Information Sciences 663 (2024) 120275

Contents lists available at ScienceDirect

Information Sciences

journal homepage: www.elsevier.com/locate/ins

Ordered weighted geometric averaging operators for basic uncertain information

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ARTICLE INFO

Keywords:
Aggregation operators
Basic uncertain information
Basic uncertain information ordered weighted averaging operators
Basic uncertain information ordered weighted geometric averaging operators
Information fusion
Ordered weighted averaging operators

ABSTRACT

Basic Uncertain Information (BUI) was proposed as an information granule that simultaneously accounts for both the value of the input and the certainty about that value. To aggregate BUI granules using Ordered Weighted Averaging (OWA) operators with bi-polar optimism–pessimism preferences, the ordered weighted averaging (BUIOWA) operators have been proposed recently. As weighted geometric mean is an alternative aggregation operator to weighted mean, the ordered weighted geometric averaging operator (OWGA) is the alternative to the OWA operator. Therefore, this study proposes basic uncertain information ordered weighted geometric averaging (BUIOWGA) operators to serve as the alternative aggregation choice to BUIOWA operators. Unlike BUIOWA operators, BUIOWGA operators have three different types, BUIOWGA type I, BUIOWGA type II, and BUIOWGA type III, with an increasing order relation. Several monotonicities, properties, and relations are proposed for the three different BUIOWGA operators and the BUIOWA operator. Finally, we present comparative analysis, numerical examples, and applications.

1. Introduction

Aggregation operators (also known as aggregation functions) [1] have been widely applied in numerous applications such as information fusion-based decision-making problems [2–5,38]. Some well-known aggregation operators include mean, geometric mean, weighted mean, weighted geometric mean, ordered weighted averaging (OWA) operator [6], and ordered weighted geometric averaging (OWGA) operator [7], to name just a few.

Aggregation operators were studied majorly for real number inputs and fuzzy granules during the past several decades [1,7–12]. As

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https://doi.org/10.1016/j.ins.2024.120275
Received 20 November 2023; Received in revised form 5 January 2024; Accepted 31 January 2024
Available online 7 February 2024
0020-0255/

uncertainty is pervasive in practice and society, aggregation operators and related aggregation methods for uncertain information have also become a hot area of research [13–17]. There are various different types and forms of uncertain information. For example, interval information, intuitionistic fuzzy information [18,42,43], vague information [19], and probability information are some well-known types. An increasing number of literature has explored various types of uncertainty in decision-making [44,46–50]. Recently, researchers proposed a new type of uncertain information called basic uncertain information (BUI) [16,17] to serve as an uncertain information paradigm. Since its inception, this new type of uncertain information has soon been applied in several areas [20–23] and some of its further extensions and variants have also been gradually developed and studied [24–27].

The ordered weighted averaging (OWA) operators, which were introduced by Yager [6], can effectively model subjective optimism–pessimism preferences. With a given input vector of real numbers, an OWA operator with a larger optimism preference, in general, will return a greater output result, and vice versa. The OWA operators have been applied in numerous applications [11,45], and some extended forms are also gradually developed [28,29]. The OWA operators can be successfully adapted for interval information, intuitionistic fuzzy information and vague information when viewed as lattice or poset structures [29]. When the inputs are BUI granules, a recent study [30] proposed a factorization-integration method to define the BUIOWA operators which can effectively and reasonably adapt OWA operators to aggregate BUI granules.

Both mean and geometric mean are important average-type aggregation operators; that is, they both return some outputs that are between the maximum and minimum of the inputs. Nevertheless, geometric mean has some special property that does not hold for (arithmetic) mean. For example, if some inputs are zero, then the geometric mean necessarily yields zero output irrespective of other inputs; that is, zero is the annihilator of the geometric mean. This property can be used in some decision-making scenarios such as the situation of "one-vote veto" where one individual disagreement will lead to the overall and final disagreement.

The OWA and OWGA operators, both being significant ordered aggregation operators, can complement each other effectively in various scenarios and practical applications. Previous research [30] has successfully introduced and analyzed the OWA operators in BUI environment, proposing and analyzing BUIOWA as a result. However, considering OWGA's equivalent importance as a counterpart to OWA, it is essential for OWGA to also be applicable in BUI environment for both theoretical advancements and practical implementations.

Therefore, the introduction and development of OWGA operator for BUI inputs are meaningful and important because it can simultaneously model, embody, and handle both the bipolar optimism–pessimism preferences and BUI granules; besides, the introduced operators have the corresponding annihilator-like elements (which plays the similar role of annihilator when with full certainty) in the form of the BUI granule $(0,1)$. Moreover, unlike BUIOWA which only has a single form, the developed operator exhibits three distinct forms. Furthermore, unexpected relationships between these forms and BUIOWA have emerged, offering significant theoretical implications in aggregation theory. Hence, this study primarily focuses on theoretical research. Additionally, the proposed operators not only find application in decision-making contexts but also possess unique characteristics and application scenarios.

The remainder of this work is organized as follows. Section 2 paraphrases some mean type aggregation operators for finite sets. Section 3 defines and constructs the three types of BUIOWGA operators. In Section 4, several related properties and relations are investigated and proved. Section 5 provides an application of the new proposal to show the usefulness and application potential. Section 6 concludes and remarks on this study.

2. Weighted mean, OWA, and OWGA for finite sets

Some notations are fixed below: $[n] = \{1,...,n\}$; $\mathbb{N}$ is the set of natural numbers. A bounded real-valued function defined on a finite set E is denoted by $x : E \to [0,1]$, which is also known as a fuzzy set on E in some different scenario; by convention, the space of all such functions defined on E is denoted by $[0,1]^E$; and for convenience such functions are sometimes written in vector or sequence form. For any non-empty subset E of $[n]$ and $x : [n] \to [0,1]$, $x|_E : E \to [0,1]$ is the restriction of $x : [n] \to [0,1]$ to $[0,1]^E$, which is defined such that $x|_E(i) = x(i)$ for any $i \in E$. Throughout this work, we will frequently consider the aggregation operators for finite sets E.

Definition 1. (*[1]*) An aggregation operator F for a finite set E is a mapping $F : [0,1]^E \to \ominus [0,1]$ such that

(i) $F(0,...,0) = 0$ and $F(1,...,1) = 1$;
(ii) $F(x) \leqslant F(y)$ whenever $x \leqslant y$ (i.e., $x(i) \leqslant y(i)$ for all $i \in E$).

For an aggregation operator $F : [0,1]^E \to [0,1]$ and any $x \in [0,1]^E$, if $\inf(x) \leqslant F(x) \leqslant \sup(x)$, then it is called a mean-type operator; if $\sup(x) \leqslant F(x)$, then it is called a disjunctive type operator; if $F(x) \leqslant \inf(x)$, then it is called a conjunctive type operator; otherwise, it is called a mixed type operator [1]. The weighted mean, weighted geometric mean, and ordered weighted averaging operator are three well-known mean type operators which are rephrased for a finite set E as follows. Note that the above rephrased definition of aggregation operator can be viewed as a special case of the recently proposed new concept of conditional aggregation operator [41].

Definition 2. A weighted mean (WM) operator for a finite set E with a normalized weight function/vector $w^E = (w_i^E)_{i \in E}$ (i.e., $\sum_{i \in E} w_i^E = 1$ and $w_i^E \geqslant 0$ for all $i \in E$), $WM_{w^E} : [0,1]^E \to [0,1]$, is defined such that

$WM_{w^E}(x) = \sum_{i \in E} w_i^E x(i)$ (when $E \neq \varnothing$)

$$WM_{w^{\varnothing}}(x) \triangleq 0 \tag{1}$$

$|E|$ denotes the cardinality of finite set E. When $w^E = (w_i^E)_{i \in E}$ is given such that for each $i \in E$, $w_i^E = 1/|E|$ ($|E| > 0$), then WM becomes the arithmetic mean operator.

Definition 3. The arithmetic mean (AM) operator for a finite set E, $AM_E : [0,1]^E \to [0,1]$, is defined such that

$$AM_E(x) = \frac{1}{|E|}\sum_{i\in E} x(i) \text{ (when } |E| > 0)$$
$$AM_\emptyset(x) \triangleq 0 \quad \text{(by convention)} \tag{2}$$

Definition 4. A weighted geometric mean (WGM) operator for a finite set E with a normalized weight function/vector $\mathbf{w}^E = (w_i^E)_{i\in E}$ ($\sum_{i\in E} w_i^E = 1$ and $w_i^E \geqslant 0$ for all $i \in E$), $WGM_{\mathbf{w}^E} : [0,1]^E \to [0,1]$, is defined such that

$$WGM_{\mathbf{w}^E}(x) = \prod_{i\in E}(x(i))^{w_i^E} \text{ (when } E \neq \emptyset)$$
$$WGM_{\mathbf{w}^\emptyset}(x) \triangleq 1 \tag{3}$$

with convention $0^0 \triangleq 1$.

When $\mathbf{w}^E = (w_i^E)_{i\in E}$ is given such that $w_i^E = 1/|E|$ ($|E| > 0$), then $WGA_{\mathbf{w}^E}(x) = \prod_{i\in E}(x(i))^{\frac{1}{|E|}}$; in this situation, the WGM operator becomes the geometric mean operator.

Definition 5. The geometric mean (GM) operator for a finite set E, $GM_E : [0,1]^E \to [0,1]$, is defined such that

$$GM_{\mathbf{w}^E}(x) = \prod_{i\in E}(x(i))^{\frac{1}{|E|}} \text{ (when } |E| > 0)$$
$$GM_\emptyset(x) \triangleq 1 \quad \text{(by convention)} \tag{4}$$

Yager's OWA operators, which are defined below, can properly model bi-polar optimism–pessimism preference in information fusion processes.

Definition 6. The ordered weighted averaging (OWA) operator for a finite set E (having cardinality $|E| = k$) with a normalized weight vector (of dimension $k = |E|$) $\mathbf{w}^{(|E|)} = \mathbf{w}^{(k)} = (w_i^{(k)})_{i=1}^k$ (i.e., $\sum_{i=1}^k w_i^{(k)} = 1$ and $w_i^{(k)} \geqslant 0$ for all $i \in [n]$), $OWA_{\mathbf{w}^{(k)}} : [0,1]^E \to [0,1]$, is defined such that

$$OWA_{\mathbf{w}^{(|E|)}}(x) = OWA_{\mathbf{w}^{(k)}}(x) = \sum_{i=1}^k w_i^{(k)} x(\sigma_E(i)) \text{ (when } |E| > 0)$$
$$OWA_{\mathbf{w}^\emptyset}(x) \triangleq 0 \quad \text{(by convention)} \tag{5}$$

where $\sigma_E : [k] \to E$ is any appropriate bijection such that $x(\sigma_E(i)) \geqslant x(\sigma_E(j))$ whenever $1 \leqslant i < j \leqslant k$. The attached normalized weight vector $\mathbf{w}^{(k)}$ is called an OWA weight vector (of dimension k). **Remark** When $\mathbf{w}^E = (w_i^E)_{i\in E}$ is given such that for each $i \in E$, $w_i^E = 1/|E|$ ($|E| > 0$), then OWA operator becomes the arithmetic mean.

Remark In Definition 2 (and Definition 4), we used a normalized vector on a general set E, $\mathbf{w}^E = (w_i^E)_{i\in E}$, but in Definition 6, we considered a normalized weight vector $\mathbf{w}^{(|E|)} = \mathbf{w}^{(k)} = (w_i^{(k)})_{i=1}^k$ which is actually described using the linear ordered set $(\{1,\ldots,k\},\leqslant)$. This is because Yager's OWA operators are defined based on linear structures, which guarantees that the involved bi-polarity can be well-based.

Remark In Definition 6, with the corresponding bijection σ_E, when $|E| > 0$ Eq. (5) has an equivalent form

$$OWA_{\mathbf{w}^{(|E|)}}(x) = \sum_{i\in E} w^{(|E|)}_{\sigma_E^{-1}(i)} x(i) \tag{6}$$

This form is called the WM expression of the OWA operator [28] since it is re-expressed as a WM operator $WM_{\mathbf{v}}(x) = \sum_{i\in E} v_i x(i)$ with the normalized weight function $\mathbf{v} = (v_i)_{i\in E}$ such that $v_i = w^{(|E|)}_{\sigma_E^{-1}(i)}$.

The extent of bi-polar optimism–pessimism preference embodied in an OWA weight vector can be measured by orness/andness.

Definition 7. (*[6]*) For an OWA weight vector $\mathbf{w}^{(n)} = (w_i^{(n)})_{i=1}^n$ of dimension n ($n \geqslant 2$), its orness and andness are defined respectively by

$$orness(\mathbf{w}^{(n)}) = \sum_{i=1}^n w_i^{(n)} \cdot \frac{n-i}{n-1}$$
$$andness(\mathbf{w}^{(n)}) = 1 - orness(\mathbf{w}^{(n)}). \tag{7}$$

If an OWA operator has a larger orness (smaller andness), in general it will have a larger output, and vice versa. For more strict statements and properties about the relationships between OWA weight vectors and their orness/andness, one can refer to Ref. [31]. In this work, when we consider some OWA weight vectors that have orness greater than 0.5, they embody some optimism attitudes (e.g., of decision makers), and when their orness degrees are lesser than 0.5, they embody some pessimism attitudes. The OWA weight vectors with orness 0.5 embody neutral attitudes, and in this work when we consider some OWA weight vectors that have orness 0.5, we only adopt the evenly allocated one with $\mathbf{w}^{(n)} = (1/n,\ldots,1/n)$.

Parameterized families of OWA weight vectors will play an important role in the later discussion since we need to discuss aggregations and weight vectors with different dimensions. We only adopt some simplified version that is enough for use, whereas for more details one may refer to Ref. [31].

Definition 8. We will say that a (parameterized) family of OWA weight vectors $\mathscr{W}\ominus^{(\alpha)} = \{w^{(n;\alpha)}\}_{n=1}^{\infty}$ ($w^{(n;\alpha)} = (w_i^{(n;\alpha)})_{i=1}^{n}$ has orness α, ($w^{(n;\alpha)} = (w_i^{(n;\alpha)})_{i=1}^{n}$), is defined such that for all $n \geqslant 2$, $orness(w^{(n;\alpha)}) = \alpha$.

Remark In later discussion, when the orness α is known without ambiguity, we will refrain from using it and write concisely with $\mathscr{W}\ominus = \{w^{(n)}\}_{n=1}^{\infty}$ ($w^{(n)} = (w_i^{(n)})_{i=1}^{n}$) to avoid clutter.

Remark In the case of $n = 1$, note that $w^{(1)} = (1)$ whose orness/andness has not been defined. However, for the convenience in the later discussion and formulating, we still include it in any family of OWA weight vectors. There are many known families of OWA weight vectors having orness α such as the recursive OWA family [32], the Binomial family and the Stancu family [33].

Naturally, the ordered weighted geometric averaging (OWGA) operator [7] can be rephrased as follows.

Definition 9. The ordered weighted geometric averaging (OWGA) for a finite set E (having cardinality $|E| = k$) with an OWA weight vector (of dimension $k = |E|$) $w^{(|E|)} = w^{(k)} = (w_i^{(k)})_{i=1}^{k}$, $OWGA_{w^{(k)}} : [0,1]^E \to [0,1]$, is defined such that

$OWGA_{w^{(|E|)}}(x) = OWGA_{w^{(k)}}(x) = \prod_{i=1}^{k} x(\sigma_E(i))^{w_i^{(k)}}$ (when $|E| > 0$).

$$OWGA_{w^{\varnothing}}(x) \triangleq 1 \quad \text{(by convention)} \tag{8}$$

where $\sigma_E : [k] \to E$ is any appropriate bijection such that $x(\sigma_E(i)) \geqslant x(\sigma_E(j))$ whenever $1 \leqslant i < j \leqslant k$. By convention, $0^0 \triangleq 1$. Remark When $w^E = (w_i^E)_{i\in E}$ is given such that for each $i \in E$, $w_i^E = 1/|E|$ ($|E| > 0$), then OWGA operator becomes the geometric mean.

Remark WM expression can be easily modified into weighted geometric mean (WGM) expression. In Definition 9, with the determined bijection σ_E, when $|E| > 0$ Eq. (8) has an equivalent form

$$OWGA_{w^{(|E|)}}(\boldsymbol{x}) = \prod_{i\in E} x(i)^{w^{(|E|)}_{\sigma_E^{-1}(i)}} \tag{9}$$

This form can be called the WGM expression of OWGA operator since it is re-expressed as a WGM operator $WGM_v(\boldsymbol{x}) = \prod_{i\in E}(x(i))^{v_i}$ with the normalized weight function $v = (v_i)_{i\in E}$ such that $v_i = w^{(|E|)}_{\sigma_E^{-1}(i)}$.

3. BUI ordered weighted geometric averaging operator

3.1. Uncertain and BUI ordered weighted averaging operators

Recently, BUIOWA operators were introduced and defined step by step [30]. We next briefly review the defining process.

In practice, usually some input values (in an input vector) are obtained with full uncertainty and the remainder input values (in that input vector) are obtained with full certainty. That is, for an input function $x : [n] \to [0,1]$ and a crisp non-empty subset $E \subseteq [n]$, the input values in $\{x(i)\}_{i\in E}$ are all with full certainty and the input values in $\{x(i)\}_{i\in[n]\setminus E}$ are all with full uncertainty.

The OWA operators are originally defined in a deterministic setting, where all input values are precisely known and certain. However, when dealing with uncertain environments, the discourse $[n]$ can be divided into two distinct subsets: one for which OWA operators should still be naturally defined (denoted as subset E), and another for which OWA and preference-based aggregation should be ignored, instead employing mean operators (denoted as subset $[n]\setminus E$). To combine these two parts effectively, weights should be assigned proportionally based on the cardinalities of each subset (i.e., $|E|/n$ and $1-(|E|/n)$ respectively). Consequently, we present the following definition of uncertain OWA aggregation.

Definition 10. (*[30]*) For any input function $x : [n] \to [0,1]$ and a subset $E \subseteq [n]$ on which the input function values are certain (with its complement $[n]\setminus E$ on which the input function values are uncertain), an uncertain ordered weighted averaging (UOWA) operator for $[n]$ with a family of OWA weight vectors having orness α, $\mathscr{W}\ominus = \{w^{(n)}\}_{n=1}^{\infty}$, is a mapping $UOWA_{\mathscr{W}} : [0,1]^{[n]} \times 2^{[n]} \to [0,1]$ such that

$$UOWA_{\mathscr{W}\ominus}(x,E) = \frac{|E|}{n} \cdot OWA_{w^{(|E|)}}\left(x|_E\right) + \left(1 - \frac{|E|}{n}\right) \cdot AM_{[n]\setminus E}\left(x|_{[n]\setminus E}\right) \tag{10}$$

with conventions $OWA_{w^{(\varnothing)}}(x|_{\varnothing}) \triangleq 0$ and $AM_{\varnothing}(x|_{\varnothing}) \triangleq 0$. The attached aggregation certainty $AC : 2^{[n]} \to [0,1]$ (aggregation uncertainty $AU : 2^{[n]} \to [0,1]$) is defined by

$$AC(E) = |E|/n AU(E) = 1 - (|E|/n) \tag{11}$$

Basic uncertain information is a recently proposed uncertainty paradigm.

Definition 11. (*[16,17]*) (i) A basic uncertain information (BUI) granule is a pair (x,c) in which $x \in [0,1]$ is an input value (which is often an evaluation value) and $c \in [0,1]$ is the certainty degree of x; and $1-c \in [0,1]$ is the uncertainty degree of x.

(ii) The set of all BUI granules is denoted by $\mathscr{B}$. A BUI function/vector of dimension n, $(x,c) : [n] \to \mathscr{B}$, is formally defined and written by $(x,c) = ((x,c)(i))_{i\in[n]} = ((x(i),c(i)))_{i\in[n]} \in \mathscr{B}^{[n]}$ where $x : [n] \to [0,1]$ is called an input value function/vector while $c : [n] \to [0,1]$ is called the certain function/vector (associated to x). (Whether (x,c) represents a BUI granule or a vector of BUI granules can be easily distinguished and identified according to context.).

(iii) There are two projection mappings in relation to BUI granules $P_x : \mathscr{B} \to [0,1]$, called value projection, and $P_c : \mathscr{B} \to [0,1]$, called certainty projection, which are defined such that $P_x((x,c)) = x$ and $P_c((x,c)) = c$ for all $(x,c) \in \mathscr{B}$.

In practice, certainty degrees may measure the degrees/extents to which decision makers are confident, sure, certain or definite of the concerned evaluation values; conversely, uncertainty degrees can measure the degrees/extents to which they are unconfident, unsure, uncertain or indefinite of the concerned evaluation values. The BUI granule $(x,1)$ indicates that the evaluation value x is with full/maximum certainty and in practice it can be regarded as equivalent to the real number x. Similarly, the BUI granule $(x,0)$ indicates that the evaluation value x is with full/maximum uncertainty (or zero/minimum certainty). For any BUI granule $(x,0)$, it is reasonable that any value between [0,1] can be considered as a true value, and hence no substantial and effective information can be obtained from $(x,0)$.

By the thought of factorization-integration of complex problems, an integral method has been devised to define BUI ordered weighted averaging operators.

Definition 12. (*[30]*) A basic uncertain information ordered weighted averaging (BUIOWA) operator for $[n]$ with a family of OWA weight vectors having orness α, $\mathscr{W}_\ominus = \{w^{(n)}\}_{n=1}^{\infty}$, is a mapping $BUIOWA_{\mathscr{W}_\ominus} : \mathscr{B}^{[n]} \to \mathscr{B}$, such that

$$BUIOWA_{\mathscr{W}_\ominus}((x,c)) = \left(\int_0^1 UOWA_{\mathscr{W}}(x,E_t)dt, \int_0^1 AC(E_t)dt\right) \tag{12}$$

where $E_t = \{k \in [n] : c(k) \geqslant t\}$. **Remark** [30] In a similar manner to the integrand of a Choquet integral, i.e., $\mu(\{k \in [n] : x(k) \geqslant t\})$, both $UOWA_{\mathscr{W}_\ominus}(x,E_t)$ and $AC(E_t)$ are composite functions of t with both domains and codomains being $[0,1]$; and from Eq. (10), observe that once t is fixed, $UOWA_{\mathscr{W}_\ominus}(x,E_t)$ is also determined. Similar to the Choquet integral, to avoid clutter we refrain from stressing that $UOWA_{\mathscr{W}}(x,E_t) = f(t)$ and $AC(E_t) = g(t)$ are two functions of t. Moreover, note that $UOWA_{\mathscr{W}}(x,E_t)$ and $AC(E_t)$ are also determined partly by certainty function c; when (x,c) is given fixed, $UOWA_{\mathscr{W}}(x,E_t)$ and $AC(E_t)$ are only affected by t and therefore the two integrals in Eq. (12) are well defined.

Remark [30] (i) By Eq. (12), we have

$$P_x(BUIOWA_{\mathscr{W}}((x,c))) = \int_0^1 UOWA_{\mathscr{W}}(x,E_t)dt$$

$$= \int_0^1 \left[\frac{|E_t|}{n} \cdot OWA_{w^{(|E_t|)}}\left(x|_{E_t}\right) + \left(1-\frac{|E_t|}{n}\right) \cdot AM_{[n]\setminus E_t}\left(x|_{[n]\setminus E_t}\right)\right]dt$$

$$= \int_0^1 \frac{|E_t|}{n} \cdot OWA_{w^{(|E_t|)}}\left(x|_{E_t}\right)dt + \int_0^1 \left(1-\frac{|E_t|}{n}\right) \cdot AM_{[n]\setminus E_t}\left(x|_{[n]\setminus E_t}\right)dt \tag{13}$$

(ii) It can be easily checked that for all $(x,c) \in \mathscr{B}^{[n]}$,

$$P_c(BUIOWA_{\mathscr{W}}((x,c))) = \int_0^1 AC(E_t)dt = \int_0^1 (|E_t|/n)dt = (1/n)\sum_{i\in[n]} c(i) \tag{14}$$

We next review the explicit weight vector for the WM expression of BUIOWA [30]. Let $(x,c) \in \mathscr{B}^{[n]}$, recall $E_t = \{i \in [n] : c(i) \geqslant t\}$ (for each $t \in [0,1]$) and $|E_t| = k_t$. Then, the value projection can be rewritten as a weighted mean

$$P_x(BUIOWA_{\mathscr{W}}((x,c))) = \int_0^1 UOWA_{\mathscr{W}}(x,E_t)dt = WA_{u^{[n]}}(x) = \sum_{i\in[n]} u_i^{[n]} x(i) \tag{15}$$

The explicit weight vector $u^{[n]} = (u_i^{[n]})_{i\in[n]}$ is normalized and can be derived by the following procedure and deduction [30].

For each $t \in [0,1]$, recall that when $E_t \neq \varnothing$, $\sigma_{E_t} : [k_t] \to E_t$ is any appropriate bijection such that $x(\sigma_{E_t}(i)) \geqslant x(\sigma_{E_t}(j))$ whenever $1 \leqslant i < j \leqslant k_t$. Then, we consider the following three situations:

(i) if both $E_t \neq \varnothing$ and $[n]\setminus E_t \neq \varnothing$ hold, we have

$$UOWA_{\mathscr{W}}(x,E_t) = \frac{k_t}{n} \cdot OWA_{w^{(|E_t|)}}\left(x|_{E_t}\right) + \left(1-\frac{k_t}{n}\right) \cdot AM_{[n]\setminus E_t}\left(x|_{[n]\setminus E_t}\right)$$

$$= \frac{k_t}{n} \cdot \sum_{i \in E_t} w^{(|E_t|)}_{\sigma^{-1}_{E_t}(i)} \cdot x|_{E_t}(i) + \left(1 - \frac{k_t}{n}\right) \cdot \sum_{i \in [n] \backslash E_t} \frac{1}{n - k_t} x|_{[n] \backslash E_t}(i)$$

$$= \sum_{i \in E_t} \frac{k_t}{n} w^{(|E_t|)}_{\sigma^{-1}_{E_t}(i)} \cdot x|_{E_t}(i) + \sum_{i \in [n] \backslash E_t} \frac{1}{n} \cdot x|_{[n] \backslash E_t}(i)$$

$$= \sum_{i \in E_t \cup ([n] \backslash E_t)} u_t(i) \cdot x(i) = \sum_{i \in [n]} u_t(i) \cdot x(i) \tag{16}$$

where $u_t : [n] \to [0,1]$ ($\sum_{i \in [n]} u_t(i) = 1$) is a normalized weight vector satisfying
$u_t(i) = \frac{k_t}{n} \cdot w^{(|E_t|)}_{\sigma^{-1}_{E_t}(i)}$ (when $i \in E_t$)

$$u_t(i) = \frac{1}{n} \quad (\text{when } i \in [n] \backslash E_t) \tag{17}$$

(ii) if $E_t = \varnothing$, then

$$UOWA_{\mathscr{W}}(x, E_t) = WM_{[n]}\left(x|_{[n]}\right) = \sum_{i \in [n]} \frac{1}{n} \cdot x|_{[n]}(i) = \sum_{i \in [n]} u_t(i) \cdot x(i)$$

where $u_t : [n] \to [0,1]$ satisfies

$$u_t(i) = \frac{1}{n} \quad (\text{for all } i \in [n]) \tag{18}$$

(iii) if $[n] \backslash E_t = \varnothing$ (i.e., $E_t = [n]$), then

$$UOWA_{\mathscr{W}}(x, E_t) = OWA_{\boldsymbol{w}^{(n)}}\left(x|_{E_t}\right) = \sum_{i \in [n]} w^{(|E_t|)}_{\sigma^{-1}_{[n]}(i)} \cdot x|_{[n]}(i) = \sum_{i \in [n]} u_t(i) \cdot x(i)$$

where u_t satisfies

$$u_t(i) = w^{(n)}_{\sigma^{-1}_{[n]}(i)} \quad (\text{for all } i \in [n]) \tag{19}$$

Since

$$P_x(BUIOWA_{\mathscr{W}}((x, c))) = \int_0^1 UOWA_{\mathscr{W}}(x, E_t) dt = \int_0^1 \left[\sum_{i \in [n]} u_t(i) \cdot x(i)\right] dt = \sum_{i \in [n]} \left(\int_0^1 u_t(i) dt\right) \cdot x(i) = \sum_{i \in [n]} u_i^{[n]} x(i)$$

Then, $\boldsymbol{u}^{[n]} = (u_i^{[n]})_{i \in [n]}$ is defined by

$$u_i^{[n]} = \int_0^1 u_t(i) dt \tag{20}$$

where $u_t(i)$ is described in Eqs. (17)–(19).

Example 1. ([30]) Given $n = 5$, and $(x,c) = ((x(i), c(i)))_{i \in [5]} = ((0.7, 0.5), (0.3, 0.8), (0.8, 0), (0.3, 0.2), (0.5, 0.5))$; that is, $x = (0.7, 0.3, 0.8, 0.3, 0.5)$ and $c = (0.5, 0.8, 0, 0.2, 0.5)$.

Suppose we adopt a family of OWA weight vectors having orness $\alpha = 0.25$, $\mathscr{W}_\ominus = \{\boldsymbol{w}^{(s)}\}_{s=1}^{\infty}$, which includes (only for $1 \leqslant s \leqslant 5$): $\boldsymbol{w}^{(1)} = (1)$, $\boldsymbol{w}^{(2)} = (0.25, 0.75)$, $\boldsymbol{w}^{(3)} = (0.1, 0.3, 0.6)$, $\boldsymbol{w}^{(4)} = (0.05, 0.15, 0.3, 0.5)$, $\boldsymbol{w}^{(5)} = (0.025, 0.075, 0.15, 0.375, 0.375)$.
The computation can use the following factorization of [0,1].
$E_t = [5] (t \in \{0\})$; $E_t = \{1, 2, 4, 5\}$ $(t \in (0, 0.2])$; $E_t = \{1, 2, 5\}$ $(t \in (0.2, 0.5])$
$E_t = \{2\} (t \in (0.5, 0.8])$; $E_t = \varnothing$ $(t \in (0.8, 1])$
Then, when $t \in \{0\}$,

$$UOWA_{\mathscr{W}}(x, E_t) = \frac{5}{5} \cdot OWA_{\boldsymbol{w}^{(5)}}\left(x|_{[5]}\right) + \left(1 - \frac{5}{5}\right) \cdot AM_{\varnothing}\left(x|_{\varnothing}\right)$$

$$= OWA_{\boldsymbol{w}^{(5)}}\left(x|_{[5]}\right) = 0.3725$$

when $t \in (0, 0.2]$,

$$UOWA_{\mathscr{W}}(x, E_t) = \frac{4}{5} \cdot OWA_{\boldsymbol{w}^{(4)}}\left(x|_{\{1,2,4,5\}}\right) + \left(1 - \frac{4}{5}\right) \cdot AM_{\{3\}}\left(x|_{\{3\}}\right)$$

$$= 0.8 \cdot OWA_{\boldsymbol{w}^{(4)}}(0.7, 0.3, 0.3, 0.5) + 0.2 \cdot AM_{\{3\}}(0.8)$$

$$= (0.8)(0.35) + (0.2)(0.8) = 0.44$$

when $t \in (0.2, 0.5]$,

$$UOWA_{\mathscr{W}}(x, E_t) = \frac{3}{5} \cdot OWA_{\boldsymbol{w}^{(3)}}\left(x|_{\{1,2,5\}}\right) + \left(1 - \frac{3}{5}\right) \cdot AM_{\{3,4\}}\left(x|_{\{3,4\}}\right)$$

$$= (0.6) \cdot OWA_{\boldsymbol{w}^{(3)}}(0.7, 0.3, 0.5) + (0.4) \cdot AM_{\{3,4\}}(0.8, 0.3)$$

$$= (0.6)(0.4) + (0.4)(0.55) = 0.46$$

when $t \in (0.5, 0.8]$,

$$UOWA_{\mathscr{W}}(x, E_t) = \frac{1}{5} \cdot OWA_{\boldsymbol{w}^{(1)}}\left(x|_{\{2\}}\right) + \left(1 - \frac{1}{5}\right) \cdot AM_{\{1,3,4,5\}}\left(x|_{\{1,3,4,5\}}\right)$$

$$= (0.2) \cdot OWA_{\boldsymbol{w}^{(1)}}(0.3) + (0.8) \cdot AM_{\{1,3,4,5\}}(0.7, 0.8, 0.3, 0.5)$$

$$= (0.2) \cdot (0.3) + (0.8) \cdot (0.575) = 0.52$$

when $t \in (0.8, 1]$,

$$UOWA_{\mathscr{W}}(x, E_t) = \frac{0}{5} \cdot OWA_{\boldsymbol{w}^{(0)}}\left(x|_{\varnothing}\right) + \left(1 - \frac{0}{5}\right) \cdot mean_{[5]}\left(x|_{[5]}\right) = mean_{[5]}\left(x|_{[5]}\right) = 0.52$$

Then,

$$P_x(BUIOWA_{\mathscr{W}}((x, c))) = \int_0^1 UOWA_{\mathscr{W}}(x, E_t)dt$$

$$= \int_{\{0\}} UOWA_{\mathscr{W}}(x, E_t)dt + \int_{(0,0.2]} UOWA_{\mathscr{W}}(x, E_t)dt + \int_{(0.2,0.5]} UOWA_{\mathscr{W}}(x, E_t)dt + \int_{(0.5,0.8]} UOWA_{\mathscr{W}}(x, E_t)dt + \int_{(0.8,1]} UOWA_{\mathscr{W}}(x, E_t)dt$$

$$= (0)(0.3725) + (0.2 - 0)(0.44) + (0.5 - 0.2)(0.46) + (0.8 - 0.5)(0.52) + (1 - 0.8)(0.52) = 0.486$$

Moreover,

$$P_c(BUIOWA_{\mathscr{W}}((x, c))) = (1/5)\sum_{i=1}^{5} c(i) = 0.4$$

As a result,

$$BUIOWA_{\mathscr{W}}((x, c)) = (0.486, 0.4)$$

The explicit weight vector $\boldsymbol{u}^{[5]} = (u_i^{[5]})_{i\in[5]}$ used for WM expression of BUIOWA can be obtained by the following process.
When $t \in \{0\}$, we can omit the computation because integration on a point will not affect the results.
When $t \in (0, 0.2]$, $E_t = \{1, 2, 4, 5\}$ and $[5]\backslash E_t = \{3\}$,
$\sigma_{E^t}(1) = 1$, $\sigma_{E^t}(2) = 5$, $\sigma_{E^t}(3) = 2$, $\sigma_{E^t}(4) = 4$ and $\sigma_{E^t}^{-1}(1) = 1$, $\sigma_{E^t}^{-1}(2) = 3$, $\sigma_{E^t}^{-1}(4) = 4$, $\sigma_{E^t}^{-1}(5) = 2$.
Therefore,

$$u_t(1) = \frac{k_t}{n} \cdot w^{(4)}_{\sigma_{E_t}^{-1}(1)} = \frac{4}{5} \cdot w_1^{(4)} = 0.04,$$

$$u_t(2) = \frac{4}{5} \cdot w_3^{(4)} = 0.24,$$

$$u_t(4) = \frac{4}{5} \cdot w_4^{(4)} = 0.4,$$

$$u_t(5) = \frac{4}{5} \cdot w_2^{(4)} = 0.12.$$

Besides, $u_t(3) = \frac{1}{5} = 0.2$.
Similarly, when $t \in (0.2, 0.5]$, $E_t = \{1, 2, 5\}$ and $[5] \backslash E_t = \{3, 4\}$.
$\sigma_{E^t}(1) = 1$, $\sigma_{E^t}(2) = 5$, $\sigma_{E^t}(3) = 2$ and $\sigma_{E^t}^{-1}(1) = 1$, $\sigma_{E^t}^{-1}(2) = 3$, $\sigma_{E^t}^{-1}(5) = 2$.
Therefore, according to Eq. (19),

$$u_t(1) = \frac{k_t}{n} \cdot w_{\sigma_{E_t}^{-1}(1)}^{(3)} = \frac{3}{5} \cdot w_1^{(3)} = 0.06,$$

$$u_t(2) = \frac{3}{5} \cdot w_3^{(3)} = 0.36,$$

$$u_t(5) = \frac{3}{5} \cdot w_2^{(3)} = 0.18.$$

Besides, $u_t(3) = u_t(4) = \frac{1}{5} = 0.2$.
When $t \in (0.5, 0.8]$, $E_t = \{2\}$ and $[5] \backslash E_t = \{1, 3, 4, 5\}$, and in this situation we easily see $u_t(i) = 0.2$ for all $i \in [5]$.
When $t \in (0.8, 1]$, $E_t = \varnothing$ and $[5] \backslash E_t = [5]$, and in this situation we also see $u_t(i) = 0.2$ for all $i \in [5]$.
Consequently, we have

$$u_1^{[5]} = \int_0^1 u_t(1)dt = (0.2)(0.04) + (0.3)(0.06) + (0.5)(0.2) = 0.126,$$

$$u_2^{[5]} = \int_0^1 u_t(2)dt = (0.2)(0.24) + (0.3)(0.36) + (0.5)(0.2) = 0.256,$$

$$u_3^{[5]} = \int_0^1 u_t(3)dt = (0.2)(0.2) + (0.3)(0.2) + (0.5)(0.2) = 0.2,$$

$$u_4^{[5]} = \int_0^1 u_t(4)dt = (0.2)(0.4) + (0.3)(0.2) + (0.5)(0.2) = 0.24,$$

$$u_5^{[5]} = \int_0^1 u_t(5)dt = (0.2)(0.12) + (0.3)(0.18) + (0.5)(0.2) = 0.178.$$

Then, $\boldsymbol{u}^{[5]} = (u_i^{[5]})_{i \in [5]} = (0.126, 0.256, 0.2, 0.24, 0.178)$.

3.2. BUI ordered weighted geometric averaging operators

With the preceding discussions and definitions in Section 2, we can define BUI ordered weighted geometric averaging (BUIOWGA) operators in some similar manner. However, unlike BUIOWA, BUIOWGA will have three different types.

We firstly define the aggregation certainty which will be the same for all the three types.

Definition 13. ([30]) For a subset $E \subseteq [n]$ on which the input function values are certain (with its complement $[n] \backslash E$ on which the input function values are uncertain). The attached aggregation certainty $AC : 2^{[n]} \rightarrow [0, 1]$ (aggregation uncertainty $AU : 2^{[n]} \rightarrow [0, 1]$) is defined by

$$AC(E) = |E|/nAU(E) = 1 - (|E|/n) \tag{21}$$

Definition 14. For any input function $x : [n] \rightarrow [0, 1]$ and a subset $E \subseteq [n]$ on which the input function values are certain (with its complement $[n] \backslash E$ on which the input function values are uncertain), an uncertain ordered weighted geometric averaging type I (UOWGA-I) operator for $[n]$ with a family of OWA weight vectors $\mathscr{W} \ominus = \{\boldsymbol{w}^{(n)}\}_{n=1}^{\infty}$, having orness α is a mapping $UOWGA - I_{\mathscr{W}} : [0, 1]^{[n]} \times 2^{[n]} \rightarrow [0, 1]$ such that

$$UOWGA - I_{\mathscr{W}}(x, E) = \frac{|E|}{n} \cdot OWGA_{\boldsymbol{w}^{(|E|)}}\left(x|_E\right) + \left(1 - \frac{|E|}{n}\right) \cdot GM_{[n] \backslash E}\left(x|_{[n] \backslash E}\right) \tag{22}$$

with conventions $OWGA_{w^{(\emptyset)}}(x|_{\emptyset}) \triangleq 1$ and $GM_{\emptyset}(x|_{\emptyset}) \triangleq 1$.

Definition 15. A basic uncertain information ordered weighted geometric averaging type I (BUIOWGA-I) operator for $[n]$ with a family of OWA weight vectors having orness α, $\mathscr{W}\ominus = \{w^{(n)}\}_{n=1}^{\infty}$, is a mapping $BUIOWGA-I_{\mathscr{W}} : \mathscr{B}^{[n]} \to \mathscr{B}$, such that

$$BUIOWGA-I_{\mathscr{W}}((x,c)) = \left(\int_0^1 UOWGA-I_{\mathscr{W}}(x,E_t)dt, \int_0^1 AC(E_t)dt\right) \tag{23}$$

where $E_t = \{k \in [n] : c(k) \geqslant t\}$.

Definition 16. For any input function $x : [n] \to [0,1]$ and a subset $E \subseteq [n]$ on which the input function values are certain (with its complement $1-(|E|/n)$ on which the input function values are uncertain), an uncertain ordered weighted geometric averaging type II (UOWGA-II) operator for $[n]$ with a family of OWA weight vectors having orness α, $\mathscr{W}\ominus = \{w^{(n)}\}_{n=1}^{\infty}$, is a mapping $UOWGA-II_{\mathscr{W}} : [0,1]^{[n]} \times 2^{[n]} \to [0,1]$ such that

$$UOWGA-II_{\mathscr{W}}(x,E) = \left(OWGA_{w^{(|E|)}}(x|_E)\right)^{\frac{|E|}{n}} \cdot \left(GM_{[n]\setminus E}\left(x|_{[n]\setminus E}\right)\right)^{1-\frac{|E|}{n}} \tag{24}$$

with conventions $\left(OWGA_{w^{(|\emptyset|)}}(x|_{\emptyset})\right)^0 \triangleq 1$ and $\left(GM_{\emptyset}(x|_{\emptyset})\right)^1 \triangleq 1$.

Definition 17. A basic uncertain information ordered weighted geometric averaging type II (BUIOWGA-II) operator for $[n]$ with a family of OWA weight vectors having orness α, $\mathscr{W}\ominus = \{w^{(n)}\}_{n=1}^{\infty}$, is a mapping $BUIOWGA-II_{\mathscr{W}} : \mathscr{B}^{[n]} \to \mathscr{B}$, such that

$$BUIOWGA-II_{\mathscr{W}}((x,c)) = \left(\int_0^1 UOWGA-II_{\mathscr{W}}(x,E_t)dt, \int_0^1 AC(E_t)dt\right) \tag{25}$$

where $E_t = \{k \in [n] : c(k) \geqslant t\}$. The third type as follows will directly use the weight vector $u^{[n]} = (u_i^{[n]})_{i\in[n]}$, which is specified by Eq. (20), as the weights for the WGM expression of BUIOWGA.

Definition 18. A basic uncertain information ordered weighted geometric averaging type III (BUIOWGA-III) operator for $[n]$ with a family of OWA weight vectors having orness α, $\mathscr{W}\ominus = \{w^{(n)}\}_{n=1}^{\infty}$, is a mapping $BUIOWGA-III_{\mathscr{W}} : \mathscr{B}^{[n]} \to \mathscr{B}$, such that

$$BUIOWGA-III_{\mathscr{W}}((x,c)) = \left(WGM_{u^{[n]}}(x), \int_0^1 AC(E_t)dt\right) \tag{26}$$

where $u^{[n]} = (u_i^{[n]})_{i\in[n]}$ is defined by Eq. (20). **Remark** Note that $u^{[n]} = (u_i^{[n]})_{i\in[n]}$ is dependent on inputs (x,c) and $\mathscr{W}\ominus = \{w^{(n)}\}_{n=1}^{\infty}$.

The computation complexity involved in Eqs. (23), (25) or (26) includes at most n times permutations, $2n$ times additions and n^2 times multiplications and power calculations.

Example 2. With the same inputs as in Example 1, partly for the purpose of comparison, in what follows we compute BUIOWGA-I, BUIOWGA-II, and BUIOWGA-III, respectively.

$(x,c) = ((x(i),c(i)))_{i\in[5]} = ((0.7,0.5),(0.3,0.8),(0.8,0),(0.3,0.2),(0.5,0.5))$; that is, $x = (0.7,0.3,0.8,0.3,0.5)$ and $c = (0.5,0.8,0,0.2,0.5)$.

Suppose we adopt a family of OWA weight vectors having orness $\alpha = 0.25$, $\mathscr{W}\ominus = \{w^{(s)}\}_{s=1}^{\infty}$, which includes (only for $1\leqslant s\leqslant 5$): $w^{(1)} = (1)$, $w^{(2)} = (0.25,0.75)$, $w^{(3)} = (0.1,0.3,0.6)$, $w^{(4)} = (0.05,0.15,0.3,0.5)$, $w^{(5)} = (0.025,0.075,0.15,0.375,0.375)$.The computation can use the following factorization of [0,1].

$E_t = [5] (t \in \{0\})$; $E_t = \{1,2,4,5\}$ $(t \in (0,0.2])$; $E_t = \{1,2,5\}$ $(t \in (0.2,0.5])$

$E_t = \{2\} (t \in (0.5,0.8])$; $E_t = \emptyset$ $(t \in (0.8,1])$

We firstly obtain $P_c(BUIOWA_{\mathscr{W}}((x,c))) = (1/5)\sum_{i=1}^{5} c(i) = 0.4$, and next we compute for the three types of BUIOWGA.

(I) The computation for $UOWGA-I_{\mathscr{W}}(x,E_t)$ and $BUIOWGA-I_{\mathscr{W}}((x,c))$:

when $t \in \{0\}$,

$$UOWGA-I_{\mathscr{W}}(x,E_t) = \frac{5}{5} \cdot OWGA_{w^{(5)}}\left(x|_{[5]}\right) + \left(1-\frac{5}{5}\right) \cdot GM_{\emptyset}(x|_{\emptyset})$$

$$= OWGA_{w^{(5)}}\left(x|_{[5]}\right)$$

when $t \in (0,0.2]$,

$$UOWGA-I_{\mathscr{W}}(x,E_t)=\frac{4}{5}\cdot OWGA_{\mathbf{w}^{(4)}}\left(x|_{\{1,2,4,5\}}\right)+\left(1-\frac{4}{5}\right)\cdot GM_{\{3\}}\left(x|_{\{3\}}\right)$$

$$\begin{aligned}&=0.8\cdot OWGA_{\mathbf{w}^{(4)}}(0.7,0.3,0.3,0.5)+0.2\cdot GM_{\{3\}}(0.8)\\&=(0.8)\left(0.7^{0.05}\cdot 0.5^{0.15}\cdot 0.3^{0.3}\cdot 0.3^{0.5}\right)+(0.2)(0.8)\\&\doteq(0.8)(0.338)+(0.2)(0.8)=0.4304\end{aligned}$$

when $t\in(0.2,0.5]$,

$$UOWGA-I_{\mathscr{W}}(x,E_t)=\frac{3}{5}\cdot OWGA_{\mathbf{w}^{(3)}}\left(x|_{\{1,2,5\}}\right)+\left(1-\frac{3}{5}\right)\cdot GM_{\{3,4\}}\left(x|_{\{3,4\}}\right)$$

$$\begin{aligned}&=(0.6)\cdot OWGA_{\mathbf{w}^{(3)}}(0.7,0.3,0.5)+(0.4)\cdot GM_{\{3,4\}}(0.8,0.3)\\&=(0.6)\left(0.7^{0.1}\cdot 0.5^{0.3}\cdot 0.3^{0.6}\right)+(0.4)\left(0.8^{0.5}0.3^{0.5}\right)\\&\doteq(0.6)(0.381)\ +\ (0.4)(0.49)=0.4246\end{aligned}$$

when $t\in(0.5,0.8]$,

$$UOWGA-I_{\mathscr{W}}(x,E_t)=\frac{1}{5}\cdot OWGA_{\mathbf{w}^{(1)}}\left(x|_{\{2\}}\right)+\left(1-\frac{1}{5}\right)\cdot GM_{\{1,3,4,5\}}\left(x|_{\{1,3,4,5\}}\right)$$

$$\begin{aligned}&=(0.2)\cdot OWGA_{\mathbf{w}^{(1)}}(0.3)+(0.8)\cdot GM_{\{1,3,4,5\}}(0.7,0.8,0.3,0.5)\\&=(0.2)(0.3)+(0.8)(0.7\cdot 0.8\cdot 0.3\cdot 0.5)^{0.25}\\&\doteq(0.2)(0.3)+(0.8)(0.538)=0.4904\end{aligned}$$

when $t\in(0.8,1]$,

$$\begin{aligned}UOWGA-I_{\mathscr{W}}(x,E_t)&=\frac{0}{5}\cdot OWGA_{\mathbf{w}^{(0)}}\left(x|_{\varnothing}\right)+\left(1-\frac{0}{5}\right)\cdot GM_{[5]}\left(x|_{[5]}\right)\\&=GM_{[5]}\left(x|_{[5]}\right)=(0.7\cdot 0.3\cdot 0.8\cdot 0.3\cdot 0.5)^{0.2}\doteq 0.479\end{aligned}$$

Then,

$$P_x(BUIOWGA-I_{\mathscr{W}}((x,c)))=\int_0^1 UOWGA_{\mathscr{W}}(x,E_t)dt$$

$$\begin{aligned}&=\int_{\{0\}}UOWGA_{\mathscr{W}}(x,E_t)dt+\int_{(0,0.2]}UOWGA_{\mathscr{W}}(x,E_t)dt+\int_{(0.2,0.5]}UOWGA_{\mathscr{W}}(x,E_t)dt\\&\quad+\int_{(0.5,0.8]}UOWGA_{\mathscr{W}}(x,E_t)dt+\int_{(0.8,1]}UOWGA_{\mathscr{W}}(x,E_t)dt\end{aligned}$$

$$=0\cdot OWGA_{\mathbf{w}^{(5)}}\left(x|_{[5]}\right)+(0.2-0)(0.4304)+(0.5-0.2)(0.4246)+(0.8-0.5)(0.4904)+(1-0.8)(0.479)=0.45638$$

Finally, $BUIOWGA-I_{\mathscr{W}}((x,c))=(0.45638,0.4)$.

(II) The computation for $UOWGA-II_{\mathscr{W}}(x,E_t)$ and $BUIOWGA-II_{\mathscr{W}}((x,c))$:

when $t\in\{0\}$,

$$UOWGA-II_{\mathscr{W}}(x,E_t)=\left(OWGA_{\mathbf{w}^{(5)}}\left(x|_{[5]}\right)\right)^{\frac{5}{5}}\cdot\left(GM_{\varnothing}\left(x|_{\varnothing}\right)\right)^{\left(1-\frac{5}{5}\right)}$$

$$=OWGA_{\mathbf{w}^{(5)}}\left(x|_{[5]}\right)$$

when $t\in(0,0.2]$,

$$UOWGA-II_{\mathscr{W}}(x,E_t)=\left(OWGA_{\mathbf{w}^{(4)}}\left(x|_{\{1,2,4,5\}}\right)\right)^{\frac{4}{5}}\cdot\left(GM_{\{3\}}\left(x|_{\{3\}}\right)\right)^{\left(1-\frac{4}{5}\right)}$$

$$\begin{aligned}&=\left(OWGA_{\mathbf{w}^{(4)}}(0.7,0.3,0.3,0.5)\right)^{0.8}\cdot\left(GM_{\{3\}}(0.8)\right)^{0.2}\\&=\left(0.7^{0.05}\cdot 0.5^{0.15}\cdot 0.3^{0.3}\cdot 0.3^{0.5}\right)^{(0.8)}\cdot(0.8)^{(0.2)}\\&\doteq(0.338)^{(0.8)}\cdot(0.8)^{(0.2)}\doteq 0.402\end{aligned}$$

when $t \in (0.2, 0.5]$,

$$UOWGA-II_{\mathscr{W}}(x,E_t) = \left(OWGA_{\boldsymbol{w}^{(3)}}\left(x|_{\{1,2,5\}}\right)\right)^{\frac{3}{5}} \cdot \left(GM_{\{3,4\}}\left(x|_{\{3,4\}}\right)\right)^{\left(1-\frac{3}{5}\right)}$$

$$= \left(OWGA_{\boldsymbol{w}^{(3)}}(0.7,0.3,0.5)\right)^{(0.6)} \cdot \left(GM_{\{3,4\}}(0.8,0.3)\right)^{(0.4)}$$
$$= \left(0.7^{0.1} \cdot 0.5^{0.3} \cdot 0.3^{0.6}\right)^{(0.6)} \cdot \left(0.8^{0.5}0.3^{0.5}\right)^{(0.4)}$$
$$\doteq \left(0.381\right)^{(0.6)} \cdot (0.49)^{(0.4)} \doteq 0.421$$

when $t \in (0.5, 0.8]$,

$$UOWGA-II_{\mathscr{W}}(x,E_t) = \left(OWGA_{\boldsymbol{w}^{(1)}}\left(x|_{\{2\}}\right)\right)^{\frac{1}{5}} \cdot \left(GM_{\{1,3,4,5\}}\left(x|_{\{1,3,4,5\}}\right)\right)^{\left(1-\frac{1}{5}\right)}$$

$$= \left(OWGA_{\boldsymbol{w}^{(1)}}(0.3)\right)^{(0.2)} \cdot \left(GM_{\{1,3,4,5\}}(0.7,0.8,0.3,0.5)\right)^{(0.8)}$$
$$= (0.3)^{(0.2)} \cdot \left((0.7 \cdot 0.8 \cdot 0.3 \cdot 0.5)^{0.25}\right)^{(0.8)}$$
$$\doteq (0.3)^{(0.2)} \cdot (0.538)^{(0.8)} \doteq 0.479$$

when $t \in (0.8, 1]$,

$$UOWGA-II_{\mathscr{W}}(x,E_t) = \left(OWGA_{\boldsymbol{w}^{(0)}}\left(x|_{\varnothing}\right)\right)^{\frac{0}{5}} \cdot \left(GM_{[5]}\left(x|_{[5]}\right)\right)^{\left(1-\frac{0}{5}\right)}$$
$$= GM_{[5]}\left(x|_{[5]}\right) = (0.7 \cdot 0.3 \cdot 0.8 \cdot 0.3 \cdot 0.5)^{0.2} \doteq 0.479$$

Then,

$$P_x(BUIOWGA-II_{\mathscr{W}}((x,c))) = \int_0^1 UOWGA-II_{\mathscr{W}}(x,E_t)dt$$

$$= \int_{\{0\}} UOWGA-II_{\mathscr{W}}(x,E_t)dt + \int_{(0,0.2]} UOWGA-II_{\mathscr{W}}(x,E_t)dt + \int_{(0.2,0.5]} UOWGA-II_{\mathscr{W}}(x,E_t)dt + \int_{(0.5,0.8]} UOWGA-II_{\mathscr{W}}(x,E_t)dt + \int_{(0.8,1]} UOWGA-II_{\mathscr{W}}(x,E_t)dt$$

$$= 0 \cdot OWGA_{\boldsymbol{w}^{(5)}}\left(x|_{[5]}\right) + (0.2-0)(0.402) + (0.5-0.2)(0.421) + (0.8-0.5)(0.479) + (1-0.8)(0.479) = 0.4462$$

Finally, $BUIOWGA-II_{\mathscr{W}}((x,c)) = (0.4462, 0.4)$.

(III) The computation for $BUIOWGA-III_{\mathscr{W}}((x,c))$:

From Example 1, we know that $\boldsymbol{u}^{[5]} = (u_i^{[5]})_{i\in[5]} = (0.126, 0.256, 0.2, 0.24, 0.178)$. Then,

$$WGM_{\boldsymbol{u}^{[n]}}(x) = (0.7)^{0.126}(0.3)^{0.256}(0.8)^{0.2}(0.3)^{0.24}(0.5)^{0.178} \doteq 0.445$$

Finally, $BUIOWGA-III_{\mathscr{W}}((x,c)) = \left(WGM_{\boldsymbol{u}^{[n]}}(x), \int_0^1 AC(E_t)dt\right) = (0.445, 0.4)$.

4. Some propositions and relations about BUI OWGA operators

Some prepositions, majorly including various monotonicities, will firstly be discussed, and then some relations between BUI OWGA operators and BUI OWA operators, including some inequalities, are analyzed.

Proposition 1. *(Monotonicities for the value projection) For any family of OWA weight vectors having orness α,* $\mathscr{W} \ominus = \{\boldsymbol{w}^{(n)}\}_{n=1}^{\infty}$*, let* $(x_1, c), (x_2, c) \in \mathscr{B}^{[n]}$ *be any two BUI functions such that the associated value functions* $x_1, x_2 : [n] \to [0,1]$ *satisfy* $x_1 \leqslant x_2$ *(*$x_1(i) \leqslant x_2(i)$ *for all* $i \in [n]$*) and they have the same associated certainty function* $c : [n] \to [0,1]$*, then*

$$P_x(BUIOWGA-I_{\mathscr{W}}((x_1,c))) \leqslant P_x(BUIOWGA-I_{\mathscr{W}}((x_2,c)))$$

$$P_x(BUIOWGA-II_{\mathscr{W}}((x_1,c))) \leqslant P_x(BUIOWGA-II_{\mathscr{W}}((x_2,c))),$$

$$P_x(BUIOWGA - III_{\mathscr{W}}((x_1,c)))\leqslant P_x(BUIOWGA - III_{\mathscr{W}}((x_2,c))).$$

Proof (i) For each $t\in[0,1]$,

$$UOWGA - I_{\mathscr{W}}(x_1,E_t) = \frac{|E_t|}{n}\cdot OWGA_{\mathbf{w}^{(|E_t|)}}\left(x_1|_{E_t}\right) + \left(1-\frac{|E_t|}{n}\right)\cdot GM_{[n]\setminus E_t}\left(x_1|_{[n]\setminus E_t}\right)$$

$$\leqslant\frac{|E_t|}{n}\cdot OWGA_{\mathbf{w}^{(|E_t|)}}\left(x_2|_{E_t}\right) + \left(1-\frac{|E_t|}{n}\right)\cdot GM_{[n]\setminus E_t}\left(x_2|_{[n]\setminus E_t}\right) = UOWGA - I_{\mathscr{W}}(x_2,E_t)$$

Then,

$$P_x(BUIOWGA - I_{\mathscr{W}}((x_1,c))) = \int_0^1 UOWGA - I_{\mathscr{W}}(x_1,E_t)dt$$

$$\leqslant\int_0^1 UOWGA - I_{\mathscr{W}}(x_2,E_t)dt = P_x(BUIOWGA - I_{\mathscr{W}}((x_2,c)))$$

(ii) For each $t\in[0,1]$,

$$UOWGA - II_{\mathscr{W}}(x_1,E_t) = \left(OWGA_{\mathbf{w}^{(|E_t|)}}\left(x_1|_{E_t}\right)\right)^{\frac{|E_t|}{n}}\cdot\left(GM_{[n]\setminus E_t}\left(x_1|_{[n]\setminus E_t}\right)\right)^{1-\frac{|E_t|}{n}}$$

$$\leqslant\left(OWGA_{\mathbf{w}^{(|E_t|)}}\left(x_2|_{E_t}\right)\right)^{\frac{|E_t|}{n}}\cdot\left(GM_{[n]\setminus E_t}\left(x_2|_{[n]\setminus E_t}\right)\right)^{1-\frac{|E_t|}{n}}$$

Then,

$$P_x(BUIOWGA - II_{\mathscr{W}}((x_1,c))) = \int_0^1 UOWGA - II_{\mathscr{W}}(x_1,E_t)dt$$

$$\leqslant\int_0^1 UOWGA - II_{\mathscr{W}}(x_2,E_t)dt = P_x(BUIOWGA - II_{\mathscr{W}}((x_2,c)))$$

(iii)$P_x(BUIOWGA - III_{\mathscr{W}}((x_1,c))) = WGA_{\mathbf{u}^{[n]}}(x_1)\leqslant WGA_{\mathbf{u}^{[n]}}(x_2) = P_x(BUIOWGA - III_{\mathscr{W}}((x_2,c)))$. □

Proposition 2. *If the family of OWA weight vectors $\mathscr{W}\ominus=\{\mathbf{w}^{(n)}\}_{n=1}^{\infty}$ satisfies $\mathbf{w}^{(n)}=(1/n,\dots,1/n)$ for all $n\in\mathbb{N}$, then for any $(x,c)\in\mathscr{B}^{[n]}$,*

$$P_x(BUIOWGA - II_{\mathscr{W}}((x,c))) = P_x(BUIOWGA - III_{\mathscr{W}}((x,c))) = GM_{[n]}(x)$$

Proof (i)$P_x(BUIOWGA - II_{\mathscr{W}}((x,c))) = GM_{[n]}(x)$

For each $t\in[0,1]$, if $E_t\neq\varnothing$ and $[n]\setminus E_t\neq\varnothing$, then

$$UOWGA - II_{\mathscr{W}}(x,E_t) = \left(OWGA_{\mathbf{w}^{(|E_t|)}}\left(x|_{E_t}\right)\right)^{\frac{|E_t|}{n}}\cdot\left(GM_{[n]\setminus E_t}\left(x|_{[n]\setminus E_t}\right)\right)^{1-\frac{|E_t|}{n}}$$

$$=\left(\prod_{i\in E_t}\left(x|_{E_t}(i)\right)^{\frac{1}{|E_t|}}\right)^{\frac{|E_t|}{n}}\cdot\left(\prod_{i\in[n]\setminus E_t}\left(x|_{[n]\setminus E_t}(i)\right)^{\frac{1}{|[n]\setminus E_t|}}\right)^{1-\frac{|E_t|}{n}}$$

$$=\prod_{i\in E_t}\left(x|_{E_t}(i)\right)^{\frac{1}{n}}\cdot\prod_{i\in[n]\setminus E_t}\left(x|_{[n]\setminus E_t}(i)\right)^{\frac{1}{n}} = \prod_{i\in[n]}(x(i))^{\frac{1}{n}} = GM_{[n]}(x)$$

if $E_t=\varnothing$, then

$$UOWGA - II_{\mathscr{W}}(x,E_t) = \left(OWGA_{\mathbf{w}^{(|\varnothing|)}}\left(x|_{\varnothing}\right)\right)^{\frac{|\varnothing|}{n}}\cdot\left(GM_{[n]\setminus\varnothing}\left(x|_{[n]\setminus\varnothing}\right)\right)^{1-\frac{|\varnothing|}{n}} = 1\cdot\left(GM_{[n]}(x)\right)^1 = GM_{[n]}(x)$$

if $[n]\setminus E_t=\varnothing$, then

$$UOWGA - II_{\mathscr{W}}(x,E_t) = \left(OWGA_{\mathbf{w}^{(|E_t|)}}\left(x|_{E_t}\right)\right)^{\frac{|E_t|}{n}}\cdot\left(GM_{[n]\setminus E_t}\left(x|_{[n]\setminus E_t}\right)\right)^{1-\frac{|E_t|}{n}}$$

$$=\left(OWGA_{\mathbf{w}^{(|[n]|)}}\left(x|_{[n]}\right)\right)^{\frac{|[n]|}{n}}\cdot\left(GM_{\varnothing}\left(x|_{\varnothing}\right)\right)^{1-\frac{|[n]|}{n}} = OWGA_{\mathbf{w}^{(n)}}(x) = GM_{[n]}(x)$$

(ii)$P_x(BUIOWGA - III_{\mathscr{W}}((x,c))) = GM_{[n]}(x)$

For each $t \in [0,1]$, if $E_t \neq \varnothing$ and $[n]\backslash E_t \neq \varnothing$, the weighted vector $u_t : [n] \to [0,1]$ defined in Eq. (17) becomes such that

$u_t(i) = \frac{k_t}{n} \cdot w^{(|E_t|)}_{\sigma^{-1}_{E_t}(i)} = \frac{k_t}{n} \cdot \frac{1}{|E_t|} = \frac{1}{n}$ (when $i \in E_t$)

$u_t(i) = \frac{1}{n}$ (when $i \in [n]\backslash E_t$)

Therefore, $u_t : [n] \to [0,1]$ satisfies $u_t(i) = \frac{1}{n}$ for all $i \in [n]$.

If $E_t = \varnothing$, then weighted vector $u_t : [n] \to [0,1]$ defined in Eq. (18) satisfies $u_t(i) = \frac{1}{n}$ for all $i \in [n]$.

If $[n]\backslash E_t = \varnothing$, then weighted vector $u_t : [n] \to [0,1]$ defined in Eq. (19) also satisfies $u_t(i) = \frac{1}{n}$ for all $i \in [n]$.

Hence, the weight vector $\boldsymbol{u}^{[n]} = (u_i^{[n]})_{i \in [n]}$ defined in Eq. (20) satisfies $u_i^{[n]} = \int_0^1 u_t(i)dt = \int_0^1 \frac{1}{n} dt = \frac{1}{n}$. Therefore, $P_x(BUIOWGA - III_{\mathscr{W}}((x,c))) = WGM_{\boldsymbol{u}^{[n]}}(x) = GM_{[n]}(x)$. □

Proposition 3. *For any family of OWA weight vectors, $\mathscr{W}\ominus = \{\boldsymbol{w}^{(n)}\}_{n=1}^{\infty}$,*

(a) *if a BUI function $(x,c) \in \mathscr{B}^{[n]}$ satisfies $c(i) = 1$ for all $i \in [n]$, then*

$$BUIOWGA - I_{\mathscr{W}}((x,c)) = BUIOWGA - II_{\mathscr{W}}((x,c)) = BUIOWGA - III_{\mathscr{W}}((x,c)) = (OWGA_{\boldsymbol{w}^{(n)}}(x), 1)$$

where $\boldsymbol{w}^{(n)}$ is the n-th weight vector defined in the family $\mathscr{W}\ominus = \{\boldsymbol{w}^{(n)}\}_{n=1}^{\infty}$.

(b) *if a BUI function $(x,c) \in \mathscr{B}^n$ satisfies $c(i) = 0$ for all $i \in [n]$, then $BUIOWGA - I_{\mathscr{W}}((x,c)) = BUIOWGA - II_{\mathscr{W}}((x,c)) = BUIOWGA - III_{\mathscr{W}}((x,c)) = (GM_{[n]}(x), 0)$.*

(c) *(idempotency for the value projection) If a BUI function $(x,c) \in \mathscr{B}^{[n]}$ satisfies $x(i) = a$ for all $i \in [n]$, then*

$$P_x(BUIOWGA - I_{\mathscr{W}}((x,c))) = P_x(BUIOWGA - II_{\mathscr{W}}((x,c))) = P_x(BUIOWGA - III_{\mathscr{W}}((x,c))) = a$$

Proof (a) On the one hand, for each $t \in [0,1]$ we have $E_t = [n]$ and therefore

$$UOWGA - I_{\mathscr{W}}(x, E_t) = \frac{|E_t|}{n} \cdot OWGA_{\boldsymbol{w}^{(|E_t|)}}\left(x|_{E_t}\right) + \left(1 - \frac{|E_t|}{n}\right) \cdot GM_{[n]\backslash E_t}\left(x|_{[n]\backslash E_t}\right)$$

$$= \frac{n}{n} \cdot OWGA_{\boldsymbol{w}^{(n)}}\left(x|_{[n]}\right) + \left(1 - \frac{n}{n}\right) \cdot GM_{\varnothing}\left(x|_{\varnothing}\right) = OWGA_{\boldsymbol{w}^{(n)}}(x)$$

$$UOWGA - II_{\mathscr{W}}(x, E_t) = \left(OWGA_{\boldsymbol{w}^{(|E_t|)}}\left(x|_{E_t}\right)\right)^{\frac{|E_t|}{n}} \cdot \left(GM_{[n]\backslash E_t}\left(x|_{[n]\backslash E_t}\right)\right)^{1-\frac{|E_t|}{n}}$$

$$= \left(OWGA_{\boldsymbol{w}^{(n)}}\left(x|_{[n]}\right)\right)^{\frac{n}{n}} \cdot \left(GM_{\varnothing}\left(x|_{\varnothing}\right)\right)^{0} = OWGA_{\boldsymbol{w}^{(n)}}(x)$$

Therefore, $P_x(BUIOWGA - I_{\mathscr{W}}((x,c))) = P_x(BUIOWGA - II_{\mathscr{W}}((x,c))) = OWGA_{\boldsymbol{w}^{(n)}}(x)$.

Since $u_t(i) = w^{(n)}_{\sigma^{-1}_{[n]}(i)}$ $(i \in [n])$ then $u_i^{[n]} = \int_0^1 u_t(i)dt = w^{(n)}_{\sigma^{-1}_{[n]}(i)}$ $(i \in [n])$. Therefore,

$$P_x(BUIOWGA - III_{\mathscr{W}}((x,c))) = WGM_{\boldsymbol{u}^{[n]}}(x) = OWGA_{\boldsymbol{w}^{(n)}}(x)$$

On the other hand,

$$P_c(BUIOWGA - I_{\mathscr{W}}((x,c))) = P_c(BUIOWGA - II_{\mathscr{W}}((x,c)))$$

$$= P_c(BUIOWGA - III_{\mathscr{W}}((x,c))) = \int_0^1 AC(E_t)dt = \int_0^1 \frac{|E_t|}{n} dt = \int_0^1 \frac{n}{n} dt = 1$$

(b) can be easily checked in some similar analyses as those used in (a).

(c) is apparent because OWGA, GM and WGM are all idempotent. □

Proposition 4. *Let $\mathscr{W}\ominus = \{\boldsymbol{w}^{(n)}\}_{n=1}^{\infty}$ ($\boldsymbol{w}^{(n)} = (w_i^{(n)})_{i=1}^{n}$) be a (parameterized) family of OWA weight vectors having orness α and satisfying $\boldsymbol{w}^{(n)} = (w_i^{(n)})_{i=1}^{n} \in (0,1]^n$ for all $n \in \mathbb{N}$. For a BUI function $(x,c) \in \mathscr{B}^{[n]}$, if there is some $i \in [n]$ such that $(x(i), c(i)) = (0,1)$, then*

$$P_x(BUIOWGA - II_{\mathscr{W}}((x,c))) = P_x(BUIOWGA - III_{\mathscr{W}}((x,c))) = 0$$

Proof For each $t \in [0,1]$,

$$UOWGA - II_{\mathscr{W}}(x, E_t) = \left(OWGA_{\boldsymbol{w}^{(|E_t|)}}\left(x|_{E_t}\right)\right)^{\frac{|E_t|}{n}} \cdot \left(GM_{[n]\backslash E_t}\left(x|_{[n]\backslash E_t}\right)\right)^{1-\frac{|E_t|}{n}}$$

Note that 0 is the annihilator of both GM and OWGA (when the associated OWA weighted vector is positive), then it is easy to check that $UOWGA - II_{\mathscr{W}}(x, E_t) = 0$ irrespective of the cardinality of E_t. Therefore, $P_x(BUIOWGA - II_{\mathscr{W}}((x,c))) = 0$.

Since $w^{(n)} = (w_i^{(n)})_{i=1}^{n} \in (0,1]^n$, then clearly $u^{[n]} \in (0,1]^{[n]}$. Note that 0 is also the annihilator of WGM (when the associated weight vector is positive), then.

$P_x(BUIOWGA - III_{\mathscr{W}}((x,c))) = WGM_{u^{[n]}}(x) = 0.$ □

Remark Proposition 4 shows a useful property that holds for BUI OWGA type II and III operators but not for BUI OWGA type I and BUI OWA operators. In some strict and conservative decision environments, if there are some individual inputs that take 0 value, then the overall aggregation result will be 0; this can both embody "average type" aggregation and model "one-vote veto".

We next show three inequalities concerning BUI OWGA operators and BUI OWA operators. First, we recall that a real-valued function $\varphi : (a,b) \to (-\infty,+\infty)$ is said to be convex (concave) if and only if for any finite number of values $x_1, x_2, ..., x_n \in (a,b)$ and any finite number of values $\lambda_1, \lambda_2, ..., \lambda_n \in [0,1]$ satisfying $\sum_{i=1}^{n} \lambda_i = 1$,

$$\varphi\left(\sum_{i=1}^{n} \lambda_i x_i\right) \leqslant \sum_{i=1}^{n} \lambda_i \varphi(x_i) \varphi\left(\sum_{i=1}^{n} \lambda_i x_i\right) \geqslant \sum_{i=1}^{n} \lambda_i \varphi(x_i) \quad (27)$$

Lemma 1. *For any finite set E ($E \neq \varnothing$) and any normalized weight function/vector $w^E = (w_i^E)_{i \in E}$, let $WM_{w^E} : [0,1]^E \to [0,1]$ be the weighted mean with w^E and $WGM_{w^E} : [0,1]^E \to [0,1]$ be the weighted geometric mean with w^E. Then for any $x \in [0,1]^{[n]}$,*

$$WGM_{w^E}(x) \leqslant WM_{w^E}(x)$$

Proof For $x \in [0,1]^{[n]}$ and $w^E = (w_i^E)_{i \in E}$, define three disjoint subsets of $[n]$: $S_+ = \{k \in [n] : x(k) > 0\}$, $S_{00} = \{k \in [n] : x(k) = w_k^E = 0\}$, and $S_{0+} = \{k \in [n] : x(k) = 0, w_k^E > 0\}$. Note that $E = S_+ \cup S_{00} \cup S_{0+}$, $S_{00} \neq [n]$, and each one of these three sets can possibly be empty set.

If $S_{0+} \neq \varnothing$, then there exist $k \in S_{0+}$ for which $(x(k))^{w_k^E} = 0^{w_k^E} = 0$, and therefore $WGM_{w^E}(x) = 0 \leqslant WM_{w^E}(x)$.

If $S_{0+} = \varnothing$, with the conventions $0^0 \triangleq 1$ and $\sum_{i \in \varnothing} w_i^E x(i) \triangleq 0$, we have

$$WGM_{w^E}(x) = \prod_{i \in E} (x(i))^{w_i^E} = \prod_{i \in S_{00}} (x(i))^{w_i^E} \prod_{i \in E \setminus S_{00}} (x(i))^{w_i^E} = \prod_{i \in E \setminus S_{00}} (x(i))^{w_i^E} = \prod_{i \in S_+} (x(i))^{w_i^E}$$

and

$$WM_{w^E}(x) = \sum_{i \in E} w_i^E x(i) = \sum_{i \in S_{00}} w_i^E x(i) + \sum_{i \in E \setminus S_{00}} w_i^E x(i) = \sum_{i \in E \setminus S_{00}} w_i^E x(i) = \sum_{i \in S_+} w_i^E x(i).$$

Since $\ln : (0,\infty) \to (-\infty,\infty)$ is concave and $f(x) = e^x$ is monotonic increasing on $(-\infty,\infty)$, then

$$WGM_{w^E}(x) = \prod_{i \in S_+} (x(i))^{w_i^E} = \exp\left(\ln \prod_{i \in S_+} (x(i))^{w_i^E}\right) = \exp\left(\sum_{i \in S_+} w_i^E \ln(x(i))\right)$$

$\leqslant \exp\left(\ln\left(\sum_{i \in S_+} w_i^E x(i)\right)\right) = \sum_{i \in S_+} w_i^E x(i) = WM_{w^E}(x).$ □

Corollary 1. *For any finite set E ($E \neq \varnothing$), let $AM_E : [0,1]^E \to [0,1]$ be the arithmetic mean and $GM_E : [0,1]^E \to [0,1]$ be the geometric mean. Then for any $x \in [0,1]^{[n]}$,*

$$GM_E(x) \leqslant AM_E(x)$$

Corollary 2. *For any finite set E ($E \neq \varnothing$) and any OWA weight vector $w^{(|E|)} = w^{(k)} = (w_i^{(k)})_{i=1}^{k}$, let $OWA_{w^{(k)}} : [0,1]^E \to [0,1]$ be the OWA operator with $w^{(k)}$ and $OWGA_{w^{(k)}} : [0,1]^E \to [0,1]$ be the OWGA operator with w^E. Then for any $x \in [0,1]^{[n]}$,*

$$OWGA_{w^{(k)}}(x) \leqslant OWA_{w^{(k)}}(x)$$

Proof Let $\sigma_E : [k] \to E$ be an appropriate bijection such that $x(\sigma_E(i)) \geqslant x(\sigma_E(j))$ whenever $1 \leqslant i < j \leqslant k$. Then, with the same σ_E and $v = (v_i)_{i \in E}$ satisfying $v_i = w^{(|E|)}_{\sigma_E^{-1}(i)}$, the WE expression of $OWA_{w^{(k)}}$ is $OWA_{w^{(|E|)}}(x) = \sum_{i \in E} w^{(|E|)}_{\sigma_E^{-1}(i)} x(i) = \sum_{i \in E} v_i x(i) = WM_v(x)$, and the WGM expression of $OWGA_{w^E}$ is $OWGA_{w^{(|E|)}}(x) = \prod_{i \in E} x(i)^{w^{(|E|)}_{\sigma_E^{-1}(i)}} = \prod_{i \in E} (x(i))^{v_i} = WGM_v(x)$. By Lemma 1, we know $WGM_v(x) \leqslant WM_v(x)$ and therefore $OWGA_{w^E}(x) \leqslant OWA_{w^E}(x)$. □

Proposition 5. *For any family of OWA weight vectors having orness α, $\mathscr{W} \ominus = \{w^{(n)}\}_{n=1}^{\infty}$, and any $(x,c) \in \mathscr{B}^{[n]}$, $P_x(BUIOWGA - I_{\mathscr{W}}((x,c))) \leqslant P_x(BUIOWA_{\mathscr{W}}((x,c)))$.*

Proof For any $(x,c) \in \mathscr{B}^{[n]}$ and any $t \in [0,1]$, recall $E_t = \{k \in [n] : c(k) \geqslant t\}$. If both $E_t \neq \varnothing$ and $E_t \neq [n]$ hold, then by Corollary 2 and Corollary 1,

$$UOWGA-I_{\mathscr{W}}(x,E_t)=\frac{|E_t|}{n}\cdot OWGA_{\mathbf{w}^{(|E_t|)}}\left(x|_{E_t}\right)+\left(1-\frac{|E_t|}{n}\right)\cdot GM_{[n]\backslash E_t}\left(x|_{[n]\backslash E_t}\right)$$

$$\leqslant\frac{|E_t|}{n}\cdot OWA_{\mathbf{w}^{(|E_t|)}}\left(x|_{E_t}\right)+\left(1-\frac{|E_t|}{n}\right)\cdot AM_{[n]\backslash E_t}\left(x|_{[n]\backslash E_t}\right)=UOWA_{\mathscr{W}\ominus}(x,E_t)$$

If $E_t=\varnothing$, then by Corollary 1,

$$UOWGA-I_{\mathscr{W}}(x,E_t)=GM_{[n]}(x)\leqslant AM_{[n]}(x)=UOWA_{\mathscr{W}\ominus}(x,E_t)$$

If $E_t=[n]$, then by Corollary 2,

$$UOWGA-I_{\mathscr{W}}(x,E_t)=OWGA_{\mathbf{w}^{(n)}}(x)\leqslant OWA_{\mathbf{w}^{(n)}}(x)=UOWA_{\mathscr{W}\ominus}(x,E_t)$$

Therefore,

$$P_x(BUIOWGA-I_{\mathscr{W}}((x,c)))=\int_0^1 UOWGA-I_{\mathscr{W}}(x,E_t)dt$$

$\leqslant\int_0^1 UOWA_{\mathscr{W}}(x,E_t)dt=P_x(BUIOWA_{\mathscr{W}}((x,c)))$. □

Proposition 6. *For any family of OWA weight vectors having orness* α, $\mathscr{W}\ominus=\{\mathbf{w}^{(n)}\}_{n=1}^{\infty}$, *and any* $(x,c)\in\mathscr{B}^{[n]}$, $P_x(BUIOWGA-II_{\mathscr{W}}((x,c)))\leqslant P_x(BUIOWGA-I_{\mathscr{W}}((x,c)))$.

Proof For any $(x,c)\in\mathscr{B}^{[n]}$ and any $t\in[0,1]$, recall $E_t=\{k\in[n]:c(k)\geqslant t\}$. If both $E_t\neq\varnothing$ and $E_t\neq[n]$ hold, then by Corollary 1,

$$UOWGA-II_{\mathscr{W}}(x,E_t)=\left(OWGA_{\mathbf{w}^{(|E_t|)}}\left(x|_{E_t}\right)\right)^{\frac{|E_t|}{n}}\cdot\left(GM_{[n]\backslash E_t}\left(x|_{[n]\backslash E_t}\right)\right)^{1-\frac{|E_t|}{n}}$$

$$\leqslant\frac{|E_t|}{n}\cdot OWGA_{\mathbf{w}^{(|E_t|)}}\left(x|_{E_t}\right)+\left(1-\frac{|E_t|}{n}\right)\cdot GM_{[n]\backslash E_t}\left(x|_{[n]\backslash E_t}\right)=UOWGA-I_{\mathscr{W}}(x,E_t).$$

If $E_t=\varnothing$, then,

$$UOWGA-II_{\mathscr{W}}(x,E_t)=GM_{[n]}(x)=UOWGA-I_{\mathscr{W}}(x,E_t)$$

If $E_t=[n]$, then,

$$UOWGA-II_{\mathscr{W}}(x,E_t)=OWGA_{\mathbf{w}^{(n)}}(x)=UOWGA-I_{\mathscr{W}}(x,E_t)$$

Therefore,

$$P_x(BUIOWGA-II_{\mathscr{W}}((x,c)))=\int_0^1 UOWGA-II_{\mathscr{W}}(x,E_t)dt$$

$\leqslant\int_0^1 UOWGA-I_{\mathscr{W}}(x,E_t)dt=P_x(BUIOWGA-I_{\mathscr{W}}((x,c)))$. □

Lemma 2. ((*Jensen's Inequality*) [34]) *Let* φ *be a convex function on* $(-\infty,\infty)$, f *an integrable function over* $[0,1]$, *and* $\varphi^\circ f$ *also integrable over [0,1]. Then*

$$\varphi\left(\int_0^1 f(t)dt\right)\leqslant\int_0^1(\varphi^\circ f)(t)dt$$

Proposition 7. For any family of OWA weight vectors having orness α, $\mathscr{W}\ominus=\{\mathbf{w}^{(n)}\}_{n=1}^{\infty}$, and any $(x,\ c)\in\mathscr{B}^{[n]}$, $P_x(BUIOWGA-III_{\mathscr{W}}((x,c)))\leqslant P_x(BUIOWGA-II_{\mathscr{W}}((x,c)))$.

Proof For each $t\in[0,1]$, let $|E_t|=k_t$. When $E_t\neq\varnothing$, let $\sigma_{E_t}:[k_t]\rightarrow E_t$ be an appropriate bijection such that $x(\sigma_{E_t}(i))\geqslant x(\sigma_{E_t}(j))$ whenever $1\leqslant i<j\leqslant k_t$.

We consider the following three situations:

(i) if both $E_t\neq\varnothing$ and $[n]\backslash E_t\neq\varnothing$ hold, then we have

$$UOWGA-II_{\mathscr{W}}(x,E_t)=\left(OWGA_{\mathbf{w}^{(|E_t|)}}\left(x|_{E_t}\right)\right)^{\frac{|E_t|}{n}}\cdot\left(GM_{[n]\backslash E_t}\left(x|_{[n]\backslash E_t}\right)\right)^{1-\frac{|E_t|}{n}}$$

$$= \left(\prod_{i \in E_t} (x|_{E_t}(i))^{w^{(k_t)}_{\sigma^{-1}_{E_t}(i)}} \right)^{\frac{k_t}{n}} \cdot \left(\prod_{i \in [n] \setminus E_t} \left(x|_{[n]\setminus E_t}(i) \right)^{\frac{1}{n-k_t}} \right)^{\left(1-\frac{k_t}{n}\right)}$$

$$= \prod_{i \in E_t} (x|_{E_t}(i))^{\frac{k_t}{n} \cdot w^{(k_t)}_{\sigma^{-1}_{E_t}(i)}} \cdot \prod_{i \in [n] \setminus E_t} \left(x|_{[n]\setminus E_t}(i) \right)^{\frac{1}{n}}$$

$$= \prod_{i \in E_t \cup ([n] \setminus E_t)} (x(i))^{u_t(i)} = \prod_{i \in [n]} (x(i))^{u_t(i)}$$

where $u_t : [n] \to [0, 1]$ is the normalized weight vector defined in Eq. (17).

(ii) if $E_t = \emptyset$, then

$$UOWGA - II_{\mathscr{W}}(x, E_t) = GM_{[n]}\left(x|_{[n]}\right) = \prod_{i \in [n]} \left(x|_{[n]}(i) \right)^{\frac{1}{n}} = \prod_{i \in [n]} (x(i))^{u_t(i)}$$

where $u_t : [n] \to [0, 1]$ is the normalized weight vector defined in Eq. (18).

(iii) if $[n] \setminus E_t = \emptyset$ (i.e., $E_t = [n]$), then

$$UOWGA_{\mathscr{W}} - II(x, E_t) = OWGA_{\boldsymbol{w}^{(n)}}\left(x|_{E_t}\right) = \prod_{i \in [n]} \left(x|_{[n]}(i) \right)^{w^{(n)}_{\sigma^{-1}_{[n]}(i)}} = \prod_{i \in [n]} (x(i))^{u_t(i)}$$

where u_t is the normalized weight vector defined in Eq. (19).

That is, $\boldsymbol{u}^{[n]} = (u_i^{[n]})_{i \in [n]}$ such that $u_i^{[n]} = \int_0^1 u_t(i)dt$ is the one defined by Eq. (20).

Then, for $(x, c) \in \mathscr{B}^{[n]}$ ($x \in [0, 1]^{[n]}$) and $\boldsymbol{u}^{[n]} = (u_i^{[n]})_{i \in [n]}$, define three disjoint subsets of $[n]$: $S_+ = \{k \in [n] : x(k) > 0\}$, $S_{00} = \{k \in [n] : x(k) = u_k^{[n]} = 0\}$, and $S_{0+} = \{k \in [n] : x_k = 0, u_k^{[n]} > 0\}$. Note that $E = S_+ \cup S_{00} \cup S_{0+}$, $S_{00} \neq [n]$, and each of the three defined sets can possibly be empty set.

If $S_{0+} \neq \emptyset$, then there exists $k \in S_{0+}$ for which $(x(k))^{u_k^{[n]}} = 0^{u_k^{[n]}} = 0$, and therefore $P_x(BUIOWGA - III_{\mathscr{W}}((x, c))) = WGA_{\boldsymbol{u}^{[n]}}(x) = 0 \leqslant P_x(BUIOWGA - II_{\mathscr{W}}((x, c)))$.

If $S_{0+} = \emptyset$, then, on one hand, we have

$$P_x(BUIOWGA - III_{\mathscr{W}}((x, c))) = WGA_{\boldsymbol{u}^{[n]}}(x) = \prod_{i \in [n]} (x(i))^{u_i^{[n]}}$$

$$= \prod_{i \in S_{00}} (x(i))^{u_i^{[n]}} \cdot \prod_{i \in [n] \setminus S_{00}} (x(i))^{u_i^{[n]}} = \prod_{i \in [n] \setminus S_{00}} (x(i))^{u_i^{[n]}}$$

on the other hand, note that for each possible $j \in S_{00}$, we must have $u_j^{[n]} = \int_0^1 u_t(j)dt = 0$ which immediately implies $u_t(j)$ (a function of t) vanishes on a subset $K \subseteq [0, 1]$ such that its Lebesgue measure $m(K) = 1$. Therefore,

$$P_x(BUIOWGA - II_{\mathscr{W}}((x, c))) = \int_0^1 UOWGA - II_{\mathscr{W}}(x, E_t)dt = \int_0^1 \prod_{i \in [n]} (x(i))^{u_t(i)} dt$$

$$= \int_K \prod_{i \in [n]} (x(i))^{u_t(i)} dt = \int_K \prod_{i \in [n] \setminus S_{00}} (x(i))^{u_t(i)} dt = \int_0^1 \prod_{i \in [n] \setminus S_{00}} (x(i))^{u_t(i)} dt$$

Note that e^t is convex on $(-\infty, \infty)$ and $\ln\left(\prod_{i \in [n] \setminus S_{00}} (x(i))^{u_t(i)}\right)$ (a function of t) is bounded. Then, by Lemma 2 we have

$$P_x(BUIOWGA - II_{\mathscr{W}}((x, c))) = \int_0^1 \prod_{i \in [n] \setminus S_{00}} (x(i))^{u_t(i)} dt$$

$$= \int_0^1 \exp\left(\ln\left(\prod_{i \in [n] \setminus S_{00}} (x(i))^{u_t(i)} \right) \right) dt = \int_0^1 \exp\left(\sum_{i \in [n] \setminus S_{00}} u_t(i) \cdot \ln(x(i)) \right) dt$$

$$\geqslant \exp\left(\int_0^1 \left(\sum_{i\in[n]\setminus S_{00}} u_t(i)\cdot \ln(x(i))\right)dt\right) = \exp\left(\sum_{i\in[n]\setminus S_{00}} \left(\left(\int_0^1 u_t(i)dt\right)\cdot \ln(x(i))\right)\right)$$

$$= \exp\left(\sum_{i\in[n]\setminus S_{00}} \left(u_i^{[n]}\cdot \ln(x(i))\right)\right) = \exp\left(\ln\left(\prod_{i\in[n]\setminus S_{00}} (x(i))^{u_i^{[n]}}\right)\right) = \prod_{i\in[n]\setminus S_{00}} (x(i))^{u_i^{[n]}}$$

$= P_x(BUIOWGA - III_{\mathscr{W}}((x,c)))$. □

5. BUI OWGA operators in uncertain decision-making problem

This section presents a decision-making process with BUI inputs and bipolar optimism–pessimism preferences to show the usefulness of BUI OWGA operators.

Suppose that a company needs to make a decision about whether or not to advertise for one product, which depends on the predicted repurchase rate of this product in the next season. The predictions are given by $n = 5$ internal experts of this company, denoted by $\{ept_i\}_{i\in[5]}$, and the predictions can be with uncertainty. Assume a BUI granule $(x(i), c(i))$ is given by each expert ept_i in which $100x(i)$% represents the repurchase rate that ept_i predicts and $c(i)$ is his/her certainty about the prediction. Then, a BUI input function is obtained with $(x, c) = ((x(i), c(i)))_{i\in[5]} \in \mathscr{B}^{[5]}$ as presented in Example 1 and Example 2.

Assume the management team is with a moderate pessimistic attitude which can be modeled with the family of OWA weight vectors having orness $\alpha = 0.25$, $\mathscr{W}_{\ominus} = \{w^{(s)}\}_{s=1}^{\infty}$, as given in Example 1 and Example 2. And suppose they believe that if there is a $r \in [5]$ such that the individual uncertain prediction $(x(r), c(r)) = (0, 1)$, which means ept_r fully believes (i.e., with the maximum certainty) that the repurchase rate will drop to zero, then the overall prediction should be expressed by a BUI granule $(0, q)$ where $q \in [0, 1]$, indicating the overall prediction for the repurchase rate is zero with more or less uncertainty. In this decision style, it is clearly more suitable to use BUI OWGA operators than BUI OWA operators as analyzed earlier. Then, by applying BUI OWGA operators to (x, c), an aggregation result (x^*, c^*) (which is also a BUI granule) can be suggested to the management team as an overall evaluation which embodies both its optimism–pessimism preference and decision style in BUI environment.

With this result, using rule-based decision-making [35–37], some evaluation thresholds can be preset. Afterwards, the aggregation result (x^*, c^*) can be compared with these thresholds, and the company will advertise the product to some different extents according to which one of the following rules the result (x^*, c^*) (the overall prediction of repurchase rate of the product and its certainty) satisfies.

In practice, decision-makers can preset various types of decision rules and thresholds according to the detailed problems. For instance, we choose the thresholds and decision rules presented below. (Rule 1) Let $x_H > 0$ represents a high repurchase rate and $c_S > 0$ is the acceptable level for the certainty degree. If both the predicted repurchase rate (x^*) and its associated certainty degree (c^*) are high enough to meet the preset levels (that is, if $x^* \geqslant x_H > 0$ and $c^* \geqslant c_S > 0$ are both true), then the decision should be not to advertise the product because they have enough confidence that the repurchase is high.

(Rule 2) Let x_L ($0 < x_L \leqslant x_H$) represent a low repurchase rate. If the predicted repurchase rate (x^*) is smaller than the low level and its associated certainty degree (c^*) is high enough to meet that satisfactory levels (that is, if both $x^* < x_L \leqslant x_H$ and $c^* \geqslant c_S$ hold), then the decision should be to spend heavily to advertise the product because the management team has enough confidence that the repurchase will be low.

(Rule 3) If all the conditions in the above two rules are not fulfilled, then the decision is to spend moderately to advertise the product because there is a lack of certainty for both a high repurchase rate prediction and a low repurchase rate prediction, and therefore a safe decision is to moderately advertise.

Suppose we preset $x_H = 0.4$, $x_L = 0.2$ and $c_H = 0.5$, and employ BUI OWGA type II operator. Then we obtain $BUIOWGA - II_{\mathscr{W}}((x,c)) = (x^*, c^*) = (0.4462, 0.4)$. By simple comparison, according to Rule 3, the decision is suggested to spend moderately to advertise the product.

6. Conclusions

The BUIOWA operators have proven to be a successful implementation of OWA operators in the BUI environment. In addition, the OWGA operators play a crucial role as counterparts to OWA operators, and our proposed BUIOWGA serves as an ideal form of OWGA operators specifically designed for the BUI environment.

We have defined and constructed three types BUIOWGA operators, namely BUIOWGA type I, BUIOWGA type II and BUIOWGA type III. The three defined operators are based on three well-defined uncertain ordered weighted geometric averaging operators, UOWGA-I, UOWGA-II and UOWGA-III, respectively. The construction method employs the methodology of factorization-integration. Hence, every type of UOWGA is based on a finite set E and then every type of BUIOWGA is therefore constructed via integral method. The results of the three types of BUIOWGA operators are all BUI values in which the certainty degrees are the same while the value parts are different from each other. We also elaborate the computation methods for the all three types of BUIOWGA operators in comparison to the computation procedure of BUIOWA operator.

Some monotonicities about the proposed operators were given and proved. We showed that there are some interrelations between the three types of BUIOWGA operators and BUIOWA operator. Concretely, for any BUI inputs (x, c), we have

$$P_x(BUIOWGA - III_{\mathscr{W}}((x,c)))\leqslant P_x(BUIOWGA - II_{\mathscr{W}}((x,c)))$$

$$\leqslant P_x(BUIOWGA - I_{\mathscr{W}}((x,c)))\leqslant P_x(BUIOWA((x,c))).$$

It is also shown that if the BUI granule $(0,1)$ is in the BUI inputs and the given family of OWA weight vectors contains only positive weight vectors, then the value projection of the resulting BUI granule is zero, which shows some similar elements to the annihilator of geometric mean (with positive weights). Therefore, the proposed BUIOWGA operators effectively capture the bipolar optimism–pessimism preference in BUI environment and possess unique properties that can be advantageous in specific decision scenarios. Consequently, both BUIOWA and BUIOWGA exhibit their own merits and limitations depending on practical application contexts, thus enabling them to complement each other when implemented. To demonstrate the computational feasibility and applicability of our proposals, we provide numerical examples as well as an application involving the prediction of product repurchase rates.

Further research directions could involve exploring the application of diverse forms of BUIOWGA in various scenarios, such as taking into account decision makers' level of tolerance. Additionally, further theoretical investigation may concentrate on the OWA and OWGA operators in generalized versions of BUI, for instance, basic uncertain linguistic information (BULI) [39,40].

CRediT authorship contribution statement

LeSheng Jin: Writing – review & editing, Writing – original draft, Visualization, Validation, Resources, Methodology, Investigation, Formal analysis, Conceptualization. **Radko Mesiar:** Writing – review & editing, Validation, Supervision, Methodology, Funding acquisition, Formal analysis. **Tapan Senapati:** Validation, Investigation, Formal analysis. **Chiranjibe Jana:** Validation, Resources. **Chao Ma:** Writing – review & editing, Formal analysis, Supervision, Resources. **Diego García-Zamora:** Writing – review & editing, Formal analysis, Supervision. **Ronald R. Yager:** Validation, Supervision.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Acknowledgment

This work is partly supported by grant: VEGA 1/0036/23, and Hubei Superior and Distinctive.
Discipline Group of "New Energy Vehicle and Smart Transportation" (Grant No. XKTD052323).

References

[1] M. Grabisch, J.L. Marichal, R. Mesiar, E. Pap, Aggregation Functions, Cambridge University Press, Cambridge, 2009. ISBN:1107013429.
[2] Z.-S. Chen, X. Zhang, R.M. Rodriguez, W. Pedrycz, L. Martinez, M.J. Skibniewski, Expertise-structure and risk-appetite-integrated two-tiered collective opinion generation framework for large scale group decision making, IEEE Trans. Fuzzy Syst. 30 (12) (2022) 5496–5510.
[3] Z.-Q. Wang, Z.-S. Chen, H. Garg, Y. Pu, K.-S. Chin, An integrated quality-function-deployment and stochastic-dominance-based decision-making approach for prioritizing product concept alternatives, Complex Intell. Syst. 8 (2022) 2541–2556.
[4] L. Jin, R. Mesiar, R.R. Yager, M. Kalina, J. Špirková, S. Borkotokey, Deriving efficacy from basic uncertain information and uncertain Choquet integral, Int. J. Gen Syst 52 (1) (2023) 72–85.
[5] L. Jin, Some consistency properties and individual preference monotonicity for weighted aggregation operators, IEEE Trans. Fuzzy Syst. 30 (6) (2022) 2113–2117.
[6] R.R. Yager, On ordered weighted averaging aggregation operators in multicriteria decision making, IEEE Trans. Syst. Man Cybern. 18 (1) (1988) 183–190.
[7] Z. Xu, Q. Da, The ordered weighted geometric averaging operator, Int. J. Intell. Syst. 17 (2002) 709–716.
[8] G. Choquet, Theory of capacities, Ann. Inst. Fourier 5 (1954) 131–295.
[9] Y. Ouyang, On fuzzy implications determined by aggregation operators, Inf. Sci. 193 (15) (2012) 153–162.
[10] M. Sugeno, Theory of fuzzy Integrals and Its Applications, Tokyo Institute of Technology, 1974 (Ph.D. thesis).
[11] R.R. Yager, J. Kacprzyk, G. Beliakov, Recent Developments on the Ordered Weighted Averaging Operators: Theory and Practice, Springer-Verlag, Berlin, 2011.
[12] J. Špirková, Weighted operators based on dissimilarity function, Inf. Sci. 281 (2014) 172–181.
[13] M. Boczek, L. Jin, M. Kaluszka, Interval-valued seminormed fuzzy operators based on admissible orders, Inf. Sci. 574 (2021) 96–110.
[14] M. Boczek, M. Kaluszka, On the extended Choquet-Sugeno-like operator, Int. J. Approx. Reason. 154 (2023) 48–55.
[15] M. Boczek, L. Jin, M. Kaluszka, The interval-valued Choquet-Sugeno-like operator as a tool for aggregation of interval-valued functions, Fuzzy Set. Syst. 448 (2022) 35–48.
[16] L. Jin, M. Kalina, R. Mesiar, S. Borkotokey, Certainty aggregation and the certainty fuzzy measures, Int. J. Intell. Syst. 33 (4) (2018) 759–770.
[17] R. Mesiar, S. Borkotokey, L. Jin, M. Kalina, Aggregation under uncertainty, IEEE Trans. Fuzzy Syst. 26 (4) (2018) 2475–2478.
[18] K.T. Atanassov, Intuitionistic fuzzy sets, Fuzzy Set. Syst. 20 (1) (1986) 87–96.
[19] W. Gau, D. Buehrer, Vague sets, IEEE Trans. Syst. Man Cybern. 23 (2) (1993) 610–614.
[20] G. Li, X. Zhang, R.R. Yager, R. Mesiar, H. Bustince, L. Jin, Comprehensive rules-based and preferences induced weights allocation in group decision making with BUI, Int. J. Comput. Intell. Syst. 15 (2022) 54, https://doi.org/10.1007/s44196-022-00116-2.
[21] Y.-Q. Xu, L. Jin, Z.-S. Chen, R.R. Yager, J. Špirková, M. Kalina, S. Borkotokey, Weight vector generation in multi-criteria decision making with basic uncertain information, Mathematics 10 (4) (2022) 572.

[22] Z.-S. Chen, et al., Sustainable building material selection: an integrated multi-criteria large group decision making framework, Appl. Soft Comput. 113 (2021) 107903.
[23] Z.-S. Chen, X. Zhang, W. Pedrycz, X.J. Wang, M.J. Skibniewski, Bid evaluation in civil construction under uncertainty: a two-stage LSP-ELECTRE III-based approach, Eng. Appl. Artif. Intel. 94 (2020) 103835.
[24] Z. Tao, Z. Shao, J. Liu, L. Zhou, H. Chen, Basic uncertain information soft set and its application to multi-criteria group decision making, Eng. Appl. Artif. Intel. 95 (2020) 103871.
[25] L. Jin, Y.-Q. Xu, Z.-S. Chen, R. Mesiar, R.R. Yager, Relative basic uncertain information in preference and uncertain involved information fusion, Int. J. Comput. Intell. Syst. 15 (2022) 12.
[26] L. Jin, R.R. Yager, Z.-S. Chen, R. Mesiar, H. Bustince, Unsymmetrical basic uncertain information with some decision-making methods, J. Intell. Fuzzy Syst. 43 (4) (2022) 4457–4463.
[27] L. Jin, R. Mesiar, R.R. Yager, S.K. Kaya, Interval basic uncertain information and related aggregations in decision making, J. Intell. Fuzzy Syst. 42 (4) (2022) 3551–3558, https://doi.org/10.3233/JIFS-211635.
[28] L. Jin, R. Mesiar, R.R. Yager, On WA expressions of Induced OWA operators and inducing function based orness with application in evaluation, IEEE Trans. Fuzzy Syst. 29 (6) (2021) 1695–1700.
[29] L. Jin, R. Mesiar, R.R. Yager, Ordered weighted averaging aggregation on convex poset, IEEE Trans. Fuzzy Syst. 27 (3) (2019) 612–617.
[30] L. Jin, Z.-S. Chen, R.R. Yager, T. Senapati, R. Mesiar, D.G. Zamora, B. Dutta, L.M. Lopez, Uncertain ordered weighted averaging operators for basic uncertain information granules, Inf. Sci. 645 (2023) 119357.
[31] X. Pu, L. Jin, R. Mesiar, R.R. Yager, Continuous Parameterized families of RIM quantifiers and quasi-preference with some properties, Inf. Sci. 481 (2019) 24–32.
[32] L. Troiano, R.R. Yager, Recursive and iterative OWA operators, Int. J. Uncertainty Fuzziness Knowl.-Based Syst. 13 (6) (2005) 579–599.
[33] A.K. Singh, A. Kishor, N.R. Pal, Stancu OWA operator, IEEE Trans. Fuzzy Syst. 23 (4) (2015) 1306–1313.
[34] H.L. Royden, P.M. Fitzpatrick, Real analysis, fourth ed., Printice-Hall Inc, Boston, 2010.
[35] L.A. Zadeh, Outline of a new approach to analysis of complex systems and decision processes, IEEE Trans. Syst. Man Cybern. 1 (1973) 28–44.
[36] T. Takagi, M. Sugeno, Fuzzy identification of systems and its application to modeling and control, IEEE Trans. Syst. Man Cybern. 15 (1985) 116–132.
[37] W. Pedrycz, Fuzzy relational equations with generalized connectives and their applications, Fuzzy Set. Syst. 10 (1–3) (1983) 185–201.
[38] L. Jin, Z.-S. Chen, J.-Y. Zhang, R.R. Yager, R. Mesiar, M. Kalina, H. Bustince, L. Martinez, Bi-polar preference based weights allocation with incomplete fuzzy relations, Inf. Sci. 621 (2023) 308–318.
[39] J.-P. Chang, Z.-S. Chen, Z.-J. Wang, L. Jin, W. Pedrycz, L. Martinez, M.J. Skibniewski, Assessing the spatial synergy between integrated urban rail transit system and urban form: a BULI-based MCLSGA model with wisdom of crowds, IEEE Trans. Fuzzy Syst. 31 (2) (2023) 434–448.
[40] Y. Yang, D.-X. Xia, W. Pedrycz, M. Deveci, Z.-S. Chen, Cross-platform distributed product online ratings aggregation approach for decision making with basic uncertain linguistic information, Int. J. Fuzzy Syst. (2023), https://doi.org/10.1007/s40815-023-01646-3.
[41] M. Boczek, L. Halčinová, O. Hutník, M. Kaluszka, Novel survival functions based on conditional aggregation operators, Inf. Sci. 580 (2021) 705–719.
[42] Z. Ali, Decision-making techniques based on complex intuitionistic fuzzy power interaction aggregation operators and their applications, J. Innov. Res. Math. Comput. Sci. 1 (1) (2022) 107–125.
[43] M.R. Khan, A. Raza, Q. Khan, Multi-attribute decision-making by using intuitionistic Fuzzy rough Aczel-Alsina prioritize aggregation operator, J. Innov. Res. Math. Comput. Sci. 1 (2) (2022) 96–123.
[44] J.C.R. Alcantud, Complemental fuzzy sets: a semantic justification of q-rung orthopair fuzzy sets, IEEE Trans. Fuzzy Syst. 26 (2023) 4262–4270.
[45] J.C.R. Alcantud, G. Santos-García, M. Akram, OWA aggregation operators and multi-agent decisions with N-soft sets, Expert Syst. Appl. 203 (2022) 117430.
[46] J.C.R. Alcantud, G. Santos-García, M. Akram, A novel methodology for multiagent decision-making based on N-soft sets, Soft. Comput. (2023), https://doi.org/10.1007/s00500-023-08522-0.
[47] M. Akram, S. Naz, S.A. Edalatpanah, S. Samreen, A hybrid decision-making framework under 2-tuple linguistic complex q-rung orthopair fuzzy Hamy mean aggregation operators, Comput. Appl. Math. 42(3) (2023) 118.
[48] A. Sarkar, S. Moslem, D. Esztergár-Kiss, M. Akram, L. Jin, T. Senapati, A hybrid approach based on dual hesitant q-rung orthopair fuzzy frank power partitioned heronian mean aggregation operators for estimating sustainable urban transport solutions, Eng. Appl. Artif. Intell. 124 (2023) 106505.
[49] P. Liu, S. Naz, M. Akram, M. Muzammal, Group decision-making analysis based on linguistic q-rung orthopair fuzzy generalized point weighted aggregation operators, Int. J. Mach. Learn. Cybern. 13 (4) (2022) 883–906.
[50] L. Jin, B. Yatsalo, L. Martínez López, T. Senapati, C. Jebari, R.R. Yager, A Weight determination model in uncertain and complex bi-polar preference environment, Int. J. Uncertainty Fuzziness Knowl. Based Syst. 31 (5) (2023) 713–727.