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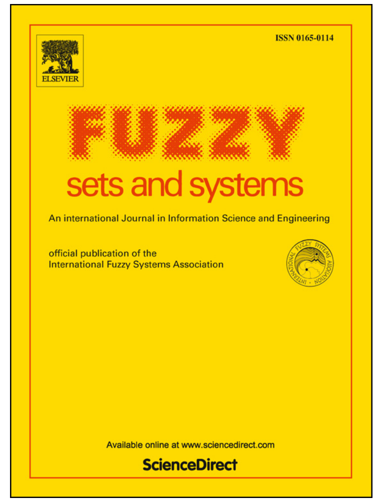

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Admissible OWA Operators for Fuzzy Numbers

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Abstract

Ordered Weighted Averaging (OWA) operators are some of the most widely used aggregation functions in classic literature, but their application to fuzzy numbers has been limited due to the complexity of defining a total order in fuzzy contexts. However, the recent notion of admissible order for fuzzy numbers provides an effective method to totally order them by refining a given partial order. Therefore, this paper is devoted to defining OWA operators for fuzzy numbers with respect to admissible orders and investigating their properties. Firstly, we define the OWA operators associated with such admissible orders and then we show their main properties. Afterward, an example is presented to illustrate the applicability of these AOWA operators in linguistic decision-making. In this regard, we also develop an admissible order for

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trapezoidal fuzzy numbers that can be efficiently applied in practice.

Keywords: OWA operators, fuzzy numbers, admissible orders, ranking for fuzzy numbers.

1. Introduction

The ordered weighted averaging (OWA) operator came to light by Yager [25] to solve multicriteria decision-making problems. In this regard, Yager's idea was to provide a family of functions whose decision lies between the case in which all the criteria must be satisfied ("and" operators), and the case in which the satisfaction of any criteria is enough ("or" operators). To do so, Yager defined the OWA operators as a normalized weighted average where the input vector is decreasingly ordered. Consequently, their coefficients are not associated directly with particular attributes, but with the magnitude of the inputs [25].

Many extensions and applications of OWA operators can be found in the decision-making literature and they continue being investigated and applied nowadays [4, 12, 24]. The classical OWA operator, as well as most of its extensions, typically operates with real numbers [8, 13, 21], but this approach may not be convenient for capturing people's perceptions in practice. Indeed, in real-world scenarios, experts usually feel much more comfortable providing their opinions using linguistic terms, which are much closer to their way of thinking [16]. To model linguistic information, the fuzzy linguistic approach has emerged as a methodology able to model the uncertainty inherent to linguistic terms by means of fuzzy linguistic variables [26]. In particular, one of the most popular approaches consists of the use of fuzzy numbers, which are a specific class of fuzzy sets, to account for the vagueness and imprecision inherent in our perceptions [7]. Even though there are some attempts to define OWA operators for fuzzy numbers in the specialized literature [11, 18], all of them neglect a fundamental aspect: in the definition of the classical OWA operator, a total order relationship, such as the one defined on the real line, is essential for its good performance.

For instance, some proposals [27, 28] rely on the fuzzy extension principle [23] to define a fuzzy OWA operator. However, this presents two major shortcomings. On the one hand, it is hard to determine if the thus-defined OWA operators result in ordering the inputs and then computing the corresponding weighted average [28]. On the other hand, it has been shown that such OWA operators may return the same result as an OWA operator based on a partial order, implying that some fuzzy numbers cannot be compared and, therefore, the domain of definition of such an OWA operator is not the entire family of fuzzy numbers [29].

Other proposals define the fuzzy OWA operator using an index-based ranking method for fuzzy sets [18]. However, such ranking methods are not based on a total order defined on the set of fuzzy numbers and usually require computing a certain scalar index [16]. The main disadvantage of the use of ranking indexes is that different fuzzy numbers could be completely indistinguishable, as it would occur with any other ranking method based on a non-antisymmetric binary relation. This is fatal for the OWA operator because, in the case of two different inputs having the same ranking index, it is not possible to decide which one goes first in the aggregation, resulting in two different possible values for the output of the OWA aggregation.

Additionally, another extended shortcoming of fuzzy OWA operators in the literature is the fact that they are exclusively defined for a specific class of fuzzy numbers such as trapezoidal fuzzy numbers (TrFN for short) or discrete fuzzy numbers [11, 28], which limits its applicability in other domains.

Therefore, to address all these limitations this proposal aims to define an OWA operator for fuzzy numbers based on the notion of admissible order introduced by Zumelzu et al. [30], which provides a total order relation that makes it possible, with a solid mathematical base, to univocally rank any family of elements within this class of fuzzy sets [30]. Hence, our goal is to obtain an OWA operator with the following characteristics:

- The inputs can be any family of fuzzy numbers, without being constrained to any specific subfamily;
- It is well-defined, i.e., for each input there is one and only one possible output;
- It performs an actual ordered weighted average with respect to the corresponding order;
- The traditional properties of OWA operators for real numbers are also satisfied by the OWA operator for fuzzy numbers.

Notice that without a total order relation on fuzzy numbers it is not possible to obtain an extension of the OWA operator with these characteristics and, until now, the only reasonable total order relation for fuzzy numbers is the notion of admissible order [30]. Thus, first, we study the necessary axioms to guarantee that the admissible order-based extension of OWA operators, the so-called Admissible OWA (AOWA) operator, is well-defined. Subsequently, we show that AOWA operators satisfy the key properties of the original OWA operators related to their averaging nature. Additionally, we introduce a general method to construct admissible orders for TrFNs. On the one hand, with this method, it is possible to overcome the limitations of traditional ranking methods and then order TrFNs according to a total order that refines the standard partial order for fuzzy numbers. On the other hand, unlike existing admissible orders [30], with our methodology it is not necessary to rely on an upper-dense sequence, which is extremely helpful from the computational point of view. Finally, we illustrate the performance of the AOWA operator for fuzzy numbers in the resolution of a linguistic multi-criteria decision problem which is modeled using TrFNs.

The remainder of this paper is as follows. In Section 2, we introduce some basic notions regarding admissible orders and fuzzy numbers. Section 3 introduces a huge family of admissible orders for TrFNs. We then define the OWA operators associated with admissible orders in Section 4, and we investigate their main properties in Section 5. In Section 6, we develop an illustrative example in which fuzzy OWA operators are applied in linguistic decision-making. Finally, in Section 7 we provide some final comments and conclusions.

2. Preliminaries

In this section, we present some definitions and results related to OWA operators, fuzzy numbers, and admissible orders.

2.1. OWA operators

The purpose of an aggregation function is to summarize several inputs to a singular output regarding the monotonicity in each variable and certain boundary conditions [9]. A special type of an n–ary aggregation function on $[0,1]$ is the ordered weighted averaging (OWA) operator, whose definition is given as follows.

Definition 2.1. [25] *Let $\omega = (\omega_1, \ldots, \omega_n)$ be a weight vector, i.e., $\omega_i \in [0,1]$, for all $i \in \{1, \ldots, n\}$, and $\sum_{i=1}^{n} \omega_i = 1$. The* OWA operator associated to ω

is the mapping $\mathrm{OWA}_\omega : [0,1]^n \to [0,1]$ *defined by*

$$\mathrm{OWA}_\omega(x_1, \ldots, x_n) = \sum_{n=1}^{n} \omega_i x_{(i)} \tag{1}$$

where $x_{(i)}$ *denotes the* i*-th largest value among* $x_1, \ldots, x_n$*.*

Example 2.1. *The mappings* $Min^{(n)}, Max^{(n)} : [0,1]^n \to [0,1]$*, given by* $Min^{(n)}(x) = \min\{x_1, \ldots, x_n\}$ *and* $Max^{(n)}(x) = \max\{x_1, \ldots, x_n\}$ *are OWA operators whose weight vectors are, respectively,* $(0, \ldots, 0, 1)$ *and* $(1, 0, \ldots, 0)$*. The arithmetic mean* $M^{(n)} : [0,1]^n \to [0,1]$*, given by* $M^{(n)}(x) = (1/n) \cdot \sum_{i=1}^{n} x_i$ *is an OWA operator with weight vector* $(1/n, \ldots, 1/n)$*.*

2.2. Fuzzy numbers

Fuzzy subsets of $\mathbb{R}$ extend the classic idea of a subset by considering characteristic functions whose codomain can be any subset of $[0,1]$. Formally,

Definition 2.2 ([19])**.** *A* fuzzy subset of $\mathbb{R}$ *is a mapping* $A : \mathbb{R} \to [0,1]$*. In addition, for* $\alpha \in]0,1]$*:*

1. *the support of* A *is the set* $supp(A) = \{x \in \mathbb{R} : A(x) > 0\}$*;*
2. *the* α-cut *(or* α-level set*) of* A *is the set* $A_\alpha = \{x \in \mathbb{R} : A(x) \geq \alpha\}$*.*

The most interesting subset of fuzzy sets is the class of fuzzy numbers, which extends the idea of real numbers.

Definition 2.3. [14, 30][1] *A* fuzzy number $A : \mathbb{R} \to [0,1]$ *is a fuzzy subset of* $\mathbb{R}$ *that satisfies the following conditions:*

1. A *is normal (i.e. there exists* $x \in \mathbb{R}$ *such that* $A(x) = 1$*).*
2. *The support of* A *is bounded.*
3. *The* α*-cuts of* A *are closed intervals for all* $\alpha > 0$*.*

From now on, $\mathcal{F}(\mathbb{R})$ *will denote the family of all fuzzy numbers.*

---

[1] Recently, there has been a debate about this notion (see [20]). In this paper, we recall the definition used in [14].

Therefore, the following statements hold:

1. Real numbers can be seen as fuzzy numbers [19].
2. A fuzzy number is upper semicontinuous [2].
3. If A is a fuzzy number with $A(r) = 1$ for a certain $r \in \mathbb{R}$, then A is non-decreasing on $(-\infty, r]$ and non-increasing on $[r, +\infty)$ [14].

Below, we introduce some examples of fuzzy numbers.

Example 2.2. *Given a closed and bounded real interval I, the lower bound will be denoted by $\underline{I}$ and the upper bound by $\overline{I}$, i.e. $I = [\underline{I}, \overline{I}]$. The family of all closed and bounded real intervals will be represented by $\mathbb{I}(\mathbb{R})$, i.e.*

$$\mathbb{I}(\mathbb{R}) = \{[\underline{I}, \overline{I}] : \underline{I}, \overline{I} \in \mathbb{R} \text{ and } \underline{I} \leq \overline{I}\}.$$

Note that each $I \in \mathbb{I}(\mathbb{R})$ can be identified with the fuzzy number $\widetilde{I} : \mathbb{R} \to [0, 1]$ defined by,

$$\widetilde{I}(x) = \begin{cases} 1, & \text{if } x \in I, \\ 0, & \text{otherwise}. \end{cases} \tag{2}$$

Because of its shape, $\widetilde{I}$ is called a rectangular fuzzy number *whenever $\underline{I} < \overline{I}$ (see the number $\widetilde{I}$ in Figure 1). Moreover, if $\underline{I} = \overline{I} = r \in \mathbb{R}$, the resulting fuzzy number $\widetilde{I} = \widetilde{r}$, so-called crisp fuzzy number, represents the characteristic function of the real number r (see the fuzzy number $\tilde{r}$ in Figure 1), i.e.,*

$$\widetilde{r}(x) = \begin{cases} 1, & \text{if } x = r, \\ 0, & \text{otherwise}. \end{cases} \tag{3}$$

$\mathcal{CFN}$ will denote the family of all crisp fuzzy numbers.

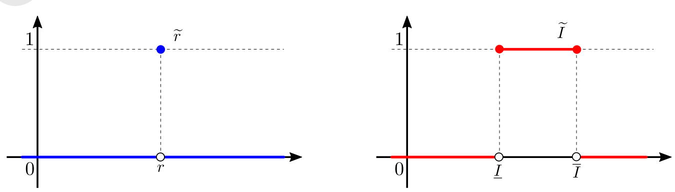

Figure 1: Plots of a crisp fuzzy number (left) and a rectangular fuzzy number (right).

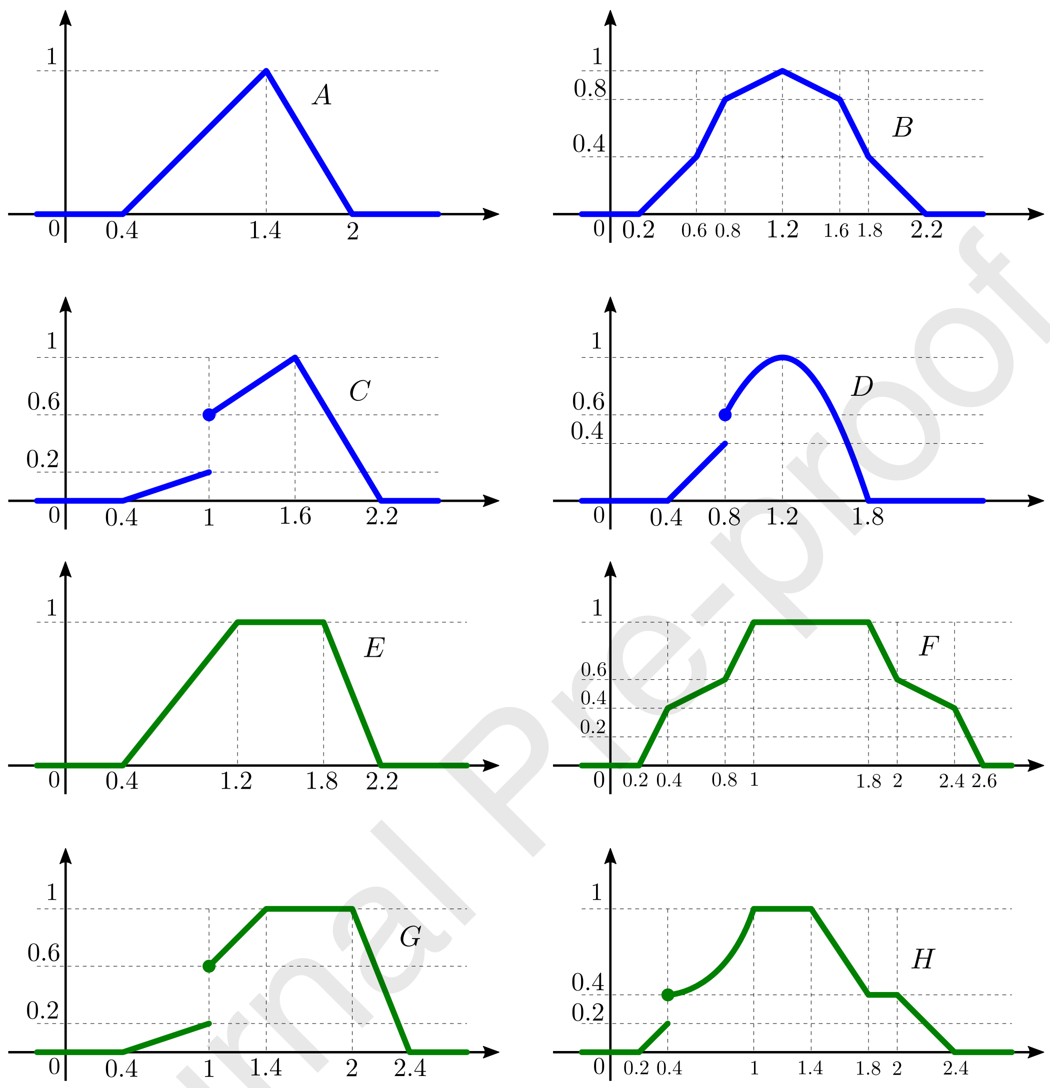

Figure 2: Plots of some fuzzy numbers.

Example 2.3. *$T \in \mathcal{F}(\mathbb{R})$ is said to be a TrFN if its membership function is* *defined by*

$$T^{(a/b/c/d)}(x) = \begin{cases} \dfrac{x-a}{b-a}, & \text{if } a \leq x < b, \\ 1, & \text{if } b \leq x \leq c, \\ \dfrac{d-x}{d-c}, & \text{if } c < x \leq d, \\ 0, & \text{if } x < a \text{ or } x > d, \end{cases}$$

for some $a \leq b \leq c \leq d$. *A TrFN* T *will be denoted by* $T^{(a/b/c/d)}$ *(see the fuzzy number* E *plotted in* Fig. 2*). Note that if* $b = c$*, then the TrFN will have a triangular shape (see the fuzzy number* A *plotted in* Fig. 2*). Also, if* $a = b$ *and* $c = d$*, then TrFN will have a rectangular shape (see fuzzy number* $\widetilde{I}$ *plotted in* Fig. 1*). Hereinafter,* $\mathcal{T}$ *stands for the set of all the TrFN whose support is contained in* $[0, 1]$*, i.e.,* $\mathcal{T} = \left\{T^{(a/b/c/d)} \;:\; 0 \leq a \leq b \leq c \leq d \leq 1\right\}$.

The classic algebraic operations defined in the real line can be extended to fuzzy numbers as follows.

Definition 2.4. [14, 19] *Let* $A, B \in \mathcal{F}(\mathbb{R})$*. Define* $A \oplus B, A \odot B, A \vee B, A \wedge B \in \mathcal{F}(\mathbb{R})$ *by*

$$(A \oplus B)(z) = \sup_{x+y=z} \min\{A(x), B(y)\} \tag{4}$$

$$(A \odot B)(z) = \sup_{x \cdot y=z} \min\{A(x), B(y)\} \tag{5}$$

$$(A \vee B)(z) = \sup_{\max\{x,y\}=z} \min\{A(x), B(y)\} \tag{6}$$

$$(A \wedge B)(z) = \sup_{\min\{x,y\}=z} \min\{A(x), B(y)\} \tag{7}$$

for all $z \in \mathbb{R}$.

Let us define the partial order '$\leq$' on $\mathcal{F}(\mathbb{R})$ as $A \leq B \iff A(x) \leq B(x)$ for all $x \in \mathbb{R}$. Then, the operations $\oplus$ and $\odot$ defined above satisfy commutativity, associativity and $\odot$ subdistributes over $\oplus$, i.e., $A \odot (B \oplus C) \leq (A \odot B) \oplus (A \odot C)$ (see [14, 19]). In addition, these operations present some desirable properties when dealing with crisp fuzzy numbers.

Proposition 2.1. [19] *Let* $A, B \in \mathcal{F}(\mathbb{R})$ *and* $\widetilde{r} \in \mathcal{CFN}$ *. Then:*

1. $A \odot \widetilde{1} = A$;

2. $\widetilde{r} \odot (A \oplus B) = (\widetilde{r} \odot A) \oplus (\widetilde{r} \odot B)$.

2.3. Admissible orders on fuzzy numbers

Given a set X, an order on X is a binary relation R which satisfies reflexivity, anti-symmetry and transitivity. Moreover, if, for any $x_1, x_2 \in X$, $x_1 R x_2$ or $x_2 R x_1$, R is said to be a total order. Otherwise, R is called a partial order.

In this context, Bustince et al. [3] introduced the admissible orders for closed subintervals of $[0,1]$ as total orders that refine the Kulisch and Miranker (KM) order [15] restricted to this set of intervals. This notion can be easily generalized for $\mathbb{I}(\mathbb{R})$ [30]. Formally, a total order $\leq^{\mathbb{I}(\mathbb{R})}$ is called an *admisible order on* $\mathbb{I}(\mathbb{R})$, if, for all $I, J \in \mathbb{I}(\mathbb{R})$, $I \leq^{\mathbb{I}(\mathbb{R})} J$ whenever $I \leq^{\mathbb{I}(\mathbb{R})}_{\mathrm{KM}} J$, where

$$I \leq^{\mathbb{I}(\mathbb{R})}_{\mathrm{KM}} J \iff \underline{I} \leq \underline{J} \text{ and } \overline{I} \leq \overline{J}. \tag{8}$$

for all $I, J \in \mathbb{I}(\mathbb{R})$.

De Miguel et al. [5] defined admissible orders on n-tuples to construct OWA operators for a special class of fuzzy sets. Finally, Zumelzu et al. [30] defined admissible orders on fuzzy numbers as follows.

Definition 2.5 (Admissible order [30])**.** *Let $Z \subseteq \mathcal{F}(\mathbb{R})$ be a class of fuzzy numbers, and consider a partial order $\preceq$ on Z. Then a total order $\trianglelefteq$ on Z is called an* admisible order *w.r.t. $\preceq$, if for all $A, B \in Z$, $A \trianglelefteq B$ whenever $A \preceq B$.*

A specific admissible order on fuzzy numbers was constructed under the notion of an upper-dense sequence.

Definition 2.6. [22] *Let $S = (\alpha_i)_{i\in\mathbb{N}}$ be a sequence in $(0,1]$. S is said to be* upper-dense *if, for every point $x \in (0,1]$ and any $\epsilon > 0$, there exists $i \in \mathbb{N}$ such that $\alpha_i \in [x, x+\epsilon)$.*

Example 2.4. *The following sequences are upper-dense sequences in $(0,1]$:*

1. *The sequence $S_b = (d_{bi})_{i\in\mathbb{N}}$ defined by*

$$d_{bi} = 2 - \frac{2i-1}{2^{\lceil \log_2(i) \rceil}},$$

where $\lceil t \rceil$ denotes the value of the ceiling function at real number t, i.e., the smallest integer greater than or equal to t. Some terms of this sequence are $d_{b1} = 1, d_{b2} = 1/2 = 0.5, d_{b3} = 3/4 = 0.75, d_{b4} = 1/4 = 0.25, d_{b5} = 7/8 = 0.875, d_{b6} = 5/8 = 0.625, d_{b7} = 3/8 = 0.375, d_{b8} = 1/8 = 0.125, d_{b9} = 15/16 = 0.9375, \ldots$

2. *Let $S_m = (m_i)_{i\in\mathbb{N}}$ be the sequence defined by $m_i = \max\{j \in \mathbb{N} : \frac{j(j+1)}{2} \leq i\}, i \in \mathbb{N}$. Then, the sequence $S_t = (t_i)_{i\in\mathbb{N}}$ defined by $t_i = \frac{i}{m_i} - \frac{m_i-1}{2}, i \in \mathbb{N}$ is upper-dense. In this case, the first terms of S_t are as follows:* $\frac{1}{1}, \frac{1}{2}, \frac{2}{2}, \frac{1}{3}, \frac{2}{3}, \frac{3}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4}, \ldots$

Using upper-dense sequences, it is possible to state a decomposition formula for fuzzy numbers.

Proposition 2.2. [30] *Let $A, B \in \mathcal{F}(\mathbb{R})$ and $S = (\alpha_i)_{i\in\mathbb{N}}$ be an upper-dense sequence in $(0, 1]$. Then, $A = B$ if and only if $A_{\alpha_i} = B_{\alpha_i}$, for all $i \in \mathbb{N}$.*

The above result guarantees that the following value is well-defined.

Definition 2.7. [30] *Let $A, B \in \mathcal{F}(\mathbb{R})$ and $S = (\alpha_i)_{i\in\mathbb{N}}$ be an upper-dense sequence in $(0, 1]$. Then, define $m(A, B)$ by*

$$m(A,B) = \begin{cases} \min\{i \in \mathbb{N} : A_{\alpha_i} \neq B_{\alpha_i}\}, & \textit{if } A \neq B, \\ 0, & \textit{otherwise.} \end{cases}$$

Given two fuzzy numbers $A, B \in \mathcal{F}(\mathbb{R})$, the value $m(A, B)$ allows defining a binary relation w.r.t. a certain order for intervals.

Definition 2.8. [30] *Let $S = (\alpha_i)_{i\in\mathbb{N}}$ be an upper-dense sequence in $(0, 1]$ and let $\preceq$ be an order on $\mathbb{I}(\mathbb{R})$. Then, define the binary relation $\trianglelefteq^S$ on $\mathcal{F}(\mathbb{R})$ by*

$$A \trianglelefteq^S B \iff A = B \textit{ or } A_{\alpha_{m(A,B)}} <^{\mathbb{I}(\mathbb{R})} B_{\alpha_{m(A,B)}}. \tag{9}$$

The following theorem states a sufficient condition on the interval order $\preceq$ to guarantee that the corresponding relation $\trianglelefteq^S$ is an admissible order on $\mathcal{F}(\mathbb{R})$ w.r.t. the Klir and Yuan (KY) partial order $\preceq_{\mathrm{KY}}$ [14] defined as

$$A \preceq_{\mathrm{KY}} B \iff A \wedge B = A \tag{10}$$

for $A, B \in \mathcal{F}(\mathbb{R})$.

Theorem 2.1. *[30] Let $S = (\alpha_i)_{i\in\mathbb{N}}$ be an upper-dense sequence in $(0,1]$. If $\preceq$ is an admissible order, then $\trianglelefteq^S$ is an admissible order on $\mathcal{F}(\mathbb{R})$ w.r.t. the KY partial order.*

In other words, any admissible order $\trianglelefteq^S$ refines the order $\leq_{\mathrm{KY}}$ on $\mathcal{F}(\mathbb{R})$. Hereinafter, we will abuse the notation and call admissible order on $\mathcal{F}(\mathbb{R})$ to any admissible order w.r.t. the KY order.

The following result provides a characterization of the KY order through the KM ordering of α-cuts.

Proposition 2.3. [14] *Given two fuzzy numbers $A, B \in \mathcal{F}(\mathbb{R})$, the following assertions are equivalent:*

1. *$A \preceq_{\mathrm{KY}} B$;*
2. *$A \vee B = B$;*
3. *$A_\alpha \leq_{\mathrm{KM}}^{\mathbb{I}(\mathbb{R})} B_\alpha$ for each $\alpha \in (0,1]$.*

Example 2.5. *Firstly, it is observed that the fuzzy numbers A and B, C and D, E and F, G and H, D and H from Example 2.3 are not comparable w.r.t. to KY partial order. However, consider an upper-dense sequence $S = (\alpha_i)_{i\in\mathbb{N}}$ in $(0,1]$ and the admissible orders $\leq_{Lex1}^{\mathbb{I}(\mathbb{R})}$, $\leq_{Lex2}^{\mathbb{I}(\mathbb{R})}$ and $\leq_{XY}^{\mathbb{I}(\mathbb{R})}$, given by [30]:*

1. *$[\underline{I},\overline{I}] \leq_{Lex1}^{\mathbb{I}(\mathbb{R})} [\underline{J},\overline{J}] \Leftrightarrow \underline{I} < \underline{J}$ or ($\underline{I} = \underline{J}$ and $\overline{I} \leq \overline{J}$); and*
2. *$[\underline{I},\overline{I}] \leq_{Lex2}^{\mathbb{I}(\mathbb{R})} [\underline{J},\overline{J}] \Leftrightarrow \overline{I} < \overline{J}$ or ($\overline{I} = \overline{J}$ and $\underline{I} \leq \underline{J}$); and*
3. *$[\underline{I},\overline{I}] \leq_{XY}^{\mathbb{I}(\mathbb{R})} [\underline{J},\overline{J}] \Leftrightarrow \underline{I}+\overline{I} < \underline{J}+\overline{J}$ or ($\underline{I}+\overline{I} = \underline{J}+\overline{J}$ and $\overline{I}-\underline{I} \leq \overline{J}-\underline{J}$),*

then for any $P, Q \in \mathcal{F}(\mathbb{R})$, we have that $\trianglelefteq_{Lex1}^S$, $\trianglelefteq_{Lex2}^S$ and $\trianglelefteq_{XY}^S$ given by:

$$P \trianglelefteq_{Lex1}^S Q \iff P = Q \text{ or } P_{\alpha_{m(P,Q)}} \leq_{Lex1}^{\mathbb{I}(\mathbb{R})} Q_{\alpha_{m(P,Q)}} \tag{11}$$

$$P \trianglelefteq_{Lex2}^S Q \iff P = Q \text{ or } P_{\alpha_{m(P,Q)}} \leq_{Lex2}^{\mathbb{I}(\mathbb{R})} Q_{\alpha_{m(P,Q)}} \tag{12}$$

$$P \trianglelefteq_{XY}^S Q \iff P = Q \text{ or } P_{\alpha_{m(P,Q)}} \leq_{XY}^{\mathbb{I}(\mathbb{R})} Q_{\alpha_{m(P,Q)}} \tag{13}$$

are admissible orders on $\mathcal{F}(\mathbb{R})$. Therefore, if S is any sequence of the Example 2.4, it follows that

$$\begin{array}{ccccc}
B \trianglelefteq^S_{Lex1} A & \text{and} & B \trianglelefteq^S_{Lex2} A & \text{and} & B \trianglelefteq^S_{XY} A \\
D \trianglelefteq^S_{Lex1} C & \text{and} & D \trianglelefteq^S_{Lex2} C & \text{and} & D \trianglelefteq^S_{XY} C \\
F \trianglelefteq^S_{Lex1} E & \text{and} & F \trianglelefteq^S_{Lex2} E & \text{and} & F \trianglelefteq^S_{XY} E \\
H \trianglelefteq^S_{Lex1} G & \text{and} & H \trianglelefteq^S_{Lex2} G & \text{and} & H \trianglelefteq^S_{XY} G \\
D \trianglelefteq^S_{Lex1} G & \text{and} & D \trianglelefteq^S_{Lex2} G & \text{and} & D \trianglelefteq^S_{XY} G \\
H \trianglelefteq^S_{Lex1} D & \text{and} & D \trianglelefteq^S_{Lex2} H & \text{and} & D \trianglelefteq^S_{XY} H
\end{array}$$

Notice that admissible orders generated by distinct upper-dense sequences may produce different rankings, even under the same interval admissible orders. Let us consider the sequence $S^*_t = (t_{i+1})_{i\in\mathbb{N}}$, where $t_i, i \in \mathbb{N}$ is defined as in Example 2.4. The first terms of S^*_t are as follows: $\frac{1}{2}, \frac{2}{2}, \frac{1}{3}, \frac{2}{3}, \frac{3}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4}, \ldots$. Under these conditions, it is easy to see that $G \trianglelefteq^{S^*_t}_{Lex1} D$ and $A \trianglelefteq^{S^*_t}_{Lex2} B$.

3. Admissible orders for trapezoidal fuzzy numbers

This section aims to present a general method to define admissible orders for the class of TrFNs. To do so, we use as a basis an interval admissible order $\leq^{\mathbb{I}(\mathbb{R})}$, and we will see that when we are restricted to TrFNs it is not necessary to rely on upper-dense sequences on $(0,1]$ to define admissible orders.

First, note that if $T^{(a/b/c/d)} \in \mathcal{T}$ and $\alpha \in (0,1]$, then each α-cut is determined as follows:

$$A_\alpha = [a + \alpha(b-a), d - \alpha(d-c)].$$

Bellow, we use this fact to show that any TrFN $T^{(a/b/c/d)}$ can be univocally determined by any two different α-cuts:

Lemma 3.1. *Let $T^{(a/b/c/d)} \in \mathcal{T}$ and denote $T^{(a/b/c/d)}_{\alpha_1} = [x_1, y_1]$, $T^{(a/b/c/d)}_{\alpha_2} = [x_2, y_2]$ for $\alpha_1 \neq \alpha_2, \alpha_1, \alpha_2 \in (0,1]$. Then,*

$$a = \frac{\alpha_2 x_1 - \alpha_1 x_2}{\alpha_2 - \alpha_1}, b = \frac{(1-\alpha_1)x_2 - (1-\alpha_2)x_1}{\alpha_2 - \alpha_1}$$
$$c = \frac{(1-\alpha_1)y_2 - (1-\alpha_2)y_1}{\alpha_2 - \alpha_1}, d = \frac{\alpha_2 y_1 - \alpha_1 y_2}{\alpha_2 - \alpha_1}.$$

Proof. The definition of α-cut yields:

$$x_1 = a + \alpha_1(b-a), x_2 = a + \alpha_2(b-a),$$
$$y_1 = d - \alpha_1(d-c), y_2 = d - \alpha_2(d-c).$$

Therefore,

$$b - a = \frac{x_1 - x_2}{\alpha_1 - \alpha_2}, d - c = \frac{y_1 - y_2}{\alpha_2 - \alpha_1},$$

and, subsequently,

$$\begin{aligned} a = x_1 - \alpha_1 \frac{x_1 - x_2}{\alpha_1 - \alpha_2} &= \frac{\alpha_1 x_2 - \alpha_2 x_1}{\alpha_1 - \alpha_2} \\ b = a + \frac{x_1 - x_2}{\alpha_1 - \alpha_2} &= \frac{(1-\alpha_2)x_1 - (1-\alpha_1)x_2}{\alpha_1 - \alpha_2} \\ d = y_1 + \alpha_1 \frac{y_1 - y_2}{\alpha_2 - \alpha_1} &= \frac{\alpha_2 y_1 - \alpha_1 y_2}{\alpha_2 - \alpha_1} \\ c = d - \frac{y_1 - y_2}{\alpha_2 - \alpha_1} &= \frac{(1-\alpha_1)y_2 - (1-\alpha_2)y_1}{\alpha_2 - \alpha_1}. \end{aligned}$$

□

The strength of the previous lemma relies on the possibility of characterizing any TrFN by using only two distinct α-cuts. Let us state this in a formal way.

Lemma 3.2. *Let $A, B \in \mathcal{T}$ be two TrFN satisfying that $A_{\alpha_1} = B_{\alpha_1}$ and $A_{\alpha_2} = B_{\alpha_2}$ for $\alpha_1 \neq \alpha_2, \alpha_1, \alpha_2 \in (0,1]$, then $A = B$.*

Proof. This result follows immediately from the previous Lemma. If the α_1-cuts and the α_2-cuts of A and B are equal, then the values of the parameters a, b, c and d for A and B must be also equal, and thus $A = B$. □

Finally, the above result can be used to define admissible orders for TrFN w.r.t. a given admissible order for intervals.

Theorem 3.1. *Let $T^{(a/b/c/d)}, T^{(a'/b'/c'/d')} \in \mathcal{T}$ and consider an admissible order for intervals $\leq_*^{\mathbb{I}(\mathbb{R})}$. Then, the binary relation $\trianglelefteq_*$ on TrFN defined by:*

$$T^{(a/b/c/d)} \trianglelefteq_* T^{(a'/b'/c'/d')} \iff$$

$$\begin{cases} a = a', b = b', c = c', d = d' \text{ or} \\ [b, c] <_*^{\mathbb{I}(\mathbb{R})} [b', c'] \text{ or} \\ b = b', c = c', [\frac{a+b}{2}, \frac{c+d}{2}] <_*^{\mathbb{I}(\mathbb{R})} [\frac{a'+b'}{2}, \frac{c'+d'}{2}] \end{cases}$$

is an admissible order on $\mathcal{T}$.

Proof. Consider an upper-dense sequence $(\tilde{\alpha}_i)_{i\in\mathbb{N}}$ on $(0,1]$. Also, consider the upper dense sequence $S = (\alpha_i)_{i\in\mathbb{N}}$ defined by:

$$\alpha_i = \begin{cases} 1 & \text{if } i = 1 \\ \frac{1}{2} & \text{if } i = 2 \\ \tilde{\alpha}_{i-2} & \text{if } i \geq 3 \end{cases}$$

At this stage, we can apply Theorem 2.1 to obtain an admissible order $\trianglelefteq^S$ on $\mathcal{F}(\mathbb{R})$ associated to S and $\leq_*^{\mathbb{I}(\mathbb{R})}$.

Now, let us analyze the restriction of the admissible order $\trianglelefteq^S$ to the class of TrFN $\mathcal{T}$. Note that for $T^{(a/b/c/d)}, T^{(a'/b'/c'/d')} \in \mathcal{T}$ we have

$$m(T^{(a/b/c/d)}, T^{(a'/b'/c'/d')}) \leq 2.$$

Indeed, if $m(T^{(a/b/c/d)}, T^{(a'/b'/c'/d')}) \geq 3$, then the α-cuts corresponding to $\alpha_1 = 1$ and $\alpha_2 = \frac{1}{2}$ would be equal, i.e., $T_1^{(a/b/c/d)} = T_1^{(a'/b'/c'/d')}$ and $T_{\frac{1}{2}}^{(a/b/c/d)} = T_{\frac{1}{2}}^{(a'/b'/c'/d')}$, which, by Lemma 3.2, implies $T^{(a/b/c/d)} = T^{(a'/b'/c'/d')}$ and thus $m(T^{(a/b/c/d)}, T^{(a'/b'/c'/d')}) = 0$, which is contradictory with the assumption of $m(T^{(a/b/c/d)}, T^{(a'/b'/c'/d')}) \geq 3$. This leads to the following mutually disjoint scenarios:

- $m(T^{(a/b/c/d)}, T^{(a'/b'/c'/d')}) = 0$. This is equivalent to $a = a', b = b', c = c', d = d'$;
- $m(T^{(a/b/c/d)}, T^{(a'/b'/c'/d')}) = 1$, which holds if and only if $[b,c] <_*^{\mathbb{I}(\mathbb{R})} [b',c']$ or $[b',c'] <_*^{\mathbb{I}(\mathbb{R})} [b,c]$;
- $m(T^{(a/b/c/d)}, T^{(a'/b'/c'/d')}) = 2$, which means that $b = b', c = c'$ and $[\frac{a+b}{2}, \frac{c+d}{2}] <_*^{\mathbb{I}(\mathbb{R})} [\frac{a'+b'}{2}, \frac{c'+d'}{2}]$ or $[\frac{a'+b'}{2}, \frac{c'+d'}{2}] <_*^{\mathbb{I}(\mathbb{R})} [\frac{a+b}{2}, \frac{c+d}{2}]$.

Hence, the restriction of $\trianglelefteq^S$ to the class of TrFN $\mathcal{T}$ is precisely the binary relation defined in the statement of this theorem, which completes the proof. □

Note that admissible orders for fuzzy numbers require checking the order of many α-cuts according to a dense sequence S that obviously has infinite terms. In practice, this can be computationally unfeasible. However, using the previous theorem we can construct admissible orders for TrFNs that only require a few comprobations on the 1-cuts and the $\frac{1}{2}$-cuts.

Example 3.1. *To show the superiority of the admissible orders for TrFN defined using this method, let us reflect on the admissible order* $\trianglelefteq_{XY}$ *defined on* $\mathcal{T}$ *associated with the interval admissible order* $\leq_{XY}^{\mathbb{I}(\mathbb{R})}$. *Below, we compare the ranking provided by* $\trianglelefteq_{XY}$ *with the one provided by the notion of magnitude [1], which is a ranking index for TrFNs defined as*

$$Mag(T^{(a/b/c/d)}) = \frac{a + 5b + 5c + d}{12}, \; \forall \, T^{(a/b/c/d)} \in \mathcal{T}.$$

Let us consider the following TrFNs, which are graphically represented in Figure 3.

$$\begin{aligned} T_1 &= T^{(0.3/0.5/0.6/0.8)} \\ T_2 &= T^{(0.3/0.45/0.65/0.8)} \\ T_3 &= T^{(0.4/0.5/0.6/0.7)} \\ T_4 &= T^{(0.4/0.45/0.65/0.7)}. \end{aligned}$$

A simple computation reveals that the magnitudes of these TrFNs are equal,

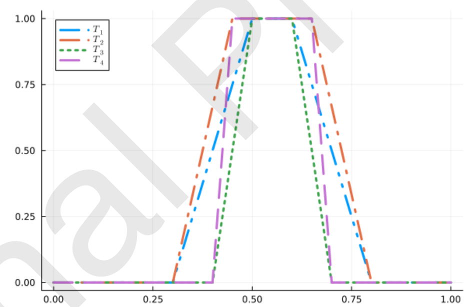

Figure 3: Graph of the TrFNs T_1, T_2, T_3 and T_4

i.e., $Mag(T_1) = Mag(T_2) = Mag(T_3) = Mag(T_4) = 0.55$. *This implies that they are completely indistinguishable from the point of view of their magnitudes and, consequently, it would not be possible to define an OWA operator using such a ranking index. Meanwhile, assume the admissible orders* $\trianglelefteq_{XY}$, $\trianglelefteq_{Lex_1}$ *and* $\trianglelefteq_{Lex_2}$ *respectively associated with the interval admissible orders* $\leq_{XY}^{\mathbb{I}(\mathbb{R})}$, $\leq_{Lex_1}^{\mathbb{I}(\mathbb{R})}$ *and* $\leq_{Lex_2}^{\mathbb{I}(\mathbb{R})}$. *A few comparisons lead to* $T_2 \vartriangleright_{XY} T_4 \vartriangleright_{XY} T_1 \vartriangleright_{XY} T_3$, $T_3 \vartriangleright_{Lex_1} T_1 \vartriangleright_{Lex_1} T_4 \vartriangleright_{Lex_1} T_2$, *and* $T_2 \vartriangleright_{Lex_2} T_4 \vartriangleright_{Lex_2} T_1 \vartriangleright_{Lex_2} T_3$. *This emphasizes the versatility of the proposed method. As final notes, let us remark that these TrFN are not even comparable when using the partial order* $\preceq_{\mathrm{KY}}$.

4. OWA operators for fuzzy numbers

This section applies the idea of admissible order to define OWA operators for fuzzy numbers.

Firstly, let us make some previous considerations. Regarding the weight vector, this paper considers that they are given as crisp fuzzy numbers. Otherwise, in case of an arbitrary fuzzy number is used as a weight vector, the associated OWA operator would present some undesired behaviors, i.e. some classical properties of the original OWA operator could not be satisfied.

Definition 4.1. *A vector* $\widehat{\omega} = (\widetilde{\omega_1}, \ldots, \widetilde{\omega_n})$ *is called a* weight vector of crisp fuzzy numbers*, whenever, for all* $j \in \{1, 2, .., n\}$*:*

1. $\widetilde{\omega}_j$ *is a crisp fuzzy number such that* $\omega_j \in [0, 1]$*,*

2. $\widetilde{\omega_1} \oplus \cdots \oplus \widetilde{\omega_n} = \widetilde{1}$*.*

Example 4.1. $\widehat{\omega} = (\widetilde{\omega_1}, \widetilde{\omega_2}, \widetilde{\omega_3}, \widetilde{\omega_4})$ *such that* $\omega_j = j \cdot 10^{-1}$*, for all* $j \in \{1, 2, 3, 4\}$*, is a weight vector of crisp fuzzy numbers.*

Furthermore, since the goal of this paper is to define OWA operators for fuzzy numbers, it is necessary to guarantee some monotonicity conditions with respect to addition and multiplication. This was discussed for the case of addition in [30] and here also is included the case of multiplication by scalars.

Definition 4.2. *An admissible order* $\leq^{\mathbb{I}(\mathbb{R})}$ *is said to be compatible with the addition and the positive scalar multiplication if for any* $I_1, I_2, J_1, J_2 \in \mathbb{I}(\mathbb{R})$ *and* $r > 0$*,* $I_1 <^{\mathbb{I}(\mathbb{R})} I_2$ *and* $J_1 <^{\mathbb{I}(\mathbb{R})} J_2$ *implies:*

1. $[\underline{I_1} + \underline{J_1}, \overline{I_1} + \overline{J_1}] <^{\mathbb{I}(\mathbb{R})} [\underline{I_2} + \underline{J_2}, \overline{I_2} + \overline{J_2}]$*; and*

2. $[\underline{I_1} \cdot r, \overline{I_1} \cdot r] <^{\mathbb{I}(\mathbb{R})} [\underline{I_2} \cdot r, \overline{I_2} \cdot r]$*.*

Definition 4.3. *Let* $\trianglelefteq$ *be an admissible order on* $\mathcal{F}(\mathbb{R})$*.* $\trianglelefteq$ *is said to be compatible with the addition (*$\oplus$*) and positive crisp fuzzy number multiplication (*$\odot$*) if, for any* $A_1, A_2, B_1, B_2 \in \mathcal{F}(\mathbb{R})$ *and* $\widetilde{r} \in \mathcal{CFN}$*, then* $A_1 \lhd A_2$*,* $B_1 \lhd B_2$*, and* $r > 0$ *implies:*

1. $A_1 \oplus B_1 \lhd A_2 \oplus B_2$*; and*

2. $A_1 \odot \widetilde{r} \lhd A_2 \odot \widetilde{r}$*.*

Proposition 4.1. *Under the same hypotheses of Theorem 2.1, the order* $\trianglelefteq^S$ *is compatible with the addition and the positive crisp fuzzy number multiplication if the order* $\leq^{\mathbb{I}(\mathbb{R})}$ *is compatible with the addition and the positive scalar multiplication.*

Proof. The proof is analogous to the one shown in [30](prop. III.3). □

Example 4.2. *The admissible orders* $\trianglelefteq^S_{Lex1}$ *and* $\trianglelefteq^S_{XY}$ *showed in Example 2.5 are compatible with the addition and the positive crisp fuzzy number multiplication.*

Now, OWA operators for fuzzy numbers based on admissible orders may be defined as follows.

Definition 4.4. *Let* $\widehat{\omega} = (\widetilde{\omega_1}, \ldots, \widetilde{\omega_n})$ *be a weight vector of crisp fuzzy numbers. Also consider an admissible order* $\trianglelefteq$ *which is compatible with the addition and the positive crisp fuzzy number multiplication. Then, the AOWA operator associated to* $\trianglelefteq$ *and* $\widetilde{\omega}$ *is the mapping* $\mathrm{AOWA}^{\widehat{\omega}}_{\trianglelefteq} : \mathcal{F}(\mathbb{R})^n \to \mathcal{F}(\mathbb{R})$ *defined by*

$$\mathrm{AOWA}^{\widehat{\omega}}_{\trianglelefteq}(A_1, \ldots, A_n) = \left(\widetilde{\omega_1} \odot A_{(1)}\right) \oplus \cdots \oplus \left(\widetilde{\omega_n} \odot A_{(n)}\right) \tag{14}$$

for all $A_1, \ldots, A_n \in \mathcal{F}(\mathbb{R})$*, where* $A_{(i)}$ *stands for i-th largest element in* $\{A_1, ..., A_n\}$ *according to the order* $\trianglelefteq$*.*

Note that Definition 2.4 guarantees that the AOWA operator is well-defined.

Example 4.3. *The mappings* $\mathrm{Min}_{\trianglelefteq}, \mathrm{Max}_{\trianglelefteq}, \mathrm{M}_{\widehat{\omega}} : \mathcal{F}(\mathbb{R})^n \to \mathcal{F}(\mathbb{R})$ *defined by*

1. $\mathrm{Min}_{\trianglelefteq}(A_1, \ldots, A_n) = A_{(n)}$,
2. $\mathrm{Max}_{\trianglelefteq}(A_1, \ldots, A_n) = A_{(1)}$, *and*
3. $\mathrm{M}_{\widehat{\omega}}(A_1, \ldots, A_n) = \left(\widetilde{\omega_1} \odot A_{(1)}\right) \oplus \cdots \oplus \left(\widetilde{\omega_n} \odot A_{(n)}\right)$

are AOWA operators for $\widehat{\omega}_* = (\widetilde{0}, \widetilde{0}, \ldots, \widetilde{1})$, $\widehat{\omega}^* = (\widetilde{1}, \widetilde{0} \ldots, \widetilde{0})$, *and* $\widehat{\omega} = \left(\widetilde{1/n}, \ldots, \widetilde{1/n}\right)$ *respectively, i.e.:*

1. $\mathrm{Min}_{\trianglelefteq} = \mathrm{AOWA}^{\widehat{\omega}_*}_{\trianglelefteq}$,
2. $\mathrm{Max}_{\trianglelefteq} = \mathrm{AOWA}^{\widehat{\omega}^*}_{\trianglelefteq}$, *and*
3. $\mathrm{M}_{\widehat{\omega}} = \mathrm{AOWA}^{\widehat{\omega}}_{\trianglelefteq}$.

5. Main properties of AOWA operators on fuzzy numbers

In this section, some classic properties of crisp OWA operators are shown in the case of AOWAs. So, in view of the original Yager's idea of this operator, we show that AOWA operators on fuzzy numbers are symmetric, monotonic, idempotent, and limited by the minimum and maximum operators. Further, other properties of OWA operators, like shift-invariance and positive homogeneity, are also demonstrated for AOWAs.

Firstly, it is shown that AOWA operators satisfy the monotonicity property. Beforehand, we clarify the relationship between monotonicity and compatibility of $\trianglelefteq$ with respect to addition and positive crisp fuzzy number multiplication.

Lemma 5.1. *Let $A, B, A_1, A_2, B_1, B_2 \in \mathcal{F}(\mathbb{R})$ and $\widetilde{r} \in \mathcal{CFN}$. Consider an admissible order $\trianglelefteq$ compatible with addition and positive crisp fuzzy number multiplication. Then, it holds that:*

1. *If $A \trianglelefteq B$, then $\widetilde{r} \oplus A \trianglelefteq \widetilde{r} \oplus B$;*
2. *If $A \trianglelefteq B$ and $r \geq 0$, then $\widetilde{r} \odot A \trianglelefteq \widetilde{r} \odot B$;*
3. *If $A_1 \trianglelefteq B_1$ and $A_2 \trianglelefteq B_2$, then $A_1 \oplus A_2 \trianglelefteq B_1 \oplus B_2$.*

Proof. Directly from the compatibility of $\trianglelefteq$ with the addition and the positive crisp fuzzy number multiplication. □

Consequently, we obtain the monotonicity of AOWA operators with respect to the order $\trianglelefteq$.

Theorem 5.1. *Under the same hypotheses of* Definition 4.4. *Let $(A_1, \ldots, A_n) \in \mathcal{F}(\mathbb{R})^n$ and $(B_1, \ldots, B_n) \in \mathcal{F}(\mathbb{R})^n$ be two ordered vectors, i.e. $A_i \trianglerighteq A_j$ and $B_i \trianglerighteq B_j$ for each $i < j$. If $B_k \trianglerighteq A_k$, for each k then*

$$\mathrm{AOWA}_{\trianglelefteq}^{\widehat{\omega}}(A_1, \ldots, A_n) \trianglelefteq \mathrm{AOWA}_{\trianglelefteq}^{\widehat{\omega}}(B_1, \ldots, B_n).$$

Proof. Since $A_{(j)} \trianglelefteq B_{(j)}$, for all $1 \leq j \leq n$, then, by Lemma 5.1(2), $(\widetilde{r_j} \odot A_{(j)}) \trianglelefteq (\widetilde{r_j} \odot B_{(j)})$, for all $1 \leq j \leq n$. Hence, by Lemma 5.1(3),

$$(\widetilde{r_1} \odot A_{(1)}) \oplus \cdots \oplus (\widetilde{r_n} \odot A_{(n)}) \trianglelefteq (\widetilde{r_1} \odot B_{(1)}) \oplus \cdots \oplus (\widetilde{r_n} \odot B_{(n)}),$$

i.e., $\mathrm{AOWA}_{\trianglelefteq}^{\widehat{\omega}}(A_1, \ldots, A_n) \trianglelefteq \mathrm{AOWA}_{\trianglelefteq}^{\widehat{\omega}}(B_1, \ldots, B_n)$. □

Corollary 5.1. *Under the same hypotheses of* Definition 4.4. *Let $A_1, \dots, A_n \in \mathcal{F}(\mathbb{R})$ and $B_1, \dots, B_n \in \mathcal{F}(\mathbb{R})$. If $B_k \trianglerighteq A_k$, for each k then*

$$\mathrm{AOWA}^{\widehat{\omega}}_{\trianglelefteq}(A_1, \dots, A_n) \trianglelefteq \mathrm{AOWA}^{\widehat{\omega}}_{\trianglelefteq}(B_1, \dots, B_n).$$

Proof. Let $\hat{A} = (A_{(1)}, \dots, A_{(n)})$ and $\hat{B} = (B_{(1)}, \dots, B_{(n)})$. Then it can be easily shown that $\hat{A}_i \trianglelefteq \hat{B}_i$. □

It is also possible to obtain a monotonicity result for AOWA operators with respect to the order $\leq$ as follows.

Lemma 5.2. *The partial order $\leq$ on fuzzy numbers is compatible with addition and positive crisp fuzzy numbers multiplication.*

Proof. Directly from Equations (4) and (5). □

Theorem 5.2. *Under the same hypotheses of* Definition 4.4. *Let $A_j, B_j \in \mathcal{F}(\mathbb{R})$, and $A_j \leq B_j$, for all $j \in \{1, ..., n\}$, then*

$$\mathrm{AOWA}^{\widehat{\omega}}_{\trianglelefteq}(A_1, \dots, A_n) \leq \mathrm{AOWA}^{\widehat{\omega}}_{\trianglelefteq}(B_1, \dots, B_n).$$

Proof. Analogous to the proof of Corollary 5.1, considering the Lemma 5.2. □

Notice that the previous results show that if the monotonicity with respect to the order $\trianglelefteq$ presented in Corollary 5.1 is required, then such order must be compatible with addition and positive fuzzy numbers multiplication. This paper considers that such monotonicity is a key property for OWA operators and for this reason we included the compatibility axioms in the AOWA definition.

In addition, averaging operators must provide an output between the minimum and the maximum. The following result states that this important property of means defined on $[0, 1]$ is also maintained for AOWA operators.

Theorem 5.3. *The least and the greatest AOWA operators, w.r.t. the admissible order $\trianglelefteq$, are* $\mathrm{Min}_{\trianglelefteq}$ *and* $\mathrm{Max}_{\trianglelefteq}$*, respectively. Formally, for any tuple $(A_1, \dots, A_n) \in \mathcal{F}(\mathbb{R})^n$*

$$\begin{aligned}\mathrm{Min}_{\trianglelefteq}(A_1, \dots, A_n) &\trianglelefteq \mathrm{AOWA}^{\widehat{\omega}}_{\trianglelefteq}(A_1, \dots, A_n)\\ &\trianglelefteq \mathrm{Max}_{\trianglelefteq}(A_1, \dots, A_n),\end{aligned}$$

for every n-ary weight vector $\widehat{\omega}$.

Proof. Let $(A_1,\dots,A_n) \in \mathcal{F}(\mathbb{R})^n$ and consider an n-ary weight vector $\widehat{\omega}$. First, let us prove that the AOWA operator is lower than the maximum operator. Since $A_{(n)} \trianglelefteq \dots \trianglelefteq A_{(1)}$, by Lemma 5.1(2) $\widetilde{\omega_j} \odot A_{(j)} \trianglelefteq \widetilde{\omega_j} \odot A_{(1)}$ for all $j \in \{1,...,n\}$. Then, by Lemma 5.1(3), $\mathrm{AOWA}^{\widehat{\omega}}_{\trianglelefteq}(A_1,\dots,A_n) = (\widetilde{\omega_1} \odot A_{(1)}) \oplus \dots \oplus (\widetilde{\omega_n} \odot A_{(n)}) \trianglelefteq (\widetilde{\omega_1} \odot A_{(1)}) \oplus \dots \oplus (\widetilde{\omega_n} \odot A_{(1)}) = \mathrm{AOWA}^{\widehat{\omega}}_{\trianglelefteq}(A_{(1)},\dots,A_{(1)}) = A_{(1)}$, which is the maximum operator. Therefore, $\mathrm{AOWA}^{\widehat{\omega}}_{\trianglelefteq}(A_1,\dots,A_n) \trianglelefteq \mathrm{Max}_{\trianglelefteq}(A_1,\dots,A_n)$.A similar argument leads to the inequality

$$\mathrm{Min}_{\trianglelefteq}(A_1,\dots,A_n) \trianglelefteq \mathrm{AOWA}^{\widehat{\omega}}_{\trianglelefteq}(A_1,\dots,A_n).$$

□

Theorem 5.4. *The least and the greatest AOWA operators, w.r.t. the partial order $\leq$, are $\mathrm{Min}_{\trianglelefteq}$ and $\mathrm{Max}_{\trianglelefteq}$, respectively. Formally, for any tuple $(A_1,\dots,A_n) \in \mathcal{F}(\mathbb{R})^n$*

$$\begin{aligned}\mathrm{Min}_{\trianglelefteq}(A_1,\dots,A_n) &\leq \mathrm{AOWA}^{\widehat{\omega}}_{\trianglelefteq}(A_1,\dots,A_n)\\ &\leq \mathrm{Max}_{\trianglelefteq}(A_1,\dots,A_n),\end{aligned}$$

for every n-ary weight vector $\widehat{\omega}$.

Proof. Analogous to the previous result, considering Lemma 5.2. □

The following statement points out that AOWAs are symmetric, i.e., invariant under permutations. This property is fundamental because the initial idea of OWA operators presumes its satisfaction.

Theorem 5.5. *Under the same hypotheses of* Definition 4.4*,*

$$\mathrm{AOWA}^{\widehat{\omega}}_{\trianglelefteq}(A_1,\dots,A_n) = \mathrm{AOWA}^{\widehat{\omega}}_{\trianglelefteq}(A_{\sigma(1)},\dots,A_{\sigma(n)}),$$

for all permutation $\sigma : \{1,\dots,n\} \to \{1,\dots,n\}$ and for any tuple $(A_1,\dots,A_n) \in \mathcal{F}(\mathbb{R})^n$.

Proof. Directly from Definition 4.4. □

Another expected property for averaging aggregation operators is the idempotency. Let us introduce a previous lemma.

Lemma 5.3. *Let $A \in \mathcal{F}(\mathbb{R})$ and $\widetilde{r}, \widetilde{s} \in \mathcal{CFN}$. Then,*

$$A \odot (\widetilde{r} \oplus \widetilde{s}) = (A \odot \widetilde{r}) \oplus (A \odot \widetilde{s}).$$

Proof. For any $z \in \mathbb{R}$, it follows that

$$
\begin{aligned}
[A \odot (\widetilde{r} \oplus \widetilde{s})]\,(z) =& [A \odot \widetilde{r+s}](z) \\
=& \sup_{x \cdot y = z} \min\{A(x), \widetilde{r+s}(y)\} \\
=& \sup_{x \cdot (r+s) = z} \min\{A(x), \widetilde{r+s}(r+s)\} \\
=& \sup_{xr+xs=z} A(x) \\
=& \sup_{xr+xs=z} \min\{A(x), A(x)\} \\
=& \sup_{xr+xs=z} \min\left\{ \left[\sup_{x \cdot u = xr} \min\{A(x), \widetilde{r}(u)\} \right], \right. \\
& \left. \left[\sup_{x \cdot v = xs} \min\{A(x), \widetilde{s}(v)\} \right] \right\} \\
=& \sup_{xr+xs=z} \min\left\{ (A \odot \widetilde{r})(xr), (A \odot \widetilde{s})(xs) \right\} \\
=& \left[(A \odot \widetilde{r}) \oplus (A \odot \widetilde{s}) \right](z).
\end{aligned}
$$

□

Corollary 5.2. *Let $A \in \mathcal{F}(\mathbb{R})$ and $\widetilde{r_1}, \ldots, \widetilde{r_n} \in \mathcal{CFN}$. Then,*

$$
A \odot (\widetilde{r_1} \oplus \cdots \oplus \widetilde{r_n}) = (A \odot \widetilde{r_1}) \oplus \cdots \oplus (A \odot \widetilde{r_n}).
$$

Finally, we show that AOWA operators are idempotent.

Theorem 5.6. *Let* $\mathrm{AOWA}^{\widehat{\omega}}_{\trianglelefteq}$ *be the operator defined in* Definition 4.4. *Then,*

$$
\mathrm{AOWA}^{\widehat{\omega}}_{\trianglelefteq}(A, \ldots, A) = A,
$$

365 *for all $A \in \mathcal{F}(\mathbb{R})$.*

Proof. It holds that

$$
\begin{aligned}
\mathrm{AOWA}^{\widehat{\omega}}_{\trianglelefteq}(A, \ldots, A) &= (\widetilde{\omega_1} \odot A) \oplus \cdots \oplus (\widetilde{\omega_n} \odot A) && \text{(by Def. 4.4)} \\
&= (A \odot \widetilde{\omega_1}) \oplus \cdots \oplus (A \odot \widetilde{\omega_n}) && \text{(by the commutativity of } \odot) \\
&= A \odot (\widetilde{\omega_1} \oplus \cdots \oplus \widetilde{\omega_n}) && \text{(by Cor. 5.2)} \\
&= A \odot \widetilde{1} && \text{(by Def. 4.1)} \\
&= A && \text{(by Prop. 2.1(1)).}
\end{aligned}
$$

Hence AOWA operators are idempotent. □

Next, it is shown that the properties describing the stability of aggregation functions with respect to changes in the used scale, i.e., shift-invariance and homogeneity, remain valid for the AOWA operators.

Theorem 5.7. *In the hypothesis of* Definition 4.4*,*

$$\mathrm{AOWA}_{\trianglelefteq}^{\widehat{\omega}}(A_1 \oplus \widetilde{\lambda}, \ldots, A_n \oplus \widetilde{\lambda}) = \mathrm{AOWA}_{\trianglelefteq}^{\widehat{\omega}}(A_1, \ldots, A_n) \oplus \widetilde{\lambda},$$

for all $\widetilde{\lambda} \in \mathcal{CFN}$ *and for any tuple* $(A_1, \ldots, A_n) \in \mathcal{F}(\mathbb{R})^n$*.*

Proof. Proposition 2.1(2) guarantees that $\widetilde{\omega_j} \odot \left(A_{(j)} \oplus \widetilde{\lambda}\right) = \left(\widetilde{\omega_j} \odot A_{(j)}\right) \oplus \left(\widetilde{\omega_j} \odot \widetilde{\lambda}\right)$, for all $j \in \{1, ..., n\}$. Considering Equation 14, Propositions 2.1(1) and associativity of $\oplus$ and Corollary 5.2, we have:

$$\begin{aligned}
&\mathrm{AOWA}_{\trianglelefteq}^{\widehat{\omega}}(A_1 \oplus \widetilde{\lambda}, \ldots, A_n \oplus \widetilde{\lambda}) = \\
&= \left[\widetilde{\omega_1} \odot \left(A_{(1)} \oplus \widetilde{\lambda}\right)\right] \oplus \cdots \oplus \left[\widetilde{\omega_n} \odot \left(A_{(n)} \oplus \widetilde{\lambda}\right)\right] \\
&= \left[\left(\widetilde{\omega_1} \odot A_{(1)}\right) \oplus \left(\widetilde{\omega_1} \odot \widetilde{\lambda}\right)\right] \oplus \cdots \oplus \left[\left(\widetilde{\omega_n} \odot A_{(n)}\right) \oplus \left(\widetilde{\omega_n} \odot \widetilde{\lambda}\right)\right] \\
&= \left[\left(\widetilde{\omega_1} \odot A_{(1)}\right) \oplus \cdots \oplus \left(\widetilde{\omega_n} \odot A_{(n)}\right)\right] \oplus \left[\left(\widetilde{\omega_1} \odot \widetilde{\lambda}\right) \oplus \cdots \oplus \left(\widetilde{\omega_n} \odot \widetilde{\lambda}\right)\right] \\
&= \left[\left(\widetilde{\omega_1} \odot A_{(1)}\right) \oplus \cdots \oplus \left(\widetilde{\omega_n} \odot A_{(n)}\right)\right] \oplus \left[\widetilde{\lambda} \odot \left(\widetilde{\omega_1} \oplus \cdots \oplus \widetilde{\omega_1}\right)\right] \\
&= \mathrm{AOWA}_{\trianglelefteq}^{\widehat{\omega}}(A_1, \ldots, A_n) \oplus \widetilde{\lambda}.
\end{aligned}$$

Hence, $\mathrm{AOWA}_{\trianglelefteq}^{\widehat{\omega}}$ is shift-invariant. □

Theorem 5.8. *Under the same hypotheses of* Definition 4.4*,*

$$\mathrm{AOWA}_{\trianglelefteq}^{\widehat{\omega}}(\widetilde{\lambda} \odot A_1, \ldots, \widetilde{\lambda} \odot A_n) = \widetilde{\lambda} \odot \mathrm{AOWA}_{\trianglelefteq}^{\widehat{\omega}}(A_1, \ldots, A_n),$$

for all $\widetilde{\lambda} \in \mathcal{CFN}$ *and for any tuple* $(A_1, \ldots, A_n) \in \mathcal{F}(\mathbb{R})^n$*.*

Proof. Commutativity and associativity of $\odot$ guarantees that $\widetilde{\omega_j} \odot \left(\widetilde{\lambda} \odot A_j\right) = \widetilde{\lambda} \odot (\widetilde{\omega_j} \odot A_j)$, for all $j \in \{1, ..., n\}$. Therefore, by Lemma 5.1(2)

$$\begin{aligned}
\mathrm{AOWA}_{\trianglelefteq}^{\widehat{\omega}}(\widetilde{\lambda} \odot A_1, \ldots, \widetilde{\lambda} \odot A_n) &= \left[\widetilde{\omega_1} \odot \left(\widetilde{\lambda} \odot A_{(1)}\right)\right] \oplus \cdots \oplus \left[\widetilde{\omega_n} \odot \left(\widetilde{\lambda} \odot A_{(n)}\right)\right] \\
&= \left[\widetilde{\lambda} \odot \left(\widetilde{\omega_1} \odot A_{(1)}\right)\right] \oplus \cdots \oplus \left[\widetilde{\lambda} \odot \left(\widetilde{\omega_n} \odot A_{(n)}\right)\right] \\
&= \widetilde{\lambda} \odot \left[\left(\widetilde{\omega_1} \odot A_{(1)}\right) \oplus \cdots \oplus \left(\widetilde{\omega_n} \odot A_{(n)}\right)\right] \\
&= \widetilde{\lambda} \odot \mathrm{AOWA}_{\trianglelefteq}^{\widehat{\omega}}(A_1, \ldots, A_n).
\end{aligned}$$

Hence, $\mathrm{AOWA}_{\trianglelefteq}^{\widehat{\omega}}$ is homogeneous. □

6. Admissible orders for trapezoidal fuzzy numbers and an application to multi-criteria decision making

This section proposes an illustrative example related to sustainability in which the AOWA operators are used in multi-criteria decision-making. To do so, we start by introducing ELICIT (Extended Comparative Linguistic Expressions with Symbolic Translation) information, which relies on the fuzzy linguistic approach [26] to model the linguistic information provided by experts. Since ELICIT information induces a bijection between the linguistic expressions and the set of TrFNs $\mathcal{T}$, we will apply the novel construction method introduced in Section 3 to generate admissible orders on $\mathcal{T}$. Finally, we show the feasibility of AOWA operators by conducting a case study.

6.1. ELICIT information

From a theoretical point of view, ELICIT expressions are generated by a context-free grammar that models comparative linguistic expressions such as "between good and very good" or "at least good" [16]. Formally, ELICIT information relies on the 2-tuple linguistic model [17], which was introduced to precisely manipulate the linguistic values by representing them as a $(s_i, \alpha) \in \overline{\mathscr{S}} := \{s_0\} \times [0, 0.5) \cup \{s_1\} \times [-0.5, 0.5) \cup \{s_2\} \times [-0.5, 0.5) \cup \ldots \cup \{s_{g-1}\} \times [-0.5, 0.5) \cup \{s_g\} \times [-0.5, 0]$, where s_i is a linguistic term in the linguistic term set $\mathscr{S} = \{s_0, s_1, \ldots, s_g\}$ ($g \in \mathbb{N}$ even) and $\alpha \in [-0.5, 0.5[$ stands for the deviation of the fuzzy membership function of the 2-tuple expression with respect to the membership function of the term s_i. The main feature of the 2-tuple linguistic approach is that any 2-tuple linguistic expression may be univocally remapped into the real interval $[0, g]$ using the bijection $\Delta^{-1} : \overline{\mathscr{S}} \to [0, g]$ defined as $\Delta^{-1}_{\mathscr{S}}(s_i, \alpha) = i + \alpha, \ \forall \ (s_i, \alpha) \in \overline{\mathscr{S}}$ [17].

Even though the 2-tuple linguistic model succeeds at modeling linguistic expressions by using the fuzzy linguistic approach, it fails at dealing with the hesitancy between different linguistic terms. For this reason, Labella et al. [16] proposed ELICIT information as a generalization of the 2-tuple approach that improves its flexibility when modeling hesitation. Formally, an ELICIT expression is denoted as $[\overline{s}_i, \overline{s}_j]_{\gamma_1,\gamma_2}$, where $\overline{s}_i, \overline{s}_j \in \overline{\mathscr{S}}, i \leq j$ are two 2-tuple terms and γ_1, γ_2 are two parameters that ensure that there is no information loss when manipulating the ELICIT expression [7]. Note that any ELICIT expression may be univocally remapped into a TrFN [7]:

Proposition 6.1 ([7]). *Let $\overline{\overline{\mathscr{S}}}$ be the set of all the ELICIT values. Then, the mapping $\zeta : \mathcal{T} \to \overline{\overline{\mathscr{S}}}$ defined as:*

$$\zeta(T^{(a/b/c/d)}) = [\overline{s}_1, \overline{s}_2]_{\gamma_1,\gamma_2},$$

where

$$\begin{aligned}\overline{s}_1 &= \Delta_S(gb), \qquad \gamma_1 = a - \max\left\{b - \frac{1}{g}, 0\right\},\\ \overline{s}_2 &= \Delta_S(gc) \quad \text{ and } \gamma_2 = d - \min\left\{c + \frac{1}{g}, 1\right\}\end{aligned}$$

is a bijection.

In order to adapt OWA operators to ELICIT information, for a real number $\omega \in [0, 1]$, let us consider the associated membership function $\tilde{\omega} : [0, 1] \to [0, 1]$ defined as

$$\widetilde{\omega}(x) = \begin{cases} 1 & \text{if } x = \omega \\ 0 & \text{otherwise} \end{cases} \quad \forall\, x \in [0, 1].$$

In this case, for a weight vector $\widehat{\omega} = (\widetilde{\omega_1}, \widetilde{\omega_2}, ..., \widetilde{\omega_n})$, $n \in \mathbb{N}$, a family of ELICIT values $T^{(a_1/b_1/c_1/d_1)}$, ..., $T^{(a_n/b_n/c_n/d_n)}$ may be aggregated according to an AOWA operator as follows:

$$\begin{aligned}&\mathrm{AOWA}^{\widehat{\omega}}_{\trianglelefteq}\left(T^{(a_1/b_1/c_1/d_1)}, ..., T^{(a_n/b_n/c_n/d_n)}\right) = \\ &= T^{\left(\sum_{i=1}^m \omega_i a_{(i)}/\sum_{i=1}^m \omega_i b_{(i)}/\sum_{i=1}^n \omega_i c_{(i)}/\sum_{i=1}^n \omega_i d_{(i)}\right)}\end{aligned}$$

where $T^{(a_{(i)},b_{(i)},c_{(i)},d_{(i)})}$ is the i-th largest value in $T^{(a_1/b_1/c_1/d_1)}, ..., T^{(a_n/b_n/c_n/d_n)}$ according to a certain admissible order $\trianglelefteq$. Note that the construction method for deriving admissible orders for TrFNs introduced in Section 3 allows inducing a total order for ELICIT values.

6.2. Case study

Sustainability aims for a regenerative production process that minimizes adverse outcomes and exploits the source material as much as possible [10]. In the implementation process of a sustainable production model, it is necessary to make decisions based on experts' recommendations that may be vague or imprecise. In this case, we consider the fast food company EasyLunch, that desires to introduce a new sustainability policy among the following:

- $A_1 \sim$ Use electric vehicles for deliveries.
- $A_2 \sim$ Replace disposable plastic cutlery with reusable steel cutlery.
- $A_3 \sim$ Install solar panels in all the restaurants.
- $A_4 \sim$ Replace the natural gas heating system with an electric heat pump.

In order to select the best alternative, the company wants to consider its $C_1 \sim$ Environmental, $C_2 \sim$ Social, and $C_3 \sim$ Economical impact [6] and, consequently, they ask an expert to evaluate the different alternatives according to these criteria by using the linguistic term set

$$\begin{aligned} S = \{s_0 \sim \text{Very inadequate}, s_1 \sim \text{Inadequate},\\ s_2 \sim \text{Slightly inadequate}, s_3 \sim \text{Irrelevant},\\ s_4 \sim \text{Slightly adequate},\\ s_5 \sim \text{Adequate}, s_6 \sim \text{Very adequate}\}\,. \end{aligned}$$

Therefore, the expert provides her opinion about the suitability of each alternative according to the above-mentioned criteria using comparative linguistic expressions. The corresponding decision matrix is as follows:

$$\begin{pmatrix} [(s_3,0.0),(s_3,0.0)]_{0.0,0.0} & [(s_0,0.0),(s_2,0.0)]_{0.0,0.0} & [(s_2,0.0),(s_4,0.0)]_{0.0,0.0} \\ [(s_2,0.0),(s_5,0.0)]_{0.0,0.0} & [(s_0,0.0),(s_5,0.0)]_{0.0,0.0} & [(s_0,0.0),(s_0,0.0)]_{0.0,0.0} \\ [(s_0,0.0),(s_6,0.0)]_{0.0,0.0} & [(s_2,0.0),(s_6,0.0)]_{0.0,0.0} & [(s_3,0.0),(s_5,0.0)]_{0.0,0.0} \\ [(s_2,0.0),(s_5,0.0)]_{0.0,0.0} & [(s_0,0.0),(s_4,0.0)]_{0.0,0.0} & [(s_5,0.0),(s_5,0.0)]_{0.0,0.0} \end{pmatrix},$$

where the item in the i-th row and j-th column corresponds to the evaluation of the alternative A_i under the criteria C_j. Note that, in spite of its notation, preference values such as $[(s_0,0.0),(s_1,0.0)]_{0.0,0.0}$ stand for the linguistic comparative expression *less than inadequate*, whereas values such as $[(s_1,0.0),(s_3,0.0)]_{0.0,0.0}$ correspond to the expression *between inadequate and irrelevant*. Using the ELICIT framework, these preferences are transformed into TrFN, obtaining the following decision matrix:

$$T = \begin{pmatrix} T^{(0.33/0.5/0.5/0.67)} & T^{(0.0/0.0/0.33/0.5)} & T^{(0.17/0.33/0.67/0.83)} \\ T^{(0.17/0.33/0.83/1.0)} & T^{(0.0/0.0/0.83/1.0)} & T^{(0.0/0.0/0.0/0.17)} \\ T^{(0.0/0.0/1.0/1.0)} & T^{(0.17/0.33/1.0/1.0)} & T^{(0.33/0.5/0.83/1.0)} \\ T^{(0.17/0.33/0.83/1.0)} & T^{(0.0/0.0/0.67/0.83)} & T^{(0.67/0.83/0.83/1.0)} \end{pmatrix}$$

Following the initial idea used by Yager to apply OWA operators in MCDM problems [25], we assume that the final score of the alternative should depend on how much it satisfies all the criteria. Therefore, we use the aforementioned AOWA operator associated with the weights $\hat{\omega} = (\widetilde{0.5}, \widetilde{0.3}, \widetilde{0.2})$ to aggregate, for each alternative, the rating of the criteria. Note that, for the TrFNs in the matrix T, the following holds:

$$\begin{aligned} &T_{1,3} \trianglerighteq_{XY} T_{1,1} \trianglerighteq_{XY} T_{1,2} \\ &T_{2,1} \trianglerighteq_{XY} T_{2,2} \trianglerighteq_{XY} T_{2,3} \\ &T_{3,3} \trianglerighteq_{XY} T_{3,2} \trianglerighteq_{XY} T_{3,1} \\ &T_{4,3} \trianglerighteq_{XY} T_{4,1} \trianglerighteq_{XY} T_{4,2}. \end{aligned}$$

It must be highlighted that $\trianglelefteq_{XY}$ is a total order constructed according to the procedure developed in Section 3 and thus the order in which these values are aggregated using the AOWA does not depend on the applied sorting algorithm.

Consequently, the TrFNs obtained after using the $\mathrm{AOWA}^{\widehat{\omega}}_{\trianglelefteq_{XY}}$ are as follows.

$$\begin{pmatrix} T^{(0.18/0.32/0.55/0.72)} \\ T^{(0.08/0.17/0.67/0.83)} \\ T^{(0.22/0.35/0.92/1.0)} \\ T^{(0.38/0.52/0.8/0.97)} \end{pmatrix},$$

and the corresponding ELICIT version is

$$\begin{pmatrix} [(s_2, -0.1), (s_3, 0.3)]_{0.03,0.0} \\ [(s_1, 0.0), (s_4, 0.0)]_{0.08,0.0} \\ [(s_2, 0.1), (s_6, -0.5)]_{0.03,0.0} \\ [(s_3, 0.1), (s_5, -0.2)]_{0.03,0.0} \end{pmatrix}.$$

Here, it should be remarked that when making the aggregation, this AOWA operator prioritizes the highest (according to the order) rating of the alternatives in any criteria, in the same way as the crisp OWA operator with weights (0.5,0.3,0.2). In other words, an alternative is considered to have a good overall performance if it has a very good performance for any of the criteria.

Finally, we can use the ELICIT XY-order induced by $\trianglelefteq_{XY}$ on the above matrix to conclude that $A_4 \succ A_3 \succ A_1 \succ A_2$ (see Fig. 4).

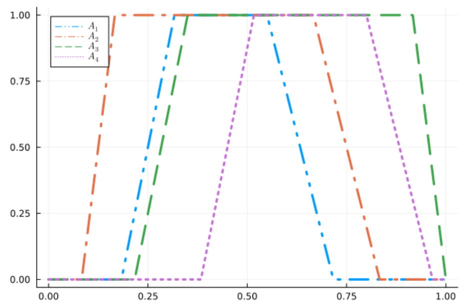

Figure 4: Membership functions of the final ELICIT ratings for the alternatives when using XY-order

7. Conclusions

The definition of a reasonable ordering method for fuzzy numbers has been an open problem for many years. However, the recently introduced notion of admissible order for fuzzy numbers provides a total order relation that allows their ranking. In this paper, we have applied this admissible order idea to extend OWA operators, which is a classic family of aggregation functions that, as we have shown in this paper, need a total order relation to be well-defined. In this perspective, we have carried out a deep study regarding the necessary axioms required to define such an OWA operator from admissible orders on fuzzy numbers. In addition, we have shown that the classic properties of OWA operators for crisp numbers remain valid in the fuzzy numbers domain. Finally, we have illustrated that our proposal can be easily applied to properly solve decision-making problems under uncertain environments. Additionally, we have provided a generic method to define admissible orders on TrFNs from admissible orders for intervals. The main advantage of this methodology is the fact that it does not need to compare α-cuts according to an upper-dense sequence, unlike in the already proposed admissible orders [30].

For future studies, we will investigate the extension of OWA operators to interval-valued fuzzy numbers. This would allow for a more comprehensive handling of uncertainty, as interval-valued fuzzy numbers can capture a wider range of possibilities. Additionally, we will focus on the application of the concept of admissible order to broaden the scope of other aggregation operators into the domain of fuzzy numbers, such as the Sugeno and Choquet integrals. Finally, from the application point of view, we will study the integration of AOWA operators into big data analytics and machine learning algorithms to enhance data fusion, feature selection, and model combination in large and complex datasets.

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Declaration of interests

☒ The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

☐The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: