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Hesitant Fuzzy Linguistic Term Sets

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Abstract

Dealing with uncertainty is always a challenging problem, different tools have been proposed to do it. Recently a new model based on hesitant fuzzy sets has been presented to manage this situation in which experts hesitate between several values to assess an indicator, alternative, variable, etc. Hesitant fuzzy sets suits the modelling of quantitative settings, however similar situations might happen in qualitative settings such that experts think of several possible linguistic values or richer expressions than a single term for an indicator, alternative, variable, etc. In this paper is introduced the concept of Hesitant Fuzzy Linguistic Term Set that will provide a linguistic and computational basis to increase the richness of linguistic elicitation based on the fuzzy linguistic approach and the use of context-free grammars.

Index Terms

Hesitant fuzzy sets, linguistic information, fuzzy linguistic approach, context-free grammar, computing with words.

I. Introduction

Problems defined under uncertain conditions are common in real world, but quite challenging due to the difficulty to model and cope with such an uncertainty. Different tools have been used to solve those problems such as probability, however in many situations uncertainty is not probabilistic in nature, but rather imprecise or vague. Hence other models as fuzzy logic and fuzzy sets theory [7], [34] have been successfully applied to handle imperfect, vague and imprecise information [23]. Nevertheless for the handling of vague and imprecise information whereby

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two or more sources of vagueness appear simultaneously, the modelling tools of ordinary fuzzy sets are limited. For this reason have been introduced different generalizations and extensions of fuzzy sets such as:

- *Type 2 fuzzy sets* [7], [22], and *type n fuzzy sets* [7] that incorporate uncertainty about the membership function in their definition.
- *Intuitionistic fuzzy sets* [1] that extends fuzzy sets by an additional degree, called degree of uncertainty.
- *Fuzzy multisets* [32] based on multisets that allow elements repeated in the set.
- *Hesitant fuzzy sets* recently introduced by Torra [27] provide a very interesting extension of fuzzy sets. It tries to manage those situations where a set of values are possible in the definition process of the membership of an element.

The previous fuzzy tools suits problems defined under quantitative situations, but often uncertainty is due to the vagueness of meanings used by experts in the problems whose nature is rather qualitative. In such situations, the fuzzy linguistic approach [35] has provided very good results in many fields and applications [2], [12], [14], [18], [24], [28]. But similarly to the fuzzy sets the use of the fuzzy linguistic approach presented some limitations mainly regarding information modelling and computational processes, called processes of Computing with Words (CW) [8], [17], [19], [21]. Different linguistic models have tried to extend and improve the fuzzy linguistic approach from both points of view:

- *The linguistic model based on type-2 fuzzy sets* representation [20], [29], [37] that represents the semantics of the linguistic terms by type-2 membership functions and use interval type-2 fuzzy sets for CW.
- *The linguistic 2-tuple model* [11] that adds a parameter to the linguistic representation so-called *symbolic translation* which keeps the accuracy in the processes of CW.
- *The proportional 2-tuple model* [31] generalizes and extends the 2-tuple model by using two linguistic terms with their proportion to model more accurately the information and perform the processes of CW.
- Other extensions based on the previous ones were introduced in [6], [16].

Revising the fuzzy linguistic approach and the different linguistic extensions and generalizations, it is observed that the modelling of linguistic information is still quite limited mainly because it is based on the elicitation of single and very simple terms that should encompass and express the information provided by the experts regarding a linguistic variable. However in different situations the experts involved in the problem defined under uncertainty cannot provide easily a single term as expression of his/her knowledge, because she/he is thinking of several terms at the same time or looking for a more complex linguistic term that usually are not defined in the linguistic term set.

Therefore, with the view of overcoming such limitations and taking into account the idea under the concept of hesitant fuzzy sets provided by Torra [27] to deal with several values in a membership function in a quantitative setting. In this paper we propose the concept of Hesitant Fuzzy Linguistic Term Set (HFLTS) based on the fuzzy linguistic approach that will serve as basis to increase the flexibility of the elicitation of linguistic information. Additionally, different computational functions and properties of HFLTS are introduced, and it is then presented how can it be used to improve the elicitation of linguistic information by using the fuzzy linguistic approach and context-free grammars.

The paper is set up as follows: Section 2 reviews briefly some preliminary concepts that will be used in the HFLTS proposal. Section 3 introduces the concept of HFLTS and several basic properties and operations to carry out processes of CW. Section 4 presents the use of HFLTS to facilitate and increase the flexibility to elicit linguistic information. Section 5 points out some concluding remarks and future research in this topic, and Appendix A contains a brief review about several necessary concepts to compare HFLTS.

II. Preliminaries

Due to the fact that our proposal is based on the fuzzy linguistic approach [35] and the hesitant fuzzy sets [27]. This section reviews their main concepts necessary to understand the proposal of HFLTS and its use.

A. Fuzzy linguistic approach

In many real decision situations is suitable and straightforward the use of linguistic information due to the nature of different aspects of the problem. In such situations one common approach to model the linguistic information is the fuzzy linguistic approach [35] that uses the fuzzy set theory [34] to manage the uncertainty and model the information.

Zadeh [35] introduced the concept of linguistic variable as *"a variable whose values are not numbers but words or sentences in a natural or artificial language"*. A linguistic value is less precise than a number, but it is closer to human cognitive processes used to solve successfully problems dealing with uncertainty. Formally a linguistic variable is defined as follows:

Definition 1: [36] A linguistic variable is characterized by a quintuple (*H,T(H),U,G,M*) in which *H* is the name of the variable; *T(H)* (or simply *T*) denotes the term set of *H*, i.e., the set of names of linguistic values of *H*, with each value being a fuzzy variable denoted generically by *X* and ranging across a universe of discourse *U* which is associated with the base variable *u*; *G* is a syntactic rule (*which usually takes the form of a grammar*) for generating the names of values of *H*; and *M* is a semantic rule for associating its meaning with each *H*, $M(X)$, which is a fuzzy subset of *U*.

To deal with linguistic variables is necessary to choose the linguistic descriptors for the term set and their semantics. Fig. 1 shows a linguistic term set with the syntax and semantics of their terms.

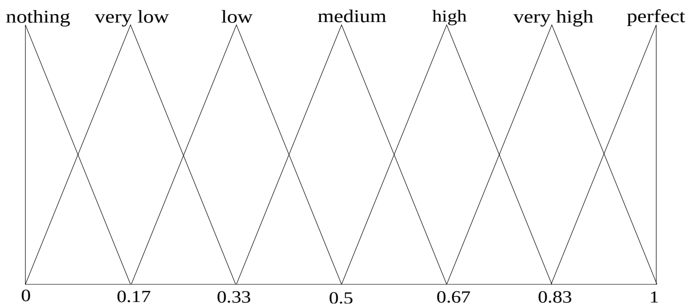

Fig. 1. A Set of 7 Terms with its Semantics

There exist different approaches to select the linguistic descriptors and different ways to define their semantics [33], [35]. The selection of the linguistic descriptors can be performed by means of:

1) *An ordered structure approach*: it defines the linguistic term set by means of an ordered structure providing the term set distributed on a scale on which a total order has been defined [9], [33]. For example, a set of seven terms, S, could be given as follows:

$$S = \{s_0 : nothing, s_1 : very\_low, s_2 : low, s_3 : medium, s_4 : high, s_5 : very\_high, s_6 : perfect\}$$

In these cases, it is usually required that there exist:

a) A negation operator Neg$(s_i) = s_j$ such that $j = g$-i (g+1 is the granularity of the term set).
b) A maximization operator: Max$(s_i, s_j) = s_i$ if $s_i \geq s_j$.
c) A minimization operator: Min$(s_i, s_j) = s_i$ if $s_i \leq s_j$.

2) *A context free grammar approach*: it defines the linguistic term set by means of a context-free grammar, G, such that the linguistic terms are sentences generated by G [3], [4], [35]. A grammar G is a 4-tuple (V_N, V_T, I, P), where V_N is the set of non-terminal symbols, V_T is the set of terminals symbols, I is the starting symbol, and P the production rules defined in an extended Backus Naur Form [4]. Among the terminal symbols of G, we can find primary terms (e.g., low, medium, high), hedges (e.g., not, much, very), relations (e.g., lower than, higher than), conjunctions (e.g., and, but), and disjunctions (e.g., or). Thus, choosing I as any non-terminal symbol and using P could be generated linguistic expressions as, $\{lower\ than\ high, greater\ than\ medium, \ldots\}$.

And the definition of their semantics can be accomplished as [33], [35] :

1) *A semantics based on membership functions and a semantic rule*: this approach assumes that the meaning of each linguistic term is given by means of a fuzzy subset defined in the interval [0,1], which is described by membership functions [4]. This semantic approach is used when the linguistic descriptors are generated by means of a context-free grammar. Thus, it contains two elements: (i) the primary fuzzy sets associated to the primary linguistic terms and (ii) a semantic rule M for providing the fuzzy sets of the non-primary linguistic terms [35].
2) *A semantics based on an ordered structure of the linguistic term set*: it introduces the semantics from the structure defined over the linguistic term set. So, the users provide their assessments by using an ordered linguistic term set [26], [33]. The distribution of a linguistic term set on scale [0,1] can be distributed symmetrically [33] or non-symmetrically [10], [26].
3) *Mixed semantics*: it assumes elements from the aforementioned semantic approaches.

B. Hesitant fuzzy sets

Torra in [27] introduced a new extension for fuzzy sets to manage those situations in which several values are possible for the definition of a membership function of a fuzzy set. Though this situation might be modelled by fuzzy multisets they are not completely adequate for these situations.

A HFS is defined in terms of a function that returns a set of membership values for each element in the domain [27]:

Definition 2: Let X be a reference set, a hesitant fuzzy set on X is a function h that returns a subset of values in [0,1].

$$h : X \to \{[0, 1]\}$$

Therefore, given a set of fuzzy sets a hesitant fuzzy set is defined as the union of their membership functions.

Definition 3: Let $M = \{\mu_1, \mu_2, ..., \mu_n\}$ be a set of n membership functions. The hesitant fuzzy set associated with M, h_M, is defined as:

$$h_M : M \to \{[0, 1]\}$$

$$h_M(x) = \bigcup_{\mu \in M} \{\mu(x)\}$$

Some basic operations with HFS were defined [27]:

Definition 4: Given a hesitant fuzzy set, h, its lower and upper bounds are:

$$h^-(x) = min\ h(x)$$

$$h^+(x) = max\ h(x)$$

Definition 5: Let, h, be a hesitant fuzzy set, its complement is defined as:

$$h^c(x) = \bigcup_{\gamma \in h(x)} \{1 - \gamma\}$$

Proposition 1: [27] The complement is involutive.

$$(h^c)^c = h$$

Definition 6: Let, h_1 and h_2, be two hesitant fuzzy sets, their union is defined as:

$$(h_1 \cup h_2)(x) = \{h \in (h_1(x) \cup h_2(x))/h \geq max(h_1^-, h_2^-)\}$$

Definition 7: Let, h_1 and h_2, be two hesitant fuzzy sets, their intersection is defined as:

$$(h_1 \cap h_2)(x) = \{h \in (h_1(x) \cap h_2(x))/h \leq min(h_1^+, h_2^+)\}$$

Definition 8: Let, h, be a hesitant fuzzy set, the envelope of h, $A_{env(h)}$, is defined as:

$$A_{env(h)} = \{x, \mu_A(x), \nu_A(x)\}$$

Being $A_{env(h)}$ the intuitionistic fuzzy set [1] of h with μ and v defined as:

$$\mu_A(x) = h^-(x)$$

$$v_A(x) = 1 - h^+(x)$$

III. Hesitant fuzzy linguistic term sets

Similarly to the situations described and managed by hesitant fuzzy sets in [27], where an expert might consider several values for defining a membership function. In the qualitative setting may happen that experts hesitate among several values to assess a linguistic variable. But the fuzzy linguistic approach is aimed to assess statically single linguistic terms to the linguistic variables. Hence it is clear when experts hesitate about several values for a linguistic variable, the fuzzy linguistic approach is very limited. As it was pointed out in the introduction several proposals to overcome such a limitation have been proposed in the literature, from the use of modifiers [5], [30] to the addition of parameters [11] or the use of two linguistic terms [31] in order to make more flexible the expressivity of linguistic information. However all of them are still limited and are not adequate to fulfil the necessities and requirements of experts in hesitant situations.

Consequently bearing in mind the idea under the hesitant fuzzy sets [27], in this section is introduced the concept of HFLTS based on the fuzzy linguistic approach and the hesitant fuzzy sets. Some basic operations of HFLTS are then defined and some properties of such operations revised.

A. Concept and Basic Operations

Definition 9: Let S be a linguistic term set, $S = \{s_0, \dots, s_g\}$, a HFLTS, H_S, is an ordered finite subset of consecutive linguistic terms of S.

Let S be a linguistic term set, $S = \{s_0, \dots, s_g\}$, we then define the empty HFLTS and the full HFLTS for a linguistic variable, x, as follows:

- Empty HFLTS: $H_S(x) = \{\}$
- Full HFLTS: $H_S(x) = S$

Any other HFLTS is formed at least with one linguistic term in S.

Example 1: Let S be a linguistic term set, $S = \{s_0 : nothing, s_1 : very\_low, s_2 : low, s_3 : medium, s_4 : high, s_5 : very\_high, s_6 : perfect\}$, different HFLTS might be:

$$H_S(x) = \{s_1 : very\_low, s_2 : low, s_3 : medium, s_4 : high\}$$

$$H_S(x) = \{s_3 : medium, s_4 : high, s_5 : very\_high, s_6 : perfect\}$$

Once it has been defined the concept of HFLTS, it is necessary the introduction of computations and operations that can be performed on them.

Let S be a linguistic term set, $S = \{s_0, \ldots, s_g\}$ and H_S, H_S^1, and H_S^2 three HFLTS:

Definition 10: The upper bound, H_{S^+}, and lower bound, H_{S^-}, of the HFLTS, H_S, are defined as:

- A *max* operator: $H_{S^+} = max(s_i) = s_j$, $s_i \in H_S$ and $s_i \leq s_j$ $\forall i$
- A *min* operator: $H_{S^-} = min(s_i) = s_j$, $s_i \in H_S$ and $s_i \geq s_j$ $\forall i$

Definition 11: The complement of HFLTS, H_S, is defined as:

$$H_S^c = S - H_S = \{s_i/s_i \in S \ and \ s_i \notin H_S\}$$

Proposition 2: The complement of a HFLTS is involutive:

$$(H_S^c)^c = H_S$$

Proof.:

Using the definition of *complement of a HFLTS*,

$$(H_S^c)^c = S - H_S^c = S - (S - H_S) = H_S$$

Definition 12: The union between two HFLTS, H_S^1 and H_S^2 is defined as:

$$H_S^1 \cup H_S^2 = \{s_i/s_i \in H_S^1 \ or \ s_i \in H_S^2\}$$

the result will be another HFLTS.

Definition 13: The intersection of two HFLTS, H_S^1 and H_S^2 is:

$$H_S^1 \cap H_S^2 = \{s_i/s_i \in H_S^1 \ and \ s_i \in H_S^2\}$$

the result of this operation is another HFLTS.

The comparison of linguistic terms is necessary in many problems and it has been always defined in the different linguistic approaches. A HFLTS is a linguistic term subset, the comparison among these elements is not simple. Therefore, we introduce the concept of *envelope* for a HFLTS in order to simplify these operations as it is showed later on.

Definition 14: The envelope of the HFLTS, $env(H_S)$, is a linguistic interval whose limits are obtained by means of upper bound (*max*) and lower bound (*min*), hence:

$$env(H_S) = [H_{S^-}, H_{S^+}], \quad H_{S^-} <= H_{S^+}$$

Example 2: Let $S = \{nothing, very\_low, low, medium, high, very\_high, perfect\}$ be a linguistic term set, and $H_S = \{high, very\_high, perfect\}$ be a HFLTS of S, its envelope is:

$H_{S^-}(high, very\_high, perfect) =$ **high**, $H_{S^+}(high, very\_high, perfect) =$ **perfect**

$$env(H_S) = [high, perfect]$$

Definition 15: The definition of the comparison between two HFLTS is based on the concept of envelope of the HFLTS, $env(H_S)$. Hence, the comparison between, H_S^1 and H_S^2 is defined as follows:

$$H_S^1(x) > H_S^2(x) \quad iff \quad env(H_S^1(x)) > env(H_S^2(x))$$

$$H_S^1(x) = H_S^2(x) \quad iff \quad env(H_S^1(x)) = env(H_S^2(x))$$

Consequently the comparison is conducted by interval values. In the Appendix *A* is briefly reviewed different approaches to compare intervals and it is then clarified how compare HFLTS.

B. Properties

To conclude this section some relevant properties of the HFLTS operations are reviewed. Let H_S^1, H_S^2 and H_S^3 be three HFLTS and $S = \{s_0, \ldots, s_g\}$, then

- *Commutativity*

$$H_S^1 \cup H_S^2 = H_S^2 \cup H_S^1$$
$$H_S^1 \cap H_S^2 = H_S^2 \cap H_S^1$$

Proof. of the union:

$\subseteq$

Let $s_i \in S$ be a linguistic value, $s_i \in H_S^1 \cup H_S^2$, then by the definition of *union*, $s_i \in H_S^1$ or $s_i \in H_S^2$, if $s_i \in H_S^2$ or $s_i \in H_S^1$, then $s_i \in H_S^2 \cup H_S^1$

$\supseteq$

Let $s_i \in H_S^2 \cup H_S^1$, then $s_i \in H_S^2$ or $s_i \in H_S^1$, if $s_i \in H_S^1$ or $s_i \in H_S^2$, then $s_i \in H_S^1 \cup H_S^2$

The demonstration of the intersection would be similar to the union.

- *Associative*

$$H_S^1 \cup (H_S^2 \cup H_S^3) = (H_S^1 \cup H_S^2) \cup H_S^3$$

$$H_S^1 \cap (H_S^2 \cap H_S^3) = (H_S^1 \cap H_S^2) \cap H_S^3$$

Proof. of the union:

$\subseteq$

Let $s_i \in S$ be a linguistic value, $s_i \in H_S^1 \cup (H_S^2 \cup H_S^3)$ then, $s_i \in H_S^1$ or $s_i \in H_S^2 \cup H_S^3$. On the second case, $s_i \in H_S^2$ or $s_i \in H_S^3$, so if $s_i \in H_S^1 \cup H_S^2$ or $s_i \in H_S^3$, then $s_i \in (H_S^1 \cup H_S^2) \cup H_S^3$

$\supseteq$

Let $s_i \in (H_S^1 \cup H_S^2) \cup H_S^3$ then, $s_i \in H_S^1 \cup H_S^2$ or $s_i \in H_S^3$. On the first case, $s_i \in H_S^1$ or $s_i \in H_S^2$, so if $s_i \in H_S^1$ or $s_i \in H_S^2 \cup H_S^3$, then $s_i \in H_S^1 \cup (H_S^2 \cup H_S^3)$

In a similar way, the associative property of the intersection can be demonstrated.

- *Distributive*

$$H_S^1 \cap (H_S^2 \cup H_S^3) = (H_S^1 \cap H_S^2) \cup (H_S^1 \cap H_S^3)$$

$$H_S^1 \cup (H_S^2 \cap H_S^3) = (H_S^1 \cup H_S^2) \cap (H_S^1 \cup H_S^3)$$

Proof. of the union:

$\subseteq$

Let $s_i \in (H_S^1 \cup H_S^2) \cap H_S^3$, then $s_i \in H_S^1 \cup H_S^2$ and $s_i \in H_S^3$. So $s_i \in H_S^1$ or $s_i \in H_S^2$.

If $s_i \in H_S^1$, then $s_i \in H_S^1 \cap H_S^3$

If $s_i \in H_S^2$, then $s_i \in H_S^2 \cap H_S^3$

Thus, $s_i \in H_S^1 \cap H_S^3$ or $s_i \in H_S^2 \cap H_S^3$, this is mean, $s_i \in (H_S^1 \cap H_S^3) \cup (H_S^2 \cap H_S^3)$

$\supseteq$

Let $s_i \in (H_S^1 \cap H_S^3) \cup (H_S^2 \cap H_S^3)$. Then $s_i \in H_S^1 \cap H_S^3$ or $s_i \in H_S^2 \cap H_S^3$. On the first case, as $s_i \in H_S^1$, then $s_i \in H_S^1 \cup H_S^2$, so $s_i \in (H_S^1 \cup H_S^2) \cap H_S^3$. On the second case, as $s_i \in H_S^2$, then $s_i \in H_S^1 \cup H_S^2$, so $s_i \in (H_S^1 \cup H_S^2) \cap H_S^3$

Similarly to the property of the union, the distributive property of the intersection can be demonstrated.

IV. Elicitation of linguistic information based on HFLTS

Across the paper it has been pointed out that the aim of the introduction of HFLTS was to improve the elicitation of linguistic information, mainly when the experts hesitate among several values to assess linguistic variables.

So far, it has been introduced the concept of HFLTS that can be directly used by the experts to elicit several linguistic values for a linguistic variable, but such elements are not similar to the human beings way of thinking and reasoning. Therefore in this section, it is proposed the definition of simple but elaborated linguistic sentences that are more similar to human beings expressions and semantically represented by means of HFLTS and generated by a context-free grammar.

A simple context-free grammar, G_H, is introduced to support the type of linguistic information that we want to allow eliciting the experts in order to increase the flexibility and expressiveness of linguistic information, denoted by ll.

Additionally to this grammar, it is necessary to define how its linguistic expressions will be represented and managed in processes of CW. To do so, it is presented a function, $E(ll)$, that transforms such linguistic information into HFLTS.

Such context-free grammar and transformation function are further detailed in the coming subsections.

A. Context-free grammar for eliciting linguistic information based on HFLTS

A context-free grammar, G, provides a way to generate linguistic terms and linguistic sentences by means of its different elements. Our objective is to define a context-free grammar, G_H, that generates simple but rich linguistic terms and sentences that can be easily represented by means of HFLTS. Therefore, the context-free grammar, G_H, is defined to generate the type of linguistic information that we want to model in hesitant situations:

Definition 16: Let G_H be a context-free grammar and $S = \{s_0, \ldots, s_g\}$ a linguistic term set. The elements of $G_H = (V_N, V_T, I, P)$ are defined as follows:

$V_N = \{\langle primary\ term \rangle, \langle composite\ term \rangle, \langle unary\ relation \rangle, \langle binary\ relation \rangle,$
$\langle conjunction \rangle\}$

$V_T = \{lower\ than, greater\ than, between, and, s_0, s_1, \ldots, s_g\}$

$I \in V_N$

The productions rules are defined in an extended Backus Naur Form such that the brackets enclose optional elements and the symbol | indicate alternative elements [4]. For G_H are the following ones:

$P = \{I ::= \langle primary\ term\rangle | \langle composite\ term\rangle$

$\langle composite\ term\rangle ::= \langle unary\ relation\rangle\langle primary\ term\rangle | \langle binary\ relation\rangle\langle primary\ term\rangle$

$\langle conjunction\rangle\langle primary\ term\rangle$

$\langle primary\ term\rangle ::= s_0|s_1|\ldots|s_g$

$\langle unary\ relation\rangle ::= lower\ than|greater\ than$

$\langle binary\ relation\rangle ::= between$

$\langle conjunction\rangle ::= and\}$

Remark 1: The *unary relation* has some limitations. If the non-terminal symbol is *lower than*, the *primary term* cannot be s_0 and if the non-terminal symbol is *greater than* the *primary term* cannot be s_g.

Remark 2: In the *binary relation* the *primary term* of the left side must be less than the *primary term* of the right side.

Example 3: Let $S = \{nothing, very\_low, low, medium, high, very\_high, perfect\}$ be a linguistic term set, some linguistic expressions obtained by means of the context-free grammar, G_H, might be:

$ll_1 = high$

$ll_2 = lower\ than\ medium$

$ll_3 = greater\ than\ low$

$ll_4 = between\ medium\ and\ very\_high$

These linguistic structures are very similar to the way that human beings express their knowledge in real situations in which they are not sure about one single value to assess such situations. Therefore, the hesitant situation is modelled by means of linguistic structures generated by the production rules, $P \in G_H$, being necessary to model semantically such information. To do so, it is proposed the use of HFLTS.

B. Transforming linguistic information of G_H in HFLTS

The transformation of the linguistic expressions, ll, produced by G_H into HFLTS is done by means of the transformation function E_{G_H}.

Definition 17: Let E_{G_H} be a function that transforms linguistic expressions, ll, obtained by G_H, into HFLTS, H_S, where S is the linguistic term set used by G_H.

$$E_{G_H} : ll \longrightarrow H_S$$

The linguistic expressions generated by using the production rules will be transformed into HFLTS in different ways according to their meaning.

- $E_{G_H}(s_i) = \{s_i / s_i \in S\}$
- E_{G_H}(less than s_i) $= \{s_j / s_j \in S \ and \ s_j \leq s_i\}$
- E_{G_H}(greater than s_i) $= \{s_j / s_j \in S \ and \ s_j \geq s_i\}$
- E_{G_H}(between s_i and s_j) $= \{s_k / s_k \in S \ and \ s_k \geq s_i \ and \ s_k \leq s_j\}$

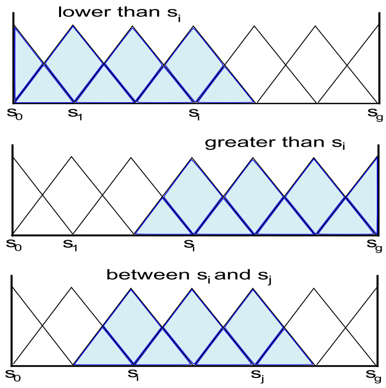

Fig. 2. HFLTS associated to the linguistic expressions

With the previous definition of E_{G_H} is easy to figure out the representation of the initial linguistic expressions, ll, into HFLTS. The Fig. 2 shows graphically these transformations.

Example 4: By using the linguistic expressions obtained in Example 3, $ll_1, ll_2, ll_3, \ and \ ll_4$ their transformation into HFLTS by E_{G_H} is:

$E_{G_H}(high) = \{high\}$

$E_{G_H}(lower \ than \ medium) = \{nothing, very\_low, low, medium\}$

$E_{G_H}(greater \ than \ low) = \{low, medium, high, very\_high, perfect\}$

$E_{G_H}(between \ medium \ and \ very\_high) = \{medium, high, very\_high\}$

V. Concluding Remarks and Future Works

There are many problems in real world that deal with vague and imprecise information. In the literature there exist different approaches to manage and model this type of information.

Recently it has been presented a new model based on hesitant fuzzy sets in which the experts hesitate among several values to assess an alternative or variable in quantitative setting. However, it might happen similar situations in a qualitative setting. This paper has introduced the concept of HFLTS to increase the flexibility and richness of linguistic elicitation based on the fuzzy linguistic approach and the use of context-free grammars to support the elicitation of linguistic information by experts in hesitant situations under qualitative settings. In addition, different computational functions and properties of HFLTS have been presented.

In the future, it will be explored the application of HFLTS to decision problems defined under uncertainty where the experts will be able to provide their assessments by using linguistic expressions based on HFLTS similar to the human beings expressions.

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Appendix A

Due to the fact that the comparison of HFLTS is based on their envelope that are intervals, in this appendix is made a brief review about several methods to compare numeric intervals that could be used in the comparison of HFLTS, but first it is revised the concept of numeric interval.

Definition 18: [13] An interval is defined by an ordered pair in brackets as:

$$A = [a_L, a_R] = \{a : a_L <= a <= a_R\}$$

where a_L is the left limit and a_R is the right limit of A.

Definition 19: [13] The interval is also denoted by its center and width as:

$$A = \langle a_C, a_W \rangle = \{a : a_C - a_W <= a <= a_C + a_W\}$$

where a_C is the center and a_W is the width of A.

From definitions 18 and 19, the center and width of an interval may be calculated as:

$$a_C = \frac{1}{2}(a_R + a_L)$$
$$a_W = \frac{1}{2}(a_R - a_L)$$

Different approaches to compare intervals have been introduced in the literature. Tanaka and Ishibuchi presented in [13] two order relations. One of them is defined by the left and right limits of an interval. This order relation is partial and there are many pairs of intervals that cannot be compared with such a relation. To overcome this limitation the authors defined a second order relation by the center and width of interval, but it is also a partial order relation. Kundu in [15] defined a fuzzy preference relation between two intervals on the real line by means of a formula that uses probability relations. The disadvantage of this approach is that it does not take into account the width of the intervals, and therefore, it could obtain that two intervals are equal, although their width were different.

Afterwards, Sengupta in [25] presented two approaches to compare any two interval numbers. Following it is presented in further detail one of them that we consider suitable to accomplish the comparison of HFLTS by using their envelopes, because it overcomes the drawbacks of Tanaka, Ishibuchi and Kundu's approaches. Such a method introduces an acceptability function which indicates the grade of acceptability regarding *the first interval is inferior to the second interval* and it is defined as follows:

Definition 20: [25] Let I be the set of all closed intervals on the real line $\Re$, and A and B two intervals, $A, B \in I$. The acceptability function $A_< : I \times I \longrightarrow [0, \infty)$ is defined as:

$$A_< = \frac{b_C - a_C}{b_W + a_W}$$

where $b_W + a_W \neq 0$, being a_C, b_C, a_W and b_W the center and width of the intervals A and B.

This grade of acceptability is a real number that represents the grade of acceptation of the interval A is inferior to the interval B and it is interpreted as:

- if $A_< = 0$ then it is not accepted that the interval A is inferior to B
- if $0 < A_< < 1$, then $A_<$ is accepted with different grades of satisfaction from zero to one.
- if $A_< >= 1$, it is absolutely truth that the interval A is inferior to B