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Power-average-operator-based hybrid multiattribute online product recommendation model for consumer decision-making

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Abstract

This study develops a power-average-operator-based hybrid multiattribute online product recommendation model that considers the consumer's risk attitude to rank categoric product options as a complement to existing recommender systems. Online production recommendation plays a key role in the development of e-commerce, and can greatly improve consumers' shopping experiences. However, few online shopping sites provide interactive decision aids for consumers such that they can articulate their preferences towards multiple selection attributes with the purpose of mitigating choice difficulty and improving decision quality. Additionally, consumers' risk attitudes to online shopping dramatically impact their product choices. In the model proposed in this paper, the risk attitude-based power average (RAPA) operator is used to integrate the risk attitude of the decision-maker into the information fusion process of multiple attribute decision-making. Subsequently, the risk attitude function, with several basic types, is introduced to quantify the risk attitude of the decision-maker for use in the RAPA operator. A proportional hesitant fuzzy 2-tuple

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linguistic term set (PHF2TLTS) is constructed by incorporating a binary of linguistic information aiming to comprehensively analyze the hybrid product information. With a focus on the information fusion process, the proportional hesitant 2-tuple linguistic RAPA operator and weighted proportional hesitant 2-tuple linguistic RAPA operator are introduced to aggregate a given set of PHF2TLTSs. The validity of the proposed model is demonstrated using an illustrative example, a comparison with existing approaches and detailed explanations of the performance differences.

KEYWORDS
multiattribute decision support, online product recommendation, power average operator, proportional hesitant fuzzy linguistic term set, risk attitude

1 | INTRODUCTION

Marketing models are central to modern marketing decision making. Modeling of marketing phenomena to support and improve marketing decisions dates back to the 1950s,[1,2] and the field of marketing decision models has been in a permanent state of development and growth ever since its incubation. With the advent of online shopping, consumers are now faced with an increasingly vast range of choices and product alternatives. As e-commerce develops, people are becoming more inclined to purchase products online for increased convenience and choice.[3] To alleviate the information overload when shopping online, and to save consumers' time and energy in product selection, researchers from diverse backgrounds have studied online product recommendation systems. Online product recommendations have been demonstrated to exert great influence on consumers' decision making and ultimately affect consumer loyalty.[4-8] Tsao[9] found that the fitness of product information is a vital factor impacting the effectiveness of recommendations. Liu et al.[10] designed novel online recommendation approaches on the basis of nonnegative matrix factorization and latent Dirichlet allocation for the purpose of predicting user preferences and dynamically adjusting the recommendation list. Choi et al.[11] proposed a hybrid online-product recommendation method integrating collaborative filtering and sequential pattern analysis to improve recommendation quality. Combining statistical parsing approaches with traditional AI rule-based technology, Chai et al.[12] built a web-based natural language assistant to help users select relevant products. Häubl and Murray[13] proposed a tool that can provide (potential) consumers with personalized product recommendations. Sheng and Zolfagharian[14] examined the complex role of consumer participation by empirically testing a theoretical model in the context of consumers using online product recommendation agents that integrated consumer participation into the technology acceptance model. Some researchers have focused on assessing consumers' acceptance and trust with online risk attitudes.[15,16] Wu et al.[17] presented a Hybrid Shilling Attack Detector to respond to hybrid shilling attacks that bias the rating profiles of recommender systems to manipulate online product recommendations. Hung[18] constructed an online recommendation system based on a modified product taxonomy and customer classification to identify customers' shopping behavior. Existing product recommendation systems have been diffusely applied in online retail stores,[19,20] hotel bookings,[21] and for online sales of products such as cell phones,[22,23] movies,[24] books,[25,26] and so on. Some additional issues in online product recommendation systems include cold-start product recommendation,[27] location-based recommendation agents,[28] and group recommendations,[29] to name just a few.

In the field of information fusion under decision-making settings, power average (PA)[30] is a useful aggregation operator that can naturally reflect the interrelationships among aggregated arguments by permitting them to support and reinforce each other.[31-33] Under this mechanism, smaller weights are automatically assigned to unduly low or high arguments, which are usually regarded as possibly "false" or "biased" inputs.[34] Due to this characteristic, the PA operator has been widely utilized in a variety of fields, such as software quality evaluation,[35] multiple attribute (group) decision-making (MADM/MAGDM),[36-38] and green product development.[39] Theoretical expansions on PA are also quite rich. Xu and Yager[34] proposed the power geometric (PG) operator, which can analogously model the interactions among aggregated data. The PG operator is considered more suitable for processing the multiplicative preference relationships. Zhou et al.[38] introduced a general form of PA operator, named the generalized power average, in which the nonlinear weight for each aggregated argument does not change with the generalized parameter. In practice, however, decision-makers undoubtedly tend to assign sufficiently small weights to "false" or "biased" elements to mitigate their influence on final aggregation results. Inspired by Zhou et al.'s study, Xiong et al.[40] proposed the variable PG operator, a generalization of the PG in which arguments' weights can vary with a nonnegative parameter. More generally, Xiong et al.[41] introduced the extended power average operator to effectively model for practical occasions where "false" or "biased" inputs may be of importance to the aggregation results.

In terms of information representational models, hesitant fuzzy sets (HFSs)[42] are widely used to deal with contexts in which experts may differ on the membership degree of an element x to a set A. For instance, one expert may assign 0.2 but the other 0.3, and they cannot convince each other to change their opinions.[42,43] The hesitancy is caused by the two experts' different opinions, and this problem can be represented by HFS $\{0.2, 0.3\}$. Rodríguez et al.[44,45] introduced an extension of HFS named hesitant fuzzy linguistic term set (HFLTS) to accommodate the linguistic hesitation information. The main applications of HFLTS are situations where an expert hesitates with several values in evaluating linguistic variables and where it is difficult for them to provide a single linguistic term as an appropriate expression of their knowledge. HFLTSs express the uncertainty of information with better effect than HFSs.[46] Xiong et al.[47] classified such differences into two aspects according to the source of hesitancy. HFLTS is an individual information representational model in which assessment information is provided by only one hesitant expert who thinks the membership degree may have a set of possible values. HFS is a group information representational model, where experts' opinions are different and they cannot convince each other to agree on a single opinion. HFLTS has been well received due to the model's characteristics, and many HFLTS research are focused on issues of its application to group decision-making (GDM).[48] Hao and Chiclana[49] embed attitude quantifiers and a new feedback mechanism into HFLTS to deal with GDM, to handle the hesitant fuzzy linguistic GDM problems. Wang[50] proposed the extended HFLTS (EHFLTS), whose underlying meaning is consistent with that of HFS.[51] Addressing the uncertainty of EHFTLS, Wei et al.[52] proposed a quantification method based on comprehensive entropy. However, these methods do not consider the proportional information of each linguistic term in a GDM context, which can represent the support for each linguistic term derived from the experts. There are also some challenges in the development of HFLTS, such as modeling complex linguistic information during decision-making.[46,53] Due to this, Wang and Hao[54] introduced a proportional 2-tuple linguistic information representational model. Based on Wang and Hao's[54] work, Zhang et al.[55] proposed a natural generalization representational model, in which all linguistic terms are assigned symbolic proportions. Wu and Xu[56] provided a particular context where all possible linguistic terms are assigned the same proportion. Chen et al.[57] further proposed the proportional hesitant fuzzy linguistic term set (PHFLTS) to simultaneously consider the linguistic assessments and their proportional information. Taking into account the fact that the 2-tuple linguistic computational model is able to avoid information loss during linguistic information aggregation,[58] this paper extends the PHFLTS with the 2-tuple linguistic computational model, and we call the extended model the proportional hesitant fuzzy 2-tuple linguistic term set (PHF2TLTS).

Decision-making plays a key role in the fields of management. When decision-makers determine the optimal alternative from multiple alternatives for multiple criteria during the process of decision-making, the process enters the field of multiattribute decision-making (MADM). With the increasing complexity of decision-making problems, a single decision-maker may be insufficient to take into account all aspects of the problem. For scientific or democratic reasons, in most cases decisions are made through GDM or multiattribute GDM (MAGDM).[59] Although GDM cannot perfectly substitute for an individual decision, it can provide assistance for decision-makers to avoid making irrational decisions. MAGDM has been applied in many fields, such as education, the environment, investment, and resource management, among others.[60] Classic MADM methods include the Technique for Order Preference by Similarity to an Ideal Solution (TOPSIS), VlseKriterijuska Optimizacija I Komoromisno Resenje (VIKOR), and Elimination and Choice Expressing Reality (ELECTRE), among others.[60] MAGDM has attracted much attention from researchers due to its decision-making characteristics, particularly in the fields of hesitation and uncertainty.[61] Chen et al.[62] proposed a hybrid MAGDM model through an extended ELECTRE method under uncertainty to select the optimal sustainable building material. Wu et al.[63] proposed an extended TOPSIS decision model on the basis of interval type-2 variables to select the optimal alternative under uncertainty. Wu et al.[64] explored the compromise solution of MAGDM through TOPSIS and VIKOR methods in the HFLTS context. However, two major challenges are identified by literature reviews on MAGDM. One is the information loss during the process of decision-making, and the other is how to scientifically integrate the opinions of decision-makers into the MAGDM.[65] Additionally, research in the field of behavior state that the psychological state of a decision-maker may greatly affect the result of decision-making, as the decision-maker is not a perfectly rational person.[66] Peng and Yang[67] proposed an algorithm to deal with fuzzy MADM problems through prospect theory and regret theory. Zhao et al.[68] constructed an integrated fuzzy MADM model considering the impact of change in the risk preference of the decision-maker on the ranking results of alternatives. The psychological state of decision-makers in MAGDM is a field worthy of further research.[61,69] The risk attitude of a decision-maker is part of their psychological state, and thus research on MAGDM under uncertainty for online product recommendation, taking into account decision-makers' risk attitude during the process, is highly important.

The main motivation of this paper is to develop a power-average-operator-based hybrid multiattribute online product recommendation model in an effort to reduce choice difficulty and improve decision quality. The main contributions of this study can be summarized as follows.

i) An extension form of the PA operator, the risk attitude-based power average (RAPA) operator, is proposed to integrate the risk attitude of the decision-maker into the information fusion process.
ii) The concept of the risk attitudefunction (RAF) is introduced to effectively quantify the risk attitude of the decision maker into the RAPA operator, and several basic types of RAF are proposed.
iii) To facilitate comprehensive analysis of hybrid product information, a novel 2-tuple linguistic representational model, PHF2TLTS, is constructed based on the PHFLTS. PHF2TLTSs are not only similar to the cognitive processes of human beings, because they encompass the quantitative proportional information hidden behind qualitative linguistic expressions,[57] but can also avoid information loss during the aggregation of linguistic term variables.[70]
iv) Another main interest of this study is the development of a distance measure, the proportional hesitant 2-tuple linguistic normalized Hamming distance, and distance-based comparison method for PHF2TLTS.
v) This study also introduces the convex-combination-based proportional hesitant 2-tuple linguistic RAPA (PH2TLRAPA) operator and weighted proportional hesitant 2-tuple linguistic RAPA (WPH2TLRAPA) operator to aggregate the proportional hesitant 2-tuple linguistic assessment information.
vi) Finally, this study provides a power-average-operator-based hybrid multiattribute online product recommendation model on the basis of the aforementioned innovative work.

This paper is organized as follows. Section 2 briefly reviews several basic concepts related to PHFLTSs and the PA operator. Section 3 presents the RAPA operator and its weighted form (known as the WRAPA operator) and investigates several of their important properties. Especially, the concept of risk attitude function (RAF) is also introduced in this section. Section 4 introduces the notion of PHF2TLTS, and the distance measure and a comparison method for PHF2TLTSs are provided in this section as well. Section 5 presents an MADM-based online product recommendation model wherein the assessment information is gathered with hybrid representational forms. Section 6 further provides a practical application of the proposed model with a detailed illustrative example. Sections 3, 4, 5, and 6 contain the main original contributions of this study. Finally, Section 7 presents the summary and conclusions of this study.

2 | PRELIMINARIES

Let $\mathfrak{R}$ be the set of real numbers and $\mathcal{N} = \{1, 2, ..., n\}$ be the set of strictly positive integers. Where there is no particular specification, this paper generally follows the conventions used in Beliakov et al.[71] and Chen et al.[72,73]

2.1 | Proportional hesitant fuzzy linguistic term set

Rodríguez et al.[44] put forward the HFLTS to accommodate qualitative settings that require efficient information elicitation methodologies. The prevalent adoption of Zadeh's[74] fuzzy linguistic approach and the computing with words (CW) paradigm promote computing with HFLTSs within the subarea of linguistic computational intelligence systems. One promising endeavor is the advancement of HFLTS-based linguistic representation models; the concept of PHFLTS, developed by Chen et al.,[57] has been proven to be particularly sophisticated in handling complex group assessment via the simultaneous consideration of expert linguistic assessments and their proportional information.

Definition 1 (Chen et al.[57]). Following Chen et al.,[57] let $\mathcal{S}$ be a linguistic term set (LTS), and $P = (p_0, p_1, ..., p_g)^{\mathrm{T}}$ be a proportional vector. A PHFLTS for a linguistic variable ϑ, namely $\mathscr{P}_{H_S}$, is an ordered finite set:

$$\mathscr{P}_{H_S}(\vartheta) = \{(s_i, p_i) | s_i \in S, i = 0, 1, ..., g\},$$

with the conditions that $\sum_{i=0}^{g} p_i \le 1$ and $0 \le p_i \le 1 (i = 0, 1, ..., g)$.

Given a PHFLTS, the proportional linguistic pairs $(s_i, p_i)(i = 0, 1, ..., g)$ in it are ranked in accordance with their ordered linguistic terms $s_i (i = 0, 1, ..., g)$.

The aggregation paradigm for PHFLTSs was presented by Chen et al.[57] with the following concept of proportional convex combination of two PHFLTSs.

Definition 2 (Chen et al.[57]). Let $\mathscr{P}^1_{H_S} = \{(s_{i_1}, p_{i_1}), (s_{i_2}, p_{i_2}), ..., (s_{i_p}, p_{i_p})\}$ and $\mathscr{P}^2_{H_S} = \{(s_{j_1}, p_{j_1}), (s_{j_2}, p_{j_2}), ..., (s_{j_q}, p_{j_q})\}$ be two PHFLTSs defined on the LTS $\mathcal{S}$ with a weighting vector $\mathbf{w} = (w_1, w_2)^{\mathrm{T}}$. A proportional convex combination of the two PHFLTSs is defined as

$$\begin{aligned}
&\mathcal{G}^2\left(w_1, \mathscr{P}^1_{H_S}, w_2, \mathscr{P}^2_{H_S}\right) = \left(w_1, \mathscr{P}^1_{H_S}\right) \odot \left(w_2, \mathscr{P}^2_{H_S}\right) \\
&= \left\{\mathcal{G}^2\left(w_1, (s_{i_m}, p_{i_m}), w_2, (s_{j_n}, p_{j_n}) | (s_{i_m}, p_{i_m}) \in \mathscr{P}^1_{H_S}, (s_{j_n}, p_{j_n}) \in \mathscr{P}^2_{H_S}\right)\right\} \\
&= \langle w_1 + w_2, \{(s_{k_1}, p_{i_1} p_{j_1}), (s_{k_2}, p_{i_2} p_{j_1}), ..., (s_{k_{mn}}, p_{i_m} p_{j_n}), ..., (s_{k_{pq}}, p_{i_p} p_{j_q})\}\rangle \\
&= \langle w_1 + w_2, \{(s_{k_1}, p_{k_{11}}), (s_{k_2}, p_{k_{21}}), ..., (s_{k_\eta}, p_{k_\eta}), ..., (s_{k_{\mathrm{H}}}, p_{k_{\mathrm{H}}})\}\rangle,
\end{aligned} \tag{1}$$

where $\eta = m \times n \in \{1, 2, ..., m \times n, ..., \mathrm{H}\}$, $\mathrm{H} = p \times q$, $w_1, w_2 \ge 0$, $w_1 + w_2 \le 1$, and

$$k_\eta = \mathrm{Min}\left\{\begin{array}{l}\mathrm{Max}\{\mathrm{Ind}((s_{i_p}, p_{i_p})), \mathrm{Ind}((s_{j_q}, p_{j_q}))\}, \\ round\left(\dfrac{w_1 p_{i_m}}{w_1 p_{i_m} + w_2 p_{j_n}} i_m + \dfrac{w_2 p_{j_n}}{w_1 p_{i_m} + w_2 p_{j_n}} j_n\right)\end{array}\right\},$$

where $\mathrm{Ind}((s_{i_p}, p_{i_p}))$ and $\mathrm{Ind}(s_{j_q}, p_{j_q})$ are indexes i_p and j_q of the proportional linguistic pairs (s_{i_p}, p_{i_p}) and (s_{j_q}, p_{j_q}), respectively.

Because the set of proportional linguistic pairs obtained via the proportional convex combination of two PHFLTSs may not be ordered and several linguistic terms may emerge more than once with different proportions, the following ordered proportional convex combination of two PHFLTSs was developed.

Definition 3 (Chen et al.[57]). Let $\mathscr{P}^1_{H_S} = \{(s_{i_1}, p_{i_1}), (s_{i_2}, p_{i_2}), ..., (s_{i_p}, p_{i_p})\}$ and $\mathscr{P}^2_{H_S} = \{(s_{j_1}, p_{j_1}), (s_{j_2}, p_{j_2}), ..., (s_{j_q}, p_{j_q})\}$ be the two PHFLTSs defined on $\mathcal{S}$ with a weighting vector $\mathbf{w} = (w_1, w_2)^{\mathrm{T}}$. An ordered proportional convex combination of the two PHFLTSs is defined as

$$\begin{aligned}\mathcal{T}\left(\mathcal{G}^2\left(w_1, \mathscr{P}^1_{H_S}, w_2, \mathscr{P}^2_{H_S}\right)\right) &= \mathcal{T}\langle w_1 + w_2, \{(s_{k_1}, p_{k_1}), (s_{k_2}, p_{k_2}), ..., (s_{k_{mn}}, p_{k_{mn}}), ..., \\ &\quad (s_{k_{pq}}, p_{k_{pq}})\}\rangle \\ &= \langle w_1 + w_2, \{(s_{i_1}, p_{i_1}), (s_{i_2}, p_{i_2}), ..., (s_{i_l}, p_{i_l}), ..., (s_{i_h}, p_{i_h})\}\rangle,\end{aligned} \tag{2}$$

where $w_1, w_2 \geq 0, w_1 + w_2 \leq 1, \mathsf{Min}(k_1, k_2, ..., k_\eta, ..., k_{\mathrm{H}}) = i_1 < i_2 < \cdots < i_h = \mathsf{Max}(k_1, k_2, ..., k_\eta, ..., k_{\mathrm{H}})$, and $p_{i_l} = \sum_{\eta=1}^{\mathrm{H}} p_{k_\eta} \chi(k_\eta, i_l)$. $\chi(k_\eta, i_l)$ is an indicator function defined by:

$$\chi(k_\eta, i_l) = \begin{cases} 1, & k_\eta = i_l, \\ 0, & k_\eta \neq i_l. \end{cases}$$

Specifically, $\langle w_1 + w_2, \{(s_{i_1}, p_{i_1}), (s_{i_2}, p_{i_2}), ..., (s_{i_h}, p_{i_h})\}\rangle$ can be simplified to $\{(s_{i_1}, p_{i_1}), (s_{i_2}, p_{i_2}), ..., (s_{i_h}, p_{i_h})\}$ on the condition that $w_1 + w_2 = 1$. For convenience, we denote

$$\mathcal{T}\mathcal{G}^2\left(w_1, \mathscr{P}^1_{H_S}, w_2, \mathscr{P}^2_{H_S}\right) = \mathcal{T}\left(\mathcal{G}^2\left(w_1, \mathscr{P}^1_{H_S}, w_2, \mathscr{P}^2_{H_S}\right)\right).$$

2.2 | PA operator

Definition 4 (Yager[30,75]). A PA operator of dimension n is a mapping PA: $\mathfrak{R}^n \to \mathfrak{R}$, according to the following formula:

$$\mathsf{PA}(A) = \frac{\sum_{i=1}^n (1 + \mathscr{T}(a_i)) a_i}{\sum_{j=1}^n (1 + \mathscr{T}(a_j))},$$

where the external support in degree (ESiD)

$$\mathscr{T}(a_i) = \sum_{l=1, l\neq i}^{n} \mathsf{Sup}(a_i, a_l), i \in \mathcal{N},$$

and $\mathsf{Sup}(a_i, a_l)$ is the support for a_i from a_l, which satisfies the following three properties:

1. $\mathsf{Sup}(a_i, a_l) \in [0, 1]$;
2. $\mathsf{Sup}(a_i, a_l) = \mathsf{Sup}(a_l, a_i)$;
3. $\mathsf{Sup}(a_i, a_l) \geq \mathsf{Sup}(x, y)$, if $|a_i - a_l| < |x - y|$.

The PA operator can be rewritten as follows:

$$\mathsf{PA}(A) = \sum_{i=1}^n \frac{(1 + \mathscr{T}(a_i))}{\sum_{j=1}^n (1 + \mathscr{T}(a_j))} a_i = \sum_{i=1}^n w_i a_i,$$

where $w_i = (1 + \mathscr{T}(a_i)) / \sum_{j=1}^n (1 + \mathscr{T}(a_j))$ is the weight of argument a_i. The PA operator therefore is a nonlinear weighted average aggregation operator, which allows the input arguments to support and reinforce each other.[30] This useful characteristic gives the PA operator the feature of discounting outliers, that is, assigning smaller weights to unduly high or low arguments. Moreover, the support measure can be interpreted as a similarity index. Yager[30] provided the following two support functions:

$$\mathrm{Sup}(a_i, a_j) = \mathbb{K}e^{-\varepsilon(a_i - a_j)^2},$$
$$\mathrm{Sup}(a_i, a_j) = \mathbb{K}(1 - |a_i - a_j|^{\varepsilon}),$$

where $\mathbb{K} \in [0, 1]$, $\varepsilon \geq 0$ and $a_i, a_j \in [0, 1]$.

3 | RAPA OPERATORS

The binary support function embedded in the PA operator is essential to its logical implementation in real-life scenarios because it establishes the weight assignment rules for a given information fusion process. This interpretation explicitly indicates that: (i) an argument $\overline{a_i}(i \in \mathcal{N})$ gains the maximum weight if and only if it has the smallest total distances to the remaining arguments, namely, $\overline{a_i} = a_{\underset{i \in \mathcal{N}}{\arg \mathrm{Max}}\{\mathscr{T}(a_i) | i \in \mathcal{N}\}}$; and (ii) the weights of the remaining arguments decrease nonlinearly with their increasing distances to the argument $\overline{a_i}$. In this sense, these observations pinpoint a sophisticated formulation of a *Limited Range Neighborhood* centered on argument $\overline{a_i}$ where those arguments contained in it are assigned higher scales of importance. The graphical explanation of the limited range neighborhood is depicted in Figure 1, in which the arguments being aggregated are the first eleven terms of the Fibonacci sequence (i.e., $A = \{1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144\}$) and the support function is $\mathrm{Sup}(a_i, a_j) = 10(1 - |a_i - a_j|/144)$ for $a_i, a_j \in A$. Figure 1 depicts the details of nonlinear weights and $\mathscr{T}(a_i)$ for aggregated arguments, in which the blue curve represents the trend of nonlinear weights by PA operator for aggregated arguments, and the red curve represents the trend of $\mathscr{T}(a_i)$ for aggregated arguments. Both curves gain their maximum values at the aggregated

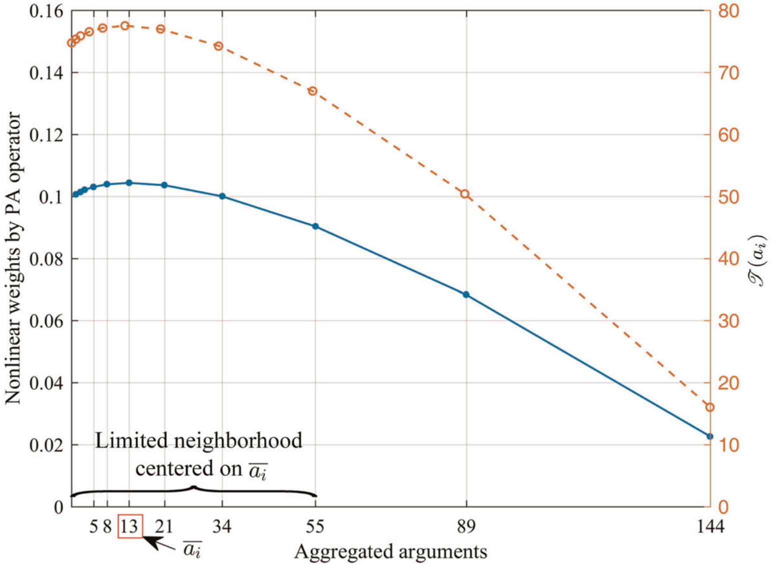

FIGURE 1 Nonlinear weights and $\mathscr{T}(a_i)$ for aggregated arguments [Color figure can be viewed at wileyonlinelibrary.com]

argument 13, which means the center of this instrumental limited range neighborhood $\overline{a_i} = a_{\arg\underset{i\in\mathcal{N}}{\mathrm{Max}}\{\mathcal{T}(a_i)|i\in\mathcal{N}\}} = 13$ has been obtained according to the aforementioned interpretation. The limited range neighborhood centered on $\overline{a_i}$ concludes the aggregated arguments $\{1, 2, 5, 8, 13, 21, 34, 55\}$, which are assigned higher scales of importance; detailed information is shown in Figure 1. The center of this instrumental limited range neighborhood is closely related to the interrelationships among the input arguments and presents significant difficulties for its reasonable implementation because the original method of binary support construction requires a considerable number of duplicated calculations that hinder its applications by requiring substantial computational efforts when very large amounts of data are involved.

Recall from the initial definition of the binary support function used in the PA operator, it is observed that when the most trusted argument is identified from a given set of aggregates, establishing a novel weight assignment rule based on the previous two observations can facilitate the simplification of the ESiD function with significant reductions in binary support calculations among all arguments. The aggregation mechanism of the PA operator interprets itself as permitting the aggregated data inputs to support and reinforce each other, based on the degrees of similarity among them. Our goal in this process is to shift the purely data-driven weight assignment paradigm into its context-dependent counterpart. The argument assigned maximum weight will be set as the referential argument to which the remaining arguments are compared, and will be treated as the most likely true aggregate that the PA operator tends to output. The weights for each of the other arguments are distributed nonlinearly.

In some real-life decision-making contexts, especially in online product recommendations, the risk attitude of the decision-maker may be important to the final decision results. This is the main motivation for extending the concept of PA.

3.1 | RAPA operator

Definition 5. Let $(a_1, a_2, \ldots, a_n)$ be a collection of distinct input arguments (the situation with repeated input arguments will be discussed later), a RAPA operator of dimension n is a mapping RAPA: $\mathfrak{R}^n \to \mathfrak{R}$, according to the following formula:

$$\mathrm{RAPA}_\sigma(a_1, a_2, \ldots, a_n) = \frac{\sum_{i=1}^n (1 + \mathcal{T}_\sigma(a_i))a_i}{\sum_{j=1}^n (1 + \mathcal{T}_\sigma(a_j))},$$

where the ESiD

$$\mathcal{T}_\sigma(a_i) = \mathrm{Sup}(a_i, a_\sigma), \quad i \in \mathcal{N},$$

with referential constant $a_\sigma \in A$ and where $\mathrm{Sup}(a, b)$ is the support for a from b, which satisfies the three properties previously described.

In Definition 5, $\mathrm{Sup}(a_i, a_\sigma), i \in \mathcal{N}$ represents the similarity between input a_i and (important limited) neighborhood center a_σ. Similar to the classical PA operator, the nonlinear weight for argument a_i is decreasing with its distance to the neighborhood center a_σ. It is worth noting that the neighborhood center in the PA operator is only determined by the relationship among the input arguments, whereas for the RAPA operator it is mainly determined by the risk attitude of the decision-maker. Let $A = (a_1, a_2, ..., a_n)$ be a collection of distinct input arguments and $\mathsf{AM}(A) = (1/n)\sum_{i=1}^{n} a_i$ be the arithmetic mean of A. When $a_\sigma \to \mathsf{AM}(A)$, the arguments near to a_σ will be assigned larger weights, whereas the other inputs will be assigned smaller weights. In this context, the arguments near to a_σ represent the main decision-making information provided by experts, and indicate that the decision-maker trusts such experts highly and her/his risk attitude is sufficiently positive. On the contrary, if a_σ is far enough away from $\mathsf{AM}(A)$, the risk attitude of the decision-maker is sufficiently negative. Consequently, the distance between a_σ and $\mathsf{AM}(A)$ can represent the risk attitude of the decision-maker.

3.2 | RAF function

Definition 6. Let A be a collection of distinct input arguments and $\mathsf{AM}(A)$ be the arithmetic mean of A. A RAF $\mathcal{RAF}_{\mathsf{AM}(A)}(a_\sigma)$, which reflects the risk attitude of the decision-maker corresponding to the RAPA operator, satisfies the following five properties:

1) $\mathcal{RAF}_{\mathsf{AM}(A)}(a_\sigma) \in [0, 1]$;
2) $\mathcal{RAF}_{\mathsf{AM}(A)}(a_\sigma) = 1$, if $a_\sigma = \mathsf{AM}(A)$;
3) $\mathcal{RAF}_{\mathsf{AM}(A)}(a_\sigma) = 0$, if $|a_\sigma - \mathsf{AM}(A)| = \mathsf{Max}_{i\in\mathcal{N}}\{|a_i - \mathsf{AM}(A)|\}$;
4) $\mathcal{RAF}_{\mathsf{AM}(A)}(a_{\sigma_1}) = \mathcal{RAF}_{\mathsf{AM}(A)}(a_{\sigma_2})$, if $|a_{\sigma_1} - \mathsf{AM}(A)| = |a_{\sigma_2} - \mathsf{AM}(A)|$;
5) $\mathcal{RAF}_{\mathsf{AM}(A)}(a_{\sigma_1}) > \mathcal{RAF}_{\mathsf{AM}(A)}(a_{\sigma_2})$, if $|a_{\sigma_1} - \mathsf{AM}(A)| < |a_{\sigma_2} - \mathsf{AM}(A)|$,

where $a_\sigma, a_{\sigma_1}, a_{\sigma_2} \in A$.

For convenience, let

$$i^{(1)} = \arg\underset{i\in\mathcal{N}}{\mathsf{Min}}\{|a_i - \mathsf{AM}(A)|\},$$

and

$$i^{(0)} = \arg\underset{i\in\mathcal{N}}{\mathsf{Max}}\{|a_i - \mathsf{AM}(A)|\}.$$

In Definition 6, a value of "1" indicates that the risk attitude of the decision-maker is completely positive, and will be obtained when a_σ has the smallest distance to $\mathsf{AM}(A)$, whereas "0" indicates her/his risk attitude is completely negative and will be obtained when a_σ has the largest distance to $\mathsf{AM}(A)$. The fourth property shows that the RAF $\mathcal{RAF}_{\mathsf{AM}(\mathbf{A})}(a_\sigma)$ is a symmetric function about $a_\sigma = a_{i^{(1)}}$. Furthermore, when a_σ approaches $a_{i^{(1)}}$, the risk attitude of the decision-maker becomes increasingly positive. The fifth property also implies that the RAF is increasing with $a_\sigma \leq \mathsf{AM}(A)$, but is decreasing with $a_\sigma > \mathsf{AM}(A)$.

Example 1. Let $A = \{1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144\}$ and $B = \{-144, -89, -55, -34, -21, -13, -8, -5, -3, -2, -1\}$ be two collections of input arguments. It is easy to verify that

$$\mathcal{RAF}_{\mathsf{AM}(A)}(a_\sigma) = \begin{cases} 1 + \dfrac{a_\sigma - \mathsf{AM}(A)}{a_{i^{(0)}} - \mathsf{AM}(A)}, & a_\sigma \leq \mathsf{AM}(A) \\ 1 - \dfrac{a_\sigma - \mathsf{AM}(A)}{a_{i^{(0)}} - \mathsf{AM}(A)}, & a_\sigma > \mathsf{AM}(A) \end{cases} \tag{3}$$

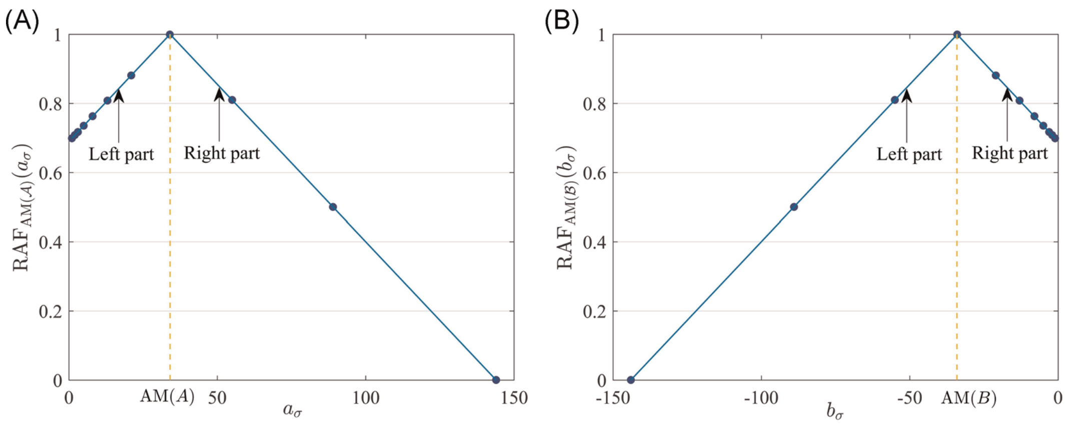

FIGURE 2 Linear RAFs corresponding to collections A and B (A) Collection A; (B) Collection B. RAF, risk attitude function [Color figure can be viewed at wileyonlinelibrary.com]

and

$$\mathcal{RAF}_{\mathsf{AM}(B)}(b_\sigma) = \begin{cases} 1 - \dfrac{b_\sigma - \mathsf{AM}(B)}{b_{i^{(0)}} - \mathsf{AM}(B)}, & b_\sigma \leq \mathsf{AM}(B) \\ 1 + \dfrac{b_\sigma - \mathsf{AM}(B)}{b_{i^{(0)}} - \mathsf{AM}(B)}, & b_\sigma > \mathsf{AM}(B) \end{cases} \tag{4}$$

are linear RAFs corresponding to collections A and B, as shown in Figure 2. $AM\,(A) = 34.1 > 0$, thus, $a_{i^{(1)}} = 34$. According to Definition 6 we know that the closer a_σ is to $a_{i^{(1)}}$, the greater the RAF value; when $0 < a_\sigma \leq AM\,(A)$, there is a positive correlation between RAF and a_σ; when $a_\sigma > AM\,(A) > 0$, there is a negative correlation between RAF and a_σ. Thus, Formula (3) is in good agreement with this case, whereas Formula (4) is the opposite of this case, and is applicable to collection B. Linear RAFs corresponding to collections A and B are shown in Figures 2A,B. These are perfectly symmetric, which verifies that $\mathcal{RAF}_{\mathsf{AM}(A)}(a_\sigma)$ is a symmetric function (i.e., the forth property of Definition 6). However, Formula (3) is not a RAF for collection B, $AM\,(B) = -34.1 < 0$, thus, $b_{i^{(1)}} = -34$, because $b_{i^{(0)}} = -144$ and then $1 + \frac{b_\sigma - \mathsf{AM}(B)}{b_{i^{(0)}} - \mathsf{AM}(B)}$ is decreasing with $b_\sigma \leq \mathsf{AM}(B)$, whereas $1 - \frac{b_\sigma - \mathsf{AM}(B)}{b_{i^{(0)}} - \mathsf{AM}(B)}$ is increasing with $b_\sigma > \mathsf{AM}(B)$. Similarly, one may easily verify that Formula (4) is not a RAF for collection A because $a_{i^{(0)}} = 144$.

Example 1 provides a linear type RAF (denoted Type I), which is relatively simple but not a continuously derivable function. Table 1 lists six basic types of RAF: linear, quadratic, sine/cosine, tangent/cotangent, exponential, and Gaussian. The basic function and properties of each type of RAF are also provided in this table. The various types of RAF provided can satisfy the demands of different cases.

Because the RAF is fully dependent on the input collection and its average mean and $i^{(0)}$, Algorithm 1 and Table 1 can be utilized to construct the RAF for a collection of distinct input arguments. In particular, because the exponential and Gaussian functions can never achieve "0" in their domains, as for point $(a_{i^{(0)}}, 0)$, a positive number ε, that is, sufficiently close to zero

TABLE 1 Basic types of risk attitude function

Serial number	Type	Basic function	Properties
I	Linear	$\mathcal{RAF}_{\mathsf{AM}(A)}(a_\sigma) = \mathbb{K}\ \lvert a_\sigma - \mathsf{AM}(A)\rvert + 1$	Continuous but non-derivable
II	Quadratic	$\mathcal{RAF}_{\mathsf{AM}(A)}(a_\sigma) = -\mathbb{K}(a_\sigma - \mathsf{AM}(A))^2 + 1, \mathbb{K} > 0$	Continuous and derivable
III	Sine/Cosine	$\mathcal{RAF}_{\mathsf{AM}(A)}(a_\sigma) = \sin(\mathbb{K}(a_\sigma - \mathsf{AM}(A)) + \pi/2)$ *or* $\mathcal{RAF}_{\mathsf{AM}(A)}(a_\sigma) = \cos(\mathbb{K}(a_\sigma - \mathsf{AM}(A)))$	Continuous and derivable
IV	Tangent/ Cotangent	$\mathcal{RAF}_{\mathsf{AM}(A)}(a_\sigma) = \tan(\mathbb{K}\lvert a_\sigma - \mathsf{AM}(A)\rvert + \pi/4)$ *or* $\mathcal{RAF}_{\mathsf{AM}(A)}(a_\sigma) = \cot(\mathbb{K}\lvert a_\sigma - \mathsf{AM}(A)\rvert + \pi/4)$	Continuous but nonderivable
V	Exponential	$\mathcal{RAF}_{\mathsf{AM}(A)}(a_\sigma) = b^{\mathbb{K}\lvert a_\sigma - \mathsf{AM}(A)\rvert}, b > 0, b \neq 1$	Continuous but nonderivable
VI	Gaussian	$\mathcal{RAF}_{\mathsf{AM}(A)}(a_\sigma) = e^{-(a_\sigma - \mathsf{AM}(A))^2/2\gamma^2}, \gamma > 0$	Continuous and derivable

is utilized to obtain an approximate result, such as $(a_{i^{(0)}}, 10^{-5})$. It is worth noting that Algorithm 1 can also generate the RAF for the special collection in which $i^{(0)} = \arg\underset{i\in\mathcal{N}}{\mathsf{Max}}\{|a_i - \mathsf{AM}(A)|\}$ has two different values. (The number of $i^{(0)}$ is always no more than two because the aggregated collection considered in this section consists of distinct input arguments.) For example, we have $i_1^{(0)} = 1$ and $i_2^{(0)} = 10$ for collection $C = \{3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$.

Algorithm 1 Framework for Generating the RAF

Input: Aggregated collection, A; Serial number of the basic type as listed in Table 1, N.
Output: RAF for collection A.

```
AM(A) ← mean(A);
if N ∈ {II,III,VI} then
    i^(0) ← arg Max_{i∈𝒩} {|a_i − AM(A)|};
    a_{i^(0)} ← a_{Min(i^(0))};
    Generate RAF: Solve the parameter of basic type N on the basis of points (AM(A), 1) and (a_{i^(0)}, 0);
else
    if AM(A) − Min{A} ≥ Max{A} − AM(A) then
        a_{i^(0)} ← Min{A};
        Generate the right part of RAF: Solve the parameter of basic type N for a_σ ≤ AM(A) on the basis of
          points (AM(A), 1) and (a_{i^(0)}, 0);
        Generate the left part of RAF: Solve the symmetric function of the right part of RAF about a_σ = AM(A)
          for a_σ > AM(A);
    else
        a_{i^(0)} ← Max{A};
        Generate the left part of RAF: Solve the parameter of basic type N for a_σ > AM(A) on the basis of
          points (AM(A), 1) and (a_{i^(0)}, 0);
        Generate the right part of RAF: Solve the symmetric function of the left part of RAF about a_σ = AM(A)
          for a_σ ≤ AM(A);
    end if
end if
```

The RAF is important to the RAPA operator because it can quantitatively determine the risk parameter $\sigma \in \mathcal{N}$ according to the risk attitude of the decision-maker. If the risk attitude value is θ, the risk parameter $\sigma \in \mathcal{N}$ can be obtained by Algorithm 2 on the basis of solving the formula $\mathcal{RAF}_{\mathsf{AM}(A)}(a_\sigma) = \theta$. A part illustration for Algorithm 2 is shown in Figure 3, which depicts the relationship between the risk attitude value and argument a_i under $\mathcal{RAF}_{\mathsf{AM}(A)}(a_\sigma)$. Among the risk attitude values, $\theta, \theta_1, \theta_2 < 0.5$, indicating the decision-maker is more inclined to a negative risk attitude, and $\theta_3, \theta_4 > 0.5$, indicating the decision-maker is more inclined to a positive risk attitude. The following four cases are included in Figure 3 (and the other cases can be analyzed similarly):

Case 1: (Line 7 in Algorithm 2, θ_1 in Figure 3.) In this case, $\mathcal{RAF}_{\mathsf{AM}(A)}(a_\sigma) = \theta$ has a unique solution and input a_{11} has the smallest distance to that solution. Consequently, $\sigma = 11$.

Case 2: (Line 12 in Algorithm 2, θ_2 in Figure 3.) In this case, $\mathcal{RAF}_{\mathsf{AM}(A)}(a_\sigma) = \theta$ has a unique solution. Inputs a_{12} and a_{13} have the same distance, which is the smallest distance to

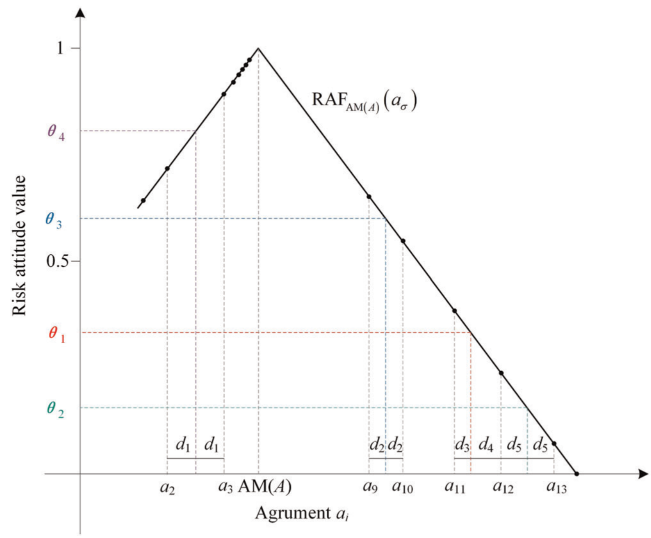

FIGURE 3 Part illustration for Algorithm 2 [Color figure can be viewed at wileyonlinelibrary.com]

the solution. Furthermore, $\theta_2 < 0.5$ indicates that the risk attitude of the decision-maker is more inclined to be negative, and therefore $\sigma = 13$ because a_{13} represents a more negative risk attitude than a_{12}.

Case 3: (Line 10 in Algorithm 2, θ_3 in Figure 3.) In this case, $\mathcal{RAF}_{\mathrm{AM}(A)}(a_\sigma) = \theta$ has a unique solution. Inputs a_9 and a_{10} have the same distance, which is the smallest distance to the solution. Moreover, $\theta_3 > 0.5$ indicates that the risk attitude of the decision-maker is more inclined to be positive, and therefore $\sigma = 9$ because a_9 represents a more positive risk attitude than a_{10}.

Case 4: (Lines 10, 19, and 20 in Algorithm 2, θ_4 in Figure 3.) In this case, $\mathcal{RAF}_{\mathrm{AM}(A)}(a_\sigma) = \theta$ has two solutions. Because the online shopping behavior of most consumers is more likely to affect the purchasing decisions of new consumers, Algorithm 2 allocates more attention to the solution $\mathcal{RAF}_{\mathrm{AM}(A)}(a_\sigma) = \theta|_{a_\sigma < \mathrm{AM}(A)}$. Consequently, similar to Case 3, $\sigma = 3$.

Algorithm 2 Solving a_σ for $\mathcal{RAF}_{\mathrm{AM}(A)}(a_\sigma) = \theta$

Input: Collection, $A = \{a_1, a_2, ..., a_n\}$; RAF for A, $\mathcal{RAF}_{\mathrm{AM}(A)}(a_\sigma)$; Risk attitude value, θ.

Output Risk attitude constant, a_σ.

1: $a^* \leftarrow$ Solve formula $\mathcal{RAF}_{\mathrm{AM}(A)}(a_\sigma) = \theta$;

2: $l(a^*) \leftarrow$ The number of a^*;

3: **if** $l(a^*) = 1$ **then**

4: *top*:

5: $l\left(a_{\arg\mathrm{Min}_{i \in \mathcal{N}}\{|a_i - a^*|\}}\right) \leftarrow$ The number of $a_{\arg\mathrm{Min}_{i \in \mathcal{N}}\{|a_i - a^*|\}}$;

(Continues)

if $l\left(a_{\underset{i \in \mathcal{N}}{\arg \mathrm{Min}}\{|a_i - a^*|\}}\right) = 1$ **then**
  $a_\sigma \leftarrow a_{\underset{i \in \mathcal{N}}{\arg \mathrm{Min}}\{|a_i - a^*|\}}$;
else
  if $\theta \geq 0.5$ **then**
    $a_\sigma \leftarrow$ the element in $a_{\underset{i \in \mathcal{N}}{\arg \mathrm{Min}}\{|a_i - a^*|\}}$ which has a smaller distance to $\mathrm{AM}(A)$;
  else
    $a_\sigma \leftarrow$ the element in $a_{\underset{i \in \mathcal{N}}{\arg \mathrm{Min}}\{|a_i - a^*|\}}$ which has a larger distance to $\mathrm{AM}(A)$;
  end if
end if
else
$l_{left} \leftarrow$ The number of arguments in A that are no more than $\mathrm{AM}(A)$;
$l_{right} \leftarrow$ The number of arguments in A that are no less than $\mathrm{AM}(A)$;
if $l_{left} \geq l_{right}$ **then**
  $a^* \leftarrow \mathrm{Min}\{a^*\}$;
  goto *top*;
else
  $a^* \leftarrow \mathrm{Max}\{a^*\}$;
  goto *top*
end if
end if

Following the definition of the RAPA operator, argument a_σ will gain the maximum nonlinear weight. This phenomenon is consistent with the real-life decision-making context because σ is the representation of the decision-maker's risk attitude and she/he tends to assign a_σ a larger weight. Similarly, the nonlinear weight for argument a_i, $i \in \mathcal{N}$ is decreasing with its distance to input a_σ. It can be interpreted that the RAPA operator is able to feed back to the decision-maker the information she/he really cares about.

Example 2. Figure 4 shows the consumer reviews of an iPhone X sold by Amazon. Consumers use a 5-star rating system to review the product, where a larger number of stars indicates a more positive the assessment of the product. As shown in Figure 4, the overall rating is 4.5 stars out of 5. We now analyze how the risk attitude of a new consumer influences her/his comprehensive assessment value for the product.

There are 107 consumer assessments of this product. Because 63% of consumers rated it 5 stars, the number of consumers giving this rating can calculated as approximately 67. Similarly, the number of consumers who rated it 1 star, 2 stars, 3 stars and 4 stars are 23, 5, 4 and 8, respectively. The assessment set for this product can therefore be represented as

$$A = \left\{ \underbrace{1, 1, \ldots, 1}_{23}, \underbrace{2, 2, \ldots, 2}_{5}, \underbrace{3, 3, \ldots, 3}_{4}, \underbrace{4, 4, \ldots, 4}_{8}, \underbrace{5, 5, \ldots, 5}_{67} \right\}.$$

Definition 5 assumes that the input collection comprises distinct arguments, so it is necessary to obtain the subset of A containing no duplicate elements, namely $B = \{1, 2, 3, 4, 5\}$. When risk parameter σ ranges from 1 to 5, the aggregated results utilized by the RAPA operator (with $\mathrm{Sup}(a, b) = (1 - |a - b|/5)$) are 3.4641, 3.6513, 3.8335, 4.0043 and 4.1626, respectively. Because

FIGURE 4 An iPhone X sold on Amazon [Color figure can be viewed at wileyonlinelibrary.com]

$\mathsf{AM}(A) = 3.8505$, argument 4 has the smallest distance from $\mathsf{AM}(A)$ and argument 1 has the largest distance from $\mathsf{AM}(A)$. Consequently, if the risk attitude of the new consumer is sufficiently positive, namely $b_\sigma = 4$, her/his comprehensive assessment value is 4.0043, whereas if her/his risk attitude is negative enough, namely $b_\sigma = 1$, the comprehensive assessment value is 3.4641. Conversely, if her/his risk attitude and product comprehensive assessment threshold values are both known, it is easy to quantitatively analyze whether she/he will purchase the product. Figure 5 shows the linear RAF and nonlinear weights for collection $B = \{1, 2, 3, 4, 5\}$, in which the nonlinear weight for each evaluation value varies from the risk attitude of the new consumer. The blue line depicts the relationship between the RAF and evaluation values, which is linear. The five dark lines depict the relationship between nonlinear weight and evaluation value; the different lines/symbols indicate different values of b_σ. The above-mentioned observations highlight that the diverse risk attitude b_σ has a meaningful impact on the decision outcomes.

Example 2 not only indicates the necessity of our extension in a real practical context, but also illustrates how to use the RAPA operator to aggregate a collection with repeated input arguments. In the calculation process, Algorithm 1 is first used to generate the RAF for collection B, in which the symmetry axis of the RAF is computed as $\mathsf{AM}(A)$. Furthermore, the risk parameter σ (or the risk attitude constant a_σ) is determined on the basis of the risk attitude of the decision-maker. Finally, the RAPA operator is utilized to calculate the comprehensive assessment value of collection A.

Next, we investigate the desirable properties of the RAPA operator. Particularly, if all supports to a_σ are the same (namely, $\mathsf{Sup}(a_i, a_\sigma) = 1, i \in \mathcal{N}$), then nonlinear weight $(1 + \mathcal{T}_\sigma(a_i))/\sum_{j=1}^{n}(1 + \mathcal{T}_\sigma(a_j)) = 1/n$ and the RAPA operator therefore reduces to the arithmetic mean:

$$\mathsf{RAPA}_\sigma(a_1, a_2, ..., a_n) = \mathsf{AM}(a_1, a_2, ..., a_n) = \frac{1}{n}\sum_{i=1}^{n} a_i.$$

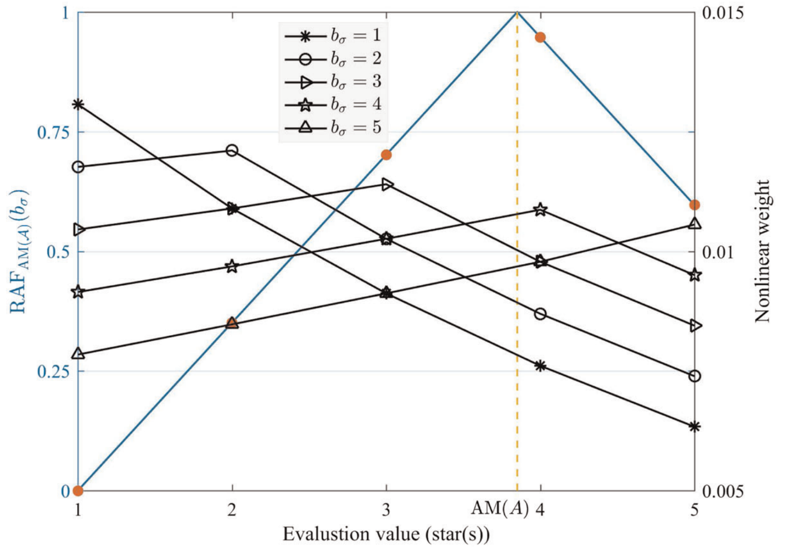

FIGURE 5 Linear risk attitude function and nonlinear weights for collection $B = \{1, 2, 3, 4, 5\}$ [Color figure can be viewed at wileyonlinelibrary.com]

Theorem 1 (Idempotency). *Let* $(a_1, a_2, ..., a_n)$ *be a collection of* n *distinct arguments, if* $a_i = a$ *for* $i \in \mathcal{N}$, *then*

$$\mathsf{RAPA}_{(\sigma)}(a_1, a_2, ..., a_n) = a.$$

Theorem 2 (Boundedness). *Let* $(a_1, a_2, ..., a_n)$ *be a collection of* n *distinct arguments, then*

$$\underset{i \in \mathcal{N}}{\mathsf{Min}}\{a_i\} \leq \mathsf{RAPA}_{(\sigma)}(a_1, a_2, ..., a_n) \leq \underset{i \in \mathcal{N}}{\mathsf{Max}}\{a_i\}.$$

Theorem 3 (Commutativity). *Let* $(a_1, a_2, ..., a_n)$ *be a collection of* n *distinct arguments, if* $(b_1, b_2, ..., b_n)$ *is any permutation of* $(a_1, a_2, ..., a_n)$, *then*

$$\mathsf{RAPA}_{(\sigma)}(a_1, a_2, ..., a_n) = \mathsf{RAPA}_{(\sigma)}(b_1, b_2, ..., b_n).$$

However, because nonlinear weight $\overline{w}_i = (1 + \mathscr{T}_\sigma(a_i))/\sum_{j=1}^{n}(1 + \mathscr{T}_\sigma(a_j))$ depends upon the input arguments, a larger weight may be assigned to a smaller argument and a larger input may be assigned a smaller weight; the RAPA operator is therefore not monotonic.

The RAPA operator is able to simultaneously incorporate the relationship among the arguments and the risk attitude of the decision-maker in the aggregation process. However, it cannot reflect the relationship originating from the importance difference of the decision attributes in MADM/MAGDM problems. For this reason, Definition 7 introduces the weighted risk attitude-based power average (WRAPA) operator.

Definition 7. Let $A = (a_1, a_2, ..., a_n)$ be a collection of distinct input arguments, a WRAPA operator of dimension n is a mapping WRAPA: $\mathfrak{R}^n \rightarrow \mathfrak{R}$, according to the following formula:

$$\mathsf{WRAPA}_\sigma(a_1, a_2, \ldots, a_n) = \frac{\sum_{i=1}^{n} w_i(1 + \mathscr{T}_\sigma(a_i))a_i}{\sum_{j=1}^{n} w_j(1 + \mathscr{T}_\sigma(a_j))},$$

where

$$\mathscr{T}_\sigma(a_i) = w_i \,\mathsf{Sup}(a_i, a_\sigma), \quad i \in \mathcal{N},$$

with risk attitude constant $a_\sigma \in A$, and $\mathsf{Sup}(a, b)$ is the support for a from b, which satisfies the three aforementioned properties.

Similarly, one may verify that the WRAPA operator is idempotent, bounded, commutative but not monotonic.

4 | RISK ATTITUDE-ORIENTED PROPORTIONAL HESITANT 2-TUPLE LINGUISTIC PA OPERATORS

Following the concept of PHFLTS and 2-tuple linguistic representation models, the PHF2TLTS can similarly be defined as follows.

Definition 8. Let $\mathcal{S} = \{s_0, s_1, \ldots, s_g\}$ be a LTS. A PHF2TLTS for a linguistic variable ϑ, namely, P_{H2T_S}, is an ordered finite set:

$$P_{H2T_S}(\vartheta) = \{((s_i, \alpha_i), p_i) | s_i \in S, \alpha_i \in [-0.5, 0.5), i = 1, 2, \ldots, q, q \geq g\},$$

with the conditions that $0 \leq p_i \leq 1 (i = 1, 2, \ldots, q, q \geq g + 1)$ and $\sum_{i=1}^{q} p_i = 1$.

Similarly, proportional 2-tuple linguistic pairs in PHF2TLTS $((s_i, \alpha_i), p_i)$ are ranked according to the ordered 2-tuple linguistic variables $(s_i, \alpha_i)(i = 1, 2, \ldots, q, q \geq g + 1)$. Different from the PHFLTS, the number within PHF2TLTS is no less than g, such as $P_{H2T_S} = \{((s_0, 0), 0), ((s_1, 0), 0.1),((s_1, 0.2), 0.2), ((s_2, 0), 0), (s_3, 0), 0.3), ((s_4, 0), 0), (((s_5, 0), 0), ((s_6, -0.1), 0.2), ((s_6, 0), 0.3)\}$, in which $g = 6$ and the number of the element contained in P_{H2T_S} is 9.

Let

$$\mathcal{S} = \left\{ \begin{matrix} s_0 = \text{very low}(VL), s_1 = \text{low}(L), s_2 = \text{moderately low}(ML), s_3 = \text{normal}(N), \\ s_4 = \text{moderately high}(MH), s_5 = \text{high}(H), s_6 = \text{very high}(VH) \end{matrix} \right\}$$

be a LTS. If the PHF2TLTS

$$P_{H2T_S} = \{((s_0, 0), 0), ((s_1, 0.2), 0.2), ((s_2, 0), 0), ((s_3, 0), 0.3), ((s_4, 0), 0), ((s_5, 0), 0), \\ ((s_6, -0.1), 0.5)\}$$

is the comprehensive product review on performance provided by consumers, it can be interpreted that 50% consumers think the performance of that product is “almost very high,” 30% consumers think it is “normal” and 20% consumers think it is “relatively low.” Due to this, only elements $((s_1, 0.2), 0.2)$, $((s_3, 0), 0.3)$, $((s_6, -0.1), 0.5)$ with nonzero proportions make practical contributions to PHF2TLTS P_{H2T_S}.

4.1 | Proportional hesitant 2-tuple linguistic normalized hamming distance (PH2TLNHD)

Let $l(A)$ be the number of elements in Set A. It is difficult to measure the distance between PHF2TLTSs $P^1_{H2T_S}$ and $P^2_{H2T_S}$ because: (i) $l(P^1_{H2T_S})$ is usually not equal to $l(P^2_{H2T_S})$; and (ii) the 2-tuple linguistic variable (s_i, α_i) contained in $P^1_{H2T_S}$ may not be contained in $P^2_{H2T_S}$. This problem can be handled by Algorithm 3, with which the number of elements in both PHF2TLTSs expands to $N_{\text{total}} = l(2TLTS^1 \bigcup 2TLTS^2)$. It is worth noting that "$\bigcup$," "$\bigcap$," and "$-$" in Algorithm 3 are the normal union, intersection and except operations for normal sets, respectively. Moreover, if $\overline{P^1_{H2T_S}} = \{((s_{i_1}, \alpha_{i_1}), p_{i_1}), ((s_{i_2}, \alpha_{i_2}), p_{i_2}), \ldots, ((s_{i_{N_{\text{total}}}}, \alpha_{i_{N_{\text{total}}}}), p_{i_{N_{\text{total}}}})\}$ and $\overline{P^2_{H2T_S}} = \{((s_{j_1}, \alpha_{j_1}), p_{j_1}), ((s_{j_2}, \alpha_{j_2}), p_{j_2}), \ldots, ((s_{j_{N_{\text{total}}}}, \alpha_{j_{N_{\text{total}}}}), p_{j_{N_{\text{total}}}})\}$ are two extended PHF2TLTSs, by utilizing Algorithm 3, $s_{i_k} = s_{j_k} \in S$ holds for all $k = 1, 2, \ldots, N_{\text{total}}$.

Algorithm 3 Adding elements for PHF2TLTS distance measure

Input: PHF2TLTSs $P^1_{H2T_S}$ and $P^2_{H2T_S}$, $t \neq q$.
Output: PHF2TLTSs, $\overline{P^1_{H2T_S}}$ and $\overline{P^2_{H2T_S}}$.

$\overline{P^1_{H2T_S}} \leftarrow P^1_{H2T_S}$;
$\overline{P^2_{H2T_S}} \leftarrow P^2_{H2T_S}$;
$2TLTS^1 \leftarrow \bigcup_{k=1}^{t} (s_{i_k}, \alpha_{i_k})$;
$2TLTS^2 \leftarrow \bigcup_{k=1}^{q} (s_{j_k}, \alpha_{j_k})$;
$2TLTS^{\text{Diff}} \leftarrow 2TLTS^1 \bigcup 2TLTS^2 - 2TLTS^1 \bigcap 2TLTS^2$;
$N \leftarrow$ number of elements in $2TLTS^{\text{Different}}$;
for $k \geq 1$ and $k \leq N$ **do**
  if $2TLTS^{\text{Diff}}_k \in 2TLTS^1$ **then**
    Add $((s_{h_k}, \alpha_{h_k}), 0)$ corresponding to $2TLTS^{\text{Diff}}_k$ into $\overline{P^2_{H2T_S}}$;
  else
    Add $((s_{h_k}, \alpha_{h_k}), 0)$ corresponding to $2TLTS^{\text{Diff}}_k$ into $\overline{P^1_{H2T_S}}$;
  end if
end for
Arrange the elements in $\overline{P^1_{H2T_S}}$ and $\overline{P^2_{H2T_S}}$ in increasing order according to the ordered 2-tuple linguistic variables $(s_{h_k}, \alpha_{h_k})(k = 1, 2, \ldots, N_{\text{total}})$, in which N_{total} is the number of elements in $2TLTS^1 \bigcup 2TLTS^2$.

Xiong et al.[47] proposed the proportional hesitant normalized Hamming distance by simultaneously taking into account the differences among membership degrees (i.e., opinion differences) and proportions (i.e., preference differences) contained in proportional hesitant fuzzy sets (PHFSs). Inspired by this idea, the PH2TLNHD is defined as follows.

Definition 9. Let $\mathcal{S}$ be a LTS, and $\overline{P^1_{H2T_S}} = \{((s_{i_1}, \alpha_{i_1}), p_{i_1}), ((s_{i_2}, \alpha_{i_2}), p_{i_2}), \ldots, ((s_{i_{N_{\text{total}}}}, \alpha_{i_{N_{\text{total}}}}), p_{i_{N_{\text{total}}}})\}$ and $\overline{P^2_{H2T_S}} = \{((s_{j_1}, \alpha_{j_1}), p_{j_1}), ((s_{j_2}, \alpha_{j_2}), p_{j_2}), \ldots, ((s_{j_{N_{\text{total}}}}, \alpha_{j_{N_{\text{total}}}}),$

$p_{j_{N_{\text{total}}}})\}$ be two extended PHF2TLTSs with respect to $P^1_{H2T_S}$ and $P^2_{H2T_S}$ obtained using Algorithm 3. The PH2TLNHD is

$$d\left(P^1_{H2T_S}, P^2_{H2T_S}\right) = \frac{1}{2N_{\text{total}}}\sum_{k=1}^{N_{\text{total}}}\left[\frac{1}{\text{Ind}(s_{i_k})+0.5}\left|\Delta^{-1}(s_{i_k},\alpha_{i_k})p_{i_k} - \Delta^{-1}(s_{j_k},\alpha_{j_k})p_{j_k}\right| + |p_{i_k} - p_{j_k}|\right], \tag{5}$$

where $N_{\text{total}} = l(\overline{P^1_{H2T_S}}) = l(\overline{P^2_{H2T_S}})$, and $\text{Ind}(s_{i_k}) \in \{0, 1, 2, ..., g\}$ is the index of linguistic term $s_{i_k} \in \mathcal{S}$. (It also can be $\text{Ind}(s_{j_k})$ because $\text{Ind}(s_{i_k}) = \text{Ind}(s_{j_k})$)

One may easily verify that the PH2TLNHD satisfies the three basic axioms of the distance measure, namely, (i) *Boundary*: $0 \leq d(P^1_{H2T_S}, P^2_{H2T_S}) \leq 1$; (ii) *Reflexivity*: $d(P^1_{H2T_S}, P^2_{H2T_S}) = 0$ if and only if $P^1_{H2T_S} = P^2_{H2T_S}$; and (iii) *Symmetry*: $d(P^1_{H2T_S}, P^2_{H2T_S}) = d(P^2_{H2T_S}, P^1_{H2T_S})$. Without loss of generality, this paper uses the PH2TLNHD to measure the distance between PHF2TLTSs.

The distance measure on PHF2TLTSs defined in Definition 9 has several advantages. First, the element $((s_{h_k}, \alpha_{h_k}), 0)$ added into $P^1_{H2T_S}$ or $P^2_{H2T_S}$ will not change their original information, because the proportion of $((s_{h_k}, \alpha_{h_k}), 0)$ is zero, as mentioned above, which means the added element does not contribute to $P^1_{H2T_S}$ or $P^2_{H2T_S}$. Furthermore, with proportion "0," the element $((s_{h_k}, \alpha_{h_k}), 0)$ added into $P^1_{H2T_S}$ or $P^2_{H2T_S}$ also has no influence on the distance measure.

4.2 | A comparison method for PHF2TLTSs

The proportion contained in PHF2TLTSs originates from statistic, which should be considered to render accurate decisions.[57,76] Accordingly, the score and deviation functions are defined as follows:

Definition 10. Let $S = \{s_0, s_1, ..., s_g\}$ be a LTS and $P_{H2T_S} = \{((s_i, \alpha_i), p_i) | s_i \in S, \alpha_i \in [-0.5, 0.5), i = 1, 2, ..., q, q \geq g\}$ be a PHF2TLTS. The score function of P_{H2T_S} is defined as

$$s(P_{H2T_S}) = \sum_{i=1}^{q}\Delta^{-1}(s_i, \alpha_i)p_i,$$

and the deviation function of P_{H2T_S} is defined as

$$t(P_{H2T_S}) = \sum_{i=1}^{q} p_i \cdot \left(\Delta^{-1}(s_i, \alpha_i) - s\left(P_{H2T_S}\right)\right)^2.$$

In particular, the score function represents the comprehensive assessment information contained in PHF2TLTSs.

Let a PHF2TLTS, $\{((s_0, 0), 0), ((s_1, 0), 0.2), ((s_2, 0), 0), ((s_3, 0), 0), ((s_4, 0), 0), ((s_5, 0), 0), ((s_6, 0), 1)\}$, be the best assessment value that an online product may obtain, which we denote by ${P_{H2T_S}}^+$. Combined with the score and deviation functions, the following definition can be utilized to compare PHF2TLTSs.

Definition 11. Let $\mathcal{S} = \{s_0, s_1, ..., s_g\}$ be a LTS, and $P^1_{H2T_S}$ and $P^2_{H2T_S}$ be two PHF2TLTSs;

(1) if $s\left(P^1_{H2T_S}\right) > s\left(P^2_{H2T_S}\right)$, then $P^1_{H2T_S} > P^2_{H2T_S}$;

(2) if $s\left(P^1_{H2T_S}\right) = s\left(P^2_{H2T_S}\right)$ and $t\left(P^1_{H2T_S}\right) < t\left(P^2_{H2T_S}\right)$, then $P^1_{H2T_S} > P^2_{H2T_S}$;

(3) if $s\left(P^1_{H2T_S}\right) = s\left(P^2_{H2T_S}\right)$ and $t\left(P^1_{H2T_S}\right) = t\left(P^2_{H2T_S}\right)$,

(a) and $d\left(\left\{P^1_{H2T_S}\right\}, P_{H2T_S}{}^{+}\right) = d\left(\left\{P^2_{H2T_S}\right\}, P_{H2T_S}{}^{+}\right)$, then $P^1_{H2T_S} = P^2_{H2T_S}$;

(b) and $d\left(\left\{P^1_{H2T_S}\right\}, P_{H2T_S}{}^{+}\right) < d\left(\left\{P^2_{H2T_S}\right\}, P_{H2T_S}{}^{+}\right)$, then $P^1_{H2T_S} > P^2_{H2T_S}$,

where $d(A, B)$ is the PH2TLNHD.

4.3 | PH2TLRAPA operator and WPH2TLRAPA operator

Based on Definitions 1 and 2, the proportional convex combination and the ordered proportional convex combination of two PHF2TLTSs can be defined as follows.

Definition 12. Let $\mathcal{S} = \{s_0, s_1, ..., s_g\}$ be a LTS, and $P^1_{H2T_S} = \{((s_{i_1}, \alpha_{i_1}), p_{i_1}), ((s_{i_2}, \alpha_{i_2}), p_{i_2}), ..., ((s_{i_p}, \alpha_{i_p}), p_{i_p})\}$ and $P^2_{H2T_S} = \{((s_{j_1}, \alpha_{j_1}), p_{j_1}), ((s_{j_2}, \alpha_{j_2}), p_{j_2}), ..., ((s_{j_q}, \alpha_{j_q}), p_{j_q})\}$ be two PHF2TLTSs with a weighting vector $bmw = (w_1, w_2)^{\mathrm{T}}, w_1, w_2 \geq 0, w_1 + w_2 \leq 1$. A proportional convex combination of two PHF2TLTSs is defined as Equation (6), in which $\eta = m \times n \in \{1, 2, ..., m \times n, ..., \mathrm{H}\}$, $\mathrm{H} = p \times q$, with the condition given in Equation (7).

$$\begin{aligned}
&\mathcal{G}^2\left(w_1, P^1_{H2T_S}, w_2, P^2_{H2T_S}\right)\\
&= \left(w_1, P^1_{H2T_S}\right) \odot \left(w_2, P^2_{H2T_S}\right)\\
&= \left\{\mathcal{G}^2\left(w_1, ((s_{i_m}, \alpha_{i_m}), p_{i_m}), w_2, ((s_{j_n}, \alpha_{j_n}), p_{j_n})|((s_{i_m}, \alpha_{i_m}), p_{i_m}) \in P^1_{H2T_S}, ((s_{j_n}, \alpha_{j_n}), p_{j_n})\right.\right.\\
&\quad \left.\left.\in P^2_{H2T_S}\right)\right\}\\
&= \langle w_1 + w_2, \{((s_{k_1}, \alpha_{k_1}), p_{i_1} p_{j_1}), ((s_{k_2}, \alpha_{k_2}), p_{i_2} p_{j_1}), ..., ((s_{k_{mn}}, \alpha_{k_{mn}}), p_{i_m} p_{j_n}), ...,\\
&\quad ((s_{k_{pq}}, \alpha_{k_{pq}}), p_{i_p} p_{j_q})\}\rangle\\
&= \langle w_1 + w_2, \{((s_{k_1}, \alpha_{k_1}), p_{k_{11}}), ((s_{k_2}, \alpha_{k_2}), p_{k_{21}}), ..., ((s_{k_\eta}, \alpha_{k_\eta}), p_{k_\eta}), ...,\\
&\quad ((s_{k_{\mathrm{H}}}, \alpha_{k_{\mathrm{H}}}), p_{k_{\mathrm{H}}})\}\rangle
\end{aligned} \tag{6}$$

$$\begin{aligned}
&(s_{k_\eta}, \alpha_{k_\eta})\\
&= \Delta\left(\mathrm{Min}\left\{\mathrm{Max}\{\Delta^{-1}(s_{i_p}, \alpha_{i_p}), \Delta^{-1}(s_{j_q}, \alpha_{j_q})\}, \frac{w_1 p_{i_m}}{w_1 p_{i_m} + w_2 p_{j_n}} \Delta^{-1}(s_{i_m}, \alpha_{i_m})\right.\right.\\
&\qquad \left.\left. + \frac{w_2 p_{j_n}}{w_1 p_{i_m} + w_2 p_{j_n}} \Delta^{-1}(s_{j_n}, \alpha_{j_n})\right\}\right)
\end{aligned} \tag{7}$$

Definition 13. Let $\mathcal{S} = \{s_0, s_1, ..., s_g\}$ be a LTS, and $P^1_{H2T_S} = \{((s_{i_1}, \alpha_{i_1}), p_{i_1}), ((s_{i_2}, \alpha_{i_2}), p_{i_2}), ..., ((s_{i_p}, \alpha_{i_p}), p_{i_p})\}$ and $P^2_{H2T_S} = \{((s_{j_1}, \alpha_{j_1}), p_{j_1}), ((s_{j_2}, \alpha_{j_2}), p_{j_2}), ..., ((s_{j_q}, \alpha_{j_q}), p_{j_q})\}$ be two PHF2TLTSs with a weighting vector $\mathbf{w} = (w_1, w_2)^{\mathrm{T}}, w_1, w_2 \geq 0, w_1 + w_2 \leq 1$. An ordered proportional convex combination of the two PHF2TLTSs is defined as in Equation (8), in which $\mathrm{Min}_{\eta=1,2,...,H}\{\Delta^{-1}(s_{k_\eta}, \alpha_{k_\eta})\} = \Delta^{-1}(s_{i_1}, \alpha_{i_1}) < \Delta^{-1}(s_{i_2}, \alpha_{i_2}) < ... < \Delta^{-1}(s_{i_h}, \alpha_{i_h}) = \mathrm{Max}_{\eta=1,2,...,H}\{\Delta^{-1}(s_{k_\eta}, \alpha_{k_\eta})\}$, and $p_{i_l} = \sum_{\eta=1}^{\mathrm{H}} p_{k_\eta}\chi(k_\eta, i_l)$. $\chi(k_\eta, i_l)$ is an indicator function defined by the following:

$$\chi(k_\eta, i_l) = \begin{cases} 1, & \Delta^{-1}(s_{k_\eta}, \alpha_{k_\eta}) = \Delta^{-1}(s_{i_l}, \alpha_{i_l}), \\ 0, & \Delta^{-1}(s_{k_\eta}, \alpha_{k_\eta}) \neq \Delta^{-1}(s_{i_l}, \alpha_{i_l}). \end{cases}$$

$$\begin{aligned} &\mathcal{T}\left(\mathcal{G}^2\left(w_1, P^1_{H2T_S}, w_2, P^2_{H2T_S}\right)\right) \\ &= \mathcal{T}\langle w_1 + w_2, \{((s_{k_1}, \alpha_{k_1}), p_{k_1}), ((s_{k_2}, \alpha_{k_2}), p_{k_2}), ..., ((s_{k_{mn}}, \alpha_{k_{mn}}), p_{k_{mn}}), ..., ((s_{k_{pq}}, \alpha_{k_{pq}}), p_{k_{pq}})\}\rangle \\ &= \langle w_1 + w_2, \{((s_{i_1}, \alpha_{i_1}), p_{i_1}), ((s_{i_2}, \alpha_{i_2}), p_{i_2}), ..., ((s_{i_l}, \alpha_{i_l}), p_{i_l}), ..., ((s_{i_h}, \alpha_{i_h}), p_{i_h})\}\rangle \end{aligned} \tag{8}$$

For convenience, we denote $\mathcal{TG}^2(w_1, P^1_{H2T_S}, w_2, P^2_{H2T_S}) = \mathcal{T}(G^2(w_1, P^1_{H2T_S}, w_2, P^2_{H2T_S}))$.

Based on the proportional convex combination and the ordered proportional convex combination of two PHF2TLTSs, we extend the RAPA and WRAPA operators proposed in Section 3.1 to accommodate the PHFLTS.

Definition 14. Let $\mathcal{S} = \{s_0, s_1, ..., s_g\}$ be a LTS, and $A = \{P^1_{H2T_S}, P^2_{H2T_S}, ..., P^n_{H2T_S}\}$ be a collection of n distinct PHF2TLTSs. The PH2TLRAPA operator is defined as follows:

$$\begin{aligned} &\mathrm{PH2TLRAPA}_\sigma\left(P^1_{H2T_S}, P^2_{H2T_S}, ..., P^n_{H2T_S}\right) \\ &= \mathcal{TG}^n\left(\overline{w}_i, P^i_{H2T_S}; i = 1, 2, ..., n\right) \\ &= \mathcal{T}\left(\left(\overline{w}_1, P^1_{H2T_S}\right) \odot \mathcal{TG}^{n-1}\left(\overline{w}_i, P^i_{H2T_S}; i = 2, 3, ..., n\right)\right), \end{aligned}$$

where

$$\overline{w_i} = \frac{1 + \mathcal{T}_\sigma\left(P^i_{H2T_S}\right)}{\sum_{j=1}^n\left(1 + \mathcal{T}_\sigma\left(P^j_{H2T_S}\right)\right)} \Bigg/ \sum_{i=1}^n \frac{1 + \mathcal{T}_\sigma\left(P^i_{H2T_S}\right)}{\sum_{j=1}^n\left(1 + \mathcal{T}_\sigma\left(P^j_{H2T_S}\right)\right)},$$

$$\mathcal{T}_\sigma\left(P^i_{H2T_S}\right) = \mathrm{Sup}\left(P^i_{H2T_S}, P^\sigma_{H2T_S}\right), \quad i \in \mathcal{N},$$

with risk attitude constant $P^\sigma_{H2T_S} \in A$, and $\mathrm{Sup}(a, b)$ is the support for a from b, which satisfies the following three properties:

1) $\mathrm{Sup}(a, b) \in [0, 1]$;
2) $\mathrm{Sup}(a, b) = \mathrm{Sup}(b, a)$;

3) $\mathrm{Sup}(a, b) \geq \mathrm{Sup}(x, y)$, if $d(a, b) < d(x, y)$ and $d(*)$ is the distance measure between PHF2TLTSs.

Definition 15. Let $\mathcal{S} = \{s_0, s_1, ..., s_g\}$ be a LTS, $A = \{P^1_{H2T_S}, P^2_{H2T_S}, ..., P^n_{H2T_S}\}$ be a collection of n distinct PHF2TLTSs, and $\boldsymbol{w} = (w_1, w_2, ..., w_n)^{\mathrm{T}}$ be a weighting vector of $P^j_{H_S}$ $(j = 1, 2, ..., n)$ with $w_j \geq 0 (j = 1, 2, ..., n)$ and $\sum_{j=1}^n w_j = 1$. The WPH2TLRAPA operator is defined as follows:

$$\begin{aligned}
&\mathrm{WPH2TLRAPA}_\sigma\left(P^1_{H2T_S}, P^2_{H2T_S}, ..., P^n_{H2T_S}\right)\\
&= \mathcal{TG}^n\left(\overline{w}_i, P^i_{H2T_S}; i = 1, 2, ..., n\right)\\
&= \mathcal{T}\left(\left(\overline{w}_1, P^1_{H2T_S}\right) \odot \mathcal{G}^{n-1}\left(\overline{w}_i, P^i_{H2T_S}; i = 2, 3, .,n\right)\right),
\end{aligned}$$

where

$$\overline{w_i} = \frac{w_i\left(1 + \mathcal{T}_\sigma\left(P^i_{H2T_S}\right)\right)}{\sum_{j=1}^n\left(1 + \mathcal{T}_\sigma\left(P^j_{H2T_S}\right)\right)} \Bigg/ \sum_{i=1}^n \frac{w_i\left(1 + \mathcal{T}_\sigma\left(P^i_{H2T_S}\right)\right)}{\sum_{j=1}^n\left(1 + \mathcal{T}_\sigma\left(P^j_{H2T_S}\right)\right)}, \tag{9}$$

$$\mathcal{T}_\sigma\left(P^i_{H2T_S}\right) = w_i \,\mathrm{Sup}\left(P^i_{H2T_S}, P^\sigma_{H2T_S}\right), \quad i \in \mathcal{N}, \tag{10}$$

with risk attitude constant $P^\sigma_{H2T_S} \in A$ and $\mathrm{Sup}(a, b)$ is the support for a from b, which satisfies the following three properties:

1) $\mathrm{Sup}(a, b) \in [0, 1]$;
2) $\mathrm{Sup}(a, b) = \mathrm{Sup}(b, a)$;
3) $\mathrm{Sup}(a, b) \geq \mathrm{Sup}(x, y)$, if $d(a, b) < d(x, y)$ and $d(*)$ is the distance measure between PHF2TLTSs.

Without loss of generality, this paper utilizes $\mathrm{Sup}(a, b) = 1 - d(a, b)$ to calculate the support for PHF2TLTSs a from b, where $d(*)$ is the distance measure between PHF2TLTSs.

5 | HYBRID MULTIATTRIBUTE ONLINE PRODUCT RECOMMENDATION FOR CONSUMER DECISION MAKING

In practice, consumers often make their purchasing decisions on the basis of online product information, such as reviews, ratings and comments.[77] This is often exploited by online retailers that design product recommendation systems with the purpose of helping customers easily identify products that meet their tastes and needs. In this process, the risk attitude of the consumer is an important factor. Using product information from an online shopping website and the risk attitude of the target consumer, this section proposes an MADM-based hybrid approach to online product recommendation with a risk attitude-oriented proportional hesitant 2-tuple linguistic PA operator. We first introduce the online product recommendation environment under a MADM framework.

- *Target consumer (Decision-maker)*: The decision-maker in our online product recommendation system is a consumer who makes her/his purchasing decision on the basis of her/his risk attitude and the product information from the online shopping website.
- *Online products (Alternatives)*: Similar to the recommendation systems used by real online shopping websites, products are recommended to the target consumer based on her/his online browsing history. For simplicity, we denote m online products (namely, the alternatives for MAGDM) as $\{A_1, A_2, ..., A_m\}$ and the index set is denoted $M = \{1, 2, ..., m\}$, in which A_i is the ith online product.
- *Online product attributes (Attributes)*: The target consumer's online browsing history is assumed to indicate his/her demand for product. This is a common and basic assumption that ensures the recommendation system will focus on useful product information. In our online product recommendation system, the following four product information types are taken into account (as the attributes for MADM): Performance (C_1), Price (C_2), Ratings (C_3), and Number of reviews (C_4). These four attributes are the main areas of concern for most decision-makers when online shopping, and any one of them may play a decisive role in the decision-maker's final choice of product. In addition, the information about product attributes presented to the consumer is limited, as shown in examples of the purchase interfaces of the online shopping websites Amazon and Best Buy, featuring cell phones, in Figures 6 and 7. "Performance" and "Price" are two common product attributes that determine whether an online product meets the basic needs of the target consumer. "Ratings" determine the target consumer's first impression of the product. Furthermore, "Number of reviews" may indirectly reflect the number of sales of the product. It is worth noting that the online product recommendation is a hybrid representative system, in which attributes C_1, C_2 and C_4 can be represented by real numbers, and C_3 can be represented by PHFLTS, which will be illustrated in detail in Section 6.
- *Decision-making information*: Decision-making matrix $\mathbf{R} = [r_{ij}]_{m\times4}$ is the product information gathered from an online shopping website and a related third party testing agency, in which r_{ij} is the assessment value of product $A_i, i \in M$ on attribute $C_j, i = 1, 2, 3, 4$. $\mathbf{w} = (w_1, w_2, w_3, w_4)^{\mathrm{T}}$ is the weighting vector of the four product attributes.

The main steps of the MADM-based hybrid data-driven online product recommendation approach with the risk attitude-oriented proportional hesitant 2-tuple linguistic PA operator are as follows.

Step 1. Determining the risk attitude value of the target consumer.

A questionnaire (see Appendix A) containing 10 questions is designed to measure the risk attitude of target consumer. Output θ is a real number in $[0, 1]$ corresponding to the range of the RAF defined in Section 3.1.

Step 2. Gathering the decision-making matrix.

Using online products selected based on the target consumer's online browsing history, the online product recommendation system gathers decision-making assessment value r_{ij} of product $A_i, i \in M$ on attribute $C_j, i = 1, 2, 3, 4$ from the online shopping website and the related third party testing agency, and hence decision-making matrix $\mathbf{R} = [r_{ij}]_{m\times4}$ is constructed.

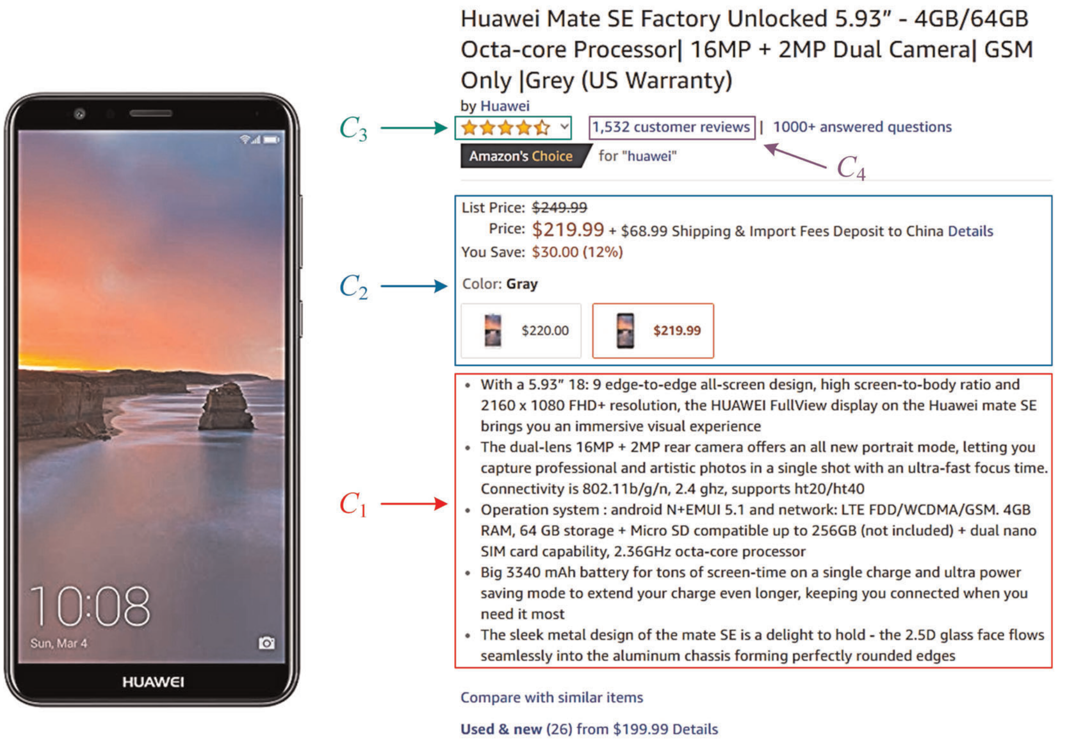

FIGURE 6 Online product information

Step 3. Obtaining the proportional hesitant fuzzy 2-tuple linguistic decision-making matrix.

For real number-type attributes, the following formula is utilized to normalize them into [0, 1]:

$$\bar{r}_{ij} = \begin{cases} \dfrac{r_{ij}}{\underset{i\in M}{\mathrm{Max}}\{r_{ij}\}}, & \text{for benifit attribute} \quad j, \\ \dfrac{\underset{i\in M}{\mathrm{Min}}\{r_{ij}\}}{r_{ij}}, & \text{for cost attribute} \quad j, \end{cases} \qquad i \in M. \tag{11}$$

After this, Definition 3 is used to transform $\bar{r}_{ij}, i \in M, j = 1, 2, 4$ into the PHF2TLTS. The value under C_2 can similarly be extended to the PHF2TLTS with Definition 4. The proportional hesitant fuzzy 2-tuple linguistic decision-making matrix with the representation of PHF2TLTS is denoted by $\mathbf{U} = [u_{ij}]_{m\times 4}$.

Step 4. Solving the attribute weighting vector $\mathbf{w} = (w_1, w_2, w_3, w_4)^{\mathrm{T}}$.

Entropy is a useful measurement tool for the degree of disorder of a system, and is widely utilized to determine weighting vectors in information fusion. We call this the entropy weight method. Subsystems of the entropy weight method with smaller entropy values tend towards more disorder and are assigned a larger weight, and vice versa. On the basis of this idea, the distance entropy weight method,[78] an extension of the entropy weight method, (i) first chooses the "positive ideal point" for the subsystem, then (ii) measures the distance between the elements in that subsystem and its "positive ideal point," and finally (iii) weights the subsystem according to the principle that a subsystem in which all elements are close to the "positive ideal point" tends towards full order, and is assigned a small weight, and vice versa.

This paper proposes the following extended distance entropy weight method (described in Steps 4.1–4.4) by considering the risk attitude of the target consumer.

Step 4.1. Choosing the "positive ideal point" u_j^* for attribute $C_j, j = 1, 2, 3, 4$ according to the risk attitude of the target consumer. In this process, collection $A = \{u_{1j}, u_{2j}, \ldots, u_{mj}\}$ is considered as the input arguments, and Algorithm 1 is utilized to generate the RAF of A, that is, $\mathcal{RA}\mathcal{F}_{\mathsf{AM}(A)}(u_{\sigma j}), j = 1, 2, 3, 4$. It is worth noting that because Algorithm 1 is defined for real numbers but $A = \{u_{1j}, u_{2j}, \ldots, u_{mj}\}$ is composed of m PHF2TLTSs, without loss of generality, we handle this by transforming PHF2TLTS into a real number by utilizing the score function defined in Definition 10. Combining this with the risk attitude value θ determined in Step 1, the score function value of the "positive ideal point" $u_j^*, j = 1, 2, 3, 4$ can be gained by Algorithm 2 and the "positive ideal point" $u_j^*, j = 1, 2, 3, 4$ can be therefore obtained.

Step 4.2. Computing the distance between $u_{ij}, i \in M, j = 1, 2, 3, 4$ and $u_j^*, j = 1, 2, 3, 4$ by Definition 9 and Algorithm 3. Specifically:

$$d_{ij} = d\left(u_{ij}, u_j^*\right), \quad i \in M, \quad j = 1, 2, 3, 4, \tag{12}$$

which constructs the assessment information distance matrix $\mathbf{D} = [d_{ij}]_{m \times 4}$.

Step 4.3. Calculating the average distance entropy

$$E_j = -\frac{1}{\ln 4} \sum_{i=1}^{m} \frac{d_{ij}}{\sum_{i=1}^{m} d_{ij}} \cdot \ln \frac{d_{ij}}{\sum_{i=1}^{m} d_{ij}} \tag{13}$$

for attribute $C_j, j = 1, 2, 3, 4$. We have made the convention $0 \cdot \ln 0 = 0$.

Step 4.4. Obtaining the distance entropy-based weight corresponding to attribute $C_j, j = 1, 2, 3, 4$ as

$$w_j = \frac{1 - E_j}{4 - \sum_{j=1}^{4} E_j}, \quad j = 1, 2, 3, 4. \tag{14}$$

Step 5. Calculating the comprehensive assessment values for online products.

Following weighting vector $\mathbf{w} = (w_1, w_2, w_3, w_4)^{\mathrm{T}}$ obtained in Step 4, the comprehensive assessment value for each online product $u_i, i \in M$ is computed using the WPHF2TLRAPA operator. In particular, the risk attitude value θ determined by Step 1 and Algorithms 1 and 2 is used to incorporate the risk attitude of the target consumer into the aggregation process. Similar to Step 4.1, the score function is utilized to solve the risk attitude constant for the WPHF2TLRAPA operator.

Step 6. Ranking online products according to $u_i, i \in M$. The larger $u_i, i \in M$, the better the respective product is considered to be.

6 | ILLUSTRATIVE EXAMPLE AND COMPARISON

Consider a consumer planning to purchase a cell phone from online shopping website Best Buy. According to their online browsing history, she/he prefers an iPhone with high performance and 64 GB of internal memory. Consequently, the MADM-based product recommendation system chooses iPhone X (64 GB) (A_1), iPhone XR (64 GB) (A_2), iPhone XS (64 GB) (A_3), and iPhone XS Max (64 GB) (A_4) as the alternatives. The basic information for each alternative with respect to performance, price, ratings, and number of reviews is shown in Figure 7, and was obtained from the web addresses shown in the footnote, accessed on February 23, 2019.

It[1] is easy to obtain the assessment values for the attributes "price" and "rating" from Figure 7. The attribute "performance" for each cell phone is highly dependent on its hardware and system optimizations, and is evaluated according to the average score from AnTuTu Benchmark software, which is a professional grading software used to judge the performance level of hardware. The average score of iPhone X is 249,030, the average score of iPhone XR is 348,072, the average score of iPhone XS is 358,517 and the average score of iPhone XS Max is 355,692. These scores are reported in Figure 8, which ranks the top 10 best performing IOS devices in January 2019 and shows their corresponding average scores.

To2 recommend the most suitable product for the consumer, we conduct the MADM-based product recommendation model proposed in Section 5, as follows:

Step 1. According to the questionnaire listed in Appendix A, the risk attitude value of the target consumer is calculated as $\theta = 0.62$.

Step 2. Obtaining the hybrid decision-making matrix $\mathbf{R} = [r_{ij}]_{4\times4}$.

The hybrid decision-making matrix $\mathbf{R} = [r_{ij}]_{4\times4}$ and information from Figures 7 and 8 are listed in Table 2. For attributes C_1, C_2, and C_4, information can be taken directly from Figures 7 and 8. For the attribute "Ratings," C_3, proportional hesitant linguistic pair (s_i, p_i), $i = 0, 1, \ldots, 4$ is composed of rating values (number of stars) and their corresponding proportions. Therefore the LTS with granularity $4 + 1 = 5$ for attribute "Ratings" is $\mathcal{S} = \{s_0 =$ "1 star," $s_1 =$ "2 stars," $s_2 =$ "3 stars," $s_3 =$ "4 stars," $s_4 =$ "5 stars" $\}$. For example, for alternative A_1 (iPhone X), the comprehensive rating is 4.8 stars and the proportion of "5 star" reviews is 84%, expressed as $(s_4, 0.84)$ using the proportional hesitant linguistic pair. Similarly, $(s_0, 0.01), \ldots, (s_3, 0.12)$ can be obtained, thus, the attribute "Rating" of A_1 can be comprehensively obtained as a set, $\{(s_0, 0.01), (s_1, 0.01), (s_3, 0.02), (s_3, 0.12), (s_4, 0.84)\}$.

Step 3. Obtaining the proportional hesitant fuzzy 2-tuple linguistic decision-making matrix. We first utilize Formula (11) to normalize the values for attributes C_1, C_2 and C_4 into $[0, 1]$, and then transform them into the PHF2TLTs by using Definition 3 with granularity $g + 1 = 7$. Similarly, the values under attribute C_3 are transformed into the PHF2TLTs by using Definition 8. The element u_{ij}, $i, j = 1, 2, 3, 4$ in the proportional hesitant fuzzy 2-tuple linguistic decision-making matrix U is computed, and is as listed below.

FIGURE 7 Product information for four alternatives: (A) iPhone X (64 GB); (B) iPhone XR (64 GB); (C) iPhone XS (64 GB); (D) iPhone XS Max (64 GB) [Color figure can be viewed at wileyonlinelibrary.com]

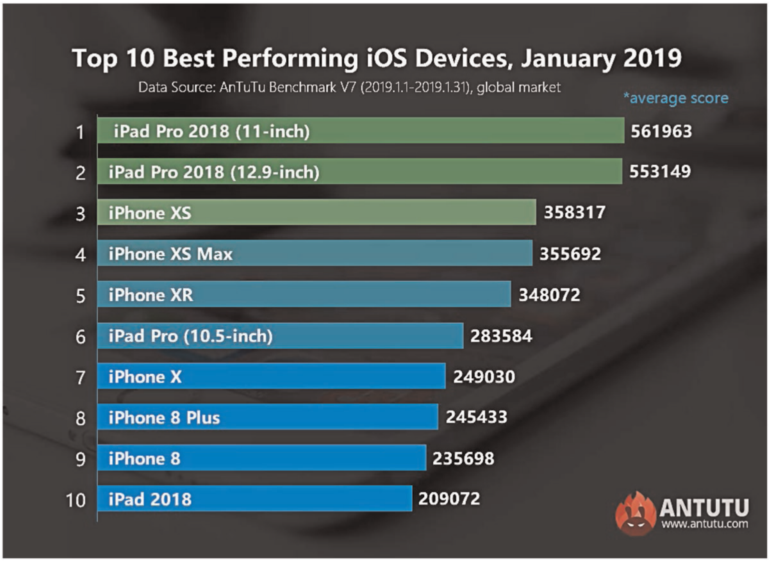

FIGURE 8 Global top 10 best performing iOS devices, January 2019 [Color figure can be viewed at wileyonlinelibrary.com]

TABLE 2 Hybrid decision-making matrix $R = [r_{ij}]_{4\times4}$

		C_1 Performance (average score)	C_2 Price (dollars)	C_3 Ratings	C_4 Number of reviews
A_1	iPhone X (64 GB)	249,030	899.99	$\{(s_0, 0.01), (s_1, 0.01), (s_2, 0.02), (s_3, 0.12), (s_4, 0.84)\}$	1049
A_2	iPhone XR (64 GB)	348,072	749.99	$\{(s_0, 0.00), (s_1, 0.01), (s_2, 0.02), (s_3, 0.13), (s_4, 0.84)\}$	470
A_3	iPhone XS (64 GB)	358,317	999.99	$\{(s_0, 0.01), (s_1, 0.01), (s_2, 0.01), (s_3, 0.09), (s_4, 0.88)\}$	222
	A_4	iPhone XS Max (64 GB)	355,692	1099.99	
				$\{(s_0, 0.01), (s_1, 0.00), (s_2, 0.02), (s_3, 0.10), (s_4, 0.87)\}$	993

- A_1 (iPhone X [64 GB]):

$$u_{11} = \{((s_0, 0), 0), ((s_1, 0), 0), ((s_2, 0), 0), ((s_3, 0), 0), ((s_4, 0.0283), 1), ((s_5, 0), 0), ((s_6, 0), 0)\},$$
$$u_{12} = \{((s_0, 0), 0), ((s_1, 0), 0), ((s_2, 0), 0), ((s_3, 0), 0), ((s_4, 0), 0), ((s_5, -1.85 \times 10^{-6}), 1), ((s_6, 0), 0)\},$$
$$u_{13} = \{((s_0, 0), 0.01), ((s_1, 0), 0.01), ((s_2, 0), 0.02), ((s_3, 0), 0.12), ((s_4, 0), 0.84)\},$$
$$u_{14} = \{((s_0, 0), 0), ((s_1, 0), 0), ((s_2, 0), 0), ((s_3, 0), 0), ((s_4, 0), 0), ((s_5, 0), 0), ((s_6, 0), 1)\}.$$

- A_2 (iPhone XR [64 GB]):

$$u_{21} = \{((s_0, 0), 0), ((s_1, 0), 0), ((s_2, 0), 0), ((s_3, 0), 0), ((s_4, 0), 0), ((s_5, 0), 0), ((s_6, -0.0286), 1)\},$$
$$u_{22} = \{((s_0, 0), 0), ((s_1, 0), 0), ((s_2, 0), 0), ((s_3, 0), 0), ((s_4, 0), 0), ((s_5, 0), 0), ((s_6, 0), 1)\},$$
$$u_{23} = \{((s_0, 0), 0.00), ((s_1, 0), 0.01), ((s_2, 0), 0.02), ((s_3, 0), 0.13), ((s_4, 0), 0.84)\},$$
$$u_{24} = \{((s_0, 0), 0), ((s_1, 0), 0), ((s_2, 0), 0), ((s_3, -0.0520), 1), ((s_4, 0), 0), ((s_5, 0), 0), ((s_6, 0), 0)\}.$$

- A_3 (iPhone XS [64 GB]):

$$u_{31} = \{((s_0, 0), 0), ((s_1, 0), 0), ((s_2, 0), 0), ((s_3, 0), 0), ((s_4, 0), 0), ((s_5, 0), 0), ((s_6, 0), 1)\},$$
$$u_{32} = \{((s_0, 0), 0), ((s_1, 0), 0), ((s_2, 0), 0), ((s_3, 0), 0), ((s_4, 0.0833), 1), ((s_5, 0), 0), ((s_6, 0), 0)\},$$
$$u_{33} = \{((s_0, 0), 0.01), ((s_1, 0), 0.01), ((s_2, 0), 0.01), ((s_3, 0), 0.09), ((s_4, 0), 0.88)\},$$
$$u_{34} = \{((s_0, 0), 0), ((s_1, 0.0450), 1), ((s_2, 0), 0), ((s_3, 0), 0), ((s_4, 0), 0), ((s_5, 0), 0), ((s_6, 0), 0)\}.$$

- A_4 (iPhone XS Max [64 GB]):

$$u_{41} = \{((s_0, 0), 0), ((s_1, 0), 0), ((s_2, 0), 0), ((s_3, 0), 0), ((s_4, 0), 0), ((s_5, 0), 0), ((s_6, -0.0073), 1)\},$$
$$u_{42} = \{((s_0, 0), 0), ((s_1, 0), 0), ((s_2, 0), 0), ((s_3, 0), 0), ((s_4, 0.0151), 0), ((s_5, 0), 0), ((s_6, 0), 0)\},$$
$$u_{43} = \{((s_0, 0), 0.01), ((s_1, 0), 0.00), ((s_2, 0), 0.02), ((s_3, 0), 0.10), ((s_4, 0), 0.87)\},$$
$$u_{44} = \{((s_0, 0), 0), ((s_1, 0), 0), ((s_2, 0), 0), ((s_3, 0), 0), ((s_4, 0), 0), ((s_5, 0), 0), ((s_6, -0.0534), 1)\}.$$

Step 4. Solving the attribute weighting vector $\mathbf{w} = (w_1, w_2, w_3, w_4)^{\mathrm{T}}$.

Step 4.1. The score matrix of the proportional hesitant fuzzy 2-tuple linguistic decision-making matrix **U** is calculated by Definition 10 as

$$\mathbf{SU} = \begin{bmatrix} 4.0283 & 5.0000 & 3.7700 & 6.0000 \\ 5.9714 & 6.0000 & 3.8000 & 2.9480 \\ 6.0000 & 4.0833 & 3.8200 & 1.0450 \\ 5.9927 & 4.0151 & 3.8200 & 5.9466 \end{bmatrix}.$$

Following Algorithm 1, Figure 9 shows the quadratic-type RAFs for the score values of alternatives under attribute $C_j, j = 1, 2, 3, 4$. Line $\mathcal{RAF}_1$ represents the quadratic-type RAFs for the score values of alternatives under attribute C_1 and the four score values under attribute C_1 are marked by the symbol "o." Similarly, the other three lines represent the RAFs for attributes C_2, C_3, and C_4.

Following Algorithm 2 and risk attitude value $\theta = 0.62$, the score values of the "positive ideal point" $u_j^*, j = 1, 2, 3, 4$ are calculated as 4.0283, 4.0151, 3.7700, and 2.9480, respectively. Consequently, the "positive ideal point" for attributes $C_j, j = 1, 2, 3, 4$ are:

$$u_1^* = \{((s_0, 0), 0), ((s_1, 0), 0), ((s_2, 0), 0), ((s_3, 0), 0), ((s_4, 0.0283), 1), ((s_5, 0), 0), ((s_6, 0), 0)\},$$

$$u_2^* = \{((s_0, 0), 0), ((s_1, 0), 0), ((s_2, 0), 0), ((s_3, 0), 0), ((s_4, 0.0151), 0), ((s_5, 0), 0), ((s_6, 0), 0)\},$$

$$u_3^* = \{((s_0, 0), 0.01), ((s_1, 0), 0.01), ((s_2, 0), 0.02), ((s_3, 0), 0.12), ((s_4, 0), 0.84)\},$$

$$u_4^* = \{((s_0, 0), 0), ((s_1, 0), 0), ((s_2, 0), 0), ((s_3, -0.0520), 1), ((s_4, 0), 0), ((s_5, 0), 0), ((s_6, 0), 0)\}.$$

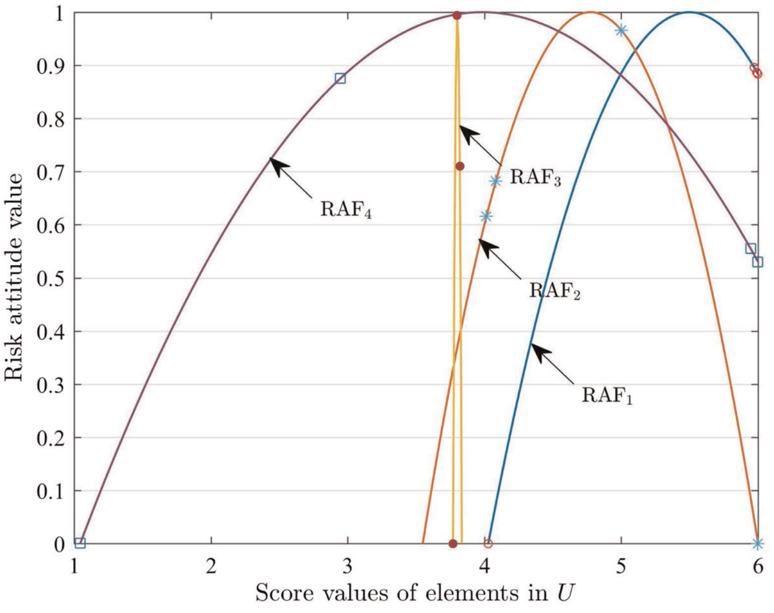

FIGURE 9 Quadratic-type RAFs for the score values of alternatives under attribute $C_j, j = 1, 2, 3, 4$. RAF, risk attitude function [Color figure can be viewed at wileyonlinelibrary.com]

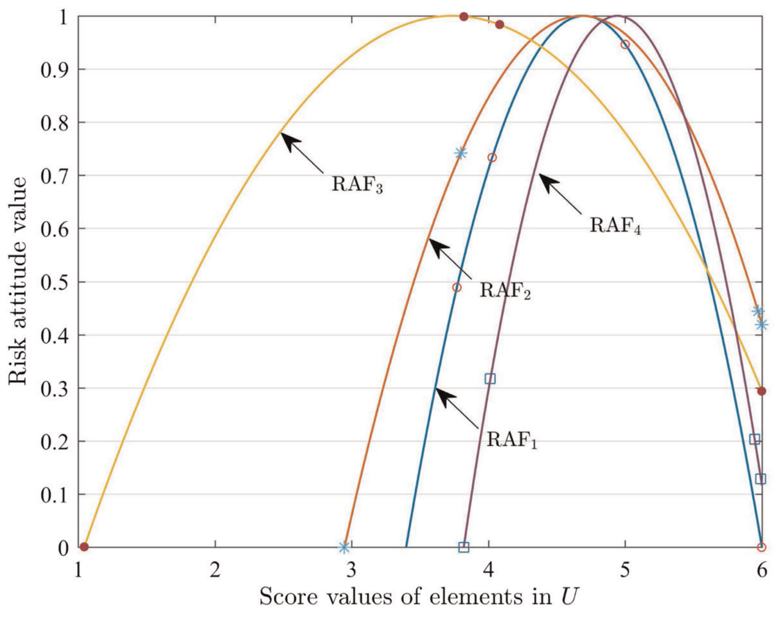

FIGURE 10 Quadratic-type RAFs for the score values of attributes with respect to alternatives $A_i, i = 1, 2, 3, 4$. RAF, risk attitude function [Color figure can be viewed at wileyonlinelibrary.com]

Step 4.2. Utilizing Definition 9 and Algorithm 3, the distances between $u_{ij}, i, j = 1, 2, 3, 4$ and the "positive ideal point" $u_j^*, j = 1, 2, 3, 4$ are denoted by the following distance matrix $\mathbf{D} = [d_{ij}]_{4\times4}$:

$$\mathbf{D} = \begin{bmatrix} 0.0000 & 0.2112 & 0.0000 & 0.2353 \\ 0.2119 & 0.2385 & 0.0029 & 0.0000 \\ 0.2386 & 0.2375 & 0.0149 & 0.1966 \\ 0.2121 & 0.0000 & 0.0110 & 0.2087 \end{bmatrix}.$$

Step 4.3. The average distance entropy for each attribute is $E_1 = 0.7913, E_2 = 0.7914, E_3 = 0.6762$, and $E_4 = 0.7904$, and therefore the weighting vector of attribute is $\mathbf{w} = (0.2195, 0.2195, 0.3406, 0.2205)^{\mathrm{T}}$.

Step 5. Calculating the comprehensive assessment values for online products.

Step 5.1. Similar to *Step* 4.1, the quadratic-type RAFs, solved by Algorithm 1, for the score values of attributes with respect to alternatives $A_i, i = 1, 2, 3, 4$ are shown in Figure 10.

Following Algorithm 2 and risk attitude value $\theta = 0.62$, the score values of the risk attitude constant for alternative A_i, $i = 1, 2, 3, 4$ are calculated as 3.7700, 3.8000, 1.0450, and 4.0151, respectively. Consequently, the risk attitude constants for alternative A_i, $i = 1, 2, 3, 4$ are:

$$\begin{aligned}
u_{1\sigma} = \{&((s_0, 0), 0), ((s_1, 0), 0), ((s_2, 0), 0), ((s_3, 0), 0), ((s_4, -0.2300), 1),\\
&((s_5, 0), 0), ((s_6, 0), 0)\},\\
u_{2\sigma} = \{&((s_0, 0), 0), ((s_1, 0), 0), ((s_2, 0), 0), ((s_3, 0), 0), ((s_4, -0.20000), 1),\\
&((s_5, 0), 0), ((s_6, 0), 0)\},\\
u_{3\sigma} = \{&((s_0, 0), 0), ((s_1, 0.0450), 1), ((s_2, 0), 0), ((s_3, 0), 0), ((s_4, 0), 0),\\
&((s_5, 0), 0), ((s_6, 0), 0)\},\\
u_{4\sigma} = \{&((s_0, 0), 0), ((s_1, 0), 0), ((s_2, 0), 0), ((s_3, 0), 0), ((s_4, 0.0151), 1),\\
&((s_5, 0), 0), ((s_6, 0), 0)\}.
\end{aligned}$$

Step 5.2. Following Formula (10), the weighted support matrix is

$$\mathbf{T} = \begin{bmatrix} 0.1678 & 0.1602 & 0.3406 & 0.1607 \\ 0.1599 & 0.1598 & 0.3406 & 0.1692 \\ 0.1698 & 0.1755 & 0.2646 & 0.2205 \\ 0.1730 & 0.2195 & 0.2604 & 0.1738 \end{bmatrix}.$$

Therefore, the weighted nonlinear weight calculated by Formula (9) is

$$\overline{\mathbf{W}} = \begin{bmatrix} 0.2095 & 0.2081 & 0.3732 & 0.2092 \\ 0.2081 & 0.2081 & 0.3732 & 0.2107 \\ 0.2114 & 0.2124 & 0.3546 & 0.2215 \\ 0.2122 & 0.2206 & 0.3538 & 0.2133 \end{bmatrix}.$$

Step 5.3. The comprehensive assessment values utilizing the WPHF2TLRAPA operator are provided as follows:

$$u_1 = \{((s_0, 0.000), 0.00), ((s_1, 0.000), 0.00), ((s_2, 0.000), 0.00), ((s_3, 0.000), 0.00), ((s_4, 0.000), 0.00), ((s_4, 0.4354), 0.00), ((s_4, 0.4742), 0.00), ((s_5, -0.4997), 0.00), ((s_5, -0.4987), 0.00), ((s_5, -0.4966), 0.00), ((s_5, 0.000), 0.00), ((s_6, -0.0435, 0.84), ((s_6, -0.0062, 0.12), ((s_6, -0.0010, 0.02), ((s_6, -5.097 \times 10^{-4}), 0.01), ((s_6, -5.090 \times 10^{-4}), 0.01), ((s_6, 0.000), 0.00)\},$$

$$u_2 = \{((s_0, 0.000), 0.00), ((s_1, 0.000), 0.00), ((s_2, 0.000), 0.00), ((s_3, -0.0520), 0.00), ((s_3, -0.0509), 0.01), ((s_3, -0.0499), 0.02), ((s_3, -0.0389), 0.13), ((s_3, 0.000), 0.00), ((s_3, 0.0243), 0.84), ((s_4, 0.000), 0.00), ((s_5, 0.000), 0.00), ((s_6, -0.3068), 0.00), ((s_6, -0.0921), 0.00), ((s_6, -0.0307), 0.00), ((s_6, -0.0286), 0.00), ((s_6, -0.0246), 0.00), ((s_6, -0.0144), 0.00), ((s_6, 0.000), 0.00)\},$$

$$u_3 = \{((s_0, 0.000), 0.00), ((s_1, 0.000), 0.00), ((s_1, 0.0450), 0.00), ((s_1, 4.6321 \times 10^{-2}), 0.01), ((s_1, 4.6322 \times 10^{-2}), 0.01), ((s_1, 4.6323 \times 10^{-2}), 0.01), ((s_1, 0.0571, 0.09), ((s_1, 0.1567), 0.88), ((s_2, 0.000), 0.00), ((s_3, 0.000), 0.00), ((s_4, 0.000), 0.00), ((s_5, -0.1119), 0.00), ((s_5, 0.000), 0.00), ((s_5, 0.0115), 0.00), ((s_5, 0.0384), 0.00), ((s_5, 0.0404), 0.00), ((s_5, 0.0425), 0.00), ((s_6, 0.000), 0.00)\},$$

$$u_4 = \{((s_0, 0.000), 0.00), ((s_1, 0.000), 0.00), ((s_2, 0.000), 0.00), ((s_3, 0.000), 0.00), ((s_4, 0.000), 0.00), ((s_4, 0.0151), 0.00), ((s_5, -0.1422), 0.00), ((s_5, -0.0293), 0.00), ((s_5, -0.0011), 0.00), ((s_5, 0.000), 0.00), ((s_5, 0.0009), 0.00), ((s_5, 0.0112), 0.00), ((s_6, -0.0847), 0.87), ((s_6, -0.0567), 0.10), ((s_6, -0.0540), 0.02), ((s_6, -0.0537), 0.01), ((s_6, -0.0534), 0.00), ((s_6, -0.0073), 0.00), ((s_6, 0.000), 0.00)\}.$$

Step 6. Following Definition 10, the scores and deviations with respect to u_i, $i = 1, 2, 3, 4$ are $s(u_1) = 4.8449$, $s(u_2) = 4.4619$, $s(u_3) = 3.4954$, $s(u_4) = 5.0055$ and $t(u_1) = 0.1133$, $t(u_2) = 0.1215$, $t(u_3) = 0.4188$, $t(u_4) = 0.0740$, respectively. Thus, the ranking of the four alternatives is $A_4 \succ A_1 \succ A_2 \succ A_3$ and product A_4, namely the iPhone XS Max, is the best choice, and our online recommendation model will recommend it to the target consumer.

6.1 | Comparison with the proportional hesitant fuzzy 2-tuple linguistic weighted averaging (PHF2TLWA) operator

To further show the applicability of the proposed operator, we solve the above product recommendation problem by utilizing the PHF2TLWA operator, which is an extension of the proportional hesitant fuzzy linguistic weighted averaging (PHFLWA) operator.[57]

Definition 16. Let $S = \{s_0, s_1, ..., s_g\}$ be a LTS, $A = \{P^1_{H2T_S}, P^2_{H2T_S}, ..., P^n_{H2T_S}\}$ be a collection of n PHF2TLTSs and $\boldsymbol{w} = (w_1, w_2, ..., w_n)^T$ be a weighting vector of $P^j_{H_S} (j = 1, 2, ..., n)$ with $w_j \geq 0 (j = 1, 2, ..., n)$ and $\sum_{j=1}^{n} w_j = 1$. The PHF2TLWA operator is defined as follows:

$$
\begin{aligned}
&\mathsf{PHF2TLWA}\left(P^1_{H2T_S}, P^2_{H2T_S}, ..., P^n_{H2T_S}\right)\\
&= \mathcal{TG}^n\left(w_i, P^i_{H2T_S}; i = 1, 2, ..., n\right)\\
&= \mathcal{T}\left(\left(w_1, P^1_{H2T_S}\right) \odot \mathcal{TG}^{n-1}\left(w_i, P^i_{H2T_S}; i = 2, 3, ..., n\right)\right).
\end{aligned}
$$

Especially, when $w_1 = w_2 = ... = w_n = 1/n$, the PHF2TLWA operator reduces to the Proportional Hesitant Fuzzy 2-Tuple Linguistic Averaging (PHF2TLA) operator.

$$
\begin{aligned}
&\mathsf{PHF2TLA}\left(P^1_{H2T_S}, P^2_{H2T_S}, ..., P^n_{H2T_S}\right)\\
&= \mathcal{TG}^n\left(1/n, P^i_{H2T_S}; i = 1, 2, ..., n\right)\\
&= \mathcal{T}\left(\left(1/n, P^1_{H2T_S}\right) \odot \mathcal{TG}^{n-1}\left(1/n, P^i_{H2T_S}; i = 2, 3, ..., n\right)\right).
\end{aligned}
$$

We solve the aforementioned product selection problem by utilizing the PHF2TLWA operator as follows:

Step 1′. Gathering the hybrid decision-making matrix $\mathbf{R} = [r_{ij}]_{4\times4}$ as listed in Table 2.

Step 2′. Obtaining the proportional hesitant fuzzy 2-tuple linguistic decision-making matrix $\mathbf{U} = [u_{ij}]_{4\times4}$, which is the same as the results listed in Step 3, above, solved by the model proposed in this paper.

Step 3′. Determining the attribute weighting vector $\mathbf{w} = (w_1, w_2, w_3, w_4)^T$. The "positive ideal point" for attribute $C_j, j = 1, 2, 3, 4$ is calculated by formula

$$u^*_j = \mathsf{PHF2TLA}(u_{1,j}, u_{2,j}, u_{3,j}, u_{4,j}), \quad j = 1, 2, 3, 4.$$

Following Formulas (12) to (14), the attribute weighting vector is similarly computed as $\mathbf{w} = (0.0137, 0.0135, 0.0056, 0.9671)^T$.

Step 4′. The comprehensive assessment values obtained by utilizing the PHF2TLWA operator are

$$u_1 = \{((s_0, 0.000), 0.00), ((s_1, 0.000), 0.00), ((s_2, 0.000), 0.00), ((s_3, 0.000), 0.00), ((s_4, 0.000), 0.00),$$
$$((s_4, 0.4354), 0.00), ((s_4, 0.4742), 0.00), ((s_5, -0.4997), 0.00), ((s_5, -0.4987), 0.00),$$
$$((s_5, -0.4966), 0.00), ((s_5, 0.000), 0.00), ((s_6, -0.0435, 0.84), ((s_6, -0.0062, 0.12),$$
$$((s_6, -0.0010, 0.02),$$
$$((s_6, -5.097 \times 10^{-4}), 0.01), ((s_6, -5.090 \times 10^{-4}), 0.01), ((s_6, 0.000), 0.00)\},$$

$$u_2 = \{((s_0, 0.000), 0.00), ((s_1, 0.000), 0.00), ((s_2, 0.000), 0.00), ((s_3, -0.0520), 0.00),$$
$$((s_3, -0.0509), 0.01),$$
$$((s_3, -0.0499), 0.02), ((s_3, -0.0389), 0.13), ((s_3, 0.000), 0.00), ((s_3, 0.0243), 0.84),$$
$$((s_4, 0.000), 0.00),$$
$$((s_5, 0.000), 0.00), ((s_6, -0.3068), 0.00), ((s_6, -0.0921), 0.00), ((s_6, -0.0307), 0.00),$$
$$((s_6, -0.0286), 0.00),$$
$$((s_6, -0.0246), 0.00), ((s_6, -0.0144), 0.00), ((s_6, 0.000), 0.00)\},$$

$$u_3 = \{((s_0, 0.000), 0.00), ((s_1, 0.000), 0.00), ((s_1, 0.0450), 0.00), ((s_1, 4.6321 \times 10^{-2}), 0.01),$$
$$((s_1, 4.6322 \times 10^{-2}), 0.01), ((s_1, 4.6323 \times 10^{-2}), 0.01), ((s_1, 0.0571, 0.09), ((s_1, 0.1567), 0.88),$$
$$((s_2, 0.000), 0.00), ((s_3, 0.000), 0.00), ((s_4, 0.000), 0.00), ((s_5, -0.1119), 0.00), ((s_5, 0.000), 0.00),$$
$$((s_5, 0.0115), 0.00), ((s_5, 0.0384), 0.00), ((s_5, 0.0404), 0.00), ((s_5, 0.0425), 0.00),$$
$$((s_6, 0.000), 0.00)\},$$

$$u_4 = \{((s_0, 0.000), 0.00), ((s_1, 0.000), 0.00), ((s_2, 0.000), 0.00), ((s_3, 0.000), 0.00), ((s_4, 0.000), 0.00),$$
$$((s_4, 0.0151), 0.00), ((s_5, -0.1422), 0.00), ((s_5, -0.0293), 0.00), ((s_5, -0.0011), 0.00),$$
$$((s_5, 0.000), 0.00),$$
$$((s_5, 0.0009), 0.00), ((s_5, 0.0112), 0.00), ((s_6, -0.0847), 0.87), ((s_6, -0.0567), 0.10),$$
$$((s_6, -0.0540), 0.02),$$
$$((s_6, -0.0537), 0.01), ((s_6, -0.0534), 0.00), ((s_6, -0.0073), 0.00), ((s_6, 0.000), 0.00)\}.$$

Step 5′. Following Definition 10, the score and deviation with respect to u_i, $i = 1, 2, 3, 4$ are $s(u_1) = 5.9627$, $s(u_2) = 3.0139$, $s(u_3) = 1.1444$, $s(u_4) = 5.9190$ and $t(u_1) = 2.0142 \times 10^{-4}$, $t(u_2) = 5.7731 \times 10^{-4}$, $t(u_3) = 0.0011$, $t(u_4) = 9.2902 \times 10^{-45}$. Thus, the ranking order is $A_1 \succ A_4 \succ A_2 \succ A_3$.

6.2 | Comparison with other group decision-making methods

To illustrate the performance of this study's proposed method, this section compares the proposed method with two other group decision making methods, the TOPSIS and VIKOR methods. The reason for performing these comparisons is because TOPSIS and VIKOR are two popular decision methods that have been applied in many research fields, such as the management of markets, management of environment and chemical engineering, among others.[79] The basic idea of TOPSIS is that it ranks the alternatives according to the degree of closeness to the ideal solution and distance from the worst solution, to determine the optimal alternative with the shortest distance to the ideal solution and farthest distance from the worst solution. VIKOR ranks plans by weighting the maximum group utility of the "majority" and the minimum individual regret of the "opponent."[64] This subsection solves the above production selection problem using the TOPSIS and VIKOR methods. Their algorithms are developed as Algorithms 4 and 5, to simplify the process of calculation.

Algorithm 4 PHF2TLTS-based TOPSIS method

Input: Hybrid decision-making matrix, $\mathbf{R} = [r_{ij}]_{m\times n}$; Proportional hesitant fuzzy 2-tuple linguistic decision-making matrix, $\mathbf{U} = [u_{ij}]_{m\times n}$, $u_{ij} = \{((s_{ij_1}, \alpha_{ij_1}), p_{ij_1}), \ldots, ((s_{ij_k}, \alpha_{ij_k}), p_{ij_k}), \ldots, ((s_{ij_K}, \alpha_{ij_K}), p_{ij_K})\}, k \in K$. The attribute weighting vector of C_j is ω_j.

Output: $d_i^+, d_i^-, C_i^*, f_j^+, f_j^-$, obtained ranking of alternatives.

for $u_{ij}^1, u_{ij}^2 \in \mathbf{U}, K_1 \neq K_2$ **do**

  run Algorithm 3

  generate N_{total}

end for

for $u_{ij} \in \mathbf{U}, i \in [1, \ldots, m], j \in [1, \ldots, n]$ **do**

  $s(u_{ij}) arrow \sum_{k=1}^{N_{total}} \Delta^{-1}(s_{ij_k}, \alpha_{ij_k})\, p_{ij_k}$

  $t(u_{ij}) \leftarrow \sum_{k=1}^{N_{total}} p_{ij_k} \cdot (\Delta(s_{ij_k}, \alpha_{ij_k}) - s(u_{ij}))^2$

end for

for $j \in [1, \ldots, n]$ **do**

  if $s(u_{ij}^1) = s(u_{ij}^2), u_{ij}^1, u_{ij}^2 \in \mathbf{U}$ **then**

    if $t(u_{ij}^1) < t(u_{ij}^2)$ **then**

      $u_{ij}^1 > u_{ij}^2$

    else

      $u_{ij}^1 < u_{ij}^2$

    end if

  end if

  $f_j^+ \leftarrow Max\{s(u_{ij})\}$

  $f_j^- \leftarrow Min\{s(u_{ij})\}$

end for

for $i \in [1, \ldots, m], j \in [1, \ldots, n]$ **do**

  $d(u_{ij}, f_j^+) \leftarrow \frac{1}{2 \times N_{total}} \sum_{k=1}^{N_{total}} \left[\frac{1}{Ind(s_{ij_k}) + 0.5} |\Delta^{-1}(s_{ij_k}, \alpha_{ij_k})\, p_{ij_k} - \Delta^{-1}(s_{j_k^+}, \alpha_{j_k^+}) p_{j_k^+}| + |p_{ij_k} - p_{j_k^+}| \right]$

  $d(u_{ij}, f_j^-) \leftarrow \frac{1}{2 \times N_{total}} \sum_{k=1}^{N_{total}} \left[\frac{1}{Ind(s_{ij_k}) + 0.5} |\Delta^{-1}(s_{ij_k}, \alpha_{ij_k})\, p_{ij_k} - \Delta^{-1}(s_{j_k^-}, \alpha_{j_k^-}) p_{j_k^-}| + |p_{ij_k} - p_{j_k^-}| \right]$

end for

for $i \in [1, \ldots, m]$ **do**

  $d_i^+ \leftarrow \sum_{j=1}^{n} \omega_j d(u_{ij}, f_j^+)$

  $d_i^- \leftarrow \sum_{j=1}^{n} \omega_j d(u_{ij}, f_j^-)$

  $C_i^* \leftarrow d_i^- / (d_i^+ + d_i^-)$

end for rank alternatives in descending order of C_i^*.

Algorithm 5 PHF2TLTS-based VIKOR method

Input: Hybrid decision-making matrix, $\mathbf{R} = [r_{ij}]_{m\times n}$; Proportional hesitant fuzzy 2-tuple linguistic decision-making matrix, $\mathbf{U} = [u_{ij}]_{m\times n}$, $u_{ij} = \left\{\left(\left(s_{ij_1}, \alpha_{ij_1}\right), p_{ij_1}\right), ..., \left(\left(s_{ij_k}, \alpha_{ij_k}\right), p_{ij_k}\right), ..., \left(\left(s_{ij_K}, \alpha_{ij_K}\right), p_{ij_K}\right)\right\}, k \in K$. The attribute weighting vector of C_j is ω_j.

Output: $d\left(u_{ij}, f_j^+\right), d\left(u_{ij}, f_j^-\right), S_i, R_i, Q_i$, obtained ranking of alternatives.

for $u_{ij}^1, u_{ij}^2 \in \mathbf{U}, K_1 \neq K_2$ **do**
  run Algorithm 3
  generate N_{total}
end for
for $u_{ij} \in \mathbf{U}, i \in [1, ..., m], j \in [1, ..., n]$ **do**
  $s(u_{ij}) \leftarrow \sum_{k=1}^{N_{total}} \Delta^{-1}\left(s_{ij_k}, \alpha_{ij_k}\right) p_{ij_k}$
  $t(u_{ij}) \leftarrow \sum_{k=1}^{N_{total}} p_{ij_k}\cdot\left(\Delta\left(s_{ij_k}, \alpha_{ij_k}\right) - s(u_{ij})\right)^2$
end for
for $j \in [1, ..., n]$ **do**
  if $s\left(u_{ij}^1\right) = s\left(u_{ij}^2\right), u_{ij}^1, u_{ij}^2 \in \mathbf{U}$ **then**
    if $t\left(u_{ij}^1\right) < t\left(u_{ij}^2\right)$ **then**
      $u_{ij}^1 > u_{ij}^2$
    else
      $u_{ij}^1 < u_{ij}^2$
    end if
  end if
  $f_j^+ \leftarrow Max\{s(u_{ij})\}$
  $f_j^- \leftarrow Min\{s(u_{ij})\}$
end for
for $i \in [1, ..., m], j \in [1, ..., n]$ **do**
  $d\left(u_{ij}, f_j^+\right) \leftarrow \frac{1}{2\times N_{total}}\sum_{k=1}^{N_{total}}\left[\frac{1}{Ind\left(s_{ij_k}\right)+0.5}\left|\Delta^{-1}\left(s_{ij_k}, \alpha_{ij_k}\right)p_{ij_k} - \Delta^{-1}\left(s_{j_k^+}, \alpha_{j_k^+}\right)p_{j_k^+}\right| + \left|p_{ij_k} - p_{j_k^+}\right|\right]$
  $d\left(u_{ij}, f_j^-\right) \leftarrow \frac{1}{2\times N_{total}}\sum_{k=1}^{N_{total}}\left[\frac{1}{Ind\left(s_{ij_k}\right)+0.5}\left|\Delta^{-1}\left(s_{ij_k}, \alpha_{ij_k}\right)p_{ij_k} - \Delta^{-1}\left(s_{j_k^-}, \alpha_{j_k^-}\right)p_{j_k^-}\right| + \left|p_{ij_k} - p_{j_k^-}\right|\right]$
  $d\left(f_j^+, f_j^-\right) \leftarrow \frac{1}{2\times N_{total}}\sum_{k=1}^{N_{total}}\left[\frac{1}{Ind\left(s_{j_k^+}\right)+0.5}\left|\Delta^{-1}\left(s_{j_k^+}, \alpha_{ij_k^+}\right)p_{ij_k^+} - \Delta^{-1}\left(s_{j_k^-}, \alpha_{j_k^-}\right)p_{j_k^-}\right| + \left|p_{j_k^+} - p_{j_k^-}\right|\right]$
end for
for $i \in [1, ..., m]$ **do**
  $S_i \leftarrow \sum_{j=1}^{n} \omega_j \frac{d\left(u_{ij}, f_j^+\right)}{d\left(f_j^+, f_j^-\right)}$
  $R_i \leftarrow Max_j \omega_j \frac{d\left(u_{ij}, f_j^+\right)}{d\left(f_j^+, f_j^-\right)}$
  $S^+ \leftarrow Max(S_i), S^- \leftarrow Min(S_i)$
  $R^+ \leftarrow Max(R_i), R^- \leftarrow Min(R_i)$
  $Q_i \leftarrow v(S_i - S^-)/(S^+ - S^-) + (1 - v)(R_i - R^-)/(R^+ - R^-)$
end for
rank alternatives in ascending order of Q_i.

The operating data used in Algorithms 4 and 5 are the same as the data used in the model proposed in this paper. Specifically: (1) the hybrid decision-making matrix is $\mathbf{R} = [r_{ij}]_{4\times4}$, and is as listed in Table 2; (2) the proportional hesitant fuzzy 2-tuple linguistic decision-making matrix is obtained as $\mathbf{U} = [u_{ij}]_{4\times4}$, and is the same as the results listed in Step 3, above, solved by the model proposed in this paper; and (3) the attribute weighting vector of C_j is computed using the PHF2TLA operator as $\mathbf{w} = (0.0137, 0.0135, 0.0056, 0.9671)^T$, and is the same as the result obtained in Section 6.1. Carrying out Algorithm 4, the ideal solution $\mathbf{f}^+ = (u_{31}, u_{22}, u_{43}, u_{14})$ and worst solution $\mathbf{f}^- = (u_{11}, u_{42}, u_{13}, u_{34})$ are obtained. Then, the matrices of $d\left(u_{ij}, f_j^+\right)$ and $d\left(u_{ij}, f_j^-\right)$ are obtained as follows:

$$\mathbf{D}^+ = \left[d\left(u_{ij}, f_j^+\right)\right]_{4\times4} = \begin{bmatrix} 2.2588\times10^{-3} & 2.2230\times10^{-3} & 0.0274\times10^{-3} & 0 \\ 0.0043\times10^{-3} & 0 & 0.0359\times10^{-3} & 0.1576 \\ 0 & 2.2376\times10^{-3} & 0.0181\times10^{-3} & 0.1585 \\ 0.0011\times10^{-3} & 2.2230\times10^{-3} & 0 & 0.5675\times10^{-3} \end{bmatrix},$$

$$\mathbf{D}^- = \left[d\left(u_{ij}, f_j^-\right)\right]_{4\times4} = \begin{bmatrix} 0 & 2.2208\times10^{-3} & 0 & 0.1585 \\ 2.2545\times10^{-3} & 2.2230\times10^{-3} & 0.0086\times10^{-3} & 0.1568 \\ 2.2588\times10^{-3} & 0.1146\times10^{-3} & 0.0366\times10^{-3} & 0 \\ 2.2577\times10^{-3} & 0 & 0.0274\times10^{-3} & 0.1580 \end{bmatrix},$$

with $\mathbf{d}^+ = (0.0045, 0.1577, 0.1608, 0.0028)^T$, $\mathbf{d}^- = (0.1608, 0.1612, 0.0023, 0.1602)^T$, and the relative closeness coefficients C_i^* are obtained as $\mathbf{C}^* = (0.9727, 0.5056, 0.0142, 0.9829)^T$. Finally, the ranking result for the PHF2TLTS-based TOPSIS method is obtained as $A_4 > A_1 > A_2 > A_3$. Similarly, Algorithm 5 is used to obtain the matrices of $f_j^+, f_j^-, d(u_{ij}, f_j^+), d(u_{ij}, f_j^-), d_i^+$ and d_i^-. The results are the same as those of the PHF2TLTS-based TOPSIS method. The values of S_i, R_i, Q_i are obtained as: $\mathbf{S} = (0.0328, 0.9690, 0.9844, 0.0170)^T$, $\mathbf{R} = (0.0137, 0.9616, 0.9671, 0.0135)^T$, $\mathbf{Q} = (0.5001, 0.9971, 1, 0)^T$. Thus, the ranking result for the PHF2TLTS-based VIKOR method is obtained as $A_4 > A_1 > A_2 > A_3$. By comparison, we can see that both of these results are the same as the result obtained using the approach proposed in this paper.

6.3 | Discussion

The ranking result obtained using the PHF2TLWA operator is $A_1 \succ A_4 \succ A_2 \succ A_3$, whereas the result obtained using the approach proposed in this paper is $A_4 \succ A_1 \succ A_2 \succ A_3$. The best alternative according to the two methods is A_1 and A_4, respectively. The main reason for this difference is that this paper's method takes the risk attitude of the decision-maker into account, whereas the method using the PHF2TLWA operator ignores this important information. In particular, without the influence of the risk attitude, the PHF2TLWA approach assigns an extremely large weight to attribute C_4, which greatly reduces the effect of other three attributes. This is not a common phenomenon in practical contexts because the PHF2TLWA approach mainly focuses on the number of reviews, and a product with more reviews (but not with higher performance, lower price or even higher ratings) will obtain a higher ranking. Consequently, the decision result gained by this paper's method appears more reliable. Further, the comparison of results of the proposed method in this study and the two popular MADM methods—TOPSIS and VIKOR—under the PHF2TLTS context show that all three decision

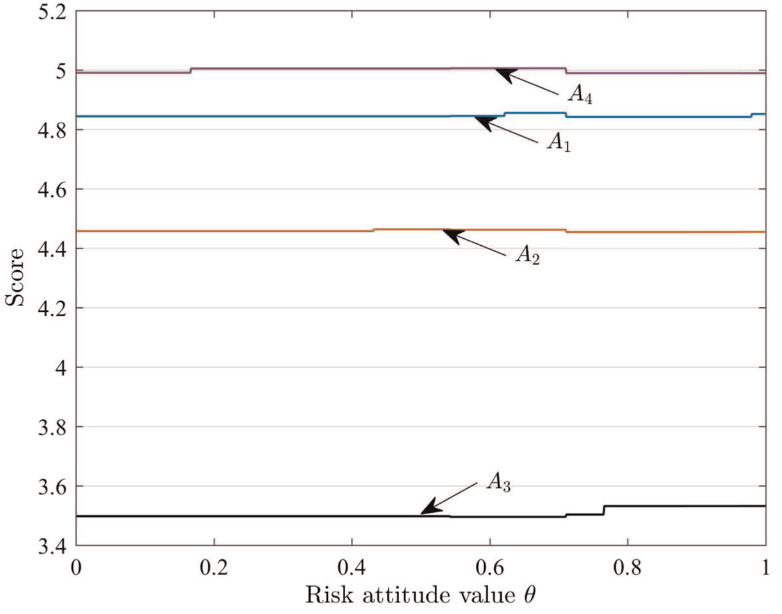

FIGURE 11 Variations of scores with respect to risk attitude value $\theta \in [0, 1]$ using the weighted proportional hesitant 2-tuple linguistic risk attitude-based power average operator [Color figure can be viewed at wileyonlinelibrary.com]

methods yield the ranking results $A_4 > A_1 > A_2 > A_3$, which verifies the validity and rationality of the method proposed in this study.

It is worth noting that both methods assign product A_3 (iPhone XS [64 GB]) the lowest ranking, although it possesses the largest performance average score and the largest ratings score (according to matrix **SU**). This is mainly because it has the lowest number of ratings, which suggests the lowest number of sales, and is far lower than the numbers for other products.

The result meaningfully reflects the importance of conformity of behavior, and this is also one important reason why the iPhone XS Max (64 GB) gains the best ranking using the WPHF2TLRAPA operator. More generally, Figure 11 shows that when the risk attitude value θ varies in $[0, 1]$, there are variations in the scores of the four alternatives obtained using the WPHF2TLRAPA operator. The figure indicates that the ranking order does not change with different $\theta \in [0, 1]$ and product A_4 always obtains the best ranking. However, the score for each alternative varies with parameter $\theta \in [0, 1]$, which implies that the risk attitude of the decision-maker meaningfully influences the decision-making result. This is the initial motivation for this paper to propose the risk attitude-oriented PA operators.

7 | CONCLUSION

With the rapid development of e-commerce, online product recommendation has become an important way of influencing decision-making by online shoppers. Online product recommendation is a topic of significant academic research that seeks to help consumers save time and energy in product selection. To extend the existing research on this topic, the present study develops a power-average-operator-based hybrid multiattribute online product recommendation model. In this model, the RAPA operator is proposed to take into account the risk attitude of the decision-maker. As the essential components of RAPA operators, this paper examines the framework for generating the RAF and the determination of risk attitude constant where four cases are involved. Algorithms 1 and 2 are presented to elicit an appropriate RAPA operator for the special collection with the input of risk attitude value, θ. Then, the PHF2TLTS is introduced to characterize the complexity and uncertainty during the process of decision-making. The PH2TLRAPA and WPH2TLRAPA operators are then proposed for the process of integrating the PH2TLTS with RAPA and WRAPA operators. In this study, a questionnaire is designed to measure the risk attitude of target consumers in a credible way. Finally, an illustrative example of recommendation of a model of iPhone and comparison with a PHF2TLWA operator indicates that the risk attitude of the decision-maker has a large influence on the decision-making result, thereby implying that our proposed model offers performance improvements in online product recommendation. This study therefore provides valuable new insights into online product recommendation.

The main contributions of this study are summarized as follows:

- The proposed RAPA operator expands the theory of PA operators by integrating the risk attitude of the decision-maker, which increases the applicability of PA operators in the field of online product recommendations.
- The first proposed RAF can effectively quantify the risk attitude of the decision-maker and incorporate it into the RAPA operator, providing quantitative information for online sales platforms to improve their marketing strategies. Several basic types have also been introduced to facilitate their application according to different cases.
- The constructed PHF2TLTS can characterize the complexity and uncertainty behind the qualitative linguistic expressions without information loss, which can further promote the development of HFLTS models.
- The proposed PH2TLRAPA and WPH2TLRAPA operators play the roles of integrating the PH2TLTS with RAPA and WRAPA operators to construct the model proposed, further extending the literature on RAPA and WRAPA operators.

Nevertheless, this study has some limitations, and there are several possible directions for future research:

- This study emphasizes the effect of the decision-maker's risk attitude on the final decision-making result. However, the study does not examine how the risk attitude of the decision-maker influences the online product recommendation result. It is necessary for further research to examine the relationship between risk attitude values and final results.
- Table 1 lists several basic types of RAF. Future research should study how to select a suitable RAF type for a specific collection to help obtain credible and accurate outcomes.
- The effect of other traits of the decision-maker (e.g., trust), as well as the interaction between traits, on the final result of decision-making should be further explored. Further research could also examine the factors that affect the risk attitude of decision-makers.

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ENDNOTES

* iPhone X (64 GB): https://www.bestbuy.com/site/apple-iphone-x-64gb-silver-sprint/600979.p?skuId=6009792; iPhone XR (64 GB): https://www.bestbuy.com/site/apple-iphone-xr-64gb-white-sprint/5801307.p?skuId=5801307; iPhone XS (64 GB): https://www.bestbuy.com/site/apple-iphone-xs-64gb-gold-sprint/6009752.p?skuId=6009752; iPhone XS Max (64 GB): https://www.bestbuy.com/site/apple-iphone-xs-max-64gb-gold-sprint/6009766.p?skuId=6009766

† This data set is available from http://www.antutu.com/en/doc/11708.htm

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APPENDIX A: EXAMPLE OF CONSUMER'S RISK ATTITUDE QUESTIONNAIRE

Survey on risk attitude for online shopping

Dear consumers,

To provide an excellent shopping experience for you, we are doing an importance rating survey on consumers' risk attitudes for online shopping. It will be greatly appreciated if you will spare us a few minutes to answer the listed ten questions carefully and independently. Thanks for your cooperation.

Instructions of the questionnaire:

There are **ten** questions. 1, 2, 4, 6, 8, 9, 10 are single answer questions and 3, 5, 7 are multiple answer questions. For each question, the score for each option is listed below, correspondingly, and you are asked to put ticking the appropriate box with respect to your preference.

No.	Titles	Contents										
	Question	What is your age range?										
1	Options	☐ 18-25	☑ 26-30	☐ 31-35	☐ 36-40	☐ 41-45	☐ 46-50	☐ 51-55	☐ 56+			
	Scores	0.8	0.6	0.3	0.1	0.2	0.4	0.5	0.7			
	Question	What is your occupation?										
2	Options	☐ Student	☐ Unemployed	☑ Employed	☐ Housewife/Househusband	☐ Others						
	Scores	0.8	0.2	0.6	0.5	0.3						
	Question	Who do you need to take care of? (multiple answer)										
3	Options	☑ Yourself	☐ Wife/Husband	☑ Parents	☐ Children							
	Scores	0.4	0.2	0.1	0.1							
	Question	How long have you been online shopping?										
4	Options	☐ Less than 1 year	☐ 1 year – 2 years	☑ More than 2 years	☐ Never							
	Scores	0.8	0.6	0.4	0.1							
	Question	How do you spend your monthly salary? (multiple answer)										
5	Options	☐ Pay off credit card	☑ Pay off car loan	☑ Pay off home loan	☑ Daily expenses							
	Scores	0.2	0.2	0.2	0.4							
	Question	How often do you do online shopping?										
6	Options	☐ Once a month	☐ 2-3 times a month	☑ More than 3 times a month	☐ Never							
	Scores	0.3	0.6	0.8	0.1							
	Question	What factors do you care more about when you shop online? (multiple answer)										
7	Options	☑ Quality	☐ Service	☑ Price	☐ Sales promotion	☐ Advertisement	☑ Sales volume					
	Scores	0.2	0.2	0.2	0.2	0.2	0.2					
	Question	Which combination of product's performance/price and price do you prefer more?										
8	Options	☐ 1 and $10	☐ 0.8 and $8	☑ 0.6 and $6	☐ 0.4 and $4	☐ 0.2 and $2						
	Scores	0.8	0.6	0.5	0.3	0.1						
	Question	The possibility that you will buy the products we recommend.										
9	Options	☐ 1	☐ 0.9	☐ 0.8	☐ 0.7	☑ 0.6	☐ 0.5	☐ 0.4	☐ 0.3	☐ 0.2	☐ 0.1	☐ 0
	Scores	1	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0
	Question	For product with uniform price, you will choose to buy it online or offline?										
10	Options	☑ Online	☐ Offline	☐ Both								
	Scores	0.8	0.2	0.5								