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Weights generation models based on acceptance degrees in decision making

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ABSTRACT

The process of determining weights for a collection of experts is an essential component in addressing collective decision-making issues. In cases where individual evaluation values are accompanied by uncertainties, it is feasible for each expert to endorse the evaluations of their peers without necessitating further interaction among the group. This study proposes innovative approaches to determining weights, primarily relying on measurements of the overall acceptance degree. Additionally, the guidelines for computing the total acceptance degrees are established. Various methods can be employed to derive and calculate the total acceptance degree of an expert based on the evaluations provided by other experts. The study at hand introduces several novel concepts, namely "parameterized family of uncertainty functions" and "uncertain system", which can be effectively utilized for the development of relevant algorithms. The mathematical properties pertaining to the proposed concepts have been scrutinized and subsequently expounded upon. A normalized weight vector can be derived directly from any vector of the obtained total acceptance degrees. Numerical examples have been provided to serve the purpose of illustration and comparison.

1. Introduction

Information fusion is typically imperative in various contexts, such as decision-making and evaluations, wherein data obtained from multiple sources necessitate amalgamation into singular values. Aggregation operators [1,2] are robust and rigorous tools that amalgamate multiple units of input data and produce a singular output. In addition to empirical data, contemporary studies have directed their attention towards diverse forms of information, as well as novel and indeterminate data categories such as interval

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information, intuitionistic fuzzy information [3], vague information [4], probabilistic information and basic uncertain information (BUI) [5,6]. Interval information and BUI are frequently utilized sources of uncertainty in practical applications.

Weighted averaging is a prevalent approach for various types of aggregation operators that rely on pre-defined normalized weights vectors [7]. For any weighted averaging operator, it is imperative to ascertain appropriate normalized weight vectors through suitable weight generation methodologies. Normalized weight vectors are essential in various intricate decision-making scenarios, including multi-criteria decision making and collective decision making, where weights indicate the comparative significance of decision criteria or various experts.

A variety of techniques can be employed to generate weights for each expert involved in collective decision making and related information fusion problems [8,9]. Two prominent approaches in this field are the Analytic Hierarchy Process (AHP) method [10] and the Bipolar Preference based Weights Generation (BPWG) method [11,12]. The BPWG approach produces distinct normalized weight vectors based on the variation in inducing information, specifically the inducing vectors. For example, the ordered weighted averaging (OWA) [11] aggregation method utilizes the magnitudes of each input value as the inducing information for aggregation, while the inducing information in time-induced aggregation is chronologic information affixed to inputs [13]. In the BPWG weights allocation mechanism, inputs with larger (or smaller) inducing variable values will be automatically assigned greater weights. The weights allocation mechanism is usually attributed to Yager's inducing weights allocations [12]. For example, for the inputs (for aggregation) $(x_i)_{i=1}^n$ which is attached with chronologic inducing information $(t_i)_{i=1}^n$ (larger t_i indicating newer input data), if a decision maker prefers to believe more on newly obtained data, then more weights can be assigned to the inputs x_i with larger attached t_i in value. It is noteworthy that sometimes the direct normalization (of inducing information) is also a BPWG method. This is due to the fact that inputs with higher inducing information tend to receive greater weights following the normalization of the associated inducing information.

The BPWG method confers notable benefits, a subset of which are outlined herein. First, the method is automated and does not require supplementary involvement or interaction from the primary decision maker or experts. Consequently, it expedites the decision-making process and mitigates decision-related stress. Second, after establishing the appropriate inducing information, a normalized weighted vector will be generated to effectively mitigate any potential conflicts among decision makers. Third, it is easily understandable, explainable and recordable, which makes it possible for further re-examination of performed decision making processes and results. Last, the proposed method is consistently feasible, unlike other methods such as AHP which may not always be feasible, and exhibits a significantly low level of computational complexity.

In the context of collective decision-making, it is justifiable to assign greater weight to the opinions of experts who possess greater levels of experience, knowledge, or authority. This can be accomplished by constructing an inducing vector based on the aforementioned importance information of the experts. Some recent literature [5,6] concerned another type of inducing information of uncertainty degrees, i.e., the uncertainty/certainty extents to which the experts provide their opinions. More formally, BUI is applied to serve as inducing information. A BUI granule is expressed by a pair form (x, c) in which $x \in [0,1]$ is an evaluation value and $c \in [0,1]$ is the certainty degree of x. In addition, we use $1 - c \in [0,1]$ to express the uncertainty degree of x. Certainty degrees may represent the degrees to which decision makers are confident, sure, certain or definite of evaluation values, while uncertainty degrees may show the extents to which they are unconfident, unsure, uncertain or indefinite of evaluation values. BUI has soon been further developed and applied [14–21].

While importance and uncertainty information are two primary types of inducing information that are typically considered in collective decision-making environments, there is value in eliciting a range of related inducing information in order to generate weights that are more accurate and reasonable from a practical perspective. At present, there exists a scarcity of such investigations in the academic literature. The objective of this study is, therefore, to examine the pertinent inducing information in interactive decision-making settings where experts are able to subjectively or objectively specify their own designated ranges for accepting the opinions of other experts. For example, apart from giving an individual evaluation value, an expert will also provide an acceptance interval $[a, b]$ such that the evaluations obtained from other experts that are within $[a, b]$ will be acceptable to him/her. The aforementioned requirement appears to be pragmatic and rational, and offers novel approaches to facilitating the creation of novel forms of inducing information, thereby enhancing the modeling of actual intersubjective and cognitive decision-making processes among involved decision actors (such as experts).

To this end, we will propose and formulate some reasonable mechanism to derive acceptance information as inducing information by formally using the recently proposed information type of cognitive interval information (CII) [22]. The environment can be considered relatively simple due to the fact that the individual acceptance area is predetermined by the designated acceptance interval in any CII granule within the CII environment. Nevertheless, the specific details regarding individual acceptance are not explicitly provided within the BUI setting. Therefore, we will also examine a range of techniques for extracting acceptance information from BUI granules in order to more accurately reflect the attitudes of professionals and to allocate weights in a logical manner. Roughly speaking, we aim to extract total acceptance degrees as inducing information from the CII and BUI granules, which have been supplied by experts. Subsequently, the derived information can be either directly normalized to form some desired weight vectors or be applied as inducing information to perform BPWG method.

We propose the following reasonable principles that serve as the basis for our further discussions of deriving acceptance information. In collective decision environment, it is reasonable that the opinion of expert A should gain acceptance from another expert B only if:

- (i) expert B accepts the opinion of expert A;

- (ii) expert A is not uncertain about the opinion of expert A himself/herself;
  and the following rule should also be satisfied:
- (iii) the acceptance to expert A (from expert B) should be more possible (or with larger extent) when expert B is more uncertain about the opinion of expert B (i.e., when the uncertainty degree of expert B is higher).

The above three rules form a basis of formulating weights allocation models and will be further explained and mathematically formulated later in this work.

The further aggregation results with applying the obtained weight vectors embody the intersubjective influences among all participating experts. A notable benefit is that the entire process of obtaining acceptance information and allocating weights does not necessitate additional negotiations or the presence of invited experts.

The remainder of this work is organized as follows. Section 2 presents a brief introduction and fixes some terminologies. Section 3 discusses a novel method to derive total acceptance degrees from a given vector of cognitive interval information, which can be directly normalized into a weight vector. Section 4 presents a method that can derive total acceptance degrees from a given vector of basic uncertain information. Still with basic uncertain information environment, in Section 5 we put forward two concepts called "parameterized family of uncertainty functions" and "uncertain system", respectively; then we present some adapted method to derive total acceptance degree. Section 6 analyzes some mathematical properties of uncertain systems and discusses their advantages in decision making. In Section 7, we discuss some limitations of uncertain systems and then propose the concept of incomplete uncertain systems in order to ameliorate the limitations. Section 8 concludes and remarks this work.

2. Preliminary

In this preparatory section, the concepts about some related different types of uncertain information will be reviewed which will help provide a more standard formulation frame to address and describe the proposed problems. We will also fix some related notational terminologies.

Definition 1. Without loss of generality, an interval information considered in this work is always defined to be the closed interval $[a,b] \subseteq [0,1]$ ($a \le b$), which implies $[a,a]$ can be understood as the real number $a \in [0,1]$.

Definition 2. A BUI granule is with a pair form (x,c) in which $x \in [0,1]$ is an *evaluation value* and $c \in [0,1]$ is the *certainty degree* of x; $1-c \in [0,1]$ is the *uncertainty degree* of x.

Definition 3. [22] A cognitive interval information (CII) granule is with the pair form $(x,[a,b]) \in [0,1] \times \mathcal{I}$ such that $x \in [a,b]$. x is an *evaluation value* and $[a,b]$ is called *acceptance interval*.

The set of all intervals of the form $[a,b] \subseteq [0,1]$ is denoted by $\mathcal{I}$. The set of all BUI granules is denoted by $\mathcal{B}$. The set of all CII granules is denoted by $\mathcal{CII}$.

Some notations and expressions, mainly in vector forms, are fixed. A real vector of dimension n is denoted by $\mathbf{x} = (x_i)_{i=1}^n \in [0,1]^n$. A fuzzy set on $[0,1]$ is for convenience denoted by a mapping $f : [0,1] \to [0,1]$; the set of all fuzzy sets on $[0,1]$ is denoted by $[0,1]^{[0,1]}$. An interval vector of dimension n is denoted by $[\mathbf{a},\mathbf{b}] = ([a_i,b_i])_{i=1}^n \in \mathcal{I}^n$. For $s \in [0,1]$, we have the definition $s[a,b] = [a,b]s = [sa,sb]$. A BUI vector is denoted by $(\mathbf{x},\mathbf{c}) = \left((x_i,c_i)\right)_{i=1}^n \in \mathcal{B}^n$, where $\mathbf{x} = (x_i)_{i=1}^n \in [0,1]^n$ tacitly means an evaluation vector while $\mathbf{c} = (c_i)_{i=1}^n \in [0,1]^n$ is the certainty vector corresponding to $\mathbf{x}$. A CII vector is denoted by $(\mathbf{x},[\mathbf{a},\mathbf{b}]) = \left((x_i,[a_i,b_i])\right)_{i=1}^n \in (\mathcal{CII})^n$, where $\mathbf{x} = (x_i)_{i=1}^n \in [0,1]^n$ represents an evaluation vector while $[\mathbf{a},\mathbf{b}] = ([a_i,b_i])_{i=1}^n \in \mathcal{I}^n$ is the vector of acceptance intervals corresponding to $\mathbf{x}$. A normalized weight vector (of dimension n ($n \ge 2$)) is with the form $\mathbf{w} = (w_i)_{i=1}^n \in [0,1]^n$ such that $\sum_{i=1}^n w_i = 1$, and the space of all such vectors is denoted by $\mathcal{W}^{(n)}$.

3. Inducing information derived in CII environment

We propose a complete weights allocation method using inducing variable of total acceptance degrees (TAD) derived from given CII granules $\left((x_i,[a_i,b_i])\right)_{i=1}^n \in (\mathcal{CII})^n$. Suppose n experts $\{E_i\}_{i=1}^n$ are invited to provide their respective opinions about the evaluation values of some object. For $i \in [n] = \{1,\cdots,n\}$, expert E_i is requested to provide an evaluation $x_i \in [0,1]$ and an acceptance interval $[a_i,b_i] \in \mathcal{I}$ (with $x_i \in [a_i,b_i]$) such that the evaluations from other experts that fall within this interval will be accepted (or supported) by expert E_i and may appropriately obtain some extent of individual acceptance from E_i. Hence, we may further assign more weights to those experts who can collect more total acceptance degrees, and vice versa. Apart from the clear reasonability, note also that this weights allocation mechanism is efficient and more workable because it needs no further interventions and interactions from the invited experts; sometimes due to time and space constraints, for example in anonymous peer review systems, it is even impossible to interact between different experts.

In detail, suppose the CII granule $(x_i,[a_i,b_i])$ is obtained by expert E_i and $(x_j,[a_j,b_j])$ is given by expert E_j. Then, we assign acceptance degree 1 to expert E_i if $x_i \in [a_j,b_j]$, indicating x_i is accepted by expert E_j since x_i fall within his/her acceptance interval $[a_j,b_j]$; otherwise, we assign acceptance degree 0 to expert E_i. This rule is taken over all $i,j \in [n]$; that is, for any $i \in [n]$ the total acceptance degree for E_i will be taken from every $j \in [n]$ and added up (including E_i himself/herself). More formally, we denote by $s(i)$ the total acceptance degree which expert E_i obtain from all other experts (including himself/herself). For $A \subset [0,1]$, let $\chi_A : [0,1] \to \{0,1\}$ be the indicator function (or characteristic function) such that $\chi_A(x) = 1$ when $x \in A$ and $\chi_A(x) = 0$ when $x \in [0,1]\backslash A$. Then, the total acceptance degree can be defined as a mapping $s : [n] \to [1,n]$ such that

$$s(i) = \sum_{j=1}^{n} \chi_{[a_j,b_j]}(x_i) \tag{1}$$

Note that in the definition of Eq. (1), $s(i) \geq \chi_{[a_i,b_i]}(x_i) = 1$.

The obtained vector $\mathbf{s} = (s(i))_{i=1}^n$ (which we call the vector of total acceptance degree) can directly generate a normalized weight vector $\mathbf{w} = (w_i)_{i=1}^n$ by simple normalization so that

$$w_i = \frac{s(i)}{\sum_{j=1}^{n} s(j)} \tag{2}$$

Note that in this relatively simpler decisional scenario, for the opinion of expert A to gain acceptance from another expert B, we only consider the fulfillment of one condition (as defined in Introduction): (i) expert B accepts the opinion of expert A, without considering the other two conditions due to no involvement of uncertainties for their own opinions.

We summarize the model as follows.

Weighting Model 1: CII environment

Input: A vector of CII granules $(\mathbf{x}, [\mathbf{a}, \mathbf{b}]) = \left((x_i, [a_i, b_i])\right)_{i=1}^n \in (CII)^n$.

Output: A normalized weight vector $\mathbf{w} = (w_i)_{i=1}^n$ with $w_i = \frac{s(i)}{\sum_{j=1}^n s(j)}$ where $s(i) = \sum_{j=1}^n \chi_{[a_j,b_j]}(x_i)$.

With the obtained normalized weight vector $\mathbf{w} = (w_i)_{i=1}^n$ we may perform required aggregation for the evaluation values $\mathbf{x} = (x_i)_{i=1}^n$ such as arithmetic weighted mean $A(\mathbf{x}) = \sum_{i=1}^n w_i x_i$. The aggregation result in decision making can be regarded as the result after considering the intersubjective influences between all the involved experts without their presences and further negotiations.

The above Weighting Model 1 has the property that enlarging the acceptance interval of any expert E_r may decrease the weight w_r he/she can obtain, which is detailed below.

Proposition 1. *In Weighting Model 1, let* $(\mathbf{x}, [\mathbf{a}, \mathbf{b}]), (\mathbf{x}, [\mathbf{a}', \mathbf{b}']) \in (CII)^n$ *be two CII vectors such that* $[a_i, b_i] = [a_i', b_i']$ *for all* $i \in [n]\backslash\{r\}$ *and* $[a_r, b_r] \subset [a_r', b_r']$, *and let* $\mathbf{w} = (w_i)_{i=1}^n$ *and* $\mathbf{w}' = (w_i')_{i=1}^n$ *be the two normalized weight vectors generated by* $(\mathbf{x}, [\mathbf{a}, \mathbf{b}])$ *and* $(\mathbf{x}, [\mathbf{a}', \mathbf{b}'])$, *respectively, then* $w_r \geq w_r'$.

Proof. Denote by $s' : [n] \to [1,n]$ the total acceptance degrees obtained with $(\mathbf{x}, [\mathbf{a}', \mathbf{b}'])$. It is observed that for all $i \in [n]\backslash\{r\}$,

$$s'(i) = \chi_{[a'_r,b'_r]}(x_i) + \sum_{j\in[n]\backslash\{r\}} \chi_{[a'_j,b'_j]}(x_i) \geq \chi_{[a_r,b_r]}(x_i) + \sum_{j\in[n]\backslash\{r\}} \chi_{[a_j,b_j]}(x_i) = \sum_{j=1}^{n} \chi_{[a_j,b_j]}(x_i) = s(i)$$

and

$$\begin{aligned} s'(r) &= \chi_{[a'_r,b'_r]}(x_r) + \sum_{j\in[n]\backslash\{r\}} \chi_{[a'_j,b'_j]}(x_r) = 1 + \sum_{j\in[n]\backslash\{r\}} \chi_{[a_j,b_j]}(x_r) \\ &= \chi_{[a_r,b_r]}(x_r) + \sum_{j\in[n]\backslash\{r\}} \chi_{[a_j,b_j]}(x_r) = s(r) \end{aligned}$$

Therefore,

$$w_r = \frac{s(r)}{\sum_{j=1}^{n} s(j)} \geq \frac{s(r)}{\sum_{j=1}^{n} s'(j)} = \frac{s'(r)}{\sum_{j=1}^{n} s'(j)} = w_r' \quad \square$$

Remark. The above Proposition 1 does not guarantee that $w_i \leq w_i'$ for any $i \in [n]\backslash\{r\}$ as the following example shows.

Example 1. Let $(\mathbf{x}, [\mathbf{a}, \mathbf{b}]) = \left((x_i, [a_i, b_i])\right)_{i=1}^4 = ((0, [0, 0.25]), (0.3, [0, 0.7]), (0.6, [0.5, 0.8]), (1, [1, 1]))$, then

$$s(1) = \sum_{j=1}^{4} \chi_{[a_j,b_j]}(0) = 2, s(2) = \sum_{j=1}^{4} \chi_{[a_j,b_j]}(0.3) = 1$$

$$s(3) = \sum_{j=1}^{4} \chi_{[a_j,b_j]}(0.6) = 2, s(4) = \sum_{j=1}^{4} \chi_{[a_j,b_j]}(1) = 1$$

Hence, $w_1 = \frac{s(1)}{\sum_{j=1}^4 s(j)} = \frac{1}{3}$, $w_2 = \frac{s(2)}{\sum_{j=1}^4 s(j)} = \frac{1}{6}$, $w_3 = \frac{s(3)}{\sum_{j=1}^4 s(j)} = \frac{1}{3}$, $w_4 = \frac{s(4)}{\sum_{j=1}^4 s(j)} = \frac{1}{6}$.

Consider

$$(\mathbf{x}, [\mathbf{a}', \mathbf{b}']) = \left((x_i, [a'_i, b'_i])\right)_{i=1}^4 = ((0, [0, 0.6]), (0.3, [0, 0.7]), (0.6, [0.5, 0.8]), (1, [1, 1])),$$

then $s'(1) = 2$, $s'(2) = 2$, $s'(3) = 3$, $s'(4) = 1$; and hence $w_1' = w_2' = 1/4 < w_1$, $w_3' = 3/8$, $w_4' = 1/8 < 1/6 = w_4$.

Apart from the method of proportionally generating normalized weight vector according to the vector of total acceptance degree $\mathbf{s} = (s(i))_{i=1}^n$, we can also take it as an inducing vector to generate normalized weight vector using bipolar preference based weights generation (BPWG) method which will not be discussed in this work.

4. Inducing information derived in BUI environment

In BUI environment, a BUI vector $(\mathbf{x}, \mathbf{c}) = (x_i, c_i)_{i=1}^n \in \mathcal{B}^n$ is known instead of a CII vector. Therefore, in order for each BUI granule (x_i, c_i) to contain some necessary acceptance information, we may firstly transform each BUI granule into an interval $[a_i, b_i]$ which contains x_i and has length $l([a_i, b_i]) = b_i - a_i = 1 - c$. Then, we can adopt a similar weights allocation method as in Weighting Model 1. For example, we can adopt the following BUI-interval transformation, $T : \mathcal{B} \to \mathcal{I}$, such that

$$T((x, c)) = c[x, x] + (1 - c)[0, 1] \tag{3}$$

Hence, we in actual have obtained a new CII vector $(\mathbf{x}, [\mathbf{a}, \mathbf{b}]) = \left((x_i, T((x_i, c_i)))\right)_{i=1}^n \in (C\mathcal{I}\mathcal{I})^n$, and then we can perform Weighting Model 1 in a similar manner. Moreover, the second rule proposed in Introduction "(ii) expert A is not uncertain about the opinion of expert A himself/herself" is now relevant to this situation since in BUI environment experts can have part certainty about their provided opinions. Therefore, in the corresponding weighting model we should consider the influence of certainty/uncertainty degrees to the obtained total acceptance degrees.

In the context of CII environment, the acceptance information is known through some given intervals. However, in a BUI environment, we should make two assumptions that are practical and reasonable in decision making. Firstly, we assume that decision makers themselves agree to apply BUI-interval transformation, which may not necessarily be communicated to the invited experts as their sole task is to provide BUI-valued evaluations. Secondly, we assume that all invited experts are genuine and will not intentionally exploit algorithmic rules to inflate their own weights (as some monotonicity properties might tempt them to reduce uncertainty for personal gain in weights if they were aware of such properties). This assumption can be readily accepted since any non-genuine expert would inevitably introduce unfair or unreasonable evaluations and decisions in almost all environments.

We still denote by $s(i)$ the total acceptance degree that expert E_i obtains from all other experts (including himself/herself). Here we will adopt a simpler while effective punishment mechanism that lower certainty degree (or higher uncertainty degree) will decrease the obtained total acceptance degree. Then, the total acceptance degree can be defined as a mapping $s : [n] \to [0, n]$ such that

$$s(i) = c_i \cdot \sum_{j=1}^n \chi_{T((x_j, c_j))}(x_i) \tag{4}$$

Remark. The product in (4) can be replaced by a transformed semicopula $\circ : [0, 1] \times [0, n] \to [0, n]$ given by $u \circ v = n \cdot S(u, v/n)$ where $S : [0, 1]^2 \to [0, 1]$ is a semicopula. Recall a semicopula [23,24] $S : [0, 1]^2 \to [0, 1]$ is a nondecreasing binary operation with respect to each coordinate such that $S(a, 1) = S(1, a) = a$ for any $a \in [0, 1]$ (i.e., binary aggregation function with neutral element 1). Then, some formulations, i.e., $s(i) = c_i \circ \sum_{j=1}^n \chi_{T((x_j, c_j))}(x_i)$, can also serve as alternative choices in practical decision making problems.

Proposition 2. *In the definition of Eq. (4), for any $i \in [n]$, $s(i) = 0$ if and only if $c_i = 0$.*

Proof. "⇒": Since for any $c_i \in [0, 1]$ we have $x_i \subseteq T((x_i, c_i))$, then $\chi_{T((x_i, c_i))}(x_i) = 1$. Therefore, if $c_i \neq 0$, then $s(i) \geq c_i \cdot \chi_{T((x_i, c_i))}(x_i) = c_i > 0$.
"⇐" is trivial. □

If the obtained TAD vector $\mathbf{s} = (s(i))_{i=1}^n$ is not zero vector (i.e., there at least exists some $i \in [n]$ for which $s(i) > 0$), then it can directly generate a normalized weight vector $\mathbf{w} = (w_i)_{i=1}^n$ by simple normalization as in Eq. (2). If $\mathbf{s} = \mathbf{0}$, we can take Laplace decision attitude to uniformly distribute the weights with defining $\mathbf{w} = (w_i)_{i=1}^n$ such that $w_i = 1/n$ for each $i \in [n]$. Alternatively, we can also perform bipolar preference based weights generation (BPWG) method with regarding the obtained TAD vector as the inducing information.

The above discussed model is summarized as follows.

Weighting Model 2: BUI environment
Input: A vector of BUI granules $(\mathbf{x}, \mathbf{c}) = \left((x_i, c_i)\right)_{i=1}^n \in \mathcal{B}^n$.
Output: A normalized weight vector $\mathbf{w} = (w_i)_{i=1}^n$ such that
when $\mathbf{s} \neq \mathbf{0}$, $w_i = \frac{s(i)}{\sum_{j=1}^n s(j)}$ where $s(i) = c_i \cdot \sum_{j=1}^n \chi_{T((x_j, c_j))}(x_i)$ and $T : \mathcal{B} \to \mathcal{I}$ is defined in Eq. (3);
when $\mathbf{s} = \mathbf{0}$, $w_i = 1/n$.

Proposition 3. *In Weighting Model 2, let $(\mathbf{x}, \mathbf{c}), (\mathbf{x}, \mathbf{c}') \in \mathcal{B}^n$ be two BUI vectors such that $c_i = c_i'$ for all $i \in [n] \backslash \{r\}$ and $c_i > c_i'$, and let $\mathbf{w} = (w_i)_{i=1}^n$ and $\mathbf{w}' = (w_i')_{i=1}^n$ be the two normalized weight vectors generated by $(\mathbf{x}, \mathbf{c})$ and $(\mathbf{x}, \mathbf{c}')$, respectively, then $w_r \geq w_r'$.*

Proof. A generalization of this proposition will be proved later. □

Example 2. Let $(\mathbf{x},\mathbf{c})=\left((x_i,c_i)\right)_{i=1}^4=((0,0.7),(0.4,0.5),(0.7,1),(1,0.25))$, then

$s(1)=c_1\cdot\sum_{j=1}^4\chi_{T((x_j,c_j))}(x_1)=0.7\left(\chi_{[0,0.7]}(0)+\chi_{[0.2,0.7]}(0)+\chi_{[0.7,0.7]}(0)+\chi_{[0.75,1]}(0)\right)=0.7$

$s(2)=c_2\cdot\sum_{j=1}^4\chi_{T((x_j,c_j))}(x_2)=0.5\left(\chi_{[0,0.7]}(0.4)+\chi_{[0.2,0.7]}(0.4)+\chi_{[0.7,0.7]}(0.4)+\chi_{[0.75,1]}(0.4)\right)=1$

$s(3)=c_3\cdot\sum_{j=1}^4\chi_{T((x_j,c_j))}(x_3)=1\left(\chi_{[0,0.7]}(0.7)+\chi_{[0.2,0.7]}(0.7)+\chi_{[0.7,0.7]}(0.7)+\chi_{[0.75,1]}(0.7)\right)=3$

$s(4)=c_4\cdot\sum_{j=1}^4\chi_{T((x_j,c_j))}(x_4)=0.25\left(\chi_{[0,0.7]}(1)+\chi_{[0.2,0.7]}(1)+\chi_{[0.7,0.7]}(1)+\chi_{[0.75,1]}(1)\right)=0.25$

Since $\mathbf{s}\neq\mathbf{0}$, then

$w_1=\frac{s(1)}{\sum_{j=1}^4 s(j)}=\frac{0.7}{0.7+1+3+0.25}=\frac{0.7}{4.95}\doteq 0.141,\ w_2=\frac{s(2)}{\sum_{j=1}^4 s(j)}=\frac{1}{4.95}\doteq 0.202$

$w_3=\frac{s(3)}{\sum_{j=1}^4 s(j)}=\frac{3}{4.95}\doteq 0.606,\ w_4=\frac{s(4)}{\sum_{j=1}^4 s(j)}=\frac{0.25}{4.95}\doteq 0.051$

With the obtained weight vector $\mathbf{w}=(0.141,0.202,0.606,0.051)$ and the known vector of evaluation values $\mathbf{x}=(0,0.4,0.7,1)$, decision makers can perform different weights averaging such as weighted arithmetic mean $WA_{\mathbf{w}}:[0,1]^4\to[0,1]$ by $WA_{\mathbf{w}}(\mathbf{x})=\sum_{i=1}^4 w_i x_i=0.556$.

5. Inducing information derived with uncertain system

The weighting model proposed in Section 4 is mainly affected by the BUI-interval transformation, $T:\mathcal{B}\to\mathcal{I}$, which can transform any BUI granule (x,c) into an $[a,b]=T((x,c))=c[x,x]+(1-c)[0,1]$, a subset of unit interval $[0,1]$. This is a reasonable and convenient weighting method in collective decision making and BUI environment without more involved computations. However, one possible shortcoming lies in the discontinuity it may cause (as analyzed later). Hence, with the help of fuzzy set theory, we may consider transforming any BUI granule (x,c) into a fuzzy set of unit interval $[0,1]$, i.e., a $[0,1]$-valued function on $[0,1]$. With the same preconditions as in Section 4, this section discusses a generalized method which is based on BUI-fuzzy transformation.

We formally introduce the concepts of parameterized family and uncertain system.

Definition 4. A family $\mathcal{F}_x=(f_t^x)_{t\in[0,1]}$ is called a parameterized family (of uncertainty functions of $[0,1]$) focused at $x\in[0,1]$, in which $f_t^x:[0,1]\to[0,1]$ are called the uncertainty functions (also known as fuzzy sets), if the following conditions are satisfied

(i) for all $t\in[0,1]$, $f_t^x(a)\le f_t^x(b)$ when $0\le a<b\le x$, and $f_t^x(a)\ge f_t^x(b)$ when $x\le a<b\le 1$;

(ii) $f_0^x=\chi_{\{x\}}$, $f_1^x=\chi_{[0,1]}$, and $f_t^x\le f_s^x$ for any $t<s$ (that is, for any $y\in[0,1]$, $f_t^x(y)\le f_s^x(y)$);

(iii) $\int_0^1 f_t^x(y)dy=t$ for all $t\in[0,1]$.

Remark. The condition (ii) implies that for any $t\in[0,1]$, $f_t^x(x)=1$.

Definition 5. A system $\mathfrak{F}=\left(\mathcal{F}_x\right)_{x\in[0,1]}=\left((f_t^x)_{t\in[0,1]}\right)_{x\in[0,1]}$ is called an uncertainty system if $\mathcal{F}_x$ is a parameterized family focused at x for all $x\in[0,1]$.

Remark. One may also view $\mathfrak{F}=\left((f_t^x)_{t\in[0,1]}\right)_{x\in[0,1]}$ directly as $\mathfrak{F}=(f_{(x,t)})_{(x,t)\in[0,1]^2}$. We adopt the above step-by-step definition is only for a clear hierarchical structure in logic, which may provide some convenience and clarity in formulating and discussing.

Definition 6. The BUI-fuzzy transformation with an uncertainty system $\mathfrak{F}=\left(\mathcal{F}_x\right)_{x\in[0,1]}=\left((f_t^x)_{t\in[0,1]}\right)_{x\in[0,1]}$ is a mapping $H:\mathcal{B}\to[0,1]^{[0,1]}$ such that $H((x,c))=f_{1-c}^x$.

Remark. BUI-fuzzy transformation actually generalizes BUI-interval transformation. Indeed, for the uncertainty system $\mathfrak{F}=\left(\mathcal{F}_x\right)_{x\in[0,1]}=\left((f_t^x)_{t\in[0,1]}\right)_{x\in[0,1]}$ such that $f_t^x=\chi_{T(x,1-t)}=\chi_{(1-t)[x,x]+t[0,1]}$ for all $t,x\in[0,1]$, we easily observe $H((x,c))=f_{1-c}^x=\chi_{c[x,x]+(1-c)[0,1]}=\chi_{T((x,c))}$.

Given a BUI vector $(\mathbf{x},\mathbf{c})=(x_i,c_i)_{i=1}^n\in\mathcal{B}^n$ together with an uncertainty system $\mathfrak{F}$, we may firstly transform each BUI granule into an uncertainty function which will serve as a similar role of judging whether or not an expert accepts the opinions of others as in Weighting Model 2. The difference lies in that in Weighting Model 2 the opinion x_i of expert E_i is accepted by expert E_j only when x_i falls within the transformed interval $T((x_j,c_j))$; that is, it either may be fully accepted or may be fully unaccepted. One advantage of fuzzy set theory is that it can make such drastic judgment threshold moderate and thus better model some real situations. Hence, using the uncertain system and BUI-fuzzy transformation (which are based on fuzzy set theory), a new weighting model can be designed as follows to formulate some acceptance degrees that can be any values within [0,1].

We still denote by $s(i)$ the total acceptance degree that expert E_i obtains from all other experts (including himself/herself); we also adopt the same punishment mechanism as in Weighting Model 2 that lower certainty degree (or higher uncertainty degree) will decrease the obtained total acceptance degree. Then, the total acceptance degree (composed of fuzzy extents) in the new model can be defined as a mapping $s:[n]\to[0,n]$ such that

$$s(i) = c_i \cdot \sum_{j=1}^{n} H((x,c))(x_i) = c_i \cdot \sum_{j=1}^{n} f_{1-c_j}^{x_j}(x_i) \tag{5}$$

The above discussed model is summarized as follows.

Weighting Model 3: BUI environment with uncertain system

Input: A vector of BUI granules $(\mathbf{x},\mathbf{c}) = \left((x_i,c_i)\right)_{i=1}^n \in \mathcal{B}^n$. An uncertain system $\mathfrak{F} = \left(\mathcal{F}_x\right)_{x\in[0,1]} = \left((f_t^x)_{t\in[0,1]}\right)_{x\in[0,1]}$.

Output: A normalized weight vector $\mathbf{w} = (w_i)_{i=1}^n$ such that

when $\mathbf{s} \neq \mathbf{0}$, $w_i = \frac{s(i)}{\sum_{j=1}^n s(j)}$ where $s(i) = c_i \cdot \sum_{j=1}^n H((x,c))(x_i) = c_i \cdot \sum_{j=1}^n f_{1-c_j}^{x_j}(x_i)$ and $H((x,c)) = f_{1-c}^x$ is defined by Definition 6;

when $\mathbf{s} = \mathbf{0}$, $w_i = 1/n$.

Example 3. An uncertain system $\mathfrak{F} = \left(\mathcal{F}_x\right)_{x\in[0,1]} = \left((f_t^x)_{t\in[0,1]}\right)_{x\in[0,1]}$ is constructed as follows.

(a1) when $x \in (0,1)$ and $t \in [0.5,1)$, f_t^x is defined such that

$f_t^x(y) = \chi_{[0,x]} \cdot \left[\frac{y}{x} + \frac{x-y}{x}(2t-1)\right] + \chi_{(x,1]} \cdot \left[\frac{1-y}{1-x} + \frac{y-x}{1-x}(2t-1)\right]$,

(a2) when $x = 1$ and $t \in [0.5,1)$, $f_t^x(y) = y + (1-y)(2t-1)$,

(a3) when $x = 0$ and $t \in [0.5,1)$, $f_t^x(y) = 1 - 2y + 2yt$,

(b1) when $x \in (0,1)$ and $t \in (0,0.5)$, f_t^x is defined such that

$f_t^x(y) = \chi_{[x-2tx,x]} \cdot \frac{y-x+2tx}{2tx} + \chi_{(x,2t(1-x)+x]} \cdot \left[1 + \frac{x-y}{2t(1-x)}\right]$,

(b2) when $x = 1$ and $t \in (0,0.5)$, $f_t^x(y) = \chi_{[1-2t,1)} \cdot \frac{y-1+2t}{2t}$,

(b3) when $x = 0$ and $t \in (0,0.5)$, $f_t^x(y) = \chi_{[0,2t]} \cdot \left[1 - \frac{y}{2t}\right]$.

Example 4. (continuation of Example 2 and Example 3)

Still let $(\mathbf{x},\mathbf{c}) = ((0,0.7),(0.4,0.5),(0.7,1),(1,0.25))$. Then, in comparison to Example 2, by Weighting Model 3 and using the system defined in Example 3 we have

$s(1) = c_1 \cdot \sum_{j=1}^4 f_{1-c_j}^{x_j}(x_1) = 0.7\left(f_{0.3}^0(0) + f_{0.5}^{0.4}(0) + f_0^{0.7}(0) + f_{0.75}^1(0)\right) = 0.7\,(1+0+1+0.5) = 1.75$,

$s(2) = c_2 \cdot \sum_{j=1}^4 f_{1-c_j}^{x_j}(x_2) = 0.5\left(f_{0.3}^0(0.4) + f_{0.5}^{0.4}(0.4) + f_0^{0.7}(0.4) + f_{0.75}^1(0.4)\right) = 0.7\left(\frac{1}{3}+1+1+0.7\right) \doteq 2.123$

$s(3) = c_3 \cdot \sum_{j=1}^4 f_{1-c_j}^{x_j}(x_3) = 1\left(f_{0.3}^0(0.7) + f_{0.5}^{0.4}(0.7) + f_0^{0.7}(0.7) + f_{0.75}^1(0.7)\right) = 0 + 0.5 + 1 + 0.85 = 2.35$

$s(4) = c_4 \cdot \sum_{j=1}^4 f_{1-c_j}^{x_j}(x_4) = 0.25\left(f_{0.3}^0(1) + f_{0.5}^{0.4}(1) + f_0^{0.7}(1) + f_{0.75}^1(1)\right) = 0.25\,(0+0+1+1) = 0.5$

Since $\mathbf{s} \neq \mathbf{0}$, then

$w_1 = \frac{s(1)}{\sum_{j=1}^4 s(j)} = \frac{1.75}{1.75+2.123+2.35+0.5} = \frac{1.75}{6.723} \doteq 0.26$, $w_2 = \frac{s(2)}{\sum_{j=1}^4 s(j)} = \frac{2.123}{6.723} \doteq 0.32$,

$w_3 = \frac{s(3)}{\sum_{j=1}^4 s(j)} = \frac{2.35}{6.723} \doteq 0.35$, $w_4 = \frac{s(4)}{\sum_{j=1}^4 s(j)} = \frac{0.5}{6.723} \doteq 0.07$

We obtain a weight vector $\mathbf{w} = (0.26, 0.32, 0.35, 0.07)$.

Proposition 4. *In Weighting Model 3, let $(\mathbf{x},\mathbf{c}),(\mathbf{x},\mathbf{c}') \in \mathcal{B}^n$ be two BUI vectors such that $c_i = c_i'$ for all $i \in [n]\backslash\{r\}$ and $c_r > c_r'$, and let $\mathbf{w} = (w_i)_{i=1}^n$ and $\mathbf{w}' = (w_i')_{i=1}^n$ be the two normalized weight vectors generated by $(\mathbf{x},\mathbf{c})$ and $(\mathbf{x},\mathbf{c}')$, respectively, with the same uncertain system $\mathfrak{F}$, then $w_r \geqslant w_r'$.*

Proof. Denote by $s' : [n] \to [1,n]$ the total acceptance degrees obtained with $(\mathbf{x},\mathbf{c}')$. We should consider the two situations below.

(a) If $\mathbf{c} \neq \mathbf{0}$ (i.e., there exists some $i \in [n]$ with $c_i > 0$), then we have that for all $i \in [n]\backslash\{r\}$,

$$s'(i) = c_i \cdot \left[f_{1-c'_r}^{x_r}(x_i) + \sum_{j\in[n]\backslash\{r\}} f_{1-c_j}^{x_j}(x_i)\right] \geqslant c_i \cdot \left[f_{1-c_r}^{x_r}(x_i) + \sum_{j\in[n]\backslash\{r\}} f_{1-c_j}^{x_j}(x_i)\right] = s(i)$$

and

$$s'(r) = c'_r \cdot \left[f_{1-c'_r}^{x_r}(x_r) + \sum_{j\in[n]\backslash\{r\}} f_{1-c_j}^{x_j}(x_r)\right] = c'_r \cdot \left[1 + \sum_{j\in[n]\backslash\{r\}} f_{1-c_j}^{x_j}(x_r)\right]$$

$$= c'_r \cdot \left[f_{1-c_r}^{x_r}(x_r) + \sum_{j\in[n]\backslash\{r\}} f_{1-c_j}^{x_j}(x_r)\right] \leqslant c_r \cdot \left[f_{1-c_r}^{x_r}(x_r) + \sum_{j\in[n]\backslash\{r\}} f_{1-c_j}^{x_j}(x_r)\right] = s(r).$$

Therefore,

$w_r = \frac{s(r)}{\sum_{j=1}^n s(j)} \geqslant \frac{s(r)}{\sum_{j=1}^n s'(j)} \geqslant \frac{s'(r)}{\sum_{j=1}^n s'(j)} = w'_r$.

(b) If $\mathbf{c}' = \mathbf{0}$ (i.e., $c'_i = 0$ for all $i \in [n]$), then it is clear $\mathbf{s}' = \mathbf{0}$ and thus $w'_i = 1/n$ for all $i \in [n]$. Since $c_r > c'_r = 0$, then $\mathbf{c} \neq \mathbf{0}$. Therefore $s(r) = c_r \cdot \sum_{j=1}^{n} f_{1-c_j}^{x_j}(x_r) \geqslant c_r \cdot f_{1-c_r}^{x_r}(x_r) = c_r > 0$, while for all $i \in [n]\backslash\{r\}$ it is trivial to see that $s(i) = 0$ (since $c_i = 0$). Then, we immediately have $w_r = 1 > 1/n = w'_r$. □

6. Some analyses for uncertain system

For the total acceptance degrees $s(i)$ as obtained in Weighting Model 1 and Weighting Model 2, when the evaluation x_i changes slightly, $s(i)$ might change significantly in sense that, as a function of x_i, it is not continuous. Precisely, $\chi_{[a_j,b_j]}$ and $\chi_{T((x_j,c_j))}$ are not continuous.

It is desired for the total acceptance degrees obtained in Eq. (5) not to have drastic variations with slight change in some other related parameters mainly including x_j, x_i and c $(1-c)$. Continuities with respect to these factors often are more ideal in decision making.

Definition 7. For a parameterized family focused at $x \in [0,1]$, $\mathcal{F}_x = (f_t^x)_{t\in[0,1]}$,

(i) if f_t^x is continuous for all $t \in (0,1]$, then $\mathcal{F}_x$ is called an *parameterized family with quasi-continuous uncertainty functions*;

(ii) if $\varphi_y : [0,1] \to [0,1]$ (with $\varphi_y(t) \triangleq f_t^x(y)$) is continuous for all $y \in [0,1]$, then $\mathcal{F}_x$ is called a *continuous parameterized family*.

Remark. It is clear that f_0^x is not continuous and therefore in the above definition we only consider $t \in (0,1]$ instead of $t \in [0,1]$.

Without loss of generality, we need to use pseudometric space $(L^1([0,1]), d)$ where $d(f,g) \triangleq \int_{[0,1]} |f-g|$. Conventionally, one should consider a collection of equivalence classes $L^1([0,1])/\cong$, in which $\cong$ is the equivalence relation so that $f \cong g$ if and only if $\int_{[0,1]} |f-g| = 0$. Hence, this defines the commonly known metric space $(L^1([0,1])/\cong, \tilde{d})$ such that $\tilde{d}([f],[g]) = d(f,g)$.

Definition 8. (i) For a system $\mathfrak{F} = (\mathcal{F}_x)_{x\in[0,1]} = \left((f_t^x)_{t\in[0,1]}\right)_{x\in[0,1]}$, if for all $t \in [0,1]$ the mappings $u_t : [0,1] \to \mathcal{F}_x$ (with $u_t(x) \triangleq f_t^x$) is continuous (with respect to $\tilde{d}$), then $\mathfrak{F}$ is called *continuous with respect to evaluation value* x.

(ii) If a system $\mathfrak{F} = (\mathcal{F}_x)_{x\in[0,1]}$ is continuous with respect to evaluation value x, and for each $x \in [0,1]$, $\mathcal{F}_x$ is a continuous parameterized family with quasi-continuous uncertainty functions, then we call $\mathfrak{F}$ continuous.

Example 5. (i) Consider the system $\mathfrak{F} = (\mathcal{F}_x)_{x\in[0,1]}$ with family $\mathcal{F}_x = (f_t^x)_{t\in[0,1]}$ satisfying $f_t^x = \chi_{(1-t)[x,x]+t[0,1]}$ for each $x \in [0,1]$, then $\mathfrak{F}$ is not continuous since $\mathcal{F}_x$ is not a continuous parameterized family and $\mathcal{F}_x$ is not with quasi-continuous uncertainty functions. However, it is easy to check that $\mathfrak{F}$ is continuous with respect to evaluation value x.

(ii) It can be easy to check that the system defined in Example 3 is continuous.

In theory, it is more interesting for a model or method to have continuities with respect to more parameters. Hence, for Definitions 7 and 8, it is desired that more types of continuity can be fulfilled.

In general, it is relatively easy to construct some systems $\mathfrak{F} = (\mathcal{F}_x)_{x\in[0,1]}$ that are continuous with respect to evaluation value x. However, this is not the case for the other two types of continuity defined in Definition 7. On one hand, in Definition 7 (i), even if $\mathcal{F}_x$ is an parameterized family with quasi-continuous uncertainty functions, f_t^x is only continuous for all $t \in [0,1]$, but not for all $t \in (0,1]$; actually f_0^x is not continuous. On the other hand, in Definition 7 (ii), if $\mathcal{F}_x$ a parameterized family with quasi-continuous uncertainty functions, it can be shown that $\{\varphi_y\}_{y\in[0,1]}$ is not equicontinuous on $[0,1]$ (precisely, not equicontinuous at 0). That is, there is an amount $\varepsilon > 0$ (which may be not tolerant for some decision makers with regard to local abrupt changing) such that for any $\delta > 0$ there always exists some $y \in [0,1]$ so that $\left|\varphi_y(0) - \varphi_y(t)\right| = \left|f_0^x(y) - f_t^x(y)\right| \geqslant \varepsilon$; roughly speaking, even if every φ_y is continuous (with respect to uncertainty degree t), this continuity in theory may be not desired enough to satisfy some decision makers for their tolerance for local abrupt changing of uncertainty in a uniform sense.

Definition 9. [25] A collection $\mathcal{F}$ of real-valued functions on a metric space (X,d) is said to be equicontinuous at the point $x \in X$ provided for each $\varepsilon > 0$, there is a $\delta > 0$ such that for every $f \in \mathcal{F}$ and $x' \in X$,

if $d(x',x) < \delta$, then $\left|f(x') - f(x)\right| < \varepsilon$.

The collection $\mathcal{F}$ is said to be equicontinuous on X provided it is equicontinuous at every point in X.

Proposition 5. *For any parameterized family with quasi-continuous uncertainty functions focused at $x \in [0,1]$, $\mathcal{F}_x = (f_t^x)_{t\in[0,1]}$, define $\varphi_y : [0,1] \to [0,1]$ (with $\varphi_y(t) \triangleq f_t^x(y)$), then the collection $\{\varphi_y\}_{y\in[0,1]}$ is not equicontinuous at 0.*

Proof. We prove by contradiction and suppose $\{\varphi_y\}_{y\in[0,1]}$ is equicontinuous at 0. There exists $1 > \varepsilon > 0$ such that for any $\delta > 0$, if $|t-0| = t < \delta$ then for any $y \in [0,1]$ it holds that $\left|\varphi_y(t) - \varphi_y(0)\right| = \left|f_t^x(y) - f_0^x(y)\right| < \varepsilon < 1$. Hence, for any $y \neq x$, we have $\left|f_t^x(y) - f_0^x(y)\right| = \left|f_t^x(y) - 0\right| = f_t^x(y) < \varepsilon < 1$ which means $\lim_{y\to x} f_t^x(y) < \varepsilon < 1$ for all $t < \delta$. Therefore, $\lim_{y\to x} f_t^x(y) < f_t^x(x) = 1$ and then f_t^x is not continuous when $0 < t < \delta$, which is in contradiction to the premise that $\mathcal{F}_x$ is with quasi-continuous uncertainty functions. □

Remark. Note that the above proposition only states that $\{\varphi_y\}_{y\in[0,1]}$ is not equicontinuous if $\mathcal{F}_x$ is a parameterized family with quasi-continuous uncertainty functions. When $\mathcal{F}_x$ is not a parameterized family with quasi-continuous uncertainty functions (i.e., f_t^x is discontinuous for some $t \in (0,1]$), then it is still possible that $\{\varphi_y\}_{y\in[0,1]}$ is equicontinuous. For example, consider $\mathcal{F}_x = (f_t^x)_{t\in[0,1]}$ such that $f_t^x = \chi_{\{x\}} + t \cdot \chi_{[0,1]\setminus\{x\}}$, then it is easy to observe that $\mathcal{F}_x$ is not with quasi-continuous uncertainty functions (since f_t^x is discontinuous for any $t \neq 1$) but that $\{\varphi_y\}_{y\in[0,1]}$ is equicontinuous.

7. Adapted weighting model with incomplete uncertain system

In this section, we propose a method that can address the discussed continuity problems in Section 6 (i.e., discontinuity and dis-equicontinuity) with the cost of sacrificing some range of uncertainties to be modeled. Put simply, we will consider some systems $\mathfrak{F} = \left(\mathcal{F}_x\right)_{x\in[0,1]} = \left((f_t^x)_{t\in[u,v]}\right)_{x\in[0,1]}$ in which the parameter t of certainty/uncertainty degree will range within a closed proper subset $[u,v] \subset [0,1]$ instead of the unit interval $[0,1]$. Technically, we will map the full certainty situation (i.e., $c = 1$) to the lower bound u and map the full uncertainty situation (i.e., $c = 0$) to the upper bound v. The advantage of this setting is that the three types of continuity as discussed previously may all achieve the corresponding equicontinuities.

Definition 10. The parameterized family focused at $x \in [0,1]$, $\mathcal{F}_x = (f_t^x)_{t\in[u,v]}$, is called *incomplete* if $[u,v] \subset [0,1]$. Moreover,

(i) if f_t^x is continuous for all $t \in [u,v]$, then $\mathcal{F}_x$ is called an *parameterized family with continuous uncertainty functions*; if $(f_t^x)_{t\in[u,v]}$ is equicontinuous, then $\mathcal{F}_x$ is called a *parameterized family with equicontinuous uncertainty functions*;

(ii) if $\varphi_y : [u,v] \to [0,1]$ (with $\varphi_y(t) \triangleq f_t^x(y)$) is continuous for all $y \in [0,1]$, then $\mathcal{F}_x$ is called a *continuous parameterized family*; if $(\varphi_y)_{y\in[0,1]}$ is equicontinuous, then $\mathcal{F}_x$ is called an *equicontinuous parameterized family*.

Definition 11. A system $\mathfrak{F} = \left(\mathcal{F}_x\right)_{x\in[0,1]} = \left((f_t^x)_{t\in[u,v]}\right)_{x\in[0,1]}$ ($[u,v] \subset [0,1]$) is called an incomplete uncertainty system if $\mathcal{F}_x$ is an incomplete parameterized family focused at x for all $x \in [0,1]$.

With the above definitions, we describe the corresponding weighting model as follows. Given a BUI vector $(\mathbf{x},\mathbf{c}) = (x_i,c_i)_{i=1}^n \in \mathcal{B}^n$ together with an incomplete uncertainty system $\mathfrak{F}$, we may still firstly transform each BUI granule into an uncertainty function which will serve as a similar role of judging whether or not (and to what degree) an expert accepts the opinions of others as in Weighting Model 3. Note that with an incomplete uncertainty system, we should also modify the BUI-fuzzy transformation as definition in Definition 6.

Definition 12. The BUI-fuzzy transformation with an incomplete uncertainty system $\mathfrak{F} = \left(\mathcal{F}_x\right)_{x\in[0,1]} = \left((f_t^x)_{t\in[u,v]}\right)_{x\in[0,1]}$ is a mapping $G : \mathcal{B} \to [0,1]^{[0,1]}$ such that $G((x,c)) = f^x_{cu+(1-c)v}$.

Remark. When $[u,v] = [0,1]$, the above BUI-fuzzy transformation $G : \mathcal{B} \to [0,1]^{[0,1]}$ degenerates into BUI-fuzzy transformation $H : \mathcal{B} \to [0,1]^{[0,1]}$.

Correspondingly, the total acceptance degree function $s : [n] \to [0,n]$ for the adapted weighting model should be modified by

$$s(i) = c_i \cdot \sum\nolimits_{j=1}^n G((x,c))(x_i) = c_i \cdot \sum\nolimits_{j=1}^n f^{x_j}_{c_j u+(1-c_j)v}(x_i) \tag{6}$$

The above discussed model is summarized as follows.

Weighting Model 4: BUI environment with incomplete uncertain system

Input: A vector of BUI granules $(\mathbf{x},\mathbf{c}) = \left((x_i,c_i)\right)_{i=1}^n \in \mathcal{B}^n$. An incomplete uncertain system $\mathfrak{F} = \left(\mathcal{F}_x\right)_{x\in[0,1]} = \left((f_t^x)_{t\in[u,v]}\right)_{x\in[0,1]}$.

Output: A normalized weight vector $\mathbf{w} = (w_i)_{i=1}^n$ such that

when $\mathbf{s} \neq \mathbf{0}$, $w_i = \frac{s(i)}{\sum_{j=1}^n s(j)}$ where $s(i) = c_i \cdot \sum_{j=1}^n G((x_j,c_j))(x_i) = c_i \cdot \sum_{j=1}^n f^{x_j}_{c_j u+(1-c_j)v}(x_i)$;

when $\mathbf{s} = \mathbf{0}$, $w_i = 1/n$.

Example 6. (continuation of Example 4)

For $[u,v] = [0.5,1]$, consider the incomplete system $\mathfrak{F} = \left(\mathcal{F}_x\right)_{x\in[0,1]} = \left((f_t^x)_{t\in[0.5,1]}\right)_{x\in[0,1]}$ in which f_t^x is defined similar to Example 3 but only with $t \in [0.5,1)$. Here

(a1) when $x \in (0,1)$ and $t \in [0.5,1)$, f_t^x is defined such that

$$f_t^x(y) = \chi_{[0,x]} \cdot \left[\frac{y}{x} + \frac{x-y}{x}(2t-1)\right] + \chi_{(x,1]} \cdot \left[\frac{1-y}{1-x} + \frac{y-x}{1-x}(2t-1)\right],$$

(a2) when $x = 1$ and $t \in [0.5, 1)$, $f_t^x(y) = y + (1 - y)(2t - 1)$,

(a3) when $x = 0$ and $t \in [0.5, 1)$, $f_t^x(y) = 1 - 2y + 2yt$.

Recall that for all $x \in [0, 1]$ we have defined (in Definition 4) $f_1^x = \chi_{[0,1]}$.

Still let $(\mathbf{x}, \mathbf{c}) = ((0, 0.7), (0.4, 0.5), (0.7, 1), (1, 0.25))$. Then, in comparison to Example 4, using Weighting Model 4 we have

$s(1) = c_1 \cdot \sum_{j=1}^4 f^{x_j}_{c_j u+(1-c_j)v}(x_1) = 0.7\left(f^0_{0.65}(0) + f^{0.4}_{0.75}(0) + f^{0.7}_{0.5}(0) + f^1_{0.875}(0)\right) = 0.7\,(1 + 0.5 + 0 + 0.75) = 1.575$,

$s(2) = c_2 \cdot \sum_{j=1}^4 f^{x_j}_{c_j u+(1-c_j)v}(x_2) = 0.5\left(f^0_{0.65}(0.4) + f^{0.4}_{0.75}(0.4) + f^{0.7}_{0.5}(0.4) + f^1_{0.875}(0.4)\right) = 0.5\left(0.72 + 1 + \frac{4}{7} + 0.85\right) \doteq 1.57$,

$s(3) = c_3 \cdot \sum_{j=1}^4 f^{x_j}_{c_j u+(1-c_j)v}(x_3) = 1\left(f^0_{0.65}(0.7) + f^{0.4}_{0.75}(0.7) + f^{0.7}_{0.5}(0.7) + f^1_{0.875}(0.7)\right) = 0.51 + 0.75 + 1 + 0.925 = 3.185$,

$s(4) = c_4 \cdot \sum_{j=1}^4 f^{x_j}_{c_j u+(1-c_j)v}(x_4) = 0.25\left(f^0_{0.65}(1) + f^{0.4}_{0.75}(1) + f^{0.7}_{0.5}(1) + f^1_{0.875}(1)\right) = 0.25\,(0.3 + 0.5 + 0 + 1) = 0.45$.

Since $\mathbf{s} \neq \mathbf{0}$, then $w_1 = \frac{s(1)}{\sum_{j=1}^4 s(j)} = \frac{1.575}{1.575+1.57+3.185+0.45} = \frac{1.575}{6.78} \doteq 0.232$, $w_2 = \frac{s(2)}{\sum_{j=1}^4 s(j)} = \frac{1.57}{6.78} \doteq 0.232$, $w_3 = \frac{s(3)}{\sum_{j=1}^4 s(j)} = \frac{3.185}{6.78} \doteq 0.47$, $w_4 = \frac{s(4)}{\sum_{j=1}^4 s(j)} = \frac{0.45}{6.78} \doteq 0.066$.

We obtain a weight vector $\mathbf{w} = (0.232, 0.232, 0.47, 0.066)$.

By slightly modifying Definition 9, we have the following equicontinuity definition for the collection $\mathcal{F} = (f_t)_{t \in I}$ (I is an index set) of functions $f_t : [0, 1] \to (Y, d)$ (where (Y, d) is a metric space).

Definition 13. A collection $\mathcal{F} = (f_t)_{t \in I}$ of functions $f_t : [0, 1] \to (Y, d)$ is said to be equicontinuous at the point $x \in [0, 1]$ provided for each $\varepsilon > 0$, there is a $\delta > 0$ such that for every $f_t \in \mathcal{F}$ and $x' \in [0, 1]$,

if $|x - x'| < \delta$, then $d\left(f(x'), f(x)\right) < \varepsilon$.

The collection $\mathcal{F}$ is said to be equicontinuous on [0,1] provided it is equicontinuous at every point in [0,1].

Definition 14. (i) For an incomplete system $\mathfrak{F} = \left(\mathcal{F}_x\right)_{x \in [0,1]} = \left((f_t^x)_{t \in [u,v]}\right)_{x \in [0,1]}$, if for all $t \in [u, v]$ the mappings $u_t : [0, 1] \to \mathcal{F}_x$ (with $u_t(x) \triangleq f_t^x$) is continuous (with respect to $\tilde{d}$), then it is called *continuous with respect to evaluation value* x; if $(u_t)_{t \in [u,v]}$ is equicontinuous (with respect to $\tilde{d}$), then $\mathfrak{F}$ is called *equicontinuous with respect to evaluation value* x.

(ii) If a system $\mathfrak{F} = \left(\mathcal{F}_x\right)_{x \in [0,1]} = \left((f_t^x)_{t \in [u,v]}\right)_{x \in [0,1]}$ is euqicontinuous with respect to evaluation value x, and for each $x \in [0, 1]$, $\mathcal{F}_x = (f_t^x)_{t \in [u,v]}$ is an equicontinuous parameterized family with equicontinuous uncertainty functions, then we call $\mathfrak{F}$ *euqicontinuous*.

Proposition 6. *The incomplete system $\mathfrak{F}$ defined in Example 6 is equicontinuous.*

Proof. We will respectively show (i) for each $x \in [0, 1]$, $\mathcal{F}_x$ is a parameterized family with equicontinuous uncertainty functions; (ii) for each $x \in [0, 1]$, $\mathcal{F}_x$ is an equicontinuous parameterized family; (iii) $\mathfrak{F}$ is equicontinuous with respect to evaluation value x.

(i) For each $x \in (0, 1)$, it is easy to observe that for any $t \in [u, v] = [0.5, 1]$ and any $y, z \in [0, 1]$, $\left|f_t^x(y) - f_t^x(z)\right| \leqslant \max\left(1/x, 1/(1 - x)\right) \cdot |y - z|$. Therefore, for each $y \in [0, 1]$ and any $\varepsilon > 0$, take $\delta = \varepsilon / \max\left(1/x, 1/(1 - x)\right)$; then for every $t \in [u, v]$ and $z \in [0, 1]$, if $|y - z| < \delta$, then $\left|f_t^x(y) - f_t^x(z)\right| \leqslant \max\left(1/x, 1/(1 - x)\right) \cdot |y - z| < \max\left(1/x, 1/(1 - x)\right) \cdot \delta = \varepsilon$. Hence, $(f_t^x)_{t \in [u,v]}$ is equicontinuous on $[0, 1]$. For $x \in \{0, 1\}$, for any $t \in [u, v]$ and any $y, z \in [0, 1]$ $(y \neq z)$ we modify the above corresponding formulation by $\left|f_t^x(y) - f_t^x(z)\right| \leqslant 0.5 \cdot |y - z|$ with the other deductions unchanged, then we can still show $(f_t^x)_{t \in [u,v]}$ is equicontinuous on $[0, 1]$. Consequently, for each $x \in [0, 1]$, $(f_t^x)_{t \in [u,v]}$ is equicontinuous on $[0, 1]$.

(ii) For each $x \in [0, 1]$, it is easy to observe that for any $y \in [0, 1]$ and any $t, s \in [u, v] = [0.5, 1]$, $\left|\varphi_y(t) - \varphi_y(s)\right| \leqslant 2\,|t - s|$. Then, for each $t \in [u, v]$ and any $\varepsilon > 0$, take $\delta = \varepsilon/2$; then for every $y \in [0, 1]$ and $s \in [u, v]$, if $|t - s| < \delta$, then $\left|\varphi_y(t) - \varphi_y(s)\right| \leqslant 2\,|t - s| < \varepsilon$. Hence, $(\varphi_y)_{y \in [0,1]}$ is equicontinuous on $[u, v]$.

(iii) For all $t \in [u, v] = [0.5, 1]$, we notice there is a relation (which can be observed from simple geometry of triangles areas) that for any $a, b \in [0, 1]$ we have

$$\tilde{d}\left(\left[u_t(a)\right], \left[u_t(b)\right]\right) \leqslant 2\left[\frac{|a - b| \cdot (1 - 2\,(t - 0.5))}{2}\right] = |a - b| \cdot (1 - 2\,(t - 0.5)) = 2\,|a - b| \cdot (1 - t) \leqslant |a - b|$$

Hence, for any $a \in [0, 1]$ and any $\varepsilon > 0$, take $\delta = \varepsilon$; then for every $t \in [u, v] = [0.5, 1]$ and any $b \in [0, 1]$, if $|a - b| < \delta$, then $\tilde{d}\left(\left[u_t(a)\right], \left[u_t(b)\right]\right) \leqslant |a - b| < \varepsilon$. Therefore, $(u_t)_{t \in [u,v]}$ is equicontinuous. □

8. Conclusions

The automatic weights allocation methods proposed in this paper are all feasible and convenient. Another significant advantage of them lies in that it is not necessary for the involved experts to interact and negotiate with each other.

We considered the uncertain decision making environment with CII and BUI. Both types of uncertain information can automatically provide intervals or uncertainty functions from which some total acceptance degrees can be derived and calculated in order to further generate some desired normalized weight vectors. All the four weighting models fulfill the three requirements for deriving the total acceptance degrees as defined in Introduction.

The intervals in CII or obtained by using BUI-interval transformation for BUI can be easier to use and involve less computations. However, the different types of continuity are harder to obtain from them. Hence, we proposed some novel concepts of "parameterized family of uncertainty functions", "uncertain system" and "incomplete uncertain system" which can help derive total acceptance degrees for each expert. Applying fuzzy set techniques, there are better continuities in Weighting Model 3 which is based on uncertain system. By scarifying some range of uncertainty functions, much better equicontinuities can be possibly obtained by using Weighting Model 4 which is based on incomplete uncertain systems. The models selection can be up to decision makers in different scenarios or with different preferences.

With the obtained total acceptance degrees, another method to generate normalized weight vectors is to apply inducing weights allocation based on Yager's ordered weighted averaging operators.

CRediT authorship contribution statement

LeSheng Jin: Conceptualization, Formal analysis, Methodology, Writing – original draft, Writing – review & editing. **Zhen-Song Chen:** Formal analysis, Funding acquisition, Project administration, Validation, Writing – review & editing. **Radko Mesiar:** Formal analysis, Methodology, Writing – review & editing. **Tapan Senapati:** Writing – review & editing. **Diego García-Zamora:** Writing – review & editing. **Luis Martínez:** Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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