U$ is defined by

$$\tilde{S}_i = \sum_{l=1}^{m} w_l \times S(o_i)(e_l), \qquad (13)$$

where $S(o_i)(e_l)$ is the D-Score of object o_i on e_l calculated by Eq. (7).

Definition 11. Let $U = \{o_1, o_2, \ldots, o_n\}$ be the universe, E be the attribute set, $A \subseteq E$ and $A = \{e_1, e_2, \ldots, e_m\}$. Let $\mathfrak{T} = \{F, A, w\}$ be a weighted fuzzy soft set where (F, A) is a fuzzy soft set over U and $w : A \longrightarrow [0, 1]$ is a weight function specifying the weight $w_l = w(e_l)$ for each attribute $e_l \in A$. For $t \in [0, 1]$, the weighted t-level D-Score of $o_i \in U$ is defined by

$$\tilde{S}_i^t = \sum_{l=1}^{m} \tilde{S}(o_i)(e_l)_t \qquad (14)$$

where $\tilde{S}(o_i)(e_l)_t = w_l \times S(o_i)(e_l)_t$ and $S(o_i)(e_l)_t$ is the t-level D-Score of object o_i on e_l calculated by Eq. (9).

Definition 12. Let $U = \{o_1, o_2, \ldots, o_n\}$ be the universe, E be the attribute set, $A \subseteq E$ and $A = \{e_1, e_2, \ldots, e_m\}$. Let $\mathfrak{T} = \{F, A, w\}$ be a weighted fuzzy soft set where (F, A) is a fuzzy soft set over U and $w : A \longrightarrow [0, 1]$ is a weight function specifying the weight $w_l = w(e_l)$ for each attribute $e_l \in A$. Let $\lambda : A \longrightarrow [0, 1]$ be a fuzzy set on A which is called a comparison threshold fuzzy set. The weighted level D-Score of $o_i \in U$ with respect to λ is defined by

$$\tilde{S}_i^\lambda = \sum_{l=1}^{m} \tilde{S}(o_i)(e_l)_\lambda \qquad (15)$$

where $\tilde{S}(o_i)(e_l)_\lambda = w_l \times S(o_i)(e_l)_\lambda$ and $S(o_i)(e_l)_\lambda$ is the level D-Score of object o_i on e_l with respect to fuzzy set λ calculated by Eq. (11).

For every $o_i \in U$, its level D-Score with respect to λ which is calculated by Eq. (12) (t-level D-Score calculated by Eq.(10)) can be regarded as its weighted level D-Score with respect to λ (weighted t-level D-Score) in which every attribute be of equal importance.

The weighted level D-Score table with respect to the fuzzy set λ (weighted t-level D-Score table) is a square table in which rows are labelled by the attributes $e_1, e_2, \ldots, e_m$, columns are labelled by the objects $o_1, o_2, \ldots, o_n$ of the universe, and the entry corresponding to attribute e_l and object o_i is $\tilde{S}(o_i)(e_l)_\lambda$ ($\tilde{S}(o_i)(e_l)_t$). The weighted level D-Score table with respect to the mid-level-comparison threshold fuzzy set λ_F^{mid}, the min-level-comparison threshold fuzzy set $\lambda_F^{\min}$ and the max-level-comparison threshold fuzzy set $\lambda_F^{\max}$ are called the weighted-mid-level, weighted-min-level and weighted-max-level D-Score table of the weighted fuzzy soft set (F, A, w), respectively. In addition, the weighted level D-Score of object $o_i \in U$ with respect to λ_F^{mid}, $\lambda_F^{\min}$ and $\lambda_F^{\max}$ in the weighted fuzzy soft set (F, A, w) are denoted by weighted-mid-level D-Score ($\tilde{S}_i^{mid}$), weighted-min-level D-Score ($\tilde{S}_i^{\min}$) and weighted-max-level D-Score ($\tilde{S}_i^{\max}$), respectively.

Let $\hat{0}$ denote the constant fuzzy set on A given by $\hat{0}(e_l) = 0$ for $\forall e_l \in A$. Then we immediately have $\tilde{S}_i^{\hat{0}} = \sum_{l=1}^{m} \tilde{S}(o_i)(e_l)_{\hat{0}} = \sum_{l=1}^{m} w_l \times S(o_i)(e_l)_{\hat{0}} = \sum_{l=1}^{m} w_l \times S(o_i)(e_l)$, which is the weighted D-Score of object $o_i \in U$.

Algorithm 4 improves Algorithm 3 to deal with the decision making problems in which weights of attributes are different based on weighted fuzzy soft set and the corresponding weighted level D-Scores of objects with respect to a fuzzy set λ (weighted t-level D-Scores). In Algorithm 4, we take the weights of attributes into consideration and compute the weighted level D-Scores instead of level D-Scores. Since λ_F^{mid}, $\lambda_F^{\min}$ and $\lambda_F^{\max}$ are actually special comparison threshold fuzzy sets for (F, A), we will not sketch them out and highlight them in Algorithm 4.

Table 14
Tabular representation of weighted fuzzy soft set (F_1, E_1, w).

	$e_1, w_1 = 0.8$	$e_2, w_2 = 0.4$	$e_3, w_3 = 0.5$	$e_4, w_4 = 0.3$
o_1	0.92	0.88	0.08	0.12
o_2	0.82	0.60	0.18	0.40
o_3	0.24	0.46	0.83	0.47
o_4	0.12	0.40	0.96	0.52

Table 15
The weighted-mid-level D-Score table of (F_1, E_1, w) with weighted-mid-level D-Scores.

	$e_1, w_1 = 0.8$	$e_2, w_2 = 0.4$	$e_3, w_3 = 0.5$	$e_4, w_4 = 0.3$	$\tilde{S}_i^{mid}$
o_1	1.6	1.2	−1	−0.9	$\tilde{S}_1^{mid} = 0.9$
o_2	1.6	0.4	−1	0	$\tilde{S}_2^{mid} = 1.0$
o_3	−1.6	−0.8	1	0.3	$\tilde{S}_3^{mid} = -1.1$
o_4	−1.6	−0.8	1	0.6	$\tilde{S}_4^{mid} = -0.8$

Algorithm 4.

[Step 1.] Input a weighted fuzzy soft set (F, A, w).
[Step 2.] Input a comparison threshold fuzzy set $\lambda : A \longrightarrow [0, 1]$ (or give a comparison threshold value $t \in [0, 1]$) for the weighted fuzzy soft set (F, A, w).
[Step 3.] Present the weighted D-Score table with respect to fuzzy set λ for the weighted fuzzy soft set (F, A, w) and compute $\tilde{S}_i^\lambda$ ($\tilde{S}_i^t$), which is the weighted level D-Score with respect to λ (weighted t-level D-Score) of o_i, $\forall i$.
[Step 4.] The optimal decision is to select o_j if $\tilde{S}_j^\lambda = \max_i \tilde{S}_i^\lambda$ (or $\tilde{S}_j^t = \max_i \tilde{S}_i^t$).
[Step 5.] If j has more than one value then any one of o_j may be chosen.

Similarly to Algorithm 3, if too many "optimal choices" are obtained by Algorithm 4, one can also go back to the second step and change the comparison threshold previously used so as to adjust the final optimal decision. The notion of weighted level D-Score provide a framework for solving decision making problems by score based method in which all the attributes may not be of equal importance.

Example 11. Suppose that there are four candidates who apply for a position in a work place, the set of candidates $U = \{o_1, o_2, o_3, o_4\}$ is characterized by a attribute set $E_1 = \{e_1, e_2, e_3, e_4\}$ which is 'e_1 = technical information' ($w_1 = 0.8$), 'e_2 = experience' ($w_2 = 0.4$), 'e_3 = training' ($w_3 = 0.5$), 'e_4 = appearance' ($w_4 = 0.3$), respectively. Thus the decision maker has a weight function $w : E_1 \longrightarrow [0, 1]$ and the fuzzy soft set (F_1, E_1) in Example 2 is changed into a weighted fuzzy soft set (F_1, E_1, w) with its tabular representation as shown in Table 14.

As an adjustable approach, the decision maker can select different comparison thresholds when dealing with the problem. If we use the mid-level threshold in this case, we obtain the weighted-mid-level D-Score table of (F_1, E_1, w) as Table 15, then the optimal decision is o_2 by Algorithm 4.

4. Improving decision making approaches based on rough soft sets

This section first discusses the limitations of existing decision making approaches based on rough soft sets, some new approaches will be then provided to overcome such limitations.

4.1. Limitations of decision making methods based on rough soft sets

There are two main limitations of the rough soft sets based decision making approaches:

- Limitation 1: The existing few decision making algorithms are far from enough to meet the various demands of applications.

Table 16
The tabular representation of $\underline{Apr}_R(\mathfrak{S})$.

	o_1	o_2	o_3	o_4	o_5	o_6	o_7	o_8
e_1	0	0	0	0	0	0	0	0
e_2	1	1	0	0	0	0	0	0
e_3	0	0	1	1	0	0	0	0
e_4	0	0	1	1	1	1	0	0
e_5	0	0	0	0	0	0	0	0

Table 17
The tabular representation of $\overline{Apr}_R(\mathfrak{S})$.

	o_1	o_2	o_3	o_4	o_5	o_6	o_7	o_8
e_1	1	1	1	1	0	0	0	0
e_2	1	1	0	0	1	1	0	0
e_3	0	0	1	1	1	1	0	0
e_4	0	0	1	1	1	1	0	0
e_5	0	0	1	1	1	1	1	1

Decision making approaches based on rough soft sets has not obtained enough attention by the researchers so far. Let us reconsider the two material selection algorithms based on rough soft sets proposed by Ma et al. in [34], one of the algorithms is used to catch the closest one in all of the materials, the other one is used to reach the most representative materials. The scope of application of these algorithms is limited. The research on rough soft sets in decision making calls for improvement by proposing more approaches to meet various practical demands.

- Limitation 2: The research on application of rough soft set in group decision making has not appeared yet.

The combination of rough set theory and soft set theory shows great potential in solving group decision making problems. However, when a group decision making problem is solved by using [26] and MSR-sets [35], every expert have to present his/her best choice alternatives. In other words, every expert has already made their own decision before carrying out the group decision making process. This strict requirement is hardly fulfilled in some real-life situations since expert may prefer provide only their assessment of alternatives/candidates in different aspects when they are short of knowledge, time or just lack of confidence. All of the researchers only concentrate on the application of soft rough set in group decision making so far, no attempt has been done in solving such problems by using rough soft set. So the problem arises that how can we make full use of the information in form of assessments on candidates with respect to different aspects provided by the decision makers during the decision process and deal with the group decision making problem by using rough soft set?

In the following parts, we will come up with two new decision making algorithms based on rough soft set which enriches the scopes of applications and conquers Limitation 1 to a certain extend. Afterwards, a group decision making algorithm based on rough soft set which successfully solve group decision making problems when the initial evaluation information provided by experts are their assessments on alternatives from different aspects, which overcomes Limitation 2 and fills the blank that this respect field is studied.

4.2. Decision making methods based on rough soft sets

In this part, some new methods will be provided by deciding a most perspective attribute or a target attribute set using the rough approximation operators on a given soft set $\mathfrak{S}=(F, A)$.

Let $U=\{o_1, o_2, \ldots, o_n\}$ be the universe of objects and E be a set of related attributes. Let $\mathfrak{S}=(F, A)$ be a soft set over U and $A=\{e_1, e_2, \ldots, e_m\} \subseteq E$. Let (U, R) be a Pawlak approximation space where R be an equivalence relation on U. For $X \subseteq U$, let $|X|$ denote the number of objects in X, let $|X|_R$ denote the number of classes in U contained in X, where the classification is determined by an equivalent relation R. Then a decision making algorithm based on rough soft sets can be presented as Algorithm 5.

Algorithm 5.

[Step 1.] Input Pawlak approximation space (U, R) and a soft set $\mathfrak{S}=(F, A)$ on U.
[Step 2.] Compute the lower and upper rough soft approximation operators $\underline{Apr}_R(\mathfrak{S})$ and $\overline{Apr}_R(\mathfrak{S})$ on S, respectively.
[Step 3.] Select a threshold λ, which satisfies the condition $\lambda \in [0, \frac{|\underline{F}_R(e_1)\cup\underline{F}_R(e_2)\ldots\underline{F}_R(e_m)|_R}{|U|_R}]$.
[Step 4.] For each attribute $e_i \in A$, calculate $\bar{F}_R(e_i)$. If there is an attribute $e_i \in A$, s.t. $\bar{F}_R(e_i)=U$, turn to Step 5; if else, turn to Step 6.
[Step 5.] Calculate

$$\frac{|\underline{F}_R(e_k)|_R}{|U|_R}=\max_{\{e_i:\bar{F}_R(e_i)=U\}}\frac{|\underline{F}_R(e_i)|_R}{|U|_R}.$$

If $\frac{|\underline{F}_R(e_k)|_R}{|U|_R} \geq \lambda$, then $\{e_k\}$ is the expected decision set; if else, turn to Step 6.
[Step 6.] For all two attributes $e_i, e_j \in A$, calculate $\bar{F}_R(e_i) \cup \bar{F}_R(e_j)$. If there are attributes $e_i, e_j \in A$, s.t. $\bar{F}_R(e_i) \cup \bar{F}_R(e_j)=U$, turn to Step 7; If else, turn to Step 8.
[Step 7.] Calculate

$$\frac{|\underline{F}_R(e_k)\cup\underline{F}_R(e_l)|_R}{|U|_R}=\max_{\{e_i,e_j\in A:\bar{F}_R(e_i)\cup\bar{F}_R(e_j)=U\}}\frac{|\underline{F}_R(e_i)\cup\underline{F}_R(e_j)|_R}{|U|_R}.$$

If $\frac{|\underline{F}_R(e_k)\cup\underline{F}_R(e_l)|_R}{|U|_R} \geq \lambda$, then $\{e_k, e_l\}$ is the expected decision set; if else, turn to Step 8.
[Step 8.] If there are q $(q<m)$ attributes $e'_{i_1}, e'_{i_2}, \ldots, e'_{i_q} \in A$ can be found satisfying $\bar{F}_R(e'_{i_1}) \cup \bar{F}_R(e'_{i_2}) \cup \ldots \bar{F}_R(e'_{i_q})=U$, then calculate

$$\frac{|\underline{F}_R(e_{i_1})\cup\underline{F}_R(e_{i_2})\cup\ldots\underline{F}_R(e_{i_q})|_R}{|U|_R}$$

$$=\max_{\{e'_{i_1},e'_{i_2},\ldots,e'_{i_q}\in A:\bar{F}_R(e'_{i_1})\cup\bar{F}_R(e'_{i_2})\cup,\ldots,\bar{F}_R(e'_{i_q})=U\}}\frac{|\underline{F}_R(e'_{i_1})\cup\underline{F}_R(e'_{i_2})\cup\ldots\underline{F}_R(e'_{i_q})|_R}{|U|_R},$$

where $e_{i_1}, e_{i_2}, \ldots, e_{i_q} \in A$, if $\frac{|\underline{F}_R(e_{i_1})\cup\underline{F}_R(e_{i_2})\cup\ldots\underline{F}_R(e_{i_q})|_R}{|U|_R} \geq \lambda$, then $\{e_{i_1}, e_{i_2}, \ldots, e_{i_q}\}$ is an expected decision set (it is worth noticing that the expected decision set may be not unique);
if else, we will check $q+1$ attributes, $q+2$ attributes, ..., $q+(m-q)$ attributes, until we find the expected decision set.

The primary motivation for designing Algorithm 5 is to select the parameters whose upper approximation cover all the objects when their lower approximation cover a specified number of object classes in U. This selection mechanism can be used in many practical situations. Here is an example to illustrate:

Example 12. Suppose that a company decides to set up a working group for the expansion of business. To choose suitable members for the working group, they make a survey of the candidates on some professional skills (referred to as o_1, o_2, o_3, o_4, o_5, o_6, o_7 and o_8). Suppose that there are five candidates $A=\{e_1, e_2, e_3, e_4, e_5\}$ and each candidate have one or more skills: $F(e_1)=\{o_1, o_4\}$, $F(e_2)=\{o_1, o_2, o_6\}$, $F(e_3)=\{o_3, o_4, o_5\}$, $F(e_4)=\{o_3, o_4, o_5, o_6\}$ and $F(e_5)=\{o_3, o_5, o_8\}$. In this case, $(o_1, o_2) \in R$, $(o_3, o_4) \in R$, $(o_5, o_6) \in R$, $(o_7, o_8) \in R$ (R represents the equivalent relationship amongst skills). Now the skills are divide into four classes/types in U.

- If a candidate is good at one skill, he/she can be expected to handle its equivalent skills quicker (in a relative short time).
- If a candidate already handles all the skills in one class before the selection, he/she can be regarded as an expert in this type of skill.

The tabular representations of soft sets $\underline{Apr}_R(\mathfrak{S})$ and $\overline{Apr}_R(\mathfrak{S})$ are obtained as Tables 16 and 17, respectively.

Since $\frac{|\underline{F}_R(e_1)\cup\underline{F}_R(e_2)\ldots\underline{F}_R(e_5)|_R}{|U|_R}=\frac{3}{4}$, in this case we can set $\lambda=\frac{1}{2} \in [0, \frac{3}{4}]$.

The calculation process is given as below:

1. For any attribute $e_i \in A$, it is obvious that $\bar{F}(e_i) \neq U$. Then we skip to two attributes situation;
2. For two attributes: $\bar{F}_R(e_1) \cup \bar{F}_R(e_5) = U$, $\bar{F}_R(e_2) \cup \bar{F}_R(e_5) = U$, it is easy to obtain $\frac{|\underline{F}_R(e_1) \cup \underline{F}_R(e_5)|_R}{|U|_R} = 0$, $\frac{|\underline{F}_R(e_2) \cup \underline{F}_R(e_5)|_R}{|U|_R} = \frac{1}{4}$. Since $\max\{0, \frac{1}{4}\} < \frac{1}{2}$, we skip to three attributes situation;
3. For three attributes:
   $\bar{F}_R(e_1) \cup \bar{F}_R(e_5) \cup \bar{F}_R(e_2) = U$, $\frac{|\underline{F}_R(e_1) \cup \underline{F}_R(e_5) \cup \underline{F}_R(e_2)|_R}{|U|_R} = \frac{|\{o_1, o_2\}|_R}{|U|_R} = \frac{1}{4}$.
   $\bar{F}_R(e_1) \cup \bar{F}_R(e_5) \cup \bar{F}_R(e_3) = U$, $\frac{|\underline{F}_R(e_1) \cup \underline{F}_R(e_5) \cup \underline{F}_R(e_3)|_R}{|U|_R} = \frac{|\{o_3, o_4\}|_R}{|U|_R} = \frac{1}{4}$.
   $\bar{F}_R(e_1) \cup \bar{F}_R(e_5) \cup \bar{F}_R(e_4) = U$, $\frac{|\underline{F}_R(e_1) \cup \underline{F}_R(e_5) \cup \underline{F}_R(e_4)|_R}{|U|_R} = \frac{|\{o_3, o_4, o_5, o_6\}|_R}{|U|_R} = \frac{1}{2}$.
   $\bar{F}_R(e_2) \cup \bar{F}_R(e_5) \cup \bar{F}_R(e_3) = U$, $\frac{|\underline{F}_R(e_2) \cup \underline{F}_R(e_5) \cup \underline{F}_R(e_3)|_R}{|U|_R} = \frac{|\{o_1, o_2, o_3, o_4\}|_R}{|U|_R} = \frac{1}{2}$.
   $\bar{F}_R(e_2) \cup \bar{F}_R(e_5) \cup \bar{F}_R(e_4) = U$, $\frac{|\underline{F}_R(e_2) \cup \underline{F}_R(e_5) \cup \underline{F}_R(e_4)|_R}{|U|_R} = \frac{|\{o_1, o_2, o_3, o_4, o_5, o_6\}|_R}{|U|_R} = \frac{3}{4}$.
   Here, $\frac{|\underline{F}_R(e_2) \cup \underline{F}_R(e_5) \cup \underline{F}_R(e_4)|_R}{|U|_R} = \max\{\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \frac{3}{4}\} = \frac{3}{4} > \frac{1}{2} = \lambda$, so we obtain that $\{e_2, e_4, e_5\}$ is the decision set.

1. If there is a skill which is not handled by anyone in the decision set, several of its equivalent skills must be handled by some memberships in the decision set. Thus, members in the decision set can be expected to handle all of the professional skills quicker (in a relative short time).
2. For at least two classes of skills, there are experts in the decision set. More classes of experts make the whole work more efficient.
3. Along with ensuring conditions 1–2, the members in the decision set will be the least, which contributes to save labor cost.
4. Along with ensuring conditions 1–3, with the same numbers of members, the number of skill classes with experts will be the most in the decision result set. Different types of experts make different types of work more efficient.

As following, another rough soft set based algorithm is proposed by selecting only one attribute whose upper approximation cover the most objects in the universe set.

Algorithm 6.

[Step 1.] Input Pawlak approximation space (U, R) and a soft set $\mathfrak{S} = (F, A)$ on U.
[Step 2.] Compute the upper rough soft approximation operator on $\mathfrak{S}$, i.e. $\bar{Apr}_R(\mathfrak{S})$.
[Step 3.] The optimal decision result is to select e_j if $|\bar{F}_R(e_j)| = \vee_{i \in \{1,2,\ldots,m\}} |\bar{F}_R(e_i)|$.
[Step 4.] If j has more than one value then any one of e_j may be chosen.

Example 13. Let us reconsider the decision making problem in Example 12. From Table 16, we can easily obtain that $|\bar{F}_R(e_4)| = \vee_{i \in \{1,2,\ldots,5\}} |\bar{F}_R(e_i)|$, so e_4 is the optimal decision by Algorithm 6. It is worth noticing that, by Algorithm 6, the optimal decision result is the candidate who has potential to handle most professional skills quicker (in a relative short time).

Remark 1. By Algorithms 5 and 6, we provide two different attempts of using rough soft sets to solve decision making problems. Different selection mechanisms make the methods have different scopes of application.

4.3. A group decision making method based on rough soft sets

Feng [39] and Zhan et al. [40] put forth approaches for group decision making problems based on soft rough sets [26] and MSR-sets [35], respectively. Benefitting from their ideas, now we will introduce an group decision making approach based on rough soft sets.

Assume that we have an expert group $G = \{T_1, T_2, \ldots, T_p\}$ consisting of p specialists to evaluate all the candidates $A = \{e_1, e_2, \ldots, e_m\}$. For each candidate, every specialist will be asked to provide an evaluation on him/her as aspect to all skills in $U = \{o_1, o_2, \ldots, o_n\}$ and will be requested to give judgement if a candidate is good at these skill or not. In this way, the judgements on all candidates with respect to all skills provided by every expert form a soft set. It is assumed that there exist some equivalent relationships between different skills. With these equivalent relationships, we can compute the rough approximations of the soft sets, the upper rough approximation of the soft set represents the low-confidence assessments provided by this expert while the lower approximation of the soft set represents the high-confidence assessments. The main character of our group decision making is that decision makers only need to provide their initial assessments of candidates (attributes) with respect to different aspects (objects), according to their knowledge/cognition of the problem, it is not necessary to provide their optimal alternatives before the group decision making process.

The evaluation result of each expert T_q ($q \in \{1, 2, \ldots, p\}$) can be described as an evaluation soft set $\mathfrak{T}_q = (F_{T_q}, A)$ over U, where $F_{T_q} : A \to P(U)$. Using rough approximations on soft set $\mathfrak{T}_q$, we can obtain two corresponding soft sets $\bar{Apr}_R(\mathfrak{T}_q) = (\bar{F}_{T_q R}, A)$ and $\underline{Apr}_R(\mathfrak{T}_q) = (\underline{F_{T_q}}_R, A)$ over U, where $\bar{F}_{T_q R} : A \to P(U)$ and $\underline{F_{T_q}}_R : A \to P(U)$.

We give a weighting vector $W' = (\eta_1, \eta_2, \ldots, \eta_p)$ such that $\eta_1 + \eta_2 + \ldots + \eta_p = 1$, where η_q ($q = 1, 2, \ldots, p$) represents the weight of expert T_q ($q = 1, 2, \ldots, p$) and can be calculated by:

$$\eta_q = \frac{S(\bar{Apr}_R(\mathfrak{T}_q), \underline{Apr}_R(\mathfrak{T}_q))}{\sum_{t=1}^{p} S(\bar{Apr}_R(\mathfrak{T}_t), \underline{Apr}_R(\mathfrak{T}_t))} \tag{16}$$

where $S(\bar{Apr}_R(\mathfrak{T}_q), \underline{Apr}_R(\mathfrak{T}_q))$ is the similarity between soft sets $\bar{Apr}_R(\mathfrak{T}_q) = (\bar{F}_{T_q R}, A)$ and $\underline{Apr}_R(\mathfrak{T}_q) = (\underline{F_{T_q}}_R, A)$. (There are a lot of formulas can be used to calculate the similarity between two softs sets.)

The weight vector $W' = (\eta_1, \eta_2, \ldots, \eta_p)$ indicates different importance degree of different experts. It is noticed that people can use a lot of ways to determine the weights of experts. If there are enough additional information in evaluating the experts, the weights of experts can even been directly specified. Here we originally apply the similarity measures between soft sets to determine the weights of experts in a group decision making problem when the weights are not specified in advance. As is shown above, the opinion of each expert is represented by a soft set. We believe that the more similar the upper approximation and the lower approximation of one's opinion (the soft set) are, the more stable and reliable his/her opinion is, and thus the larger his/her weight should be.

Then the evaluation result of the whole expert group G could be formulated in terms of fuzzy sets:

$$\mu_{\mathfrak{T}'} : A \to [0, 1], e_i \mapsto \mu_{\mathfrak{T}'}(e_i) = (\frac{1}{n}) \sum_{q \in \{1,2,\ldots,p\}} \eta_q \times |F_{T_q}(e_i)|$$

where $i = 1, 2, \ldots, n$. Similarly, we can obtain two other fuzzy sets $\mu_{\underline{\mathfrak{T}}'}$ and $\mu_{\bar{\mathfrak{T}}'}$ in U, which are respectively given by

$$\mu_{\underline{\mathfrak{T}}'} : A \to [0, 1], e_i \mapsto \mu_{\underline{\mathfrak{T}}'}(e_i) = (\tfrac{1}{n}) \sum_{q \in \{1,2,\ldots,p\}} \eta_q \times |\underline{F_{T_q}}_R(e_i)|$$

$$\mu_{\bar{\mathfrak{T}}'} : A \to [0, 1], \quad e_i \mapsto \mu_{\bar{\mathfrak{T}}'}(e_i) = (\tfrac{1}{n}) \sum_{q \in \{1,2,\ldots,p\}} \eta_q \times |\bar{F}_{T_q R}(e_i)|$$

where $i = 1, 2, \ldots, n$.

Then, we can construct a fuzzy soft sets to gather together the above fuzzy evaluation results. Let $C = \{L, M, H\}$ be a set of attributes, where L, M and H represent three kinds of confidence, respectively. Then we can define a fuzzy soft set $\mathfrak{F} = (G, C)$ over U, where $G : C \to F(U)$ is given by $G(L) = \mu_{\bar{\mathfrak{T}}'}$, $G(M) = \mu_{\mathfrak{T}'}$ and $G(H) = \mu_{\underline{\mathfrak{T}}'}$.

Now we give a weighting vector $W = (w_L, w_M, w_H)$ such that $w_L + w_M + w_H = 1$, we define

$$\nu(e_k) = (w_L) \times G(L)(e_k) + (w_M) \times G(M)(e_k) + (w_H) \times G(H)(e_k) \quad (17)$$

which is called the weighted evaluation value of the candidate $e_k \in A$. Finally we can select the attribute e_j such that $\nu(e_j) = \max(\nu(e_k))$ ($k = 1, 2, \ldots, m$.) as the most preferred candidate. Now we present the decision making method based on rough soft sets by Algorithm 7.

Algorithm 7.

[Step 1.] Input Pawlak approximation space (U, R) and soft sets $\mathfrak{T}_1 = (F_{T_1}, A)$, $\mathfrak{T}_2 = (F_{T_2}, A), \ldots, \mathfrak{T}_p = (F_{T_p}, A)$ on U.
[Step 2.] For $\forall q \in \{1, 2, \ldots, p\}$, compute the lower and upper rough approximations on soft set $\mathfrak{T}_q$, i.e., $\overline{Apr}_R(\mathfrak{T}_q) = (\overline{F_{T_q}}_R, A)$ and $\underline{Apr}_R(\mathfrak{T}_q) = (\underline{F_{T_q}}_R, A)$, respectively.
[Step 3.] Compute the weighting vector $W' = (\eta_1, \eta_2, \ldots, \eta_p)$ by Eq.(16).
[Step 4.] Compute the corresponding fuzzy sets $\mu_{\bar{\mathfrak{T}}'}$, $\mu_{\mathfrak{T}'}$ and $\mu_{\underline{\mathfrak{T}}'}$.
[Step 5.] Construct a fuzzy soft set $\mathfrak{F} = (G, C)$ using $\mu_{\bar{\mathfrak{T}}'}$, $\mu_{\mathfrak{T}'}$ and $\mu_{\underline{\mathfrak{T}}'}$.
[Step 6.] The optimal decision is to select e_j if $\nu(e_j) = \vee_{k \in \{1,2,\ldots,m\}} \nu(e_k)$.

Example 14. Suppose that we have an expert group $G = \{T_1, T_2, T_3, T_4\}$ consisting of 4 specialists and our goal is to choose an optimal candidate from a candidates set $A = \{e_1, e_2, \ldots, e_5\}$. For each candidate, every specialist will be asked to provide an evaluation as respect to all the professional skills in $U = \{o_1, o_2, \ldots, o_n\}$ and will be requested to give judgement if the candidate is good at these skill or not. In this case, the professional skills in U are divided into three classes/types: $(o_1, o_2) \in R$, $(o_3, o_4) \in R$ and $(o_5, o_6, o_7) \in R$ (R represent the type of some skills are equivalent). The evaluation result of all the candidates provided by expert T_q ($q = 1, 2, \ldots, 4$) can be described as a soft set $\mathfrak{T}_q = (F_{T_q}, A)$ over U. Using rough approximations on soft sets, the tabular representations of soft sets $\overline{Apr}_R(\mathfrak{T}_q) = (\overline{F_{T_q}}_R, A) (q = 1, 2, \ldots, 4)$ and $\underline{Apr}_R(\mathfrak{T}_q) = (\underline{F_{T_q}}_R, A)$ ($q = 1, 2, \ldots, 4$) over U are obtained as (Table 18).

Finally, we can calculate the fuzzy soft set $\mathfrak{F} = (G, C)$. Assume that the weighting vector for confidence $W = (0.25, 0.5, 0.25)$ and to calculate the similarity between two soft sets (F, A) and (G, B) we use $S((F, A), (G, B)) = \frac{|A \cap B|}{|A \cup B|} \cdot \frac{\sum_{e \in A \cap B} |F(e) \cap G(e)|}{\sum_{e \in A \cap B} |F(e) \cup G(e)|}$ [41]. It is easy to obtain that

$$S(\overline{Apr}_R(\mathfrak{T}_1), \underline{Apr}_R(\mathfrak{T}_1)) = 0.5625,\ S(\overline{Apr}_R(\mathfrak{T}_2), \underline{Apr}_R(\mathfrak{T}_2)) = 0.625,$$
$$S(\overline{Apr}_R(\mathfrak{T}_3), \underline{Apr}_R(\mathfrak{T}_3)) = 0.32,\ S(\overline{Apr}_R(\mathfrak{T}_4), \underline{Apr}_R(\mathfrak{T}_4)) = 0.375.$$

Then, we can obtain $W' = (0.299, 0.332, 0.170, 0.199)$ by Eq. (16), the weighted evaluation value can be calculated by Eq. (17). Tabular representation of the fuzzy soft set $\mathfrak{F} = (G, C)$ with evaluation values is given by Table 19. Hence e_2 should be the most preferred candidate.

5. Conclusions

Fuzzy set theory, rough set theory and soft set theory are three relatively independent and closely related mathematical tools for dealing with uncertainty [42]. Based on the combination of these theories, various hybrid models, including fuzzy soft set theory and rough soft set theory, have been obtained to handle the vagueness in practical problems. In this paper, we focus on the application of fuzzy soft set theory and rough soft set theory in decision making. A classical fuzzy soft based decision making approach is improved to deal with decision making problems that contain updating information so that attributes need to be added/deleted in the fuzzy soft sets. We also present a new adjustable fuzzy soft sets based decision making approach by introducing comparison thresholds and corresponding level D-Score tables of fuzzy soft sets. This new approach has the potential to be extended to the intuitionistic fuzzy soft sets, interval-valued fuzzy soft sets situations, etc. Based on rough

Table 18
The tabular representations for soft sets in Example 14.

	o_1	o_2	o_3	o_4	o_5	o_6	o_7
Table for soft set $\overline{Apr}_R(\mathfrak{T}_1)$							
e_1	0	0	1	1	0	0	0
e_2	1	1	0	0	1	1	1
e_3	0	0	1	1	1	1	1
e_4	0	0	1	1	0	0	0
e_5	0	0	1	1	0	0	0
Table for soft set $\mathfrak{T}_1$							
e_1	0	0	0	1	0	0	0
e_2	1	1	0	0	1	1	1
e_3	0	0	1	0	1	1	0
e_4	0	0	1	1	0	0	0
e_5	0	0	1	1	0	0	0
Table for soft set $\underline{Apr}_R(\mathfrak{T}_1)$							
e_1	0	0	0	0	0	0	0
e_2	1	1	0	0	1	1	1
e_3	0	0	0	0	0	0	0
e_4	0	0	1	1	0	0	0
e_5	0	0	1	1	0	0	0
Table for soft set $\overline{Apr}_R(\mathfrak{T}_2)$							
e_1	1	1	1	1	0	0	0
e_2	1	1	0	0	1	1	1
e_3	0	0	1	1	1	1	1
e_4	0	0	1	1	1	1	1
e_5	0	0	1	1	1	1	1
Table for soft set $\mathfrak{T}_2$							
e_1	1	1	1	1	0	0	0
e_2	1	1	0	0	1	1	0
e_3	0	0	1	1	1	1	0
e_4	0	0	1	1	1	1	0
e_5	0	0	1	1	1	1	1
Table for soft set $\underline{Apr}_R(\mathfrak{T}_2)$							
e_1	1	1	1	1	0	0	0
e_2	1	1	0	0	0	0	0
e_3	0	0	1	1	0	0	0
e_4	0	0	1	1	0	0	0
e_5	0	0	1	1	1	1	1
Table for soft set $\overline{Apr}_R(\mathfrak{T}_3)$							
e_1	1	1	1	1	0	0	0
e_2	1	1	0	0	1	1	1
e_3	1	1	1	1	0	0	0
e_4	1	1	1	1	1	1	1
e_5	0	0	1	1	1	1	1
Table for soft set $\mathfrak{T}_3$							
e_1	1	1	1	1	0	0	0
e_2	0	1	0	0	1	1	0
e_3	1	0	1	1	0	0	0
e_4	1	0	1	0	1	1	0
e_5	0	0	1	1	0	0	1
Table for soft set $\underline{Apr}_R(\mathfrak{T}_3)$							
e_1	1	1	1	1	0	0	0
e_2	0	0	0	0	0	0	0
e_3	0	0	1	1	0	0	0
e_4	0	0	0	0	0	0	0
e_5	0	0	1	1	0	0	0
Table for soft set $\overline{Apr}_R(\mathfrak{T}_4)$							
e_1	1	1	1	1	0	0	0
e_2	1	1	0	0	1	1	1
e_3	0	0	1	1	1	1	1
e_4	0	0	0	0	1	1	1
e_5	1	1	1	1	1	1	1
Table for soft set $\mathfrak{T}_4$							
e_1	1	0	1	1	0	0	0
e_2	1	0	0	0	1	1	1
e_3	0	0	1	1	1	0	0
e_4	0	0	0	0	1	1	0
e_5	0	1	1	1	1	0	1
Table for soft set $\underline{Apr}_R(\mathfrak{T}_4)$							
e_1	0	0	1	1	0	0	0
e_2	0	0	0	0	1	1	1
e_3	0	0	1	1	0	0	0
e_4	0	0	0	0	0	0	0
e_5	0	0	1	1	0	0	0

Table 19
The tabular representation of $\mathfrak{F} = (G, C)$.

	e_1	e_2	e_3	e_4	e_5
L	0.486	0.714	0.690	0.578	0.643
M	0.415	0.590	0.476	0.429	0.538
H	0.344	0.394	0.200	0.180	0.428
$v(e_i)$	0.415	0.572	0.461	0.404	0.537

soft sets, some new algorithms are also provided to solve decision making and group decision making problems, different algorithms have different scopes of application. These original rough soft sets based approaches have the potential to be extended to the generation models of rough soft sets situations. In further research, the generation models of rough soft set theory and their corresponding application in decision making is an interesting issue to be addressed. The time complexity analysis of all the algorithms in the current work can be found in Appendix A.

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Appendix A. Complexity analysis of Algorithms 1–7

The complexity analysis of the algorithms in the current work are listed as follows:

- Algorithm 1: For calculating each entry of the comparison table from the fuzzy soft set, the complexity of running $|A|$ comparisons is $O(|A|)$, there are $|U|^2$ entries in the comparison table, hence the complexity of computing the comparison table is $O(|A||U|^2)$. The complexity of computing each score of each object by using the comparison score is $O(2|U|) = O(|U|)$, afterwards the complexity of selecting the max value is also $O(|U|)$. Thus, the complexity of Algorithm 1 is $O(|A||U|^2) + O(|U|) + O(|U|) = O(|A||U|^2)$.
- Algorithm 2: For calculating each entry of the D-Score table from the initial fuzzy soft set, the complexity of running $|A|$ comparisons is $O(|U|)$, there are $|U||A|$ entries in the comparison table, hence the complexity of computing the D-Score table is $O(|A||U|^2)$. The complexity of computing each D-Score of each object by using the D-Score table is $O(|A|)$, afterwards the complexity of selecting the max value is also $O(|U|)$. Thus, the complexity of Algorithm 2 is $O(|A||U|^2) + O(|A|) + O(|U|) = O(|A||U|^2)$.
- Algorithm 3: Compared to Algorithm 2, in Algorithm 3 we only introduce a threshold value when doing the comparisons to obtain the corresponding D-score Table, so the time complexity of Algorithm 3 is the same as Algorithm 2, that is, $O(|A||U|^2)$.
- Algorithm 4: The time complexity of Algorithm 4 is the same as Algorithm 3, that is, $O(|A||U|^2)$.
- Algorithm 5: For all $e_j \in A$, the time complexity of computing $\bar{F}_R(e_j)$ and $\underline{F}_R(e_j)$ from $F(e_j)$ is $O(|U|)$. There are $|A|$ parameters, therefore the complexity of computing the rough soft set from a given soft set $\mathfrak{S} = (F, A)$ is $O(|U||A|)$. The second step is to select a threshold λ manually, in which to compute the upper bound the time complexity is $O(|U|)$. The time complexity of the worst case to find the decision result is $O(c^1_{|A|} + c^2_{|A|} + \ldots + c^{|A|}_{|A|}) = O(2^{|A|})$. It is easy to obtain the time complexity of Algorithm 5 is $O(|U||A| + 2^{|A|})$.

  It is determined by the time complexity of Algorithm 5 that this algorithm is only suitable for decision making problems in which the number of attributes is relative small, which is a limitation of both Algorithm 5 in the current work and the Algorithm 9 in [34]. In the future it is worth paying attention to the further improvement of these algorithms to make them more feasible for large scale of data sets.
- Algorithm 6: For all $e_j \in A$, the time complexity of computing $\bar{F}_R(e_j)$ from $F(e_j)$ is $O(|U|)$. There are $|A|$ parameters, therefore the complexity of computing the upper approximation of a given soft set $\mathfrak{S} = (F, A)$ is $O(|U||A|)$. The complexity of selecting the max value of $\bar{F}_R(e_j)$, $e_j \in A$ is $O(|A|)$. It is easy to obtain the complexity of Algorithm 6 is $O(|U||A| + |A|) = O(|U||A|)$.
- Algorithm 7: For all $e_j \in A$, the time complexity of computing $\underline{F_{T_q}}_R(e_j)$ and $\bar{F}_{T_q R}(e_j)$ from $F_{T_q}(e_j)$ is $O(|U|)$. There are $|A|$ parameters, therefore the complexity of computing a rough soft set from a given soft set $\mathfrak{T}_q = (F_{T_q}, A)$ is $O(|U||A|)$. The complexity of obtaining all rough soft sets from all soft sets provided by experts $G = \{T_1, T_2, \ldots, T_p\}$ is $O(|U||A||G|)$. And the time complexity of computing each row of the fuzzy soft set from all rough soft sets is $O(|U||G||A|)$, three rows is $O(3|U||A||G|) = O(|U||A||G|)$, afterwards for computing $v(e_j)$, $e_j \in A$ from the fuzzy soft set is $O(3|A|) = O(|A|)$. The complexity of the last step to catch the largest value is obvious $O(|A|)$. The complexity of Step 3 has been ignored since it depends on the way for computing the similarity measures of soft sets. When the weights of experts are predefined, Step 3 should be skipped. Thus, the time complexity of Algorithm 7 is $O(|U||A||G|) + O(|U||A||G|) + O(|A|) + O(|A|) = O(|U||A||G|)$.

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