Computational and Applied Mathematics (2024) 43:383
https://doi.org/10.1007/s40314-024-02885-2

On the admissibility of the alpha-order for fuzzy numbers

Diego García-Zamora[1] · Anderson Cruz[2] · Fernando Neres[3] · Antonio Francisco Roldán López de Hierro[4] · Regivan H. N. Santiago[5] · Humberto Bustince[6]

Received: 2 January 2024 / Revised: 19 June 2024 / Accepted: 31 July 2024 /
Published online: 23 August 2024

Abstract

The concept of alpha-order has been recently introduced as a binary relation that allows ranking fuzzy sets with bounded support and whose β-cuts are closed subsets with a finite number of connected components. This paper explores some key properties of such alpha-order. Specifically, we will focus on admissibility, a fundamental property for any ranking method defined for fuzzy numbers since it guarantees that such a ranking method is consistent with the usual order on the real line.

Keywords Alpha-order · Order for fuzzy sets · Admissible order · Ranking fuzzy numbers

1 Introduction

The definition of orders within fuzzy set classes is crucial in both fuzzy set theory and fuzzy logic. These ranking methods play a significant role in various areas of study, particularly in decision-making theory when dealing with imprecise data, such as expert opinions. Therefore, since the past half-century, numerous techniques for ordering fuzzy numbers have emerged, such as the ones in (Herencia and Lamata 1999; Buckley and Eslami 2004).

The commonly accepted idea of order between fuzzy sets is the point-wise ordering of functions. However, this notion of ordering is quite stringent and cannot be applied to any pairs of fuzzy sets. This ongoing challenge has led to the search for more general order relations that have the key property of *totality*, i.e., the ability to compare any pair of fuzzy sets (see Takáč 2016; López et al. 2018, 2022 for more details).

Among other proposals, admissible orders for fuzzy numbers, introduced by Zumelzu et al. (2022), are highlighted because they allow constructing total order relations that refine specific partial orders. Admissibility is an essential property when defining any ranking method for fuzzy numbers (García-Zamora et al. 2024). Intuitively, a binary relation for fuzzy numbers is admissible if it refines the α-cut-based partial order defined in Klir and Yuan (1995). In particular, admissibility implies that such a binary relation orders pairs of crisp fuzzy numbers in the same way as the partial order induced by the real line order.

Anderson Cruz, Fernando Neres, Antonio Francisco Roldán López de Hierro, Regivan H. N. Santiago and Humberto Bustince contributed to this work.

Extended author information available on the last page of the article

Recently, the notion of alpha-order was introduced as a total pre-order that allows ranking normal fuzzy sets with bounded support and whose α-cuts are closed subsets with a finite number of connected components (Neres et al. 2024). In that sense, alpha-orders provide a more lenient form of order based on membership functions that can be applied to ranking fuzzy sets, which are not necessarily fuzzy numbers.

Despite the merits of the original contribution presented in Neres et al. (2024), there is still a lack of research devoted to delving into the mathematical properties of the alpha-order. Therefore, this research focuses on further analyzing this total pre-order. To do so, we will first analyze how the alpha-order can be simplified to the domain of triangular and trapezoidal fuzzy numbers, enhancing the ranking method's applicability. Afterward, we will explore the alpha-order admissibility and look into this property's theoretical and practical implications.

The remainder of this proposal is set up as follows. Section 2 presents some basic concepts on fuzzy sets and aggregation functions as well as the alpha-order and its already-known properties. Section 3 deeps the study of alpha-ordering in the context of symmetric triangular and trapezoidal fuzzy numbers. In turn, Sect. 4 studies the admissibility of alpha-orders. Finally, Sect. 5 concludes the contribution.

2 Basic notions

In this section, we present some definitions and results related to fuzzy sets as well as the central notions related to admissible orders and alpha-orders.

2.1 Fuzzy Sets

This section introduces some classical notions regarding fuzzy sets.

Definition 2.1 (*Zadeh* 1965) A fuzzy set on $\mathbb{R}$ is a mapping $A : \mathbb{R} \to [0, 1]$. Given a fuzzy set $A : \mathbb{R} \to [0, 1]$ and $\beta \in \;]0, 1]$. Then the support of A is the set $supp(A) = \{x \in \mathbb{R} : A(x) > 0\}$ and the *β-cut* of A is the set $A_\beta = \{x \in \mathbb{R} : A(x) \geq \beta\}$.

To simplify the notation, in this study, we will focus on a particular class of fuzzy sets on the real line. The reason behind this is the fact that the alpha-order was originally defined on such a class:

Definition 2.2 (*Neres et al.* 2024) Let $\mathscr{C}$ be the family of all fuzzy sets A on $\mathbb{R}$ that satisfy:

- there exists $x_0 \in \mathbb{R}$ such that $A(x_0) = 1$;
- $supp(A)$ is a bounded subset of $\mathbb{R}$;
- for each $\beta \in \;]0, 1]$, the β-cut A_β is a closed subset of $\mathbb{R}$ with a finite number of connected components.

Of course, it is possible to consider the classical operations defined for fuzzy sets on the class $\mathscr{C}$.

Definition 2.3 (*Klir and Yuan* 1995; *Nguyen* 2019) Let $A, B \in \mathscr{C}$. The addition, product, maximum, and minimum operations are defined as follows:

$$(A \oplus B)(z) = \sup_{x+y=z} \min\{A(x), B(y)\};$$
$$(A \odot B)(z) = \sup_{x \cdot y=z} \min\{A(x), B(y)\};$$

$$(A \vee B)(z) = \sup_{\max\{x,y\}=z} \min\{A(x), B(y)\};$$
$$(A \wedge B)(z) = \sup_{\min\{x,y\}=z} \min\{A(x), B(y)\}.$$

A fuzzy number is a fuzzy set $A \in \mathscr{C}$ representing a generalization of a real number $r \in \mathbb{R}$. Formally, $A \in \mathscr{C}$ is said to be a fuzzy number if the β-cuts of A are closed intervals, for all $\beta \in \,]0, 1]$ (Nguyen 2019). From now on, $\mathcal{FN}$ will denote the family of all fuzzy numbers.

Given a closed real interval I, we will denote its lower bound by $\underline{I}$ and its upper bound by $\overline{I}$, i.e. $I = [\underline{I}, \overline{I}]$. The family of all closed real intervals will be denoted by $\mathbb{I}(\mathbb{R})$, i.e.

$$\mathbb{I}(\mathbb{R}) = \{[\underline{I}, \overline{I}] : \underline{I}, \overline{I} \in \mathbb{R} \text{ and } \underline{I} \leq \overline{I}\}.$$

Example 2.1 (Zumelzu et al. 2022, Corollary II.1) Note that we can associate each $I \in \mathbb{I}(\mathbb{R})$ with the fuzzy number $\widetilde{I} : \mathbb{R} \to [0, 1]$ defined by

$$\widetilde{I}(x) = \begin{cases} 1, & \text{if } x \in I, \\ 0, & \text{otherwise.} \end{cases}$$

If $\underline{I} = \overline{I} = r \in \mathbb{R}$, the obtained fuzzy number is the crisp fuzzy number $\widetilde{I} = \widetilde{r}$ which represents the characteristic function of the real number r, i.e.,

$$\widetilde{r}(x) = \begin{cases} 1, & \text{if } x = r, \\ 0, & \text{otherwise.} \end{cases}$$

2.2 Admissible orders for fuzzy numbers

Generally, given a set X, a binary relation R is called a partial pre-order on X if it satisfies the following statements:

- xRx for all $x \in X$ (Reflexivity);
- If $x_1 R x_2$ and $x_2 R x_3$, then $x_1 R x_3$ for all $x_1, x_2, x_3 \in X$ (Transitivity).

Additionally, if $x_1 R x_2$ and $x_2 R x_1$ implies $x_1 = x_2$ for any $x_1, x_2 \in X$ (Anti-symmetry), then R is called a partial order on X. Furthermore, a binary relation R on X is said to be total if for any $x_1, x_2 \in X$, then $x_1 R x_2$ or $x_2 R x_1$. Therefore, a total reflexive and transitive binary relation will be called total pre-order, whereas a total reflexive, anti-symmetric, and transitive binary relation will be called total order.

In the context of fuzzy numbers, one of the most successful ranking methods is the idea of admissible orders, which derives from the concept of admissible orders for intervals introduced in Bustince et al. (2013).

Definition 2.4 (*Zumelzu et al.* 2022) Let $X \subseteq \mathcal{FN}$, and consider a partial order $\preceq$ on X. Then, a total order $\trianglelefteq$ on X is called an *admissible order* with respect to (w.r.t.) $\langle X, \preceq \rangle$, if: for all $A, B \in X$, $A \trianglelefteq B$ whenever $A \preceq B$.

In this contribution, the admissible orders $\trianglelefteq$ w.r.t. the Klir–Yuan (KY) partial order $\preceq_{\text{KY}}$ on $\mathcal{FN}$, given by Klir and Yuan (1995), will be simply called admissible orders. Such a partial order is defined as

$$A \preceq_{\text{KY}} B \iff A \wedge B = A.$$

In this context, the following result provides a characterization of the KY order through the Kulisch–Miranker (KM) ordering of β-cuts

Proposition 2.1 *(Klir and Yuan 1995) Given the fuzzy numbers A and B, the following assertions are equivalent:*

(i) $A \preceq_{\mathrm{KY}} B$;
(ii) $A \vee B = B$;
(iii) $A_\beta \leq_{\mathrm{KM}} B_\beta$, *for each* $\beta \in\ ]0, 1]$;

We recal here that the KM order on $\mathbb{I}(\mathbb{R})$ (Kulisch and Miranker (1981), p.85) is given as

$$[a, b] \leq_{\mathrm{KM}} [c, d] \iff a \leq c \text{ and } b \leq d.$$

for any $[a, b], [c, d] \in \mathbb{I}(\mathbb{R})$.

A specific admissible order on fuzzy numbers $\trianglelefteq^S$ was constructed using upper dense sequences by Zumelzu et al. (2022). A sequence $S = (\alpha_i)_{i\in\mathbb{N}} \subset\ ]0, 1]$. is said to be *upper dense* if, for every point $x \in\ ]0, 1]$ and any $\epsilon > 0$, there exists $i \in \mathbb{N}$ such that $\alpha_i \in [x, x+\epsilon]$.

Definition 2.5 (*Zumelzu et al.* 2022) Let $A, B \in \mathcal{FN}$ and consider $S = (\alpha_i)_{i\in\mathbb{N}}$ an upper dense sequence in $]0, 1]$. Then, define $m(A, B)$ by

$$m(A, B) = \begin{cases} \min\{i \in \mathbb{N} : A_{\alpha_i} \neq B_{\alpha_i}\}, & \text{if } A \neq B, \\ 0, & \text{otherwise.} \end{cases}$$

The following result allows defining admissible orders $\trianglelefteq^S$ on $\mathcal{FN}$ from admissible orders on $\mathbb{I}(\mathbb{R})$.

Theorem 2.1 (Zumelzu et al. 2022) *Let* $S = (\alpha_i)_{i\in\mathbb{N}}$ *be an upper dense sequence in* $]0, 1]$. *If* $\leq$ *is an admissible order on* $\mathbb{I}(\mathbb{R})$, *i.e., it is a total order that refines* $\leq_{KM}$, *then the relation* $\trianglelefteq^S$ *defined as*

$$A \trianglelefteq^S B \iff A = B \text{ or } A_{\alpha_{m(A,B)}} < B_{\alpha_{m(A,B)}} \tag{1}$$

is an admissible order on $\mathcal{FN}$ *w.r.t.* $\preceq_{KY}$.

Example 2.2 (Zumelzu et al. 2022) Let $S = (\alpha_i)_{i\in\mathbb{N}}$ be an upper dense sequence in $]0, 1]$ and the admissible orders $\leq_{Lex1}$ and $\leq_{XY}$ on $\mathbb{I}(\mathbb{R})$, given by:

1. $[a, b] \leq_{Lex1} [c, d] \iff a < c$ or ($a = c$ and $b \leq d$); and
2. $[a, b] \leq_{XY} [c, d] \iff a + b < c + d$ or ($a + b = c + d$ and $b - a \leq d - c$).

Then, $\trianglelefteq^S_{Lex1}$ and $\trianglelefteq^S_{XY}$ are admissible orders on $\mathcal{FN}$.

2.3 The alpha-order

In this section, we present the notion of alpha-order defined on the class $\mathscr{C}$. Hereinafter, we write $\mathrm{card}(X)$ to refer to the cardinality of a crisp set X. Given $n \in \mathbb{N}$, let $J_n = \{1, 2, \ldots, n\}$. A *weighting vector* is a vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n) \in [0, 1]^n$ such that $\omega_1+\omega_2+\cdots+\omega_n = 1$. Let Ω_n be the family of all weighting vectors in $[0, 1]^n$, and let us denote $\min_+(\omega) = \min(\{\omega_1, \omega_2, ..., \omega_n\}\backslash\{0\})$, for $\omega \in \Omega_n$.

A *membership degree vector* is a vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in [0, 1]^n$. Let $\Delta_n =]0, 1]^n$ be the family of all membership degree vectors with nonnull components. Let us denote $\mathbf{0}_n = (0, 0, \ldots, 0) \in \mathbb{R}^n$.

Now, consider the following auxiliary functions for weighting vectors.

- $\nu : \Omega_n \to \mathcal{P}(J_n)$ given by $\nu(\omega) = \{ j \in J_n : \omega_j = \min_+(\omega) \}$;

- $\overline{\nu} : \Omega_n \to \mathcal{P}(J_n)$ given by $\overline{\nu}(\omega) = \{ j \in J_n : j \notin \nu(\omega) \text{ and } \omega_j > 0 \}$;
- $s : \Omega_n \to \mathbb{R}$ given by $s(\omega) = \sum_{j \in \nu(\omega)} \omega_j$.

where $\mathcal{P}(X)$ means the family of all subsets of X.
For each $\alpha \in \Delta_n$, consider the following mappings:

- $\lambda_\alpha : \Omega_n \to \mathcal{P}([0, 1])$ given by $\lambda_\alpha(\omega) = \{ \alpha_j : j \in \nu(\omega) \}$;
- $\zeta_\alpha : \Omega_n \to \mathcal{P}(J_n)$ given by $\zeta_\alpha(\omega) = \{ j \in J_n : \alpha_j = \min(\lambda_\alpha(\omega)) \}$;
- $\overline{\zeta}_\alpha : \Omega_n \to \mathcal{P}(J_n)$ given by $\overline{\zeta}_\alpha(\omega) = \{ j \in J_n : j \notin \zeta_\alpha(\omega) \text{ and } \omega_j > 0 \}$;
- $s_{\zeta_\alpha} : \Omega_n \to \mathbb{R}$ given by $s_{\zeta_\alpha}(\omega) = \sum_{j \in \zeta_\alpha(\omega)} \omega_j$.

Example 2.3 Let $\alpha = (0.2, 0.4, 0.6, 0.8)$ and $\omega = (0.1, 0.3, 0.5, 0.1)$ be a membership vector and a weight vector, respectively. Then, $\nu(\omega) = \{1, 4\}$, $\overline{\nu}(\omega) = \{2, 3\}$, $s(\omega) = \omega_1 + \omega_4 = 0.2$, $\lambda_\alpha(\omega) = \{\alpha_1, \alpha_4\} = \{0.2, 0.8\}$, $\zeta_\alpha(\omega) = \{1\}$, $\overline{\zeta}_\alpha(\omega) = \{2, 3, 4\}$ and $s_{\zeta_\alpha}(\omega) = \omega_1 = 0.1$.

Using the previous auxiliary functions, the weight distribution functions are defined as follows.

Definition 2.6 (*Neres et al.* 2024) The mapping $\upsilon : \Omega_n \cup \{\mathbf{0}_n\} \to \Omega_n \cup \{\mathbf{0}_n\}$ is the *first weight distribution* which is given by:

$$\upsilon(\omega)(j) = \begin{cases} 0, & \text{if } \omega_j = 0 \text{ or } j \in \nu(\omega), \\ \omega_j + \dfrac{s(\omega)}{\text{card}(\overline{\nu}(\omega))}, & \text{otherwise,} \end{cases}$$

and, for $\alpha \in \Delta_n$, the mapping $\psi_\alpha : \Omega_n \cup \{\mathbf{0}_n\} \to \Omega_n \cup \{\mathbf{0}_n\}$ is the *second weight distribution* defined by:

$$\psi_\alpha(\omega)(j) = \begin{cases} 0, \text{ if } \omega_j = 0 \text{ or } j \in \zeta_\alpha(\omega), & \\ \omega_j + \dfrac{s_{\zeta_\alpha}(\omega)}{\text{card}(\overline{\zeta}_\alpha(\omega))}, & \text{otherwise,} \end{cases}$$

where $\upsilon(\omega)(j)$ is the j-th component of the vector $\upsilon(\omega)$ and $\psi_\alpha(\omega)(j)$ is the j-th component of the vector $\psi_\alpha(\omega)$.

Example 2.4 Let α and ω be the vectors of Example 2.3. Then,

$$\upsilon(\omega) = (\upsilon(\omega)(1), \upsilon(\omega)(2), \upsilon(\omega)(3), \upsilon(\omega)(4)) = (0, 0.4, 0.6, 0),$$

and

$$\psi_\alpha(\omega) = (\psi_\alpha(\omega)(1), \psi_\alpha(\omega)(2), \psi_\alpha(\omega)(3), \psi_\alpha(\omega)(4)) = (0, 0.\overline{3}, 0.5\overline{3}, 0.1\overline{3}).$$

In order to apply the previous mappings recurrently, we denote by υ^0 to the identity on $\Omega_n \cup \{\mathbf{0}_n\}$, by υ^1 to υ and by υ^k to the composition $\upsilon \circ \upsilon^{k-1}$ for all $k \geq 2$. Similarly, ψ_α^0 will represent the identity on $\Omega_n \cup \{\mathbf{0}_n\}$, ψ_α^1 will be ψ_α and ψ_α^k will be the composition $\psi_\alpha \circ \psi_\alpha^{k-1}$ for all $k \geq 2$.

Example 2.5 Let α and ω be the vectors of Example 2.3. Then,

$$\begin{aligned} \upsilon^2(\omega) &= \upsilon(\upsilon(\omega)) = \upsilon(0, 0.4, 0.6, 0) = (0, 0, 1, 0), \\ \upsilon^3(\omega) &= \upsilon(\upsilon^2(\omega)) = \upsilon(0, 0, 1, 0) = (0, 0, 0, 0), \\ \upsilon^k(\omega) &= (0, 0, 0, 0), \text{ for all } k > 3 \end{aligned}$$

and

$$\begin{aligned}\psi_\alpha^2(\omega) &= \psi_\alpha(\psi_\alpha(w)) = \psi_\alpha(0, 0.\overline{3}, 0.5\overline{3}, 0.1\overline{3}) = (0, 0.3\overline{9}, 0.5\overline{9}, 0),\\ \psi_\alpha^3(\omega) &= \psi_\alpha(\psi_\alpha^2(w)) = \psi_\alpha(0, 0.3\overline{9}, 0.5\overline{9}, 0) = (0, 0, 1, 0),\\ \psi_\alpha^4(\omega) &= \psi_\alpha(\psi_\alpha^3(w)) = \psi_\alpha(0, 0, 1, 0) = (0, 0, 0, 0),\\ \psi_\alpha^k(\omega) &= (0, 0, 0, 0),\ \text{for all } k > 4.\end{aligned}$$

Let $\beta \in\ ]0, 1]$ and $A \in \mathscr{C}$. Then, $A_\beta = I_1^{A,\beta} \cup I_2^{A,\beta} \cup \ldots \cup I_{m(A,\beta)}^{A,\beta}$, where $I_j^{A,\beta} = [\,a_j^{A,\beta}, b_j^{A,\beta}\,]$ for each $j \in \{1, 2, \ldots, m(A, \beta)\}$. Let E_β^A be the multiset of extreme points of the connected components, i.e. $E_\beta^A = \{\,a_1^{A,\beta}, b_1^{A,\beta}, a_2^{A,\beta}, b_2^{A,\beta}, \ldots, a_{m(A,\beta)}^{A,\beta}, b_{m(A,\beta)}^{A,\beta}\,\}$. By avoiding repetitions in E_β^A, we can consider the set $C_\beta^A = \{\,c_1^{A,\beta}, c_2^{A,\beta}, \ldots, c_{n(A,\beta)}^{A,\beta}\,\}$, where $n(A, \beta) \leq 2m(A, \beta)$. We can assume that C_β^A is increasingly ordered, which leads to the vector $U^{A,\beta} = (c_1^{A,\beta}, c_2^{A,\beta}, \ldots, c_{n(A,\beta)}^{A,\beta}) \in \mathbb{R}^{n(A,\beta)}$.

At this point, we need to aggregate the extreme points of the connected components of the β-cuts in order to obtain a sort of average value. Since the elements in $\mathscr{C}$ are not necessarily fuzzy numbers, we need a generic definition of aggregation function that allows modifying the number of elements to be aggregated in each β-cut.

Definition 2.7 A function $\mathcal{A} : \cup_{n\in\mathbb{N}}\mathbb{R}^n \to \mathbb{R}$ is said to be an *extended aggregation function on the real line* if it satisfies the following conditions:

(i) for each $n \in \mathbb{N}$, the restriction $\mathcal{A}|_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}$ is a non-decreasing function on each argument (that is, if $t_1, t_2, \ldots, t_n, s_1, s_2, \ldots, s_n \in \mathbb{R}$ are such that $t_i \leq s_i$ for each $i \in \{1, 2, \ldots, n\}$, then $\mathcal{A}|_{\mathbb{R}^n}(t_1, t_2, \ldots, t_n) \leq \mathcal{A}|_{\mathbb{R}^n}(s_1, s_2, \ldots, s_n)$); and

(ii) if $n = 1$, then $\mathcal{A}|_{\mathbb{R}}$ is the identity mapping on $\mathbb{R}$.

Example 2.6 1. $\overline{\mathrm{M}} : \cup_{n\in\mathbb{N}}\mathbb{R}^n \to \mathbb{R}$ such that $\overline{\mathrm{M}}|_{\mathbb{R}^n}(c_1, c_2, \ldots, c_n) = \frac{c_1+c_2+\cdots+c_n}{n}$, for each $n \in \mathbb{N}$;
2. $\overline{\mathrm{Min}} : \cup_{n\in\mathbb{N}}\mathbb{R}^n \to \mathbb{R}$ such that $\overline{\mathrm{Min}}|_{\mathbb{R}^n}(c_1, c_2, \ldots, c_n) = \min\{c_1, \ldots, c_n\}$, for each $n \in \mathbb{N}$;
3. $\overline{\mathrm{Max}} : \cup_{n\in\mathbb{N}}\mathbb{R}^n \to \mathbb{R}$ such that $\overline{\mathrm{Max}}|_{\mathbb{R}^n}(c_1, c_2, \ldots, c_n) = \max\{c_1, \ldots, c_n\}$, for each $n \in \mathbb{N}$.

Given $A \in \mathscr{C}$ and $\beta \in\ ]0, 1]$. We denote by $\mathcal{A}_\beta^A$ the aggregated value of the components of the vector $U^{A,\beta}$, i.e.

$$\mathcal{A}_\beta^A = \mathcal{A}|_{\mathbb{R}^{n(A,\beta)}}\left(c_1^{A,\beta}, c_2^{A,\beta}, \ldots, c_{n(A,\beta)}^{A,\beta}\right).$$

Now, we weigh the values $\mathcal{A}_\beta^A$ according to the above-defined weight distributions.

Definition 2.8 (*Neres et al.* 2024) Given $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, the k-valuation of A relative to υ and to ψ_α are defined as the real numbers:

$$\varrho_{\alpha,\omega,\upsilon,\mathcal{A}}^k(A) = \sum_{j=1}^{n}\left[\upsilon^k(\omega)(j) \cdot \mathcal{A}_{\alpha_j}^A\right] \quad\text{and}\quad \varrho_{\alpha,\omega,\psi_\alpha,\mathcal{A}}^k(A) = \sum_{j=1}^{n}\left[\psi_\alpha^k(\omega)(j) \cdot \mathcal{A}_{\alpha_j}^A\right].$$

Whenever α, ω and $\mathcal{A}$ are clear in context, we denote $\varrho_\upsilon^k = \varrho_{\alpha,\omega,\upsilon,\mathcal{A}}^k$ and $\varrho_{\psi_\alpha}^k = \varrho_{\alpha,\omega,\psi_\alpha,\mathcal{A}}^k$. Finally, we define the alpha-order relation.

Definition 2.9 (*Neres et al.* 2024) Given $m \in J_{n-1}^0 = J_{n-1} \cup \{0\}$, let $\overset{\upsilon}{\underset{m}{=}}$, $\overset{\psi_\alpha}{\underset{m}{=}}$ and $\overset{\alpha}{=}$ be the binary relations on $\mathscr{C}$ defined as follows:

- $A \overset{\upsilon}{\underset{m}{=}} B$ iff $\varrho_\upsilon^k(A) = \varrho_\upsilon^k(B)$ for each $k \in J_m^0$;
- $A \overset{\psi_\alpha}{\underset{m}{=}} B$ iff $\varrho_{\psi_\alpha}^k(A) = \varrho_{\psi_\alpha}^k(B)$ for each $k \in J_m^0$;
- $A \overset{\alpha}{=} B$ iff $A \overset{\upsilon}{\underset{n-1}{=}} B$ and $A \overset{\psi_\alpha}{\underset{n-1}{=}} B$.

where $A, B \in \mathscr{C}$. The relation "$\overset{\alpha}{=}$" will be called the *alpha-equality*.

Definition 2.10 (*Neres et al.* 2024) Given $A, B \in \mathscr{C}$, we define the relation "$\overset{\alpha}{<}$" as follows

- $$A \overset{\upsilon}{<} B \quad \text{iff} \quad \begin{cases} \text{either } \varrho_\upsilon^0(A) < \varrho_\upsilon^0(B) \\ \text{or } \exists\, k_0 \in J_{n-1} \text{such that} A \overset{\upsilon}{\underset{m}{=}} B \text{for} \\ \text{each } m < k_0 \text{ and } \varrho_\upsilon^{k_0}(A) < \varrho_\upsilon^{k_0}(B); \end{cases}$$
- $$A \overset{\psi_\alpha}{<} B \ \text{iff} \ \begin{cases} \text{either } \varrho_{\psi_\alpha}^0(A) < \varrho_{\psi_\alpha}^0(B) \\ \text{or } \exists\, k_0 \in J_{n-1} \text{such that} A \overset{\psi_\alpha}{\underset{m}{=}} B \text{for} \\ \text{each } m < k_0 \text{ and } \varrho_{\psi_\alpha}^{k_0}(A) < \varrho_{\psi_\alpha}^{k_0}(B); \end{cases}$$
- $A \overset{\alpha}{<} B$ iff $A \overset{\upsilon}{<} B$ or $(A \overset{\upsilon}{\underset{n-1}{=}} B$ and $A \overset{\psi_\alpha}{<} B)$.

Definition 2.11 (*Neres et al.* 2024) Let $\overset{\alpha}{\leq}$ be the binary relation on $\mathscr{C}$ defined, for each $A, B \in \mathscr{C}$, as follows:

$$A \overset{\alpha}{\leq} B \quad \text{iff} \quad (A \overset{\alpha}{<} B \ \text{ or } \ A \overset{\alpha}{=} B \ ).$$

The relation "$\overset{\alpha}{\leq}$" will be called the *alpha-order* or the *local order in* $\mathscr{C}$ *according to* α.

The alpha-order satisfies the following key properties.

Proposition 2.2 *(Neres et al.* 2024*) The following statements hold.*

1 *The binary relation "$\overset{\alpha}{=}$" is an equivalence relation on $\mathscr{C}$.*
2 *The relation "$\overset{\alpha}{<}$" is non-reflexive, transitive, and asymmetric (i.e. $A \overset{\alpha}{<} B$ and $B \overset{\alpha}{<} A$ cannot be simultaneously true).*
3 *The binary relation "$\overset{\alpha}{\leq}$" is a total relation on $\mathscr{C}$ that satisfies* reflexivity, transitivity, *and alpha-antisymmetry, i.e. $A \overset{\alpha}{\leq} B$ and $B \overset{\alpha}{\leq} A \implies A \overset{\alpha}{=} B$.*

Example 2.7 Let A, B and C be three fuzzy sets such that (Fig. 1)

$$A(x) = \begin{cases} 2x - 0.2, & \text{if } 0.1 < x \leq 0.6, \\ -2x + 2.2, & \text{if } 0.6 < x < 1.1, \\ 0, & \text{if } x \leq 0.1 \ \text{ or } \ x \geq 1.1 \end{cases}$$

$$B(x) = \begin{cases} 2x - 1.4, & \text{if } 0.7 < x \leq 1.2, \\ -2x - 3.4, & \text{if } 1.2 < x \leq 1.4, \\ 2x - 2.2, & \text{if } 1.4 < x \leq 1.6, \\ -2x + 4.2, & \text{if } 1.6 < x < 2.1, \\ 0, & \text{if } x \leq 0.7 \ \text{ or } \ x \geq 2.1 \end{cases}$$

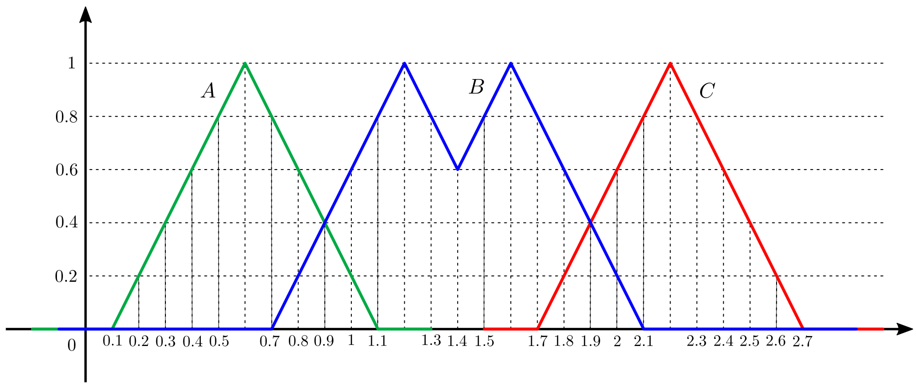

Fig. 1 Plots of the fuzzy sets A, B and C given in Example 2.7

Table 1 0-valuations of A and B in Example 2.7 with respect to υ

j	α_j	ω_j	$\upsilon^0(\omega)(j)$	$\overline{\mathrm{M}}^A_{\alpha_j}$	$\overline{\mathrm{M}}^B_{\alpha_j}$	$\varrho^0_{\alpha,\omega,\upsilon,\overline{\mathrm{M}}}(A)$	$\varrho^0_{\alpha,\omega,\upsilon,\overline{\mathrm{M}}}(B)$
1	0.2	0.1	0.1	0.6	1.4	0.6	1.4
2	0.4	0.3	0.3	0.6	1.4		
3	0.6	0.5	0.5	0.6	1.4		
4	0.8	0.1	0.1	0.6	1.4		

Table 2 0-valuations of B and C in Example 2.7 with respect to υ

j	α_j	ω_j	$\upsilon^0(\omega)(j)$	$\overline{\mathrm{M}}^B_{\alpha_j}$	$\overline{\mathrm{M}}^C_{\alpha_j}$	$\varrho^0_{\alpha,\omega,\upsilon,\overline{\mathrm{M}}}(B)$	$\varrho^0_{\alpha,\omega,\upsilon,\overline{\mathrm{M}}}(C)$
1	0.2	0.1	0.1	1.4	2.2	1.4	2.2
2	0.4	0.3	0.3	1.4	2.2		
3	0.6	0.5	0.5	1.4	2.2		
4	0.8	0.1	0.1	1.4	2.2		

Table 3 0-valuations of A and C in Example 2.7 with respect to υ

j	α_j	ω_j	$\upsilon^0(\omega)(j)$	$\overline{\mathrm{M}}^A_{\alpha_j}$	$\overline{\mathrm{M}}^C_{\alpha_j}$	$\varrho^0_{\alpha,\omega,\upsilon,\overline{\mathrm{M}}}(A)$	$\varrho^0_{\alpha,\omega,\upsilon,\overline{\mathrm{M}}}(C)$
1	0.2	0.1	0.1	0.6	2.2	0.6	2.2
2	0.4	0.3	0.3	0.6	2.2		
3	0.6	0.5	0.5	0.6	2.2		
4	0.8	0.1	0.1	0.6	2.2		

$$C(x) = \begin{cases} 2x - 3.4, & \text{if } 1.7 < x \leq 2.2, \\ -2x + 5.4, & \text{if } 2.2 < x < 2.7, \\ 0, & \text{if } x \leq 1.7 \text{ or } x \geq 2.7 \end{cases}$$

Note that $A, B, C \in \mathscr{C}$, but B is not a fuzzy number. Let us consider an alpha-ordering for these fuzzy sets in which the membership vector is $\alpha = (0.2, 0.4, 0.6, 0.8)$, the weight vector is $\omega = (0.1, 0.3, 0.4, 0.2)$ and the extended aggregation function on $\mathbb{R}$ is $\overline{\mathrm{M}}$.

From Tables 1, 2 and 3, one concludes that $A \overset{\alpha}{<} B$, $B \overset{\alpha}{<} C$ and $A \overset{\alpha}{<} C$. Therefore, $A \overset{\alpha}{\leq} B$, $B \overset{\alpha}{\leq} C$ and $A \overset{\alpha}{\leq} C$.

Note that for the alpha-order considered in Example 2.7 it was not necessary to use the second weight distribution function to define the ranking between the fuzzy sets. However, whenever $A, B \in \mathscr{C}$ are υ-equal (i.e. $A \overset{\upsilon}{=} B$), it becomes necessary to use the second weight distribution function to define the ranking of such fuzzy sets.

Additionally, as shown in Example 2.7, fuzzy sets may model some information that is not according to the fuzzy number's geometrical point of view. So, the fact that alpha-orders can rank elements of $\mathscr{C}$ enhances the theory. For more examples of ranking of fuzzy sets using the theory of alpha-ordering and justifications to apply it on $\mathscr{C}$, see Neres et al. (2024).

3 Alpha-ordering for triangular and trapezoidal fuzzy numbers

In this section, we aim to futher illustrate the behavior of the alpha-order when applied to triangular and trapezoidal fuzzy numbers. To do so, we first show how we can use the membership function of a trapezoidal fuzzy number to define simple alpha-orderings, enhancing the applicability of this ranking method. Then, we provide some insights regarding how alpha-orders rank pairs of triangular and trapezoidal fuzzy numbers that satisfy a specific symmetry condition.

First, let us recall that trapezoidal fuzzy numbers are fuzzy numbers $T = Tr(a, b, c, d) : \mathbb{R} \to [0, 1]$ with $a \leq b \leq c \leq d$, defined by

$$T(x) = \begin{cases} \frac{x-a}{b-a}, & \text{if } x \in \ ]a, b[\\ 1, & \text{if } x \in [b, c] \\ \frac{d-x}{d-c}, & \text{if } x \in \ ]c, d[\\ 0, & \text{otherwise.} \end{cases}$$

Of note, in the particular case in which $b = c$, the trapezoidal fuzzy number is called a triangular fuzzy number instead. The following result shows that, if we restrict to the set of all trapezoidal fuzzy numbers $\mathcal{T}$, it is possible to simplify the definition of alpha-orders considerably:

Theorem 3.1 *Let us fix $\omega_1 > \omega_2 > 0$ such that $\omega_1 + \omega_2 = 1$ and consider, for any $T = Tr(a, b, c, d) \in \mathcal{T}$*

$$M_0(T) = \frac{\omega_1}{2}(b + c) + \frac{\omega_2}{4}(a + b + c + d),$$
$$M_1(T) = \frac{b + c}{2}.$$

Then, the binary relation $\leq_{\mathcal{T}}$ defined by

$$T_1 \leq_{\mathcal{T}} T_2 \iff \begin{cases} M_0(T_1) < M_0(T_2) \\ \text{or} \\ M_0(T_1) = M_0(T_2) \text{ and } M_1(T_1) \leq M_1(T_2) \end{cases}$$

is an alpha-order.

Proof Let us consider $\alpha = (\frac{1}{2}, 1)$ and $\omega = (\omega_1, \omega_2)$ satisfying $\omega_1 > \omega_2 > 0$ and $\omega_1+\omega_2 = 1$. Then, we have that

$$\nu(\omega) = \zeta_\alpha(\omega) = \{2\} \qquad \overline{\nu}(\omega) = \overline{\zeta}_\alpha(\omega) = \{1\}$$
$$\lambda_\alpha(\omega) = \{1\} \qquad s(\omega) = s_{\zeta_\alpha}(\omega) = \omega_2.$$

Consequently, $\upsilon(\omega) = \psi_\alpha(\omega) = (1, 0)$. Now, notice that the β-cuts of a trapezoidal fuzzy number $T = Tr(a, b, c, d) \in \mathcal{T}$ are given by

$$T_\beta = [a + \beta(b - a), d - \beta(d - c)].$$

Therefore, for the extended aggregation $\overline{\mathrm{M}}$, one has that

$$\overline{\mathrm{M}}^T_\beta = \frac{1}{2}(a + d + \beta(-a + b + c - d)), \text{ for all } \beta \in\ ]0, 1].$$

In particular, $\overline{\mathrm{M}}^T_1 = \frac{b+c}{2}$ and $\overline{\mathrm{M}}^T_{\frac{1}{2}} = \frac{a+b+c+d}{4}$, which implies that

$$\varrho^0_{\alpha,\omega,\upsilon,\overline{\mathrm{M}}}(T) = \varrho^0_{\alpha,\omega,\psi_\alpha,\overline{\mathrm{M}}}(T) = \omega_1\overline{\mathrm{M}}^T_1 + \omega_2\overline{\mathrm{M}}^T_{\frac{1}{2}} = M_0(T)$$
$$\varrho^1_{\alpha,\omega,\upsilon,\overline{\mathrm{M}}}(T) = \varrho^1_{\alpha,\omega,\psi_\alpha,\overline{\mathrm{M}}}(T) = \overline{\mathrm{M}}^T_1 = M_1(T).$$

Given this, the resulting alpha-order can be described as follows:

$$T_1 \overset{\alpha}{\leq} T_2 \iff \begin{cases} \varrho^0_{\alpha,\omega,\upsilon,\overline{\mathrm{M}}}(T_1) < \varrho^0_{\alpha,\omega,\upsilon,\overline{\mathrm{M}}}(T_2) \\ \text{or} \\ \varrho^0_{\alpha,\omega,\upsilon,\overline{\mathrm{M}}}(T_1) = \varrho^0_{\alpha,\omega,\upsilon,\overline{\mathrm{M}}}(T_2) \text{ and } \varrho^1_{\alpha,\omega,\upsilon,\overline{\mathrm{M}}}(T_1) \leq \varrho^1_{\alpha,\omega,\upsilon,\overline{\mathrm{M}}}(T_2) \end{cases}$$

for any two trapezoidal fuzzy numbers $T_1 = Tr(a_1, b_1, c_1, d_1)$ and $T_2 = Tr(a_2, b_2, c_2, d_2)$, which is exactly the binary relation described in the statement of the theorem. □

Now, to further illustrate the performance of the alpha-ordering, we present some results on the alpha-ordering between symmetric triangular and trapezoidal fuzzy numbers centered on the same point.

Proposition 3.1 *Let us set $c \in \mathbb{R}$ and $d, d' > 0$ and consider the triangular fuzzy numbers $A = Tr(c - d, c, c, c + d)$ and $B = Tr(c - d', c, c, c + d')$ (see Fig. 2). Then, it holds that:*

(i) If $0 < d' < d$, then $\overline{\mathrm{Min}}^A_\beta < \overline{\mathrm{Min}}^B_\beta$, $\overline{\mathrm{M}}^A_\beta = \overline{\mathrm{M}}^B_\beta = c$ and $\overline{\mathrm{Max}}^B_\beta < \overline{\mathrm{Max}}^A_\beta$ for all $\beta \in\ ]0, 1]$;
(ii) If $0 < d' = d$, then $\overline{\mathrm{Min}}^A_\beta = \overline{\mathrm{Min}}^B_\beta$, $\overline{\mathrm{M}}^A_\beta = \overline{\mathrm{M}}^B_\beta = c$ and $\overline{\mathrm{Max}}^A_\beta = \overline{\mathrm{Max}}^B_\beta$ for all $\beta \in\ ]0, 1]$;
(iii) If $0 < d < d'$, then $\overline{\mathrm{Min}}^B_\beta < \overline{\mathrm{Min}}^A_\beta$, $\overline{\mathrm{M}}^A_\beta = \overline{\mathrm{M}}^B_\beta = c$ and $\overline{\mathrm{Max}}^A_\beta < \overline{\mathrm{Max}}^B_\beta$ for all $\beta \in\ ]0, 1]$.

Proof Straightforward. □

Corollary 3.1 *Let $A = Tr(c - d, c, c, c + d)$, $B = Tr(c - d', c, c, c + d')$, $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a membership vector such that $\alpha_j > 0$ for all $j \in J_n$ and $\omega = (\omega_1, \ldots, \omega_n)$ be a weight vector. Then it holds that:*

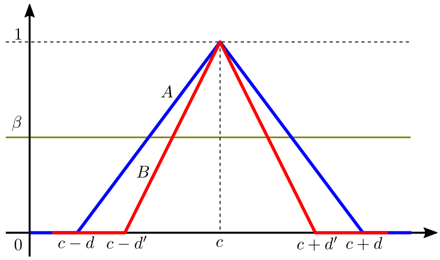

Fig. 2 Symmetric triangular fuzzy numbers centered on the same point

(i) *If* $0 < d' < d$, *then* $A \overset{\alpha}{<} B$ *with respect to* $\overline{\text{Min}}$, $A \overset{\alpha}{=} B$ *with respect to* $\overline{\text{M}}$ *and* $B \overset{\alpha}{<} A$ *with respect to* $\overline{\text{Max}}$*;*

(ii) *If* $0 < d' = d$, *then* $A \overset{\alpha}{=} B$ *with respect to* $\overline{\text{Min}}$, $\overline{\text{M}}$ *and* $\overline{\text{Max}}$*;*

(iii) *If* $0 < d < d'$, *then* $B \overset{\alpha}{<} A$ *with respect to* $\overline{\text{Min}}$, $A \overset{\alpha}{=} B$ *with respect to* $\overline{\text{M}}$ *and* $A \overset{\alpha}{<} B$ *with respect to* $\overline{\text{Max}}$.

Proof It follows directly from Proposition 3.1. □

Now, we study the performance of the alpha-ordering when comparing symmetric triangular and trapezoidal fuzzy numbers centered on the same point.

Proposition 3.2 *Let* $A = Tr(c-d, c, c, c+d)$ *and* $B = Tr(c-d'', c-d', c+d', c+d'')$ *(see Fig. 3). Then, it holds that:*

(a) *If* $0 < d' < d'' < d$ *and* $q = 1 - \dfrac{d'}{d+d'-d''}$, *then:*

(i) $\overline{\text{M}}_\beta^A = \overline{\text{M}}_\beta^B = c$ *for all* $\beta \in \,]0, 1]$*;*

(ii) $\overline{\text{Min}}_1^B < \overline{\text{Min}}_1^A$, $\overline{\text{Max}}_1^A < \overline{\text{Max}}_1^B$, $\overline{\text{Min}}_q^A = \overline{\text{Min}}_q^B = -r$ *and* $\overline{\text{Max}}_q^A = \overline{\text{Max}}_q^B = r$, *where* $r = c + \dfrac{d' \cdot d}{d+d'-d''}$*;*

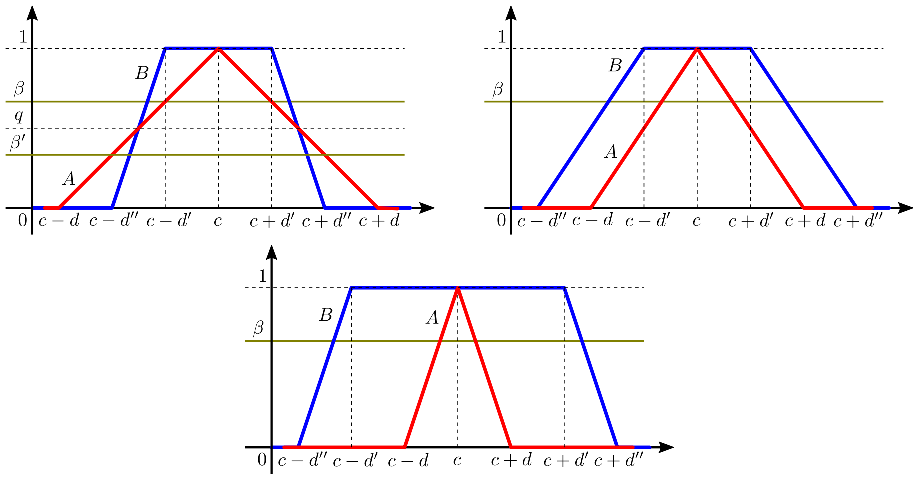

Fig. 3 Symmetric triangular and trapezoidal fuzzy numbers centered on the same point

(iii) $\overline{\text{Min}}_\beta^A < \overline{\text{Min}}_\beta^B$ *and* $\overline{\text{Max}}_\beta^B < \overline{\text{Max}}_\beta^A$ *for all* $\beta \in\ ]0, q[$;
(iv) $\overline{\text{Min}}_\beta^B < \overline{\text{Min}}_\beta^A$ *and* $\overline{\text{Max}}_\beta^A < \overline{\text{Max}}_\beta^B$ *for all* $\beta \in\ ]q, 1[$.

(b) If $0 < d' < d < d''$ *or* $0 < d < d' < d''$*, then:*

(i) $\overline{\text{M}}_\beta^A = \overline{\text{M}}_\beta^B$ *for all* $\beta \in\ ]0, 1]$;
(ii) $\overline{\text{Min}}_\beta^B < \overline{\text{Min}}_\beta^A$ *and* $\overline{\text{Max}}_\beta^A < \overline{\text{Max}}_\beta^B$ *for all* $\beta \in\ ]0, 1]$.

Proof Straightforward. □

Corollary 3.2 *Let* $A = Tr(c-d, c, c, c+d)$, $B = Tr(c-d'', c-d', c+d', c+d'')$, $\alpha = (\alpha_1, \ldots, \alpha_n)$ *be a membership vector such that* $\alpha_j > 0$ *for all* $j \in J_n$ *and* $\omega = (\omega_1, \ldots, \omega_n)$ *be a weight vector. Then, it holds that:*

(a) If $0 < d' < d'' < d$ *and* $q = 1 - \dfrac{d'}{d+d'-d''}$*, then:*

(i) $A \overset{\alpha}{=} B$ *with respect to* $\overline{\text{M}}$;
(ii) if $\alpha_{j_0} = 1$ *for some* $j_0 \in J_n$ *and* $\alpha_j = q$ *for all* $j \neq j_0$*, then* $B \overset{\alpha}{<} A$ *with respect to* $\overline{\text{Min}}$;
(iii) if $\alpha_{j_0} = 1$ *for some* $j_0 \in J_n$ *and* $\alpha_j = q$ *for all* $j \neq j_0$*, then* $A \overset{\alpha}{<} B$ *with respect to* $\overline{\text{Max}}$;
(iv) if $\alpha_j \in\ ]0, q[$ *for all* $j \in J_n$*, then* $A \overset{\alpha}{<} B$ *with respect to* $\overline{\text{Min}}$ *and* $B \overset{\alpha}{<} A$ *with respect to* $\overline{\text{Max}}$;
(v) if $\alpha_j \in\ ]q, 1[$ *for all* $j \in J_n$*, then* $B \overset{\alpha}{<} A$ *with respect to* $\overline{\text{Min}}$ *and* $A \overset{\alpha}{<} B$ *with respect to* $\overline{\text{Max}}$.

(b) If $0 < d' < d < d''$ *or* $0 < d < d' < d''$*, then* $A \overset{\alpha}{=} B$ *with respect to* $\overline{\text{M}}$, $B \overset{\alpha}{<} A$ *with respect to* $\overline{\text{Min}}$ *and* $A \overset{\alpha}{<} B$ *with respect to* $\overline{\text{Max}}$.

Proof It follows directly from Proposition 3.2. □

Finally, we provide a comparison between symmetric trapezoidal fuzzy numbers centered on the same point.

Proposition 3.3 *Let* $A = Tr(c-d', c-d, c+d, c+d')$ *and* $B = Tr(c-l', c-l, c+l, c+l')$ *(see Fig. 4). Then, it holds that:*

(a) If $0 < d < l < l' < d'$ *and* $q = \dfrac{d'-l'}{d'-l'+l-d}$*, then*

(i) $\overline{\text{M}}_\beta^A = \overline{\text{M}}_\beta^B = c$ *for all* $\beta \in\ ]0, 1]$;
(ii) $\overline{\text{Min}}_1^B < \overline{\text{Min}}_1^A$, $\overline{\text{Max}}_1^A < \overline{\text{Max}}_1^B$, $\overline{\text{Min}}_q^A = \overline{\text{Min}}_q^B = -r$ *and* $\overline{\text{Max}}_q^A = \overline{\text{Max}}_q^B = r$, *where* $r = c + \dfrac{l \cdot d' - l' \cdot d}{d' - d - l' + l}$;
(iii) $\overline{\text{Min}}_\beta^A < \overline{\text{Min}}_\beta^B$ *and* $\overline{\text{Max}}_\beta^B < \overline{\text{Max}}_\beta^A$ *for all* $\beta \in\ ]0, q[$;
(iv) $\overline{\text{Min}}_\beta^B < \overline{\text{Min}}_\beta^A$ *and* $\overline{\text{Max}}_\beta^A < \overline{\text{Max}}_\beta^B$ *for all* $\beta \in\ ]q, 1[$.

(b) If $0 < d < l < d' < l'$ *or* $0 < d < d' < l < l'$*, then*

(i) $\overline{\text{M}}_\beta^A = \overline{\text{M}}_\beta^B$ *for all* $\beta \in\ ]0, 1]$;
(ii) $\overline{\text{Min}}_\beta^B < \overline{\text{Min}}_\beta^A$ *and* $\overline{\text{Max}}_\beta^A < \overline{\text{Max}}_\beta^B$ *for all* $\beta \in\ ]0, 1]$.

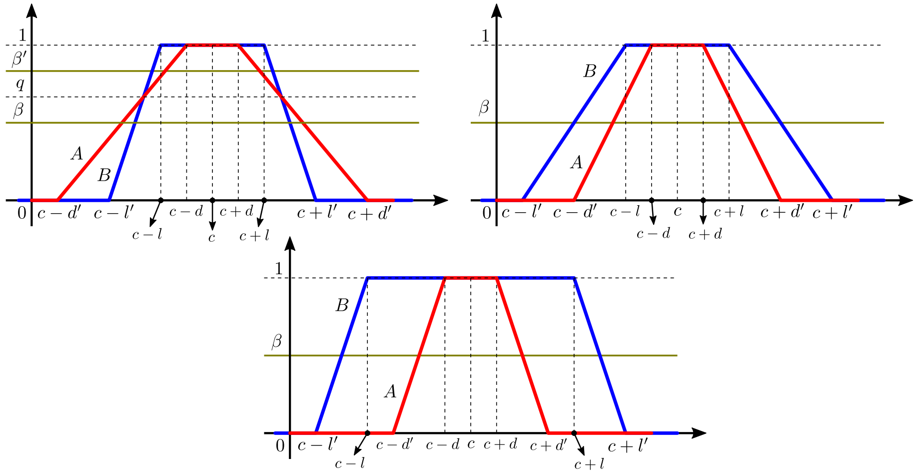

Fig. 4 Symmetric trapezoidal fuzzy numbers centered on the same point

Proof Straightforward. □

Corollary 3.3 *Let* $A = Tr(c-d', c-d, c+d, c+d')$*,* $B = Tr(c-l', c-l, c+l, c+l')$*,* $\alpha = (\alpha_1, \ldots, \alpha_n)$ *be a membership vector such that* $\alpha_j > 0$ *for all* $j \in J_n$ *and* $\omega = (\omega_1, \ldots, \omega_n)$ *be a weight vector. Then, it holds that:*

(a) If $0 < d < l < l' < d'$ *and* $q = \dfrac{d'-l'}{d'-l'+l-d}$*, then:*

- *(i)* $A \overset{\alpha}{=} B$ *with respect to* $\overline{\mathrm{M}}$*;*
- *(ii) if* $\alpha_{j_0} = 1$ *for some* $j_0 \in J_n$ *and* $\alpha_j = q$ *for all* $j \neq j_0$*, then* $B \overset{\alpha}{<} A$ *with respect to* $\underline{\mathrm{Min}}$*;*
- *(iii) if* $\alpha_{j_0} = 1$ *for some* $j_0 \in J_n$ *and* $\alpha_j = q$ *for all* $j \neq j_0$*, then* $A \overset{\alpha}{<} B$ *with respect to* $\underline{\mathrm{Max}}$*;*
- *(iv) if* $\alpha_j \in\ ]0, q[$ *for all* $j \in J_n$*, then* $A \overset{\alpha}{<} B$ *with respect to* $\overline{\mathrm{Min}}$ *and* $B \overset{\alpha}{<} A$ *with respect to* $\underline{\mathrm{Max}}$*;*
- *(v) if* $\alpha_j \in\ ]q, 1[$ *for all* $j \in J_n$*, then* $B \overset{\alpha}{<} A$ *with respect to* $\overline{\mathrm{Min}}$ *and* $A \overset{\alpha}{<} B$ *with respect to* $\underline{\mathrm{Max}}$.

(b) If $0 < d < l < d' < l'$ *or* $0 < d < d' < l < l'$*, then* $A \overset{\alpha}{=} B$ *with respect to* $\overline{\mathrm{M}}$*,* $B \overset{\alpha}{<} A$ *with respect to* $\overline{\mathrm{Min}}$ *and* $A \overset{\alpha}{<} B$ *with respect to* $\overline{\mathrm{Max}}$.

Proof It follows directly from Proposition 3.3. □

4 Admissibility of the alpha-order

Admissibility is a key property of total orders for fuzzy numbers since it guarantees that the considered ranking order refines the KY partial order. Although admissibility was originally defined for total orders, here we are interested in analyzing it for the case of alpha-orderings. For this reason, we start this section by providing a more general definition of admissibility:

Definition 4.1 Let $X \subseteq \mathcal{FN}$ and consider a partial order $\preceq$ on X. Then, a total pre-order $\trianglelefteq$ is called an admissible pre-order w.r.t. $\langle X, \preceq \rangle$, if, for all $A, B \in X$, $A \trianglelefteq B$ whenever $A \preceq B$. When $\preceq$ is the KY partial order, we will say that $\trianglelefteq$ is an admissible pre-order.

Equivalently, Proposition 2.1 guarantees that a total pre-order $\trianglelefteq$ on $\mathcal{FN}$ is admissible if it respects the partial order induced by the KM order on the β-cuts, i.e., if $A \trianglelefteq B$ whenever $A_\beta \preceq_{KM} B_\beta$ for all $\beta \in\ ]0, 1]$.

In this line, let us illustrate that the alpha-ordering acts as the KM order when it is restricted to the class of the fuzzy numbers $\widetilde{I}$ where $I \in \mathbb{I}(\mathbb{R})$.

Proposition 4.1 *Let* $I, J \in \mathbb{I}(\mathbb{R})$. *If* $I \leq_{\text{KM}} J$, *then* $\widetilde{I} \overset{\alpha}{\leq} \widetilde{J}$.

Proof Let $\alpha \in \Delta_n$ be a membership vector, $\omega \in \Omega_n$ be a weight vector and $\mathcal{A} : \cup_{n\in\mathbb{N}}\mathbb{R}^n \to \mathbb{R}$ be an extended aggregation function of the real line. For each $\beta \in\ ]0, 1]$, one has that $A_\beta = I = [\underline{I}, \overline{I}]$. Since $I \leq_{\text{KM}} J$ and $\mathcal{A}|_{\mathbb{R}^2}$ is non-decreasing on each argument, then, it follows that $\mathcal{A}^{\widetilde{I}}_\beta = \mathcal{A}|_{\mathbb{R}^2}(\underline{I}, \overline{I}) \leq \mathcal{A}|_{\mathbb{R}^2}(\underline{J}, \overline{J}) = \mathcal{A}^{\widetilde{J}}_\beta$ for each $\beta \in\ ]0, 1]$. Hence, one obtains that

$$\varrho^k_\upsilon(\widetilde{I}) = \sum_{j=1}^{n}[\upsilon^k(\omega)(j) \cdot \mathcal{A}^{\widetilde{I}}_{\alpha_j}] \leq \sum_{j=1}^{n}[\upsilon^k(\omega)(j) \cdot \mathcal{A}^{\widetilde{J}}_{\alpha_j}] = \varrho^k_\upsilon(\widetilde{I})$$

and

$$\varrho^k_{\psi_\alpha}(\widetilde{I}) = \sum_{j=1}^{n}[\psi^k_\alpha(\omega)(j) \cdot \mathcal{A}^{\widetilde{I}}_{\alpha_j}] \leq \sum_{j=1}^{n}[\psi^k_\alpha(\omega)(j) \cdot \mathcal{A}^{\widetilde{J}}_{\alpha_j}] = \varrho^k_{\psi_\alpha}(\widetilde{J}),$$

for all $k \in J^0_{n-1}$. Thereby, one concludes that $\widetilde{I} \overset{\alpha}{=} \widetilde{J}$ or $\widetilde{I} \overset{\alpha}{<} \widetilde{J}$, i.e. $\widetilde{I} \overset{\alpha}{\leq} \widetilde{J}$. □

Note that the admissibility of a partial pre-order is related to a refinement of the KY order, which is defined on the set of fuzzy numbers. Therefore, we restrict the study of the admissibility of the alpha-order to the class $\mathcal{FN}$. Taking this into account, let us prove that the pre-order $\overset{\alpha}{\leq}$ is admissible.

Theorem 4.1 *The binary relation* $\overset{\alpha}{\leq}$ *is an admissible pre-order on* $\mathcal{FN}$.

Proof From the discussion in Sect. 2, we already know that the relation $\overset{\alpha}{\leq}$ is a total pre-order in $\mathcal{FN}$. Now, let $A, B \in \mathcal{FN}$ such that $A \preceq_{\text{KY}} B$. Then, by Proposition 2.1, $A_\beta \leq_{\text{KM}} B_\beta$, where $A_\beta = [a_\beta, b_\beta]$ and $B_\beta = [c_\beta, d_\beta]$, for all $\beta \in\ ]0, 1]$ because A and B are fuzzy numbers. Thus $a_\beta \leq c_\beta$ and $b_\beta \leq d_\beta$, for all $\beta \in\ ]0, 1]$. Since $\mathcal{A}|_{\mathbb{R}^2}$ is non-decreasing on each argument, then $\mathcal{A}|_{\mathbb{R}^2}(a_\beta, b_\beta) \leq \mathcal{A}|_{\mathbb{R}^2}(c_\beta, d_\beta)$, i.e. $\mathcal{A}^A_\beta \leq \mathcal{A}^B_\beta$, for all $\beta \in\ ]0, 1]$. Therefore,

$$\varrho^k_{\alpha,\omega,\upsilon,\mathcal{A}}(A) = \sum_{j=1}^{n}[\upsilon^k(\omega)(j) \cdot \mathcal{A}^A_{\alpha_j}] \leq \sum_{j=1}^{n}[\upsilon^k(\omega)(j) \cdot \mathcal{A}^B_{\alpha_j}] = \varrho^k_{\alpha,\omega,\upsilon,\mathcal{A}}(B)$$

and

$$\varrho^k_{\alpha,\omega,\psi_\alpha,\mathcal{A}}(A) = \sum_{j=1}^{n}[\psi^k_\alpha(\omega)(j) \cdot \mathcal{A}^A_{\alpha_j}] \leq \sum_{j=1}^{n}[\psi^k_\alpha(\omega)(j) \cdot \mathcal{A}^B_{\alpha_j}] = \varrho^k_{\alpha,\omega,\psi_\alpha,\mathcal{A}}(B),$$

for all $k \in \{0, 1, \ldots, n-1\}$. Hence, $A \overset{\alpha}{\leq} B$. □

Corollary 4.1 *Let $A, B \in \mathcal{FN}$. If $A \vee B = B$, then $A \overset{\alpha}{\leq} B$.*

Proof It follows directly from Proposition 2.1 and Theorem 4.1. □

To emphasize the relationship between alpha-orderings and admissible orders, we show the following result.

Theorem 4.2 *Let $S = (\beta_i)_{i\in\mathbb{N}}$ be an upper dense sequence in $]0, 1]$. Consider the binary relations $\trianglelefteq^S_{\mathrm{KM}}$ and $\overset{\alpha}{\leq}$ both restricted to $\mathcal{FN}$. Then, for each $A, B \in \mathcal{FN}$, there is a membership vector $\alpha = (\alpha_1, \ldots, \alpha_n)$ such that $A \overset{\alpha}{\leq} B$ whenever $A \trianglelefteq^S_{\mathrm{KM}} B$.*

Proof Let $A, B \in \mathcal{FN}$ such that $A \trianglelefteq^S_{\mathrm{KM}} B$. Then, either $A = B$ or $A_{\beta_{m(A,B)}} <_{\mathrm{KM}} B_{\beta_{m(A,B)}}$. Suppose $A = B$. As $\overset{\alpha}{=}$ is reflexive, so, $A \overset{\alpha}{=} B$ and, therefore $A \overset{\alpha}{\leq} B$. Now, assume that $A_{\beta_{m(A,B)}} <_{\mathrm{KM}} B_{\beta_{m(A,B)}}$, and let $A_{\beta_{m(A,B)}} = [\underline{A_{\beta_{m(A,B)}}}, \overline{A_{\beta_{m(A,B)}}}]$ and $B_{\beta_{m(A,B)}} = [\underline{B_{\beta_{m(A,B)}}}, \overline{B_{\beta_{m(A,B)}}}]$. Then, $\underline{A_{\beta_{m(A,B)}}} \leq \underline{B_{\beta_{m(A,B)}}}$ and $\overline{A_{\beta_{m(A,B)}}} \leq \overline{B_{\beta_{m(A,B)}}}$. Since $\mathcal{A}|_{\mathbb{R}^2}$ is non-decreasing on each argument, then, it follows that $\mathcal{A}|_{\mathbb{R}^2}\left(\underline{A_{\beta_{m(A,B)}}}, \overline{A_{\beta_{m(A,B)}}}\right) \leq \mathcal{A}|_{\mathbb{R}^2}\left(\underline{B_{\beta_{m(A,B)}}}, \overline{B_{\beta_{m(A,B)}}}\right)$, i.e. $\mathcal{A}^A_{\beta_{m(A,B)}} \leq \mathcal{A}^B_{\beta_{m(A,B)}}$. On the other hand, from the minimality of $m(A, B)$, one has that $A_{\beta_i} = B_{\beta_i}$, for all $i \in \{1, \ldots, m(A, B) - 1\}$. Hence, it follows that $\mathcal{A}^A_{\beta_i} = \mathcal{A}^B_{\beta_i}$, for all $i \in \{1, \ldots, m(A, B) - 1\}$. Therefore, $\mathcal{A}^A_{\beta_i} \leq \mathcal{A}^B_{\beta_i}$, for all $i \in \{1, \ldots, m(A, B)\}$.

Thereby, if $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a membership vector such that $\alpha_j \in \{\beta_1, \ldots, \beta_{m(A,B)}\}$ for each $j \in \{1, \ldots, n\}$, then

$$\varrho^k_{\alpha,\omega,\upsilon,\mathcal{A}}(A) = \sum_{j=1}^{n}[\upsilon^k(\omega)(j) \cdot \mathcal{A}^A_{\alpha_j}] \leq \sum_{j=1}^{n}[\upsilon^k(\omega)(j) \cdot \mathcal{A}^B_{\alpha_j}] = \varrho^k_{\alpha,\omega,\upsilon,\mathcal{A}}(B)$$

and

$$\varrho^k_{\alpha,\omega,\psi_\alpha,\mathcal{A}}(A) = \sum_{j=1}^{n}[\psi^k_\alpha(\omega)(j) \cdot \mathcal{A}^A_{\alpha_j}] \leq \sum_{j=1}^{n}[\psi^k_\alpha(\omega)(j) \cdot \mathcal{A}^B_{\alpha_j}] = \varrho^k_{\alpha,\omega,\psi_\alpha,\mathcal{A}}(B),$$

for all $k \in \{0, 1, \ldots, n - 1\}$. Hence $A \overset{\alpha}{\leq} B$. □

Finally, let us illustrate the importance of the admissibility in a ranking process.

Example 4.1 Let us consider a decision-making problem modeled through fuzzy numbers. For instance, we can use the linguistic model described in García-Zamora et al. (2022). Assume that for the final ranking of the alternatives, the aggregated performance values of the alternatives are given by the trapezoidal fuzzy numbers (see Fig. 5):

$$A = Tr(0.2, 0.3, 0.3, 0.4)$$
$$B = Tr(0.3, 0.4, 0.5, 0.6)$$
$$C = Tr(0.5, 0.5, 0.7, 0.7)$$
$$D = Tr(0.8, 0.8, 0.8, 0.8)$$

Since these trapezoidal fuzzy numbers are ordered according to the KY partial order, i.e., $A \leq_{KY} B \leq_{KY} C \leq_{KY} D$, the admissibility of the alpha-ordering guarantees that any alpha-ordering will satisfy $A \overset{\alpha}{\leq} B \overset{\alpha}{\leq} C \overset{\alpha}{\leq} D$, which is the most rational way to order these

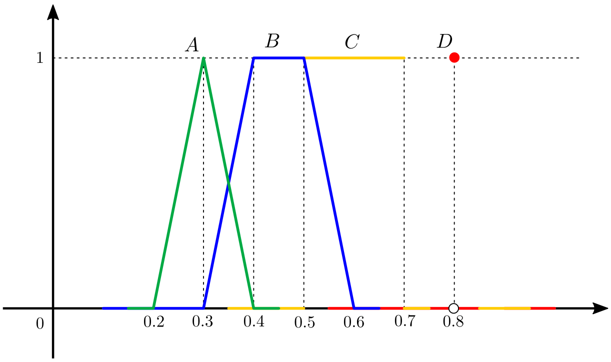

Fig. 5 Plots of the trapezoidal fuzzy numbers A, B, C and D given in Example 4.1

trapezoidal fuzzy numbers in view of their shapes in Fig. 5. In particular, we can consider the alpha-ordering $\leq_{\mathcal{T}}$ provided by Theorem 3.1 with $\omega_1 = \frac{3}{4}$ and $\omega_2 = \frac{1}{4}$. For these values, we obtain

$$M_0(A) = 0.30$$
$$M_0(B) = 0.45$$
$$M_0(C) = 0.60$$
$$M_0(D) = 0.80$$

which implies the alpha-ordering $A \leq_{\mathcal{T}} B \leq_{\mathcal{T}} C \leq_{\mathcal{T}} D$, as expected.

5 Conclusions and final remarks

The recent introduction of admissible orders for fuzzy numbers (Zumelzu et al. 2022) has addressed the challenge of finding a reasonable ordering method in this domain. Meanwhile, the alpha-ordering has been introduced (Neres et al. 2024) to solve the problem of ordering fuzzy sets that are not fuzzy numbers. This contribution has further studied the properties of the alpha-ordering, paying special attention to its admissibility. Concretely, the main contributions of this proposal are:

- We have introduced a simplified alpha-ordering for trapezoidal fuzzy numbers,
- We have provided some insights on the performance of the alpha-ordering when comparing symmetric triangular and trapezoidal fuzzy numbers centered on the same point,
- We have shown that the alpha-ordering is an admissible pre-order.

In future studies, we will analyze how to determine the parameters of the alpha-ordering according to the specific real-world application and we will try to generalize operators that depend on the order, such as Ordered Weighted Average (OWA) (Yager 1988).

Acknowledgements This study was funded by National Council for Scientific and Technological Development (CNPq-Brazil) within Project 312899/2021-1 and 403167/2022-1, by the project PID2022-136627NB-I00 of the Spanish Government. Also, A.F. Roldán López de Hierro is grateful to Ministerio de Ciencia e Innovación by Project PID2020-119478GB-I00.

Data availability No data was used for this research.

Declarations

Conflict of interest Authors declare that they have no conflict of interest.

References

Buckley JJ, Eslami E (2004) Fuzzy ordering of fuzzy numbers. Int J Uncertain Fuzziness Knowl Based Syst 12(01):105–114. https://doi.org/10.1142/s0218488504002680

Bustince H, Fernandez J, Kolesárová A et al (2013) Generation of linear orders for intervals by means of aggregation functions. Fuzzy Sets Syst 220:69–77. https://doi.org/10.1016/j.fss.2012.07.015. (**theme: Aggregation functions**)

García-Zamora D, Labella Á, Rodríguez RM et al (2022) A linguistic metric for consensus reaching processes based on elicit comprehensive minimum cost consensus models. IEEE Trans Fuzzy Syst 31:1676–1688

García-Zamora D, Cruz A, Neres F et al (2024) Admissible OWA operators for fuzzy numbers. Fuzzy Sets Syst 480:108863. https://doi.org/10.1016/j.fss.2024.108863

Herencia JA, Lamata MT (1999) A total order for the graded numbers used in decision problems. Int J Uncertain Fuzziness Knowl Based Syst 07(03):267–276. https://doi.org/10.1142/s0218488599000209

Klir GJ, Yuan B (1995) Fuzzy sets and fuzzy logic: theory and applications. Prentice Hall, Upper Saddle River

Kulisch UW, Miranker WL (1981) Computer arithmetic in theory and practice. Academic Press, Boca Raton

López Roldán, de Hierro AF, Roldán C, Herrera F (2018) On a new methodology for ranking fuzzy numbers and its application to real economic data. Fuzzy Sets Syst 353:86–110. https://doi.org/10.1016/j.fss.2018.04.003

López Roldán, de Hierro AF, Tíscar MÁ, Roldán C et al (2022) A fuzzy methodology for approaching fuzzy sets of the real line by fuzzy numbers. Fuzzy Sets Syst 435:55–77. https://doi.org/10.1016/j.fss.2021.08.024

Neres F, Santiago RHN, López Roldán, de Hierro AF et al (2024) The alpha-ordering for a wide class of fuzzy sets of the real line: the particular case of fuzzy numbers. Comput Appl Math. https://doi.org/10.1007/s40314-023-02516-2

Nguyen H (2019) A first course in fuzzy logic. CRC Press, Taylor & Francis Group, Boca Raton

Takáč Z (2016) OWA operator for discrete gradual intervals: implications to fuzzy intervals and multi-expert decision making. Kybernetika. https://doi.org/10.14736/kyb-2016-3-0379

Yager R (1988) On ordered weighted averaging aggregation operators in multicriteria decision making. IEEE Trans Syst Man Cybern 18(1):183–190. https://doi.org/10.1109/21.87068

Zadeh L (1965) Fuzzy sets. Inf Control 8(3):338–353. https://doi.org/10.1016/S0019-9958(65)90241-X

Zumelzu N, Bedregal B, Mansilla E et al (2022) Admissible orders on fuzzy numbers. IEEE Trans Fuzzy Syst 30(11):4788–4799. https://doi.org/10.1109/tfuzz.2022.3160326

Authors and Affiliations

Diego García-Zamora[1] · Anderson Cruz[2] · Fernando Neres[3] · Antonio Francisco Roldán López de Hierro[4] · Regivan H. N. Santiago[5] · Humberto Bustince[6]

✉ Diego García-Zamora
dgzamora@ujaen.es

✉ Anderson Cruz
anderson@imd.ufrn.br

Fernando Neres
fernandoneres@ufersa.edu.br

Antonio Francisco Roldán López de Hierro
aroldan@ugr.es

Regivan H. N. Santiago
regivan@dimap.ufrn.br

Humberto Bustince
bustince@unavarra.es

[1] Department of Computing, University of Jaén, Jaén, Jaén, Spain

[2] Metropóle Digital Institute, Universidade Federal do Rio Grande do Norte, Natal, Rio Grande do Norte, Brazil

[3] Department of Technology and Science, Universidade Federal Rural do Semi-Árido, Caraúbas, Rio Grande do Norte, Brazil

[4] Department of Statistics and Operations Research, Universidad de Granada, Granada, Granada, Spain

[5] Department of Informatics and Applied Mathematics, Universidade Federal do Rio Grande do Norte, Natal, Rio Grande do Norte, Brazil

[6] Departamento de Estadística, Informática y Matemáticas, Universidad Pública de Navarra, Pamplona, Navarra, Spain