Universidad de Jaén

Escuela de Doctorado

TESIS DOCTORAL

TOMA DE DECISIONES INTELIGENTES BAJO INCERTIDUMBRE

PRESENTADA POR:

DIEGO GARCÍA ZAMORA

DIRIGIDA POR:

LUIS MARTÍNEZ LÓPEZ

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Diego García Zamora: *Toma de decisiones inteligentes bajo incertidumbre,*
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La memoria titulada Toma de decisiones inteligentes bajo incertidumbre, que presenta D. Diego García Zamora para optar al grado de doctor, ha sido realizada dentro del Programa de Doctorado en Tecnologías de la Información y la Comunicación de la Universidad de Jaén bajo la dirección del doctor D. Luis Martínez López. Para su evaluación, esta memoria se presenta como un compendio de trabajos publicados, acogiéndose y ajustándose a lo establecido en el punto 2 del artículo 25 del Reglamento de los Estudios de Doctorado de la Universidad de Jaén aprobado en febrero de 2012 y modicado en febrero de 2019.

En Jaén, a 16 de Mayo de 2023

Doctorando   Director

Fdo. Diego García Zamora   Fdo. Luis Martínez López

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ÍNDICE GENERAL

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ACRÓNIMOS

TD Toma de Decisiones

TDG Toma de Decisiones en Grupo

PAC Procesos para Alcanzar el Consenso

TDGGE Toma de Decisiones en Grupo a Gran Escala

PACGE Procesos para Alcanzar el Consenso en Gran Escala

CCM Consenso de Coste Mínimo

Parte I

MEMORIA DE TESIS DOCTORAL

1

INTRODUCCIÓN

En el primer capítulo de esta memoria, se proporciona una introducción que tiene como objetivo contextualizar la investigación realizada en esta tesis doctoral. Primero, se aborda el área de investigación sobre la que se enfoca la tesis, las motivaciones y las hipótesis que han impulsado su realización. Posteriormente, se presentan los objetivos que se pretenden alcanzar y, finalmente, se describe la estructura general de la tesis.

1.1 CONTEXTO INICIAL DE LA INVESTIGACIÓN

La Toma de Decisiones (TD) suele entenderse como un proceso cognitivo que involucra diversos procesos mentales y de razonamiento para elegir la opción más apropiada entre varias posibles soluciones en una situación determinada [23]. La TD se vuelve más compleja en situaciones de incertidumbre que requieren considerar diversos tipos de costes, conflictos y grandes volúmenes de datos [36]. Para abordar problemas de esta complejidad, se desarrollan técnicas inteligentes de decisión adaptados específicamente al problema en cuestión [5]. De forma general, se puede decir que los procesos de TD tienen ciertas fases comunes [18], tales como las siguientes (ver Fig. 1.1).

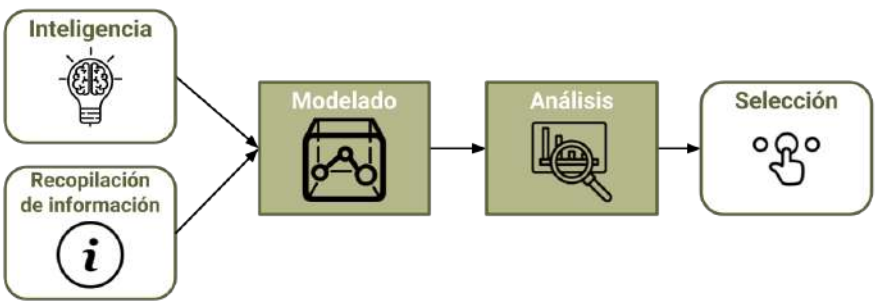

Fig. 1.1: Esquema de un proceso de TD

INTELIGENCIA. Observar el mundo real para identificar el problema.

RECOPILACIÓN DE INFORMACIÓN. Obtener datos, conocimiento y preferencias relacionados con el problema.

MODELADO. Definir un marco de trabajo que establezca la estructura del problema, las preferencias y la incertidumbre.

ANÁLISIS. Estudiar y combinar la información según los objetivos, restricciones y resultados considerados en la fase de selección.

SELECCIÓN. Explotar los resultados del análisis para seleccionar una alternativa/solución al problema.

En la actualidad, cada vez es más común utilizar procesos de TD dirigidos por datos (data-driven) y modelos cuantitativos que pueden limitar la participación de expertos humanos que habitualmente toman parte en los procesos de decisión utilizando información cualitativa [1]. A pesar de esto, en muchos campos que requieren decisiones inteligentes, eficientes y eficaces en condiciones de incertidumbre, la participación de decisores humanos sigue siendo esencial [37]. De hecho, para abordar situaciones de decisión en las que no hay datos o información objetiva disponible sobre el problema en cuestión, los enfoques de TD guiados por expertos (expert-driven) siguen siendo indispensables [41]. Sin embargo, para integrar de forma eficiente el conocimiento experto en la TD es necesario tener en cuenta ciertas limitaciones asociadas a la condición humana.

En primer lugar, para garantizar una TD eficaz conviene atender al modelado de las preferencias de los expertos. A diferencia de cuando se utilizan datos numéricos, usar opiniones dadas por expertos requiere considerar tanto aspectos psicológicos involucrados en el proceso de obtención de las preferencias como el modelado de la incertidumbre asociada a las mismas [14, 22].

En segundo lugar, es necesario considerar que, al contrario de lo que ocurre cuando se utilizan modelos dirigidos por datos, cuando a los expertos se les pide que expresen sus preferencias acerca de las posibles soluciones al problema de TD, éstos tendrán una visión parcial sobre el mismo atendiendo no sólo a su formación y experiencia, sino también a sus propios intereses [24]. Para eliminar los posibles sesgos en el proceso de TD originados por estos factores es recomendable el uso de técnicas de Toma de Decisiones en Grupo (TDG), que tienen en cuenta simultáneamente las opiniones de varios expertos [32].

Finalmente, cuando varios expertos participan en un problema de TDG es de esperar que, en cierto sentido, todos ellos busquen ver reflejadas sus opiniones particulares en la decisión final, lo que puede llevar a situaciones de conflicto entre los miembros del grupo [9]. Los Procesos para Alcanzar el Consenso (PAC) tienen como objetivo garantizar un nivel de acuerdo en las decisiones obtenidas en problemas de TDG [40]. Generalmente, se dice que un PAC es un proceso iterativo, habitualmente coordinado por un moderador humano, cuyo objetivo es aumentar el grado de acuerdo entre los miembros del grupo durante múltiples rondas de discusión si es necesario.

En los problemas de TDG clásicamente participaban un número reducido de expertos. Sin embargo, los avances tecnológicos de los últimos tiempos permiten involucrar a muchos más decisores en la resolución de los problemas de TDG [25]. En este contexto, ha surgido recientemente un gran interés en los denominados problemas de Toma de Decisiones en Grupo a Gran Escala (TDGGE), así como sus Procesos para Alcanzar el Consenso en Gran Escala (PACGE), cuyo objetivo inicial es resolver problemas de TDG en los que intervienen un gran número de decisores. Considerar muchos decisores en los procesos de TD permite tener en cuenta más experiencias, conocimiento y perspectivas, lo cual puede conducir a una decisión más completa que tenga en cuenta una mayor variedad de factores y posibles resultados. Consecuentemente, los grandes grupos pueden tomar decisiones con una mayor diversidad y mejor documentadas que las decisiones tomadas por un único individuo o un pequeño grupo.

1.2 MOTIVACIÓN

La TDGGE ofrece una capacidad mayor para resolver problemas reales de TDG en los que participan un gran número de decisores, frente la TDG clásica en los que el número se limita a un conjunto reducido de los mismos. Sin embargo, aumentar el número de decisores involucrados supone también un aumento de la complejidad del proceso de TD, lo cual requiere el desarrollo de nuevos modelos, métodos y herramientas para mejorar y analizar estos problemas bajo las nuevas premisas. Esto es especialmente importante para garantizar la efectividad de los procesos de TDGGE y abordar nuevos desafíos emergentes como los siguientes:

1. Modelado no lineal de preferencias. Tradicionalmente, se ha considerado que los expertos proporcionan sus opiniones de forma lineal [10]. Sin embargo, estudios psicológicos recientes sostienen que usar deformaciones no lineales de sus preferencias podría mejorar los resultados de la TD [6].

2. Procesos de TDGGE que pueden y/o deben involucrar cientos o miles de expertos. En la literatura especializada, la TDGGE se ha definido como un problema de TDG en el que intervienen más de una veintena de expertos [5]. Sin embargo, la filosofía de este tipo de problemas, por los avances tecnológicos actuales y la demanda social, debe enfocarse a problemas en los que puedan tomar parte cientos, miles e incluso más decisores [37]. Por tanto, es necesario mejorar los procesos de TDGGE para poder manejar eficientemente las opiniones en grupos realmente grandes, como los usuarios de un marketplace o de una red social.

3. Métricas para PACGE. En la literatura relacionada existen múltiples propuestas para PACGE, pero es difícil discriminar de manera objetiva cuál es la más adecuado para resolver un problema de TDGGE concreto. Consecuentemente, es necesario desarrollar métricas capaces de determinar el rendimiento de los PACGE de forma objetiva [15].

Estas limitaciones y desafíos en el ámbito de la TDGGE bajo incertidumbre nos llevaron a formular las siguientes hipótesis al inicio de esta investigación:

1. El uso de modelos de representación no lineal de las opiniones de los expertos permitirá mejorar los resultados de los problemas de TDG y sus PAC.

2. La optimización de los procesos de TDGGE actuales permitirá abordar de forma eficiente problemas con un gran número de decisores, mejorando los resultados y campos de aplicación actuales.

3. La definición de métricas para los PACGE permitirá una evaluación más eficaz del rendimiento tanto de los PAC existentes como de las nuevas propuestas.

1.3 OBJETIVOS

Teniendo en cuenta las motivaciones derivadas de las limitaciones en la literatura actual de TDGGE y partiendo de las hipótesis iniciales, el propósito central de esta tesis doctoral consiste en mejorar los procesos de TDGGE en presencia de incertidumbre y sus PACGE mediante el empleo de herramientas y modelos matemáticos. Con ello se busca superar las deficiencias metodológicas actuales y contribuir a una mayor precisión y robustez en la TD en este campo. En consecuencia, se proponen los siguientes objetivos:

1. Definir un marco metodológico para modelar el comportamiento no lineal de los expertos al proporcionar sus opiniones que permita corregir las desviaciones derivadas de la psicología humana.

2. Optimizar procesos de TDGGE para abordar problemas de TD que requieran de la participación de un gran número de decisores (cientos, miles, ...).

3. Crear métricas para PACGE que establezcan estándares de rendimiento para el alcance de consenso.

1.4 ESTRUCTURA

De acuerdo a lo establecido en el artículo 25 punto 2 de la normativa actual de los Estudios de Doctorado en la Universidad de Jaén (RD. 99/2011), esta tesis doctoral consistirá en una compilación de artículos publicados por el estudiante de doctorado. El objetivo de esta compilación es alcanzar los objetivos previamente establecidos en la sección anterior. Específicamente, esta memoria se compone de seis artículos, publicados o aceptados, en revistas internacionales Q1 indexadas en la base de datos Journal Citation Reports (JCR).

El resto de la memoria consta de los siguientes capítulos:

CAPÍTULO 2. Se presentan los conceptos fundamentales relacionados con la temática de la tesis doctoral. Se describen los problemas de TDG, prestando especial atención a la TDGGE, y se analizan las ventajas y limitaciones de los modelos de decisión existentes. También, se expone la necesidad de los PACGE para alcanzar soluciones consensuadas.

CAPÍTULO 3. Se describen los seis artículos que conforman esta memoria, resaltando los resultados logrados y las conclusiones obtenidas en cada uno de ellos.

CAPÍTULO 4. Está integrado por las seis propuestas mencionadas anteriormente.

CAPÍTULO 5. Se identifican las principales conclusiones la tesis doctoral y se sugieren posibles campos de investigación para futuros trabajos.

De forma adicional se ha incluido un Apéndice (Apéndice A) que presenta un resumen en inglés de la investigación realizada, con el fin de cumplir con los requisitos necesarios para obtener la Mención Internacional de Doctorado. Para finalizar, se ha incluido una recopilación bibliográfica de los artículos más relevantes relacionados con esta memoria.

2

CONCEPTOS TEÓRICOS Y ANTECEDENTES

En este capítulo se presenta un resumen breve de los conceptos teóricos y antecedentes relevantes para la investigación presentada en esta memoria. Concretamente, se abordan los conceptos básicos sobre la TDG y sus PAC, así cómo un resumen del estado del arte de la TDGGE, revisando las principales propuestas en la literatura, así como las limitaciones existentes en TDGGE tanto desde el punto de vista de la TDG como de los PAC. Los contenidos de este capítulo se desarrollan con mayor profundidad en la Sección 4.1, que se corresponde con una revisión sistemática sobre TDGGE realizada y publicada durante mi investigación y en la que dichos conceptos y cuestiones relacionadas con ellos se muestran en mayor detalle.

2.1 TOMA DE DECISIONES EN GRUPO Y PROCESOS DE ALCANCE DE CONSENSO

Un problema de TDG se presenta cuando se necesita que varios individuos elijan la mejor de entre varias alternativas para solucionar un problema determinado [10]. Formalmente, los problemas de TDG se modelan como un par (D, A) en el que D es un conjunto finito de m decisores

$$D = \{dm_1, dm_2, \ldots, dm_m\},$$

a los que se les pide evaluar n alternativas

$$A = \{a_1, a_2, \ldots, a_n\},$$

con el objetivo de seleccionar la mejor solución para el problema de TD. Por lo general, la resolución de este tipo de problemas consta de dos pasos principales [28] (ver Fig. 2.1).

AGREGACIÓN. Para representar la opinión global del grupo, las preferencias de los decisores se combinan mediante un operador de agregación en una única preferencia colectiva.

EXPLOTACIÓN. Se elige una o varias alternativas como solución al problema.

Fig. 2.1: Esquema de un problema TDG

En el pasado, Butler y Rothstein [4] propusieron varias reglas para resolver el proceso de TDG, como la regla de la mayoría, la regla de la minoría o el recuento de Borda. No obstante, al emplear estas reglas, es posible que algunos decisores no estén completamente satisfechos con la solución elegida ya que sus opiniones pueden no haber sido debidamente consideradas en la elección colectiva final. Con el fin de abordar las posibles discrepancias entre las opiniones de los decisores, se suele incorporar un PAC en la resolución de los problemas de TDG. Un PAC es un proceso dinámico e iterativo en el que los decisores discuten entre sí y ajustan sus opiniones iniciales para lograr una mayor consenso dentro del grupo. Por lo general, estos procesos son supervisados por un moderador que se encarga de proporcionar a los decisores la información necesaria sobre el estado de las negociaciones para eliminar conflictos. En términos generales, un PAC se compone de cuatro pasos principales, según se ha descrito en la literatura [24] (ver Fig. 2.2).

RECOPILACIÓN DE PREFERENCIAS. Los decisores proporcionan sus evaluaciones sobre las alternativas utilizando estructuras de preferencia.

CÓMPUTO DEL NIVEL DE CONSENSO . Se utilizan medidas de consenso para obtener el nivel actual de acuerdo en el grupo.

CONTROL DEL CONSENSO. Se compara el nivel actual de acuerdo en el grupo con un nivel de consenso deseado previamente establecido. Si el grupo logra alcanzar el nivel deseado, el PAC finaliza y se procede con el proceso de selección para elegir la mejor alternativa. En caso contrario, se lleva a cabo otra ronda. Para evitar que el proceso se prolongue indefinidamente, se establece un límite en el número de rondas permitidas.

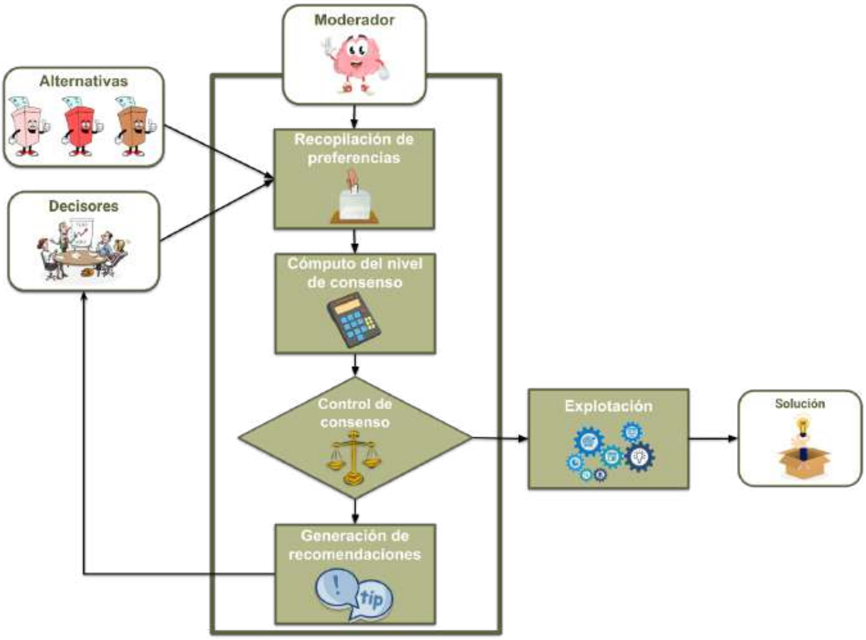

Fig. 2.2: Esquema de un PAC

GENERACIÓN DE RECOMENDACIONES. El moderador identifica a los decisores y a las opiniones que causan mayor conflicto en el grupo y les sugiere la forma de modificarlas para eliminar dicho conflicto.

2.2 TOMA DE DECISIONES EN GRUPO A GRAN ESCALA

Históricamente, los problemas de TDG y sus PAC han involucrado solo a un número reducido de decisores. No obstante, los avances tecnológicos recientes, como el Big Data [21] y el comercio electrónico [35], junto con las demandas de la sociedad moderna para abordar problemas críticos como situaciones de emergencia [34] o sostenibilidad [17], han dado lugar a nuevas situaciones de decisión que requieren la participación de un mayor número de decisores en los procesos de TD. En este contexto, surge la denominada TDGGE, que, de acuerdo a la definición clásica, se refiere a los problemas de TDG en los que participan un gran número de decisores (habitualmente se ha definido en la literatura especializada como veinte o más decisores) [7].

Fig. 2.3: Esquema de un problema de TDGGE

Tang et al. [33] y Labella et al. [12] señalan que la participación de un gran número de decisores con diferentes puntos de vista y preferencias requiere considerar nuevos aspectos en el proceso general de resolución de problemas de TDG (ver Fig. 2.3):

REDUCCIÓN DE LA DIMENSIÓN. Para manejar la gran cantidad de información involucrada en los modelos de TDGGE, se utilizan mecanismos para reducir la dimensionalidad de los datos.

PONDERACIÓN Y AGREGACIÓN DE INFORMACIÓN. Están relacionadas con la tarea de determinar adecuadamente la importancia de cada uno de los decisores que participan en el proceso de decisión y fusionar sus opiniones de manera efectiva.

GESTIÓN DE COMPORTAMIENTOS Se refiere a la necesidad de incluir mecanismos para detectar y manejar a los decisores que no colaboran en el proceso de TD, con el fin de evitar que estos decisores afecten negativamente el resultado final.

GESTIÓN DE COSTES. Es necesario considerar los recursos humanos, económicos y de tiempo requeridos para desarrollar modelos que tengan como objetivo gestionar la participación de cientos, miles o incluso millones de decisores en el proceso de decisión.

ANÁLISIS DE REDES SOCIALES. En grupos grandes, es hay que considerar cómo las relaciones entre los decisores (como la confianza o la reputación) influyen en el proceso de decisión.

CONSENSO. A medida que aumenta el número de personas involucradas en una decisión, la probabilidad de desacuerdo también aumenta. Por lo tanto, es esencial desarrollar PACGE para grandes grupos con el fin de llegar a soluciones acordadas.

La TDGGE se basa en el esquema clásico de TDG (ver Fig. 2.1), que consta de dos fases: agregación y explotación. Sin embargo, la fase de agregación en TDGGE es mucho más compleja ya que se tienen en cuenta varios aspectos para fusionar los valores originales de las opiniones de los decisores. Gracias a esta combinación de técnicas, es posible proponer una amplia variedad de esquemas TDGGE. Algunas de las más importantes se describen a continuación.

1. Palomares et al. [25] y Dong et al. [8] introducen modelos de consenso que consideran la gestión de comportamientos no cooperativos y la reducción de la dimensión para la ponderación y agregación de información.

2. En su trabajo, Zhang et al. [42] utilizan un proceso de agregación lingüística para abordar problemas de TDGGE multiatributo bajo incertidumbre.

3. Liu et al. [16] proponen un modelo de consenso que integra mecanismos para controlar el coste de cambio de opinión de los decisores y utilizan análisis de redes sociales para determinar la importancia de los mismos.

4. Lu et al. [19] introducen un proceso de consenso que combina técnicas de análisis de redes sociales y clustering para realizar la reducción de dimensiones y determinar la influencia de los decisores en un proceso de decisión. Este proceso también tiene en cuenta el costo asociado con el cambio de preferencias de los decisores.

5. Shi et al. [29] utilizan técnicas de gestión de comportamiento y costos en su modelo de consenso, el cual también incorpora una reducción de dimensión con pesos adaptativos.

2.3 PRINCIPALES LIMITACIONES DE LA LITERATURA EN TDGGE

Actualmente, la TDGGE es un tema candente entre los investigadores de diversas áreas (investigación operativa, decisiones políticas, informática, gestión, ingeniería, marketing, etc. [15, 25, 37]) . Sin embargo, los cimientos de la TDGGE se basan en supuestos heredados de su uso extendido, y no en fundamentos sólidos de carácter teórico o práctico. Para abordar los nuevos desafíos que enfrenta la investigación en el área, esta sección tiene como objetivo discutir las principales deficiencias de la TDGGE.

En líneas generales, las principales limitaciones de la TDGGE parten de su propia definición. La definición extendida en la literatura especializada considera que la TDGGE consiste en "problemas de TDG con más veinte expertos". Esta definición parece estar completamente obsoleta, especialmente teniendo en cuenta que con las nuevas tecnologías permiten considerar situaciones de decisión que involucren cientos o miles de decisores. Por tanto, es necesario revisar esta definición para garantizar no sólo la aplicabilidad en la práctica de los modelos, sino también para poder comparar los procesos de manera justa.

Además, en la literatura especializada es habitual encontrar técnicas de TDG aplicadas directamente en TDGGE sin estudiar su viabilidad y rendimiento [27]. En lo que respecta al desempeño de los modelos revisados en la Sección 4.1, una crítica importante es la falta de estudios que demuestren el buen rendimiento de las técnicas clásicas de TDG en contextos a gran escala que involucren cientos o miles de decisores. Aunque estos métodos se han aplicado exitosamente en contextos con cincuenta o menos decisores, no hay evidencia de que sean igualmente efectivos en situaciones de decisión con un mayor número de decisores. Se necesitan estudios rigurosos para evaluar la viabilidad de estas técnicas en contextos a gran escala, y en caso necesario, adaptar las propuestas existentes para abordar problemas TDGGE.

Otra consecuencia de la definición de TDGGE como "TDG con veinte o más expertos" es su abuso en publicaciones sobre el tema que no están dirigidos a resolver problemas del mundo real. La gran mayoría de las propuestas revisadas en la sección 4.1 se limitan a probar la validez de los correspondientes modelos sobre ejemplos que consideran menos de cincuenta expertos, sin dar estudios objetivos ni verificables de su rendimiento a la hora de manejar miles de decisores (ver Fig. 2.4). En este sentido, es necesario tener un nivel de exigencia y calidad mucho mayor en cuanto a las condiciones en las que se prueba la validez de un método.

Además, es práctica habitual utilizar un sesgo en las medidas de desempeño y usar aquellas más convenientes para destacar las ventajas de los modelos propuestos al compararlos con otros, pero no hay

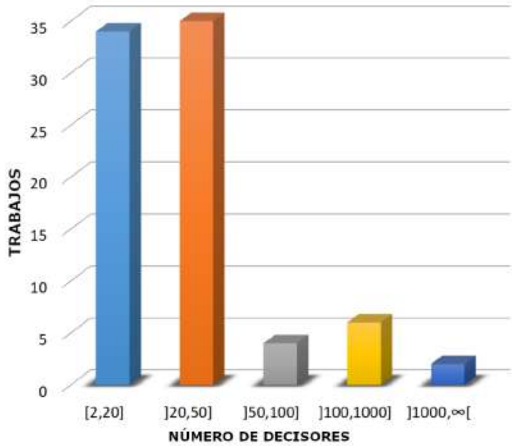

Fig. 2.4: Trabajos revisados en la Sección 4.1 según el número de decisores.

métricas objetivas que permitan presentar de manera justa tanto los aspectos positivos como los negativos de tales modelos. Un nuevo enfoque con esta capacidad ha sido propuesto recientemente para los modelos de consenso en TDG [15], pero no existen propuestas para TDGGE. Es importante desarrollar nuevas métricas para abordar otros problemas que permitan analizar objetivamente diferentes características de los métodos TDGGE.

En lo que respecta a las estructuras de preferencia, conviene destacar el elevado número de éstas que se pueden encontrar en la literatura (ver Sección 4.1). Sin embargo, considerar estructuras de preferencia excesivamente complejas aumenta significativamente el número de variables del problema de TDGGE, lo que conlleva un mayor consumo de recursos [27]. Por tanto, en TDGGE, las mejoras en el modelado de preferencias no deberían de implicar un incremento de las variables del problema, sino que deberían enfocarse a un mejor modelado de la psicología humana. En este sentido, también indicar que las propuestas existentes para la representación de preferencias suelen asumir que los decisores proporcionan sus preferencias de forma lineal. Sin embargo, estudios recientes sugieren que al utilizar escalas no lineales para reasignar las preferencias de los decisores, se obtienen soluciones colectivas más realistas desde un punto de vista psicológico [6, 20]. Por lo tanto, es necesario llevar a cabo estudios más profundos sobre el impacto de las escalas no lineales en TDGGE.

3 DISCUSIÓN DE LOS RESULTADOS

En este capítulo, se presentará un resumen de las propuestas que conforman esta memoria de investigación, así como los resultados y conclusiones derivados de ellas.

3.1 ANÁLISIS CRÍTICO SOBRE LA TEMÁTICA

El objetivo del primer trabajo de esta memoria, incluido en la Sección 4.1, es servir como estado del arte actualizado para que los investigadores comprendan mejor el concepto de TDGGE, presenten propuestas orientadas a abordar nuevos retos en el área relacionados con nuevos desarrollos tecnológicos como Big Data o redes sociales, y presten más atención a la validez de sus modelos en estos contextos.

En tal trabajo se ha llevado a cabo una revisión sistemática de la literatura existente sobre TDGGE, siguiendo las indicaciones propuestas por Kitchenham et al. [11] para el desarrollo de análisis bibliográficos en Ingeniería del Software. Utilizando esta metodología, se han revisado las propuestas existentes desde cuatro perspectivas diferentes: Estructura de Preferencias, Reglas de Decisión de Grupo, Evaluación de la Calidad y Aplicaciones. Estos puntos de vista contienen las palabras clave más relevantes en la literatura sobre TDGGE y representan los diferentes pasos a considerar al proponer modelos. Dado que el análisis realizado ha revelado varias limitaciones importantes en la investigación actual en el área, este artículo también proporciona un análisis crítico detallado de estas malas prácticas encontradas en la literatura, así como algunas indicaciones sobre cómo reorientar la investigación futura hacia una TDGGE más realista, relacionada con proponer metodologías para hacer frente a situaciones de decisión que implican un gran número de decisores.

En líneas generales, es importante destacar que la definición de modelos teóricos y la comprobación de su rendimiento en ejemplos muy simples (toy examples), en los que se consideran entre veinte y cincuenta decisores, que difícilmente se podrían aplicar en situaciones prácticas reales si no especifican explícitamente el número de decisores que son capaces de gestionar y demuestran su buen rendimiento en estos contextos. En un ámbito aplicado como la TDGGE, los investigadores deberían centrar los estudios futuros en abordar problemas del mundo real que involucren a un gran grupo de decisores (por ejemplo, Netflix gestiona 209 millones de suscripciones de pago) en lugar de proponer modelos "a gran escala" que funcionan con veinte decisores y no clarifican cómo sería su desempeño si se aumentase esa cifra.

Con el objetivo de aportar mayor transparencia a los procesos de TDGGE, en este trabajo también se propone la definición de modelos de $m-$TDGGE como aquellas propuestas que pueden manejar eficientemente al menos m decisores. Esto no sólo ofrece una visión más justa del rendimiento de cada propuesta, sino que además permite diferenciar a modelos orientados a manejar pocos cientos de decisores de modelos diseñados para manejar millones.

3.2 PREFERENCIAS NO LINEALES

En la actualidad, los PAC son un pilar fundamental de la investigación en TDG. Aunque existen diversas propuestas de PAC en la literatura, éstos suelen asumir escalas lineales para las preferencias de los expertos [10]. Sin embargo, estudios recientes sugieren que el uso de escalas no lineales pueden mejorar los resultados de la TDG [6, 20].

En el trabajo presentado en la Sección 4.2 se explora el uso de escalas no lineales para definir modelos de preferencia más realistas a partir de las preferencias originales de los expertos, incluso en situaciones a gran escala. En ese trabajo se ha realizado un estudio exhaustivo de las propiedades analíticas de tales escalas no lineales y se han obtenido las principales características matemáticas de las funciones que pueden ser adecuadas para adaptar las preferencias de los expertos de acuerdo a este factor psicológico. A estas funciones las hemos llamado Amplificaciones de Valores Extremos (EVAs, de Extreme Values Amplifications) y permiten reasignar relaciones de preferencia difusas dadas en una escala lineal a una escala no lineal, aumentando la distancia entre los valores extremos y disminuyendo la distancia entre los valores intermedios. Además, se ha enunciado la definición dual de las Reducciones de Valores Extremos (EVR, de

Extreme Values Reductions), que reducen la distancia entre valores extremos y amplifican la distancia entre los intermedios.

Se ha introducido un método general para construir EVAs y EVRs y se han propuesto varias familias de EVAs. Se ha comprobado que el uso de las escalas no lineales proporcionadas por los EVAs mejora el rendimiento de los modelos de consenso utilizados en el estudio. Concretamente, además de obtener resultados más realistas desde el punto de vista psicológico, las simulaciones muestran que el enfoque EVA reduce el número medio de rondas necesarias para alcanzar el consenso en ambos modelos y aumenta el nivel de consenso.

3.3 agregación de valores extremos en pac

En TDG, es necesario combinar las preferencias de los expertos en una fase de agregación para obtener una opinión colectiva antes de pasar a la fase de explotación [28]. En la literatura de operadores de agregación, los operadores OWA (de Ordered Weighed Average, en inglés) destacan porque permiten fusionar información de acuerdo a la magnitud de los valores que se quieren agregar [38]. Para calcular los pesos correspondientes, se han propuesto varias alternativas, incluyendo el método propuesto por Yager, que se basa en el uso de una familia biparamétrica de cuantificadores lingüísticos difusos lineales [39].

En el trabajo desarrollado en la Sección 4.3 se prueba que, aunque el método propuesto por Yager [39] es sencillo y efectivo, presenta importantes inconvenientes en cuanto a la elección de los parámetros. Por ejemplo, las agregaciones podrían producir resultados sesgados (medida de orness [2] no igual a 0.5) o incluso no agregar suficiente información (medida de entropía [2] baja). Además, el operador OWA construido a partir de estos cuantificadores ignora completamente los valores más extremos en el proceso de agregación, lo que podría resultar en agregaciones no realistas.

Estas agregaciones sesgadas son un gran inconveniente en aplicaciones del mundo real como los PAC [10, 23], pues un operador OWA cuya orness es mayor que 0.5 tendería a priorizar los valores extremos cercanos a 1 respecto a los cercanos a 0, lo cual no es razonable ya que estos valores deberían ser igual de importantes. Además, un consenso teórico que ignore por completo los valores más extremos no sería realista. Puesto que también se ha demostrado que la información menos extrema tiene un efecto cohesionador y facilita el acuerdo entre expertos [30, 31], en el trabajo de la Sección 4.3 se exploran nuevas formas de generar pesos OWA que prioricen la información intermedia por delante de los datos extremos, como lo hacen los cuantificadores lingüísticos difusos lineales, pero teniendo en cuenta más información en el proceso de agregación y evitando agregaciones sesgadas en los resultados.

Para superar estas limitaciones, se ha propuesto el operador EVR-OWA que utiliza EVR como cuantificadores lingüísticos difusos. Este operador OWA tiene en cuenta los valores más extremos pero da más importancia a los intermedios. Además, las agregaciones realizadas por los operadores EVR-OWA son mejores para ciertas aplicaciones del mundo real como los modelos de consenso para TDG [10], puesto que estos operadores agregan las preferencias de forma no sesgada y permiten tener en cuenta más información en el proceso de agregación.

La propuesta del operador EVR-OWA no solo proporciona un método sencillo y general para obtener ponderaciones OWA, sino que también proporciona una caracterización que relaciona aquellas familias de pesos para OWA simétricas, positivas y que priorizan valores intermedios, con los EVRs.

3.4 OPTIMIZACIÓN DE PROCESOS DE TDGGE

Los modelos de Consenso de Coste Mínimo (CCM), basados en la resolución de problemas de optimización convexa, son PAC automáticos que no necesitan de un mecanismo de recomendación, por lo que son especialmente interesantes para TDGGE [3, 27]. Estos modelos minimizan el coste de modificar las preferencias de los expertos para alcanzar un consenso mutuo y establecen que la distancia entre las preferencias individuales modificadas y la opinión colectiva debe estar limitada por un umbral $\varepsilon > 0$. Recientemente, se han propuesto los modelos de CCM integrales, que añaden una restricción adicional relacionada con un umbral de consenso $\gamma \in [0, 1]$ asociado a una medida de consenso.

El trabajo presentado en la Sección 4.4 analiza la relación entre las restricciones mencionadas en los modelos CCM integrales desde dos perspectivas diferentes. La primera se basa en desigualdades y permite determinar cotas simples para relacionar los parámetros ε y

γ. La segunda perspectiva se basa en la Teoría de Politopos Convexos y proporciona algoritmos que calculan cotas más precisas y complejas para relacionar estos parámetros.

Puesto que en los modelos CCM integrales los valores de los parámetros se fijan a priori, el método propuesto permite identificar las configuraciones de parámetros que pueden simplificar el modelo de optimización, eliminando aquellas restricciones que son redundantes y, consecuentemente, mejorando notablemente la eficiencia de estos modelos en TDGGE.

3.5 MODELOS CCM GENERALIZADOS PARA TDGGE

Los modelos CCM han sido ampliamente utilizados para obtener consenso en problemas de TDG. Sin embargo, la relación entre las extensiones previas de estos modelos aun no ha sido estudiada, lo que limita su aplicación práctica. En el artículo presentado en la sección 4.5, se presenta una reformulación de los modelos CCM utilizando la Teoría de Conjuntos Difusos para abordar problemas de TDG. El enfoque propuesto, llamado FZZ-MCC, ofrece una comprensión más clara de los modelos CCM y sus extensiones, así como una metodología rigurosa y flexible para abordar diversos tipos de problemas de TDG. Además, se demuestra la aplicabilidad del enfoque FZZ-MCC a través de tres ejemplos prácticos relacionados con la democracia electrónica, la selección de personal y la selección de proveedores ecológicos.

El enfoque FZZ-MCC introduce tres ventajas principales:

- Notación rigurosa y unificada basada en Conjuntos Difusos que permite generalizar estudios previos sobre CCM.
- Generalización de nociones clásicas relativas a TDG tales como estructura de preferencias, medida de consenso o función de coste.
- Flexibilidad para adaptar el esquema FZZ-MCC para abordar diversas situaciones de decisión.

Esta propuesta explota la flexibilidad del enfoque FZZ-MCC para proponer varios modelos totalmente nuevos basados en CCM:

- Se define un modelo FZZ-MCC para hacer frente a un escenario de democracia electrónica que implica la planificación urbana mediante la gestión de miles de preferencias a través de modelos FZZ-MCC y Relaciones de Preferencias Multiplicativas.

- Un modelo FZZ-MCC se utiliza para persuadir eficientemente a un comité de contratación para seleccionar a un gerente en particular. Para ello, se analiza el coste asociado y se conduce a los decisores hacia un acuerdo sobre la solución objetivo predefinida.
- Se propone un modelo híbrido FZZ-MCC que combina valoraciones en una base de datos con las comparaciones por pares de los directivos, integrando conocimiento experto y datos en un problema de selección de proveedores ecológicos.

Además, todos estos modelos se han propuesto en términos de funciones objetivo y restricciones lineales y basadas en valores absolutos, lo que facilita su linealización para mejorar tanto su precisión como los aspectos de eficiencia computacional, que son esenciales para hacer frente a problemas de TDGGE.

3.6 MÉTRICA PARA MODELOS DE CONSENSO LINGÜÍSTICOS A GRAN ESCALA

Aunque los PAC basados en información lingüística han sido objeto de una amplia investigación y se han propuesto numerosas soluciones en la literatura especializada, no existe una métrica objetiva para comparar estos modelos y decidir cuál es el mejor para cada problema de decisión.

En el trabajo desarrollado en la Sección 4.6 introducimos una métrica para evaluar el rendimiento de los PAC lingüísticos que tiene en cuenta tanto el grado de consenso resultante como el costo de modificar las opiniones iniciales de los participantes.

Esta métrica se basa en un modelo lingüístico de CCM que utiliza información ELICIT (Extended Comparative Linguistic Expressions with Symbolic Translation) [14] para modelar la indecisión de los participantes y garantizar procesos de computación con palabras precisos. Además, esta métrica está definida en base a un modelo de optimización lineal para acelerar el modelo computacional y mejorar su precisión, pudiendo así ser aplicada en procesos de TDGGE con varios miles de expertos en pocos segundos.

La métrica propuesta para evaluar PAC lingüísticos compara el coste óptimo necesario para lograr el nivel de consenso deseado con los cambios realizados por el PAC. Si el grado de consenso logrado por el PAC es inferior al umbral deseado, la métrica calificará el PAC como poco efectivo. En el caso de que el PAC supere el umbral de consenso, la métrica le dará mayor o menor puntuación atendiendo a la magnitud de las modificaciones innecesarias en las preferencias realizadas por el PAC.

Esta métrica también se ha utilizado para evaluar el rendimiento de dos modelos de consenso lingüístico definidos en la literatura especializada [13, 26] y demostrar así su aplicabilidad en la práctica.

4 PUBLICACIONES

Este capítulo incluye las publicaciones que forman parte de la tesis doctoral, de acuerdo con lo establecido en el artículo 25, punto 2, de la normativa actual para los Estudios de Doctorado de la Universidad de Jaén, que se refiere al programa RD. 99/2011. Concretamente, en este capítulo se presentan seis artículos científicos, publicados o aceptados, en revistas internacionales indexadas en la base de datos JCR (Journal Citation Reports), la cual es producida por Clarivate Analytics.

4.1 TOMA DE DECISIÓN EN GRUPO A GRAN ESCALA: REVISIÓN SISTEMÁTICA

ESTADO. Publicado.

TÍTULO. Large-scale group decision making: A systematic review and a critical analysis.

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Large-Scale Group Decision Making: A Systematic Review and a Critical Analysis

Diego García-Zamora, Álvaro Labella, Weiping Ding, *Senior Member, IEEE*, Rosa M. Rodríguez, *Member, IEEE*, and Luis Martínez, *Senior Member, IEEE*

***Abstract*—The society in the digital transformation era demands new decision schemes such as e-democracy or based on social media. Such novel decision schemes require the participation of many experts/decision makers/stakeholders in the decision processes. As a result, large-scale group decision making (LSGDM) has attracted the attention of many researchers in the last decade and many studies have been conducted in order to face the challenges associated with the topic. Therefore, this paper aims at reviewing the most relevant studies about LSGDM, identifying the most profitable research trends and analyzing them from a critical point of view. To do so, the Web of Science database has been consulted by using different searches. From these results a total of 241 contributions were found and a selection process regarding language, type of contribution and actual relation with the studied topic was then carried out. The 87 contributions finally selected for this review have been analyzed from four points of view that have been highly remarked in the topic, such as the preference structure in which decision-makers' opinions are modeled, the group decision rules used to define the decision making process, the techniques applied to verify the quality of these models and their applications to real world problems solving. Afterwards, a critical analysis of the main limitations of the existing proposals is developed. Finally, taking into account these limitations, new research lines for LSGDM are proposed and the main challenges are stressed out.**

***Index Terms*—Challenges, large-scale consensus models, large-scale group decision making (LSGDM), systematic review.**

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D. García-Zamora, Á. Labella, R. M. Rodríguez, and L. Martínez are with the Department of Computer Science, University of Jaén, Jaén 23071, Spain (e-mail: dgzamora@ujaen.es; alabella@ujaen.es; rmrodrig@ujaen.es; martin@ujaen.es).

W. Ding is with the School of Information Science and Technology, Nantong University, Nantong 226019, China (e-mail: dwp9988@163.com).

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I. Introduction

THESE days, decision-making processes entirely guided by data and quantitative modeling are being widely used, and the participation of human experts who usually manage qualitative information is either ignored or relegated to second place [1], [2]. However, considering the commitment, cost and relevance of human stakeholders in economic, social or learning research, the use of expert-guided decision-making methods, also known as group decision making (GDM) models in specialized literature, is still essential in several areas [3], [4], especially when agreed solutions are required [5], [6].

On the other hand, the digitization era and application of novel technologies to all human-beings tasks have implied a transition towards new ways to solve real-world problems. In the decision making field, the GDM problems have evolved from a few decision-makers (DMs) involved in the solving process to numerous of them, emerging the large-scale group decision making (LSGDM) [7], [8]. E-democracy technologies [9], [10], e-marketplaces [11], [12], social media [13], earthquake shelter selection [14] or water resource management [15], [16] are just a few examples of new decision making situations involving an increasing number of DMs, making of LSGDM an important topic in recent years [17]. This emergence has caused multiple changes and challenges in the approach for solving these new types of GDM problems. Initially, four research trends were pointed out and mainly developed in LSGDM [18]:

1) Clustering Methods in LSGDM: Dimension reduction has a key role in the resolution of LSGDM problems, since managing large decision groups may be tough or even impossible because of resources limitations. Clustering large groups into smaller and more manageable subgroups, usually based on the similarity between DMs' preferences, has been used as a satisfactory solution to overcome this dimension reduction issue [18], [19].

2) Large-Scale Consensus Reaching Processes (LSCRPs): Conflicting and polarized opinions are even more common in LSGDM than in classical GDM due to the participation of many DMs. If the conflicts are not addressed, the decision process may fail and affect negatively on the society. LSCRPs are applied to smooth out disagreements and increase the level of accordance in the group [18], [20], [21].

3) LSGDM Methods: The resolution of decision problems is usually carried out by ad hoc decision methods. These methods aim, in general terms, at obtaining a ranking of the alternatives of the problem by applying a set of algorithmic/optimization steps. LSGDM methods introduce new features regarding the classical decision methods for GDM in order to meet the new challenges posed by large-scale problems [22], [23].

4) LSGDM Support Systems: The enormous complexity associated with the LSGDM problems makes their resolution difficult by the DMs. The LSGDM support systems are software tools that aim at helping DMs along the decision process by providing additional information and reducing uncertainty related to LSGDM problems [24], [25].

The recent impact of LSGDM in the specialized literature has given place to many proposals, which have been reviewed by several authors. For instance, Labella *et al*. [20] analyze the performance of classical consensus models focused on solving GDM process with a few experts in LSGDM problems, concluding that these models are not able to deal with the challenges related to the large-scale context. Zhang *et al*. [26] review the consensus models with feedback mechanism based on minimum adjustments proposed in the literature from two different contexts, classical GDM problems and complex GDM problems which include large-scale contexts. Ding *et al*. [17] develop a taxonomy for the existing literature and discuss future research directions under a perspective based on Artificial Intelligence, whereas Tang and Liao [27] analyze the state of the art in order to provide an analysis from Big Data point of view.

However, although these reviews propose some classifications for the existing literature from different points of view, the necessary critical analysis of the existing literature is usually neglected, which has implied a deviation of the original purpose of the topic related to apply decision models in groups with a huge number of stakeholders. For instance, the classical definition of LSGDM itself (GDM with more than 20 DMs [27]) may be inadequate for current society demands because it assumes that just 20 DMs are a large group, whereas nowadays real-world decision situations may require much bigger groups (Netflix recommendation system deals with more than 200 million users). To this regard, it is usual to find lots of LSGDM proposals in the literature which test the performances of their methods by using toy examples in which just 20–50 DMs are considered. Undoubtedly, this is a prominent source of papers, but it is far away from solving real-world problems, which should be the main goal of the research in a purely applied area like this.

Hence, the main motivation of this survey is to analyze the current state of the art about the existing trends obtained from our literature analysis, but also to provide a comprehensive view about LSGDM and a critical discussion about the main limitations of present proposals, in order to redirect current research towards new trends which face the real world needs demanded by large-scale contexts.

Therefore, this contribution is devoted to answer the following research questions:

Q1: What are the most relevant studies addressing LSGDM?

Q2: What is the current state-of-the-art regarding LSGDM?

Q3: What are the limitations of the current contributions?

Q4: What are the most promising new trends in LSGDM for future research?

Consequently, the four major contributions of this proposal are summarized as follows: first a systematic review about the current state of the art of LSGDM is performed in order to point out the most relevant papers and trends in the area, which are then studied from different points of view, according to the different steps that conform a classical LSGDM process, namely i) the preference structure used to model DMs' opinions, ii) the internal group decision rules used to model the decision process, iii) the mechanisms to evaluate the quality of the proposed model and, finally, iv) the application of the models to solve real world LSGDM problems. Subsequently, it is provided a deep critical analysis based on these four perspectives regarding the way that researches have developed so far their methods, and faced the different challenges demanded by LSGDM problems. Eventually, future research lines about LSGDM are discussed keeping in mind this critique and pointing out how to overcome it.

The remaining of this contribution is set up as follows. In Section II, the main concepts related to LSGDM are introduced. Section III describes the search process adopted to identify relevant studies on the topic. Section IV introduces the results obtained from the search process related to LSGDM. Afterwards, Section V exposes a critique vision about the current researches based on LSGDM. Additionally, Section VI provides a discussion about the future challenges and trends on LSGDM. Finally, Section VII draws some conclusions.

II. Background

A GDM problem is a decision situation in which several DMs are required to decide one or several alternatives as solution for the given problem [28], [29]. Formally, such problems are modeled by a pair (D,X) in which D is a finite set of DMs

$$D=\{dm_1,dm_2,\ldots,dm_m\}$$

which are asked to judge a finite set of alternatives

$$X=\{x_1,x_2,\ldots,x_n\}$$

with the aim of choosing the best solution for the problem. Traditionally, the resolution process of these problems mainly consists of two steps [30] (see Fig. 1):

1) Aggregation: The DMs' preferences are grouped, by using an aggregation operator, into a single collective preference that represents the overall group's opinion.

2) Exploitation: One or several alternatives are selected as solution of the problem.

Formerly, Butler and Rothstein [31] introduced several rules to guide the resolution process such as majority, minority, or Borda count. However, when using these kinds of rules, some DMs may not feel satisfied with the chosen solution because their opinions may not have been sufficiently considered in the final collective choice. To deal with these discrepancies among DMs' opinions, CRPs were added as an additional phase in the resolution process of a GDM problem. A CRP is a dynamic and iterative process in which DMs discuss each other and change their initial opinions in order to bring closer

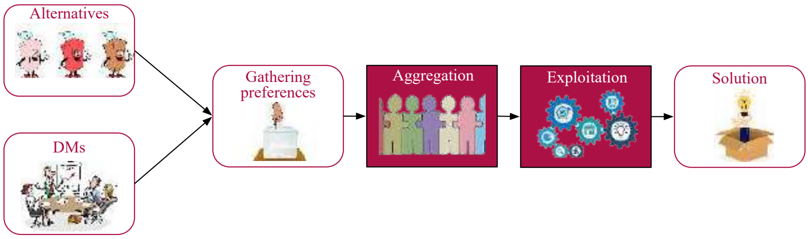

Fig. 1. Scheme of a GDM problem.

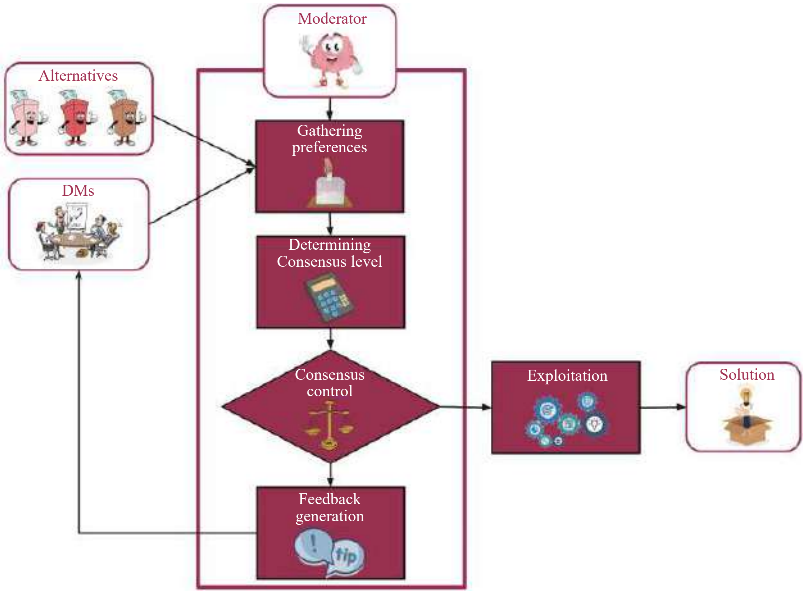

Fig. 2. Scheme of a CRP.

their views and increase the agreement within the group. These processes are usually supervised by a moderator, who is responsible for providing DMs with the proper feedback about the state of the negotiation. In broad terms, a CRP consists of four steps [32] (see Fig. 2):

1) Gathering Preferences: DMs provide their assessments over the alternatives by using preference structures.

2) Consensus Measuring: The current level of agreement in the group is derived by using consensus measures [33].

3) Consensus Control: The current level of agreement is compared with a predefined desired level of consensus for the group. If the group achieves such a desired level, the CRP finishes and the process to select the best alternative starts, otherwise another consensus round is accomplished. In order to avoid endless processes, the number of rounds is limited.

4) Feedback Generation: The moderator identifies the DMs whose opinions are the furthest away from the group and recommends that they change them.

Classically, GDM problems and their CRPs have considered just a small numbers of DMs, however, new technological advances such as Big Data [34] or e-commerce [35] and the emergent society demands to deal with problems like emergency situations [36] or sustainability [37] have given place to new large-scale contexts requiring the participation of more DMs in the decision process, which has attracted the attention of many researchers. In this context, LSGDM has arisen as those GDM problems in which 20 or more DMs take part in the decision process [17].

According to Tang and Liao [27] and Labella *et al.* [20] the involvement of numerous DMs with different views and preferences inevitably implies to consider new aspects in the general resolution scheme of GDM problems (see Fig. 3):

1) Dimension Reduction: These models usually include mechanisms to manage the large amount of information.

2) Weighting and Aggregation of Information: Related to properly determine the importance of the DMs participating in the process and fuse their opinions,

3) Behavior Management: A mechanism to detect and manage uncooperative DMs should be considered to avoid these DMs harm the decision process,

4) Cost Management: The human, economic and time resources required for developing models which aim at managing hundreds, thousands, or millions of DMs,

5) Social Network Analysis (SNA): When large groups are

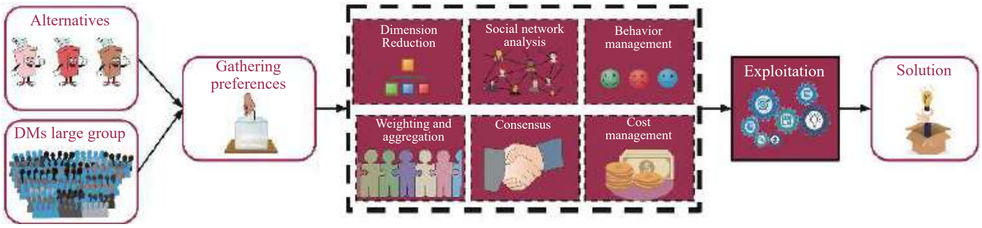

Fig. 3. Scheme of a LSGDM problem.

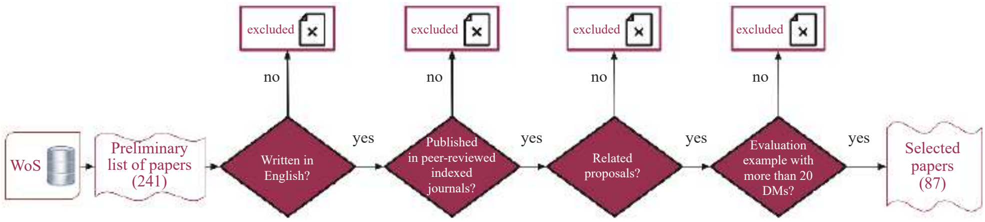

Fig. 4. Study selection process.

considered it is necessary to take into account how the relationships among DMs (trust or reputation) influence the decision process.

6) Consensus: The larger the number of DMs, the greater the probability of disagreement. Therefore, new consensus models dealing with large groups are key to reach agreed solutions.

Consequently, LSGDM inherits two phases of the classic scheme of GDM (see Fig. 1), namely the gathering of preferences and the exploitation phase, but the aggregation phase becomes much more complex because several of the aforementioned aspects may be taken into account in the fusion of the original values of DMs' opinions. The combination of these techniques allows proposing a huge variety of LSGDM schemes. Some of the most relevant ones are listed as follows:

- Palomares *et al*. [7] and Dong *et al*. [38] propose consensus models which take into account the management of the uncooperative behaviors and the dimension reduction to weight and aggregate the information.
- Zhang *et al*. [8] deal with multi-attribute LSGDM problems by using a linguistic aggregation process.
- Xu *et al*. [14], and Wu and Xu [39] introduce consensus models which also develop a dimension reduction to weight and aggregate the original preferences.
- Liu *et al*. [40] develop a consensus model which includes mechanisms to control the cost of moving DMs' opinions and use SNA to derive the importance of the DMs.
- Lu *et al*. [41] present a CRP which combines SNA and clustering to perform the dimension reduction and determine the influence of DMs in a decision process which also takes into account the cost of moving DM's preferences.
- Shi *et al*. [42] apply behavior and cost management techniques in a consensus model, which also performs a dimension reduction with adaptive weights.

III. Methodology

This study aims at reviewing the main concepts regarding LSGDM, by showing the relations among them and their future perspectives. To do so, the guidelines proposed by Kitchenham and Charters [43] to develop a systematic review in Software Engineering have been taken into consideration and adapted to our topic.

To obtain the documents that conform the state of the art of LSGDM, we have selected as data source the WoS database because maybe it is the most prestigious scientific bibliographic database. Even though others like Scopus are also relevant, in our case we make decision about WoS because, when comparing the results between both databases, the extra results obtained by Scopus were marginal regarding our aim. Our search strategy consisted of performing two different queries. In the first one, the keywords "Large-scale" and "Group Decision Making" were used as topic, whereas the second one used the keywords "Group Decision Making" as topic and also asked for the words "Large-scale" to appear in the title of the papers. As a result of these searches, which were done on 29th April 2021, a collection of 241 papers was found.

After that, a study selection process (see Fig. 4) was carried out in order to discard non-relevant proposals. The contributions which either were written in a language different from English or not published in peer-reviewed indexed journals were excluded. In addition, we also discarded those contributions non-related to the topic or which developed the quality evaluation of the proposed models by using examples involving problems that are not LSGDM at all because of the number of DMs. The number of papers which passed this filter was 87.

These 87 contributions have been published on 33 different journals, most of them belonging to the Computer Science & Artificial Intelligence category. The journals in which more contributions have been published are *Knowledge-Based*

Systems (12), *Information Fusion* (11), *IEEE Transactions on Fuzzy Systems* (8) and *Information Sciences* (7). Fig. 5 illustrates the complete journals' distribution of the reviewed papers. Table I shows the 5 most highly cited papers found in our search. The temporal distribution of the reviewed papers is shown in Fig. 6. The first proposal in our database related to LSGDM, from the interpretation of this review paper, was published in 2011. In such a contribution, Carvalho *et al.* [24] proposed a decision support system for LSGDM contexts and defined "large groups" as those groups with 10–20 individuals. In the subsequent years, just a few contributions were published until 2017 and the majority of papers have been published between 2018 and 2021, making LSGDM a hot topic in recent years.

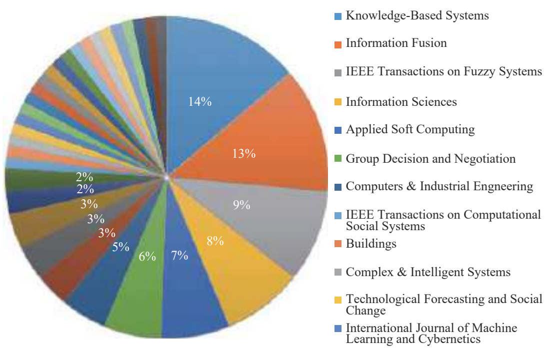

Fig. 5. Journal distribution of the reviewed contributions.

In order to identify the most relevant keywords in the topic for the data extraction process, a bibliography visualization tool has been applied to our database. On the one hand, Fig. 7 shows the main keywords used in the selected LSGDM literature and their connections so that the size of each node represents its occurrence.

These keywords are also classified into several colored categories so that those with a closer connection are represented with the same color. This figure allows to easily identify the most prolific trends related to LSGDM. For instance, it can be appreciated that *consensus* is one of the most important research lines within LSGDM but also as one of the trends with more links to other keywords such as *feedback mechanism* or *consensus level*. In addition, this figure also shows some keywords involving weighting and reduction dimension techniques such as *cluster* or *clustering method* and *social networks*. It should be highlighted the distinction among *expert* and *DM* terms because the former is related to GDM, whereas the latter is more related to LSGDM. The proper use of these terms may be key to make differences between GDM and LSGDM, though many researches use both interchangeably.

On the other hand, Fig. 8 shows the publications mean per year regarding several key topics. According to this figure, the most recent interest in LSGDM seems to be the validness and quality of the proposed models related to terms such as *comparative analysis*, *feasibility* or *validity*.

In addition to this automatized review of keywords, a manual abstract analysis was performed in order to provide a more comprehensive view of the current state of the art. From this manual research, we have identified some other keywords which have been used as complement to the ones obtained in the automatized search. Finally, to synthesize all the information, the resulting list of keywords has been organized in four blocks (see Fig. 9) according to the step of the model resolution process that these keywords belong to:

1) Preference Structures: This block includes the keywords related to the modelling of DMs' preferences and their characteristics.

2) Group Decision Rules: This block is related to the different formal processes applied to solve an LSGDM problem.

3) Evaluation of Quality: This block is devoted to group those keywords regarding the measure of quality and validness of the proposed LSGDM approaches by means of metrics, comparative analysis or use of datasets.

4) Application to Real-World Contexts: The keywords in this block deal with the applicability of the proposed LSGDM approaches to real-world LSGDM situations and the use of LSGDM support systems.

IV. Main Results

This section analyzes the main research trends related to LSGDM to provide a clear view of the topic by providing a taxonomy of the studied contributions according to the aforementioned four points of view, namely *Preference Structure*, *Group Decision Rules*, *Evaluation of Quality* and *Real World Problems* (see Fig. 9). To do so, first each block is introduced by providing a detailed description of its main specificities and then the 87 contributions obtained from our search in the WoS database are classified according to such four points of view. By using the results obtained in this section, a critical analysis of the studied contributions and several possible new research trends will be, respectively, provided in Sections V and VI.

A. Preference Structure

This block is devoted to classify the studied contributions according to the way in which the information is elicited from DMs [44]. The concept of *Preference Structure* in decision making in general and in LSGDM in particular is referred to the format in which DMs give their opinions. DMs could be asked to provide their opinions by following different formal rules which, in turn, give place to several preference structures. Since the chosen preference structure will determine the nature of the input of any GDM model, it is key to properly select these structures according to the faced decision situation.

There are several relevant features related to preference structures to keep in mind:

1) Type of Information: One of the most relevant features in preference structures is the type of information in which DMs are allowed to give their opinions, which may be of different natures. The analyzed proposals from the 87 papers essentially use three types of information when modelling DMs'

TABLE I
HIGHLY CITED PAPERS

Title	Reference	Journal	Year	Citations
A consensus model to detect and manage noncooperative behaviors in large-scale group decision making	Palomares *et al.* [7]	IEEE Transactions on Fuzzy Systems	2014	265
Managing multigranular linguistic distribution assessments in large-scale multiattribute group decision making	Zhang *et al.* [8]	IEEE Transactions on Fuzzy Systems Man Cybernetic-Systems	2017	218
A consensus model for large-scale group decision making with hesitant fuzzy information and changeable clusters	Wu and Xu [39]	Information Fusion	2018	181
A self-management mechanism for noncooperative behaviors in large-scale group consensus reaching processes	Dong *et al.* [38]	IEEE Transactions on Fuzzy Systems	2018	129
A two-stage consensus method for large-scale multi-attribute group decision making with an application to earthquake shelter selection	Xu *et al.* [14]	Computer & Industrial Engineering	2018	123

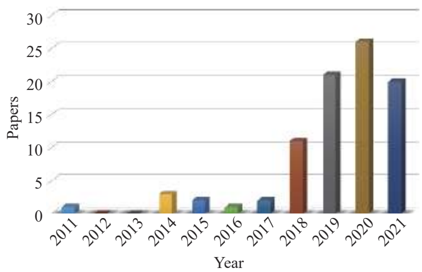

Fig. 6. Temporal distribution of the reviewed contributions (April 2021).

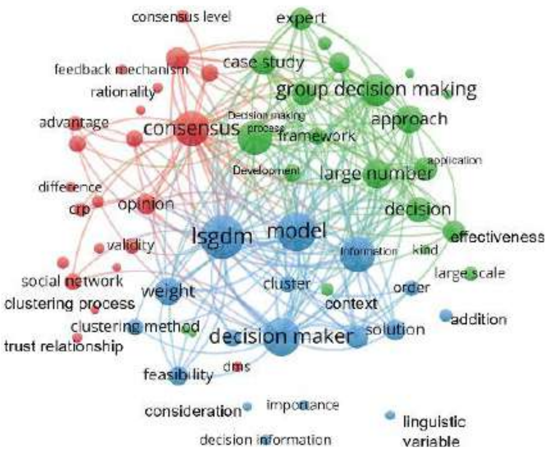

Fig. 7. Keywords related to LSGDM.

preferences, namely, numeric, linguistic and heterogeneous:

i) Numeric: Some proposals consider that the information is given numerically by using preference structures such as fuzzy preference relations (FPRs) [45], multiplicative preference relations (MPRs) [46], hesitant fuzzy preference relations (HFPRs) and so on. Apart from these, there are other numeric structures such as preference orderings [47] or utility functions [48].

ii) Linguistic: Other contributions allow DMs to express their preferences by using linguistic information, which is very useful to model the uncertainty inherent in LSGDM

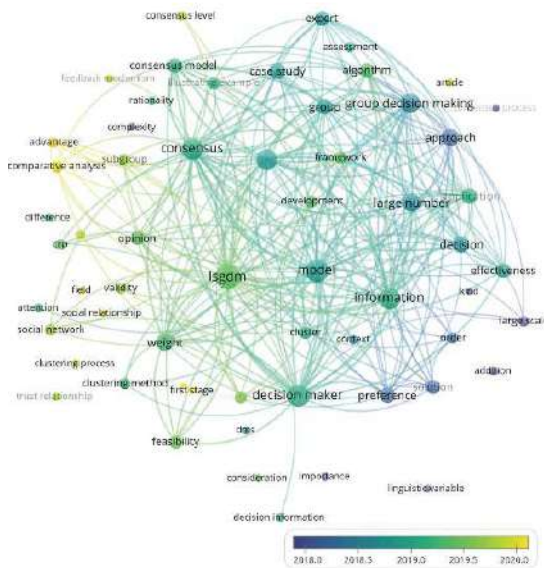

Fig. 8. Mean per year publications related to LSGDM keywords.

problems due to their complexity. In this sense, there are many types of linguistic preference structures such as linguistic preference relations (LPRs) or hesitant fuzzy linguistic preference relations (HFLPRs), whose elements are represented by linguistic terms belonging to a predefined linguistic term set.

iii) Heterogeneous: Finally, some papers consider situations in which DMs may provide their opinions by using different types of preference structures, numeric or linguistic. By using heterogeneous information, each DM may use the most suitable preference structure according to her/his necessity, which provides more flexibility to the elicitation task.

2) Personalized Semantics: On the other hand, especially in large-scale contexts, the DMs participating in the decision may possess different backgrounds or use different scales to express their preferences. Therefore, an interesting research area related to preference structures is the management of DMs' *personalized individual semantics*, which is devoted to deal with the different knowledges and subjectivities of DMs

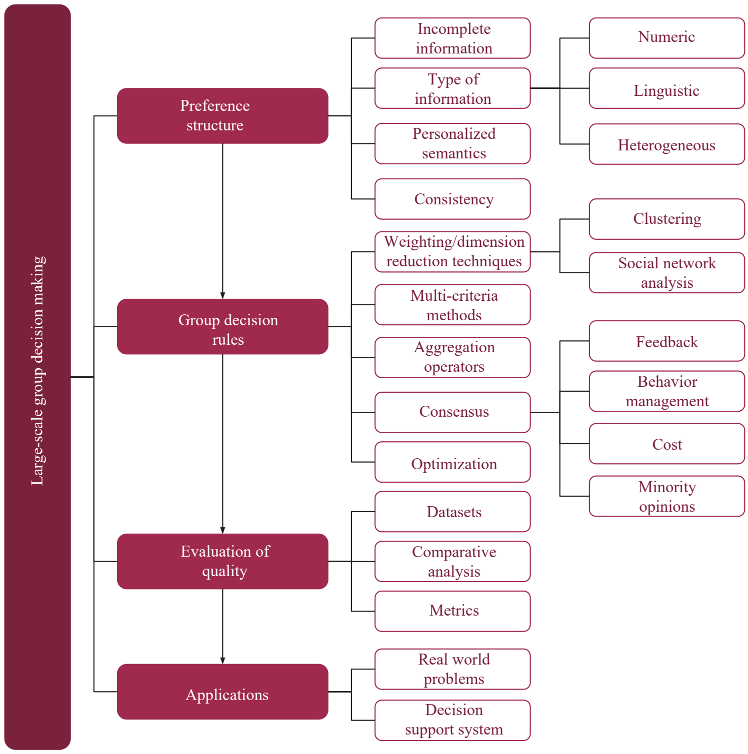

Fig. 9. Found keywords classified according to their relations.

when expressing their opinions [49]–[51].

3) Consistency: Other research line is devoted to study the *consistency* of the DMs' preferences [22], [52]–[54], since sometimes the information provided by these DMs may be contradictory and lead to unreliable results.

4) Incomplete Information: The last identified research line focuses on dealing with *incomplete* preference structures, since limitation of knowledge over the alternatives or the time pressure could lead to circumstances in which DMs may not provide all the necessary preference values [53], [55]–[57].

Table II classifies the revised contributions according to the type of preference structures used to model the DMs' preferences and their type of information, and Table III shows the acronyms of such preference structures.

B. Group Decision Rules

Group Decision Rules block analyses the contributions according to the internal performance of the models. Initially, GDM was based on using certain classic rules [31] such as the Majority Rule, Borda Count, or Unanimity in order to fuse the individual preferences of the respective DMs into one single collective opinion. Nowadays, these few methods have evolved into many rules, which provide several frameworks to achieve the same goal. These rules cover a wide spectrum of possibilities, such as methods to reach agreed solutions obtained by simulating a discussion process or proposals which evaluate alternatives taking into consideration different conflicting criteria. Therefore, when designing an LSGDM model, it is essential to carefully select these rules according to the needs of the faced problem.

1) Aggregation Operators: The importance of selecting the adequate aggregation operator cannot be neglected [105] since the main differences among the GDM models are usually related to the way in which the information is combined. In spite of this, just a few articles in our database [44], [106] focus exclusively on proposing new aggregation operators for large-scale contexts.

2) Multi-Criterion Decision Making: The analysis of the proposals in our database reveals that the use of classic multi-criterion group decision methods to solve LSGDM problems is widely extended. Among these approaches, one of the most common is the *technique for order of preference by similarity to ideal solution* (TOPSIS) [51], [57] based on the idea that the best chosen alternative for a decision problem should have the shortest geometric distance regarding the ideal solution and the largest geometric distance regarding the negative anti-ideal solution, being the ideal solution the one that maximizes benefit criteria and minimizes cost criteria and the anti-ideal solution the one that maximizes cost criteria and minimizes benefit criteria. There are also approaches that use the *multi-objective optimization on the basis of a ratio analysis plus the full multiplicative form* (MULTIMOORA) [37], which obtains a final ranking by aggregating the results of the ternary ranking methods Ratio systems, Reference Point approach and

TABLE II
PROPOSALS CLASSIFIED ACCORDING TO THE PREFERENCE STRUCTURE USED IN LSGDM

Type of Information	Nature	Preference structure	References
Numeric	Discrete	Utility vector	[41], [58]
		Numerical vector	[19], [42], [59], [60]
		Decision matrix	[14], [23], [52], [55], [61]–[65]
		FPR	[7], [18], [20], [25], [38], [39], [56], [66]–[72]
		MPR	[22], [73]
		IFS	[40]
	Continuous	HFPR	[74]
		QRIVOFN	[37]
		IVIFS	[75], [76]
	Probabilistic	BPA	[77]–[79]
Linguistic	Discrete	2-tuple	[36]
		Decision matrix	[21], [35], [57], [80]–[83]
		LDA	[8], [15]
		LDPR	[49]
		LPR	[50], [53], [54]
		DHLPR	[84]
	Continuous	HFLPR	[16], [85], [86]
		HFLTs	[87]–[91]
		CIVLTS	[92]
		IT2FS	[51], [93]–[95]
	Probabilistic	PLTS	[96]–[99]
Heterogeneous	-	Two or more	[24], [44], [100]–[104]

TABLE III
ACRONYM FOR THE IDENTIFIED PREFERENCE STRUCTURES USED IN LSGDM

Preference structure	Acronym
Fuzzy preference relation	FPR
Multiplicative preference relation	MPR
Linguistic preference relation	LPR
Linguistic distribution preference relation	LDPR
Linguistic distribution assessments	LDA
Double hierarchy linguistic preference relation	DHLPR
Hesitant fuzzy preference relation	HFPR
Hesitant fuzzy linguistic preference relation	HFLPR
Hesitant fuzzy linguistic terms sets	HFLTS
Intuitionistic fuzzy set	IFS
Interval-valued intuitionistic fuzzy sets	IVIFS
Interval type-2 fuzzy sets	IT2FS
Probabilistic linguistic terms sets	PLTS
Continuous interval-valued linguistic terms sets	CIVLTS
Q-rung interval-valued orthopair fuzzy numbers	QRIVOFN
Basic probability assignments	BPA

Full Multiplicative Form or the *ELimination Et Choix Traduisant la REalité* (ELECTRE) III [89], an outranking method based on pairwise comparisons (every option is compared to all other options) which is able to provide a total/partial order of the alternatives by using pseudo-criteria and outranking degrees.

3) Weighting and Dimension Reduction Techniques: In large-scale contexts, it is essential to be able to manage at the same time thousands of DMs' opinions to achieve a solution. Therefore, it is necessary to use dimension reduction techniques to reduce the resource consumption or specific weighting processes to determine the importance of each DM. Several dimension reduction techniques have been identified in the analyzed contributions:

i) Clustering: This technique consists of reducing the dimension of DMs by grouping those with a similar performance into the same subgroups/clusters. In the literature, we can find well-known clustering methods such as fuzzy C-means [7], [15] or K-means [39], [41] but also other novel clustering methods such as grey clustering, fuzzy equivalence and others techniques.

ii) SNA: Another widely accepted method is the use of tools to reduce the data sparsity related to DMs' preferences through SNA techniques [40], [60]. These kinds of proposals are based on the graph theory and allow weighting DMs by taking into account human factors such as the trust relations among them.

iii) Clustering and SNA: Some proposals combine clustering and SNA to produce several independent subnetworks of DMs according to the relations among them [86], [93].

iv) Others: Besides clustering and SNA, it is possible to find other weighting and dimension reduction techniques in the literature, which are usually based on mathematical programming [57], [73].

The main contributions related to weighting and dimension reduction techniques are shown in Table IV.

4) Consensus Models: Some real world situations require an agreement among a large number of DMs. Traditionally, researchers have faced these situations by proposing consensus models for a few DMs in GDM. However, these models have proven to be inappropriate to deal with LSGDM problems [20] because of the peculiarities of these contexts. The main consensus models identified in the analyzed proposals are shown in Table V.

i) Feedback: Even though classical consensus models assume the role of a moderator to analyze the state of the consensus process and provide recommendations to the DMs, in contexts in which hundreds or thousands of DMs take part both the moderator figure and *feedback* mechanisms [16], [18], [39], [42] are obsolete due to the fact that they are too time-consuming and not feasible in practice. Therefore, large-scale consensus proposals are devoted to replace both with automatic mechanisms to provide recommendations and analyze the level of consensus achieved. The use of mathematical optimization techniques is widely extended in the literature related to this regard.

ii) Behavior management: The large number of DMs in large-scale contexts increases the probability of dealing with DMs who refuse to adjust or make changes in their preferences. For this reason, it is necessary to include

TABLE IV
MAIN PROPOSALS ACCORDING TO THEIR WEIGHTING AND DIMENSION REDUCTION TECHNIQUES

Reference	Technique	Details	Citations
Palomares *et al.* [7]	Fuzzy C-means	A consensus model which implements a clustering framework to manage non cooperative behaviors	252
Wu and Xu [39]	K-means	Consensus model in which the clusters are allowed to change and uses a possibility distribution based hesitant fuzzy element to represent each cluster's opinion	172
Wu and Liu [95]	Fuzzy equivalence	An interval type-2 fuzzy equivalence clustering analysis is used in a multi-criterion large-scale decision making problem	54
Dong *et al.* [38]	Grey clustering	Consensus framework to manage non cooperative behaviors in which the weights are dynamically generated	123
Liu *et al.* [40]	SNA	Trust relationship-based conflict detection and elimination decision making model applicable for LSGDM problems in social network contexts	92
Wu *et al.* [60]	SNA	Two-stage trust network partition algorithm is proposed to reduce the complex of LGDM problems	51
Tian *et al.* [93]	Clustering and SNA	A SNA based decision framework for addressing problems with incomplete interval type-2 fuzzy information	54
Ren *et al.* [86]	Clustering and SNA	A consensus model to manage minority opinions with SNA for micro-grid planning	25
Song and Li [57]	Others	LGDM model to handle incomplete multi-granular linguistic information and which ranks alternatives by an extended TOPSIS method	49

TABLE V
MAIN CONSENSUS MODELS

Reference	Consensus measure	Feedback	Behavior management	Minority opinions	Citations
Palomares *et al.* [7]	Similarity between DMs	Parametric change directions	√	×	252
Wu and Xu [39]	Similarity between centroids	Parametric change directions	×	×	172
Dong *et al.* [38]	Similarity between DMs	Random change directions	√	×	123
Li *et al.* [50]	Similarity between DMs and collective	Parametric change directions	×	×	119
Xu *et al.* [14]	Similarity between DMs and collective	Automatic-formula	×	×	118
Quesada *et al.* [71]	Similarity between DMs	Parametric change directions	√	×	100
Liu *et al.* [40]	Similarity between DMs	Automatic-optimization	×	×	92
Xiao *et al.* [49]	Similarity between DMs and collective	Automatic optimization	×	×	51
Ren *et al.* [86]	Similarity between groups and collective	Parametric change directions	×	√	25
Gou *et al.* [84]	Similarity between DMs and collective	Parametric change directions	√	√	23

mechanisms to face these *uncooperative behaviors* in order to prevent the failure of the consensus process.

iii) Cost: Cost refers to the price (economical or attitudinal) of changing DMs' opinions [40]–[42]. For instance, some widely used consensus models are the so-called minimum cost consensus models [26] whose aim is to provide a feasible consensual solution by changing the initial DMs' opinions as few as possible.

iv) Minority opinions: The coalition of large groups in large-scale contexts may cause ignoring minority group opinions that are just as valid as the first. Even though these differing opinions are often referred to as obstacles to decision-making, several proposals study how to properly manage the importance given to these minority opinions [21], [84], [86].

5) Optimization Models: Due to their flexibility, the use of *mathematical programming* techniques is also pretty popular among researchers. Therefore, it is usual to find models which rely on optimization models to complete missing information [53], managing individual semantics [49], translating preference structures [15], for weighting determination [52], [54], [57] defining groups [73], in SNA [97], or in consensus models [22], [36], [41].

C. Evaluation of Quality

After designing the rules which define an LSGDM method, it is necessary to test the feasibility of the proposal when dealing with a specific decision problem. Consequently, the block *Evaluation of quality* is devoted to study the reviewed papers according to the mechanisms used by researchers in order to show the feasibility of their models. In the studied literature, there are essentially three kinds of methods to show

TABLE VI
MAIN EVALUATION TECHNIQUES

Reference	Method	Details	Citations
Zhang *et al.* [8]	Theoretical comparisons	Due to the impossibility of carrying out computational experiments, the authors show the advantages of their proposal before other existing models.	209
Wu and Xu [39]	Experimental comparisons	Evaluates the model by comparing it with a similar proposal which used the same illustrative example by studying the weighting of clusters and the consensus level.	172
Dong *et al.* [38]	Theoretical comparisons	Compares the proposal with other previous methods by highlighting its advantages and limitations.	123
Li *et al.* [50]	Theoretical comparisons	Provides a brief taxonomy of the existing large-scale consensus models and develops a discussion related to frame the model in this taxonomy.	119
Xu *et al.* [14]	Experimental comparisons	Compares the proposal with a similar method by analysing the weighting of the attributes and the consensus level.	118
Wu *et al.* [51]	Experimental comparisons	Compare the proposal with another model by using the same illustrative example by showing the similarities and differences of both proposals from the ranking and rating of the alternatives point of view.	107
Liu *et al.* [103]	Dataset	Evaluates the model regarding the consensus level and number of rounds by using an illustrative example.	72

TABLE VII
MAIN APPLICATIONS TO REAL-WORLD PROBLEMS

Reference	Area	Details	DMs	Citations
Zhang *et al.* [15]	Water management	Framework with linguistic information based on optimization which is applied to the selection of the best sustainable disinfection technique for wastewater reuse projects.	20	164
Xu *et al.* [14]	Emergency situation	Two-stage method to support the consensus reaching process for large-scale multi-attribute group decision making problems applied to earthquake shelter selection.	25	118
Gou *et al.* [16]	Water management	Consensus model with double hierarchy hesitant fuzzy linguistic preference relations applied to evaluate Sichuan water resource management.	20	103
Song and Li [57]	Suppliers selection	Model to handle incomplete multi-granular linguistic information applied to a sustainable supplier selection problem.	30	49
Chao *et al.* [101]	Finances	Method with non-cooperative behaviors and heterogeneous preferences applied to financial inclusion.	52	43
Wu *et al.* [35]	E-commerce	Linguistic model for multi-attribute LSGDM applied to the customer decision for e-commerce service.	50	33
Ren *et al.* [86]	Energy	Consensus model to manage minority opinions with SNA applied to micro-grid planning in Ali district in Tibet.	25	25

the good performance of the proposed models (see Table VI).

1) Experimental Comparisons: The majority of the consulted references use experimental comparisons, consisting of testing the performance of the proposed models by comparing them with other techniques through different simulations [14], [39], [51]. However, there are not widely extended metrics to compare these models, on the contrary, a huge number of different measures can be found in the reviewed literature, such as the final ranking of the alternatives, the cost incurred to achieve a solution, consensus degree, number of discussion rounds, and so on.

2) Theoretical Comparisons: Other authors propose theoretical comparisons in which the advantages of their models over others are discussed [8], [38], [50].

3) Datasets: Finally, other proposals just provide the results of testing their models in a certain dataset, which may be obtained from real DMs or created manually by the authors [68], [79], [103].

D. Applications

Decision making is a natural activity of human beings' life and covers multiple disciplines in society related to management, education, or healthcare. Therefore, *Application* block is focused on analyzing the proposals from the point of view of their implementation to solve concrete problems. Consequently, this subsection reviews how the different LSGDM proposals in the specialized literature are enforced by taking into account two main groups of applications:

1) Real-World Problems: This group resembles those applications related to using LSGDM models in real-world situations. The flexibility of LSGDM techniques to deal with all kinds of situations has allowed researchers to provide solutions for many problems (see Table VII). For instance, it has been applied to solve health-related problems [90] such as COVID-19 pandemic [96], [100] and other emergency situations [14], [36], [54]. In addition, the recent interests of society in sustainability problems have led to studies related to green suppliers selection [37], [57], energy [86], [107] or water management [15], [16], [89]. Furthermore, it is also possible to find applications of LSGDM in technological environments [34], [79], [108].

2) Decision Support Systems: Decision support system refers to those software applications whose aim is to assist DMs to make proper choices when facing decision situations. Several LSGDM support systems have been found in the review such as LaSca [24], which stands out because of the flexibility in which DMs can "decide how to decide", MENTOR [25] which is a graphical tool to study the evolution of the preferences during an LSGDM process and DeciTrustNET [109] which takes into account trust and reputation in social networks.

V. Critical Analysis of the LSGDM State of Art

Once we have a clear view of the current state of the art of LSGDM, it is necessary to devote one section to provide a critical analysis of it in LSGDM. First, it is provided a general critique regarding the vagueness of several notions related to LSGDM. Afterwards, the main trends related to LSGDM identified in the bibliographic analysis are discussed from the four blocks considered in Section IV.

A. General Critique: LSGDM Foundations

Undoubtedly, LSGDM is today a hot topic among researchers in Computer Science area. In spite of this, the main notions regarding this topic do not have any theoretical or practical support, but they are based on assumptions which have been inherited through years because of their wide extended use, which, in the end, has implied a deviation from the initial purpose of LSGDM. Consequently, this subsection is devoted to discuss all of these definitions and redirect them to face the new challenges demanded by society.

1) Definition of LSGDM: Even though LSGDM should be devoted to deal with decision situations in which thousands or millions of DMs take part, the analysis of the existing literature shows that researchers have abused of the "20 or more experts" definition [17] to publish papers in the topic which are not necessarily focused on solving any real-world problem nor society demand.

According to Carvalho *et al.* [24], the oldest reference found in our search, this definition seems to be motivated by the fact that finding 20 experts in a certain area who want to participate in the decision process is a difficult task to carry out in practice, especially if they are expected to meet in the same room. However, the origin of this boundary of 20 DMs is not clear. When justifying the number of DMs which bounds the notion of LSGDM, some proposals refer to even older works from the early 2000s, which are usually hard to retrieve because they have been published on nonindexed research journals, and others do not provide any justification or cite. As a consequence, the vast majority of the reviewed papers validate their approaches by using examples with 50 or fewer experts referring to this definition (see Fig. 10).

However, new technological advances allow us to consider the preferences of a huge number of DMs and this former definition for LSGDM seems to be inadequate for the current situation. Furthermore, this definition introduces a certain ambiguity when considering a model whose performance is limited to 50 DMs and another proposal which can deal with 500 DMs to be *the same*. On the one hand, the formal aspects of both problems do not have necessarily to be similar, and

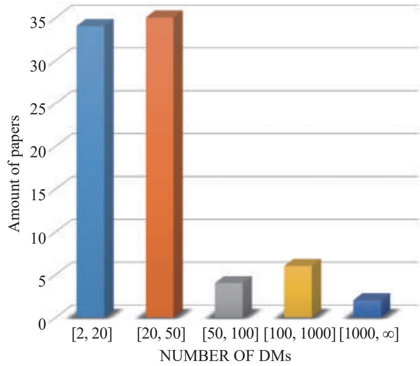

Fig. 10. Contributions are classified according to the number of DMs used in their examples.

neither the methods and techniques used to properly model these decision situations. On the other hand, this ambiguity may result in redundant proposals in which a GDM model in which 19 DMs are considered, could be easily transformed into an *LSGDM model* by using the same proposal in a problem which requires of 20 DMs.

In order to overcome this problem, we propose the use of the following definition:

Definition 1 (m-Large-Scale Group Decision Making Model): An m-large-scale group decision making (m-LSGDM) model is a method which has proven to be able to efficiently manage LSGDM situations involving m DMs.

Remark 1: It should be noted that to consider a model as an m-LSGDM, the respective authors must provide a sustained proof of its good performance when dealing with these kinds of problems.

This nomenclature not only provides a clear vision of what authors intend with their proposals, but also a taxonomy regarding the performance threshold of each contribution. In addition, this allows to easily identify the most suitable models to solve a specific LSGDM problem.

2) Ambiguity in the Notion of Expert: Another controversial terminology is the use of the term *expert* to name the participants of an LSGDM problem, because it does not seem to be reasonable to ask a million people to be an expert in a concrete area. In spite of this, many contributions use the terms expert/decision maker/stakeholders interchangeably. Therefore, the term "expert" should be replaced by other terms such as *stakeholder* or *DM* when dealing with large-scale decision situations, especially those in which hundreds or thousands of DMs are required.

3) Consensus in LSGDM: The notion of consensus in large-scale contexts regarding millions of DMs seems to be unclear. Classic literature states that a fundamental assumption for CRPs is the fact that all the DMs agree to change their preferences in order to get a collective agreement [110]. However, this collective agreement may not be the goal of the DMs which participate in large-scale decision situations and considering the same assumption could be too optimistic. Therefore, in large-scale situations, the philosophy behind the idea of consensus should not assume a will for agreement, but a personal interest in achieving a collective solution which harms each DM as little as possible: when millions of DMs take part in a decision situation in which consensus is desired (for example, e-democracy), this will of consensus should be understood as a will of maximizing the personal satisfaction of each individual with respect to the desired consensual solution.

B. Preference Structure

1) Elevate Number of Preference Structures: The most remarkable feature is the fact that there are too many preference structures proposed in the literature. For the sake of providing more flexibility for DMs, researchers have developed different types of preference structures. However, even though this purpose is noble, we have found no proposals related to the comparison of the performance of the different preference structures in LSGDM contexts, which could lead to imprecise results or redundant proposals in which only variation is given by changing the type of preference structure used. To overcome this drawback, rigorous studies are necessary to decide which preference structure is most suitable for a certain problem.

2) Heterogeneous Knowledge: It should also be highlighted the fact that in problems in which thousands of DMs take part, the differences in their knowledge could be considerable. However, due to the majority of the reviewed proposals consider *toy examples* (less than 50 experts) to validate the proposed model, this issue is often neglected, and these differences are not considered. When dealing with LSGDM problems in which a larger number of DMs are involved, they should be allowed to express their preferences by using flexible expression domains and their influence in the decision process must be related to their degree of knowledge about the topic.

3) Inconsistency: Another key aspect related to the preference structures is the consistency of the information given by DMs. However, this issue is usually not considered in the reviewed proposals, which could lead to contradictory results. To avoid this issue, it is necessary to evaluate the consistency from DMs' opinions (before and after the decision process) to guarantee reliable solutions, especially in real LSGDM problems in which the high complexity and uncertainty may increase the probability of the occurrence of this phenomenon.

4) Incompleteness: Finally, it is possible that because of the lack of knowledge, time limitations, or simply human errors, some values of the preferences are missing, especially in LSGDM problems in which complexity is high and hundreds of DMs, usually not experts in the topic, take part in the decision process. Although this fact is rarely taken into account by researchers, new nontrivial mechanisms to manage these missing values should be proposed to generate complete preferences as complete as possible.

C. Group Decision Rules

1) Extension of Classic GDM Techniques to LSGDM: In the revised proposals no reviews about the performance of classic multi-criterion GDM methods (TOPSIS, AHP, PCA,...), weighting mechanisms, or dimension reduction techniques in large-scale contexts in which hundreds or thousands of DMs take part have been found. Even though they have proved to be effective when dealing with 20−50 DMs, there is no guarantee of their good performance for larger groups [27] and it seems that these methods have been directly imported into LSGDM contexts without any proof of their feasibility. It has been already proved that classic CRPs are not suitable for dealing with LSGDM problems [20], because these techniques do not perform a reduction of the dimension and neglect the consideration of DMs' behaviors. Therefore, to guarantee the good performance of other classic GDM techniques in large-scale contexts, it is necessary to previously develop a depth study regarding the feasibility of these models in several scenarios in which different numbers of DMs are considered and, in case they are not suitable for dealing with, study the possibility of extending these methods to LSGDM.

2) Feedback and Moderator in Large-Scale Consensus: Regarding consensus models, the use of the terminology *process* when referring to consensus models seems to be obsolete. On the one hand, the role of the human moderator is unfeasible to develop in large-scale contexts due to time and resource limitations. On the other hand, simulating different discussion rounds in which feedback is provided to the DMs to influence their opinions could lead to endless situations. However, some reviewed contributions inherit the original concept of CRP and apply these ideas to propose consensus models which consider either the moderator figure or feedback mechanisms. This could be feasible when dealing with 20–50 DMs, but it is a nonsense when considering thousands of them. Therefore, the classic idea of consensus model as an iterative discussion process should be replaced by automatic algorithms which do not necessarily involve discussion rounds, human moderators, nor the approval of DMs to change their opinions.

3) Non-Cooperative Behaviors in Consensus Models: In addition, when thousands or millions of DMs take part in a decision problem, it should not be supposed that all of them agree to reach a collective agreement because they may have different interests and, consequently, form groups according to their personal profits. According to our bibliography analysis, some proposals already include techniques to detect and manage these uncooperative behaviors, but their use is not extended and those authors who take into account such mechanisms usually apply them to solve simple problems involving 50 or fewer DMs.

D. Evaluation of Quality

1) Toy Examples: The main critique in this subsection is related to the widely spread use of *toy examples* to study the performance of the proposed models (see Fig. 11). The majority of the reviewed papers claim to propose LSGDM models, but just solve cases in which less than 50 DMs are considered and there is no information about the performance of these models when thousands of DMs are required (see Fig. 10). In this regard, it is necessary to be more demanding with the conditions in which the validity of a method is tested.

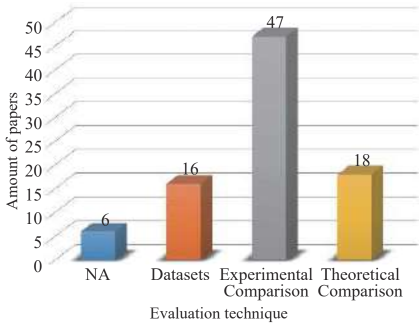

Fig. 11. Contributions are classified according to their evaluation technique.

Solving a problem from a concrete dataset is not enough to guarantee the good performance of the proposals in any context. Since models dealing with 20–50 DMs do not have to be similar to those which deal with several millions, researchers should clarify from the beginning the volume of DMs which their proposal is able to manage (see Definition 1) and also make sure that the models are stable by carrying out several simulations with different values of the preferences.

2) Global Metrics: Regarding the simulations from the previous paragraph, there is no universal way to develop them. Usually, researchers use a convenient measure to highlight the best properties of their models when making comparisons with others, but there are no global metrics which allow researchers to do a fair balancing by showing both positive and negative aspects of the models. Recently, a metric with this property was proposed [111] for consensus models, but it is key to introduce new ones for other problems to analyze different features of the LSGDM methods such as the proper selection of the preference structures according to the problem to solve and the DMs, the robustness of the final alternatives ranking or the understanding degree of the results.

3) Accessibility to the Existing Models: Currently, there is no easy way to get access to the models proposed by other authors, since there are no common repositories in which authors can upload their proposals, making it quite complex to make comparisons among several approaches. To facilitate comparisons among different models, a common platform should be developed to allow researchers to test and upload their proposals.

E. Applications

1) Real World Problems: The majority of the revised studies are oriented to introduce abstract methods and the proposed models are used to solve simple toy examples with no interest to society. Especially in a purely applied area like LSGDM, the main purpose of research should be facing real world problems instead of being deviated towards publication goals.

2) LSGDM Support Systems: Finally, the inherent complexity in real world LSGDM problems makes it difficult to approach their resolution by users who are not experts in the area. Under these circumstances, the use of LSGDM support systems is mandatory to facilitate the entire decision process. However, there is an evident lack of LSGDM support systems to facilitate the resolution of LSGDM problems and appropriate user-friendly software should be developed.

VI. New Trends on LSGDM

Section IV was devoted to analyzing the current state of the art of LSGDM and in Section V we developed a critical analysis of the main drawbacks in the area. In this critique, several limitations regarding the researching in the topic have been highlighted, which must be addressed for the sake of the quality of current and future researches in the topic. Therefore, this section provides a discussion about the future challenges and trends on LSGDM according to our bibliographic and critical analysis. The remaining of this section will be based on the four block scheme shown in Section IV.

A. Preference Structure

Regarding preference structures, the main issue which is usually neglected in the literature is the fact that there are too many preference structures. It is required a deep analysis of if some of them are redundant and about which ones are better for representing DMs opinions in a certain LSGDM problem, especially taking into account that some preference structures, such as FPRs, add more variables to the LSGDM problem, which implies more complexity and resource consumption.

Besides, the reviewed proposals consider preferences modeled by using linear preferences. However, a recent study [112] has shown that when using nonlinear scales to remap the DMs' preferences the consensus models improve and the obtained collective solution for the decision problem is also more realistic from a psychological point of view. Therefore, further studies regarding the impact of these nonlinear scales in LSGDM would be desirable.

B. Group Decision Rules

When dealing with the internal performance of the reviewed models, the most remarkable critique is related to the nonexistence of studies to guarantee the good performance of classic GDM techniques in large-scale contexts in which hundreds or thousands of DMs are involved [27]. Researchers have been directly applying these methods in contexts in which 50 or fewer DMs are considered, but there is no proof about if they will also present a good performance when more DMs are involved in the decision situation. Rigorous studies about the feasibility of these techniques in large-scale contexts are required and, if necessary, these proposals should be extended to deal with LSGDM problems.

Additionally, an interesting research line for this block could be proposing hybrid models in which it is necessary to combine the knowledge of a group of DMs and the information obtained from a large database of users' preferences, Internet of Things (IoT) devices and so on in order to provide realistic solutions for real world problems.

C. Evaluation of Quality

In order to prove the validity of the reviewed techniques, authors usually test their models by using *toy examples* which consider less than 50 DMs. Although it matches the original definition of LSGDM [17], this way of evaluating the performance of a proposal does not seem to be appropriate for nowadays society in which some problems require of taking into account the preferences of millions of users. In this contribution, we have proposed the definition of *m*-LSGDM addressing those models which are able to manage decision situations in which *m* DMs are required. This notion allows to easily classify both the existing and new proposals according to the number of DMs which are designed to deal with. In this regard, it is necessary to test classical models in more demanding contexts which require of standard datasets with hundreds or thousands of DMs in order to avoid ambiguous proposals whose performance in contexts with more than 50 experts is unclear. In addition, global metrics (none of them were found in our search) which allow comparing models should be proposed and used by researchers to show the quality of their models. Furthermore, it would be interesting to develop a universal research platform composed by the different existing LSGDM models in order to facilitate the accessibility of these proposals and the comparisons among them. Therefore, a new research line focuses on the performance analysis of the LSGDM models and their validness from an objective point of view seems to be primordial.

D. Applications

Finally, even though GDM is a purely applied topic, the reviewed proposals usually consist of providing theoretical models which are applied to solve easy examples. The main interest of the area should be devoted to solving real world problems, instead of proposing more models whose performance is just studied for 50 or fewer DMs. Using LSGDM models in Big Data environments or designing new LSGDM Support Systems devoted to e-democracy could be prominent research lines regarding this issue. In addition, it would be interesting to consider the application of other Artificial Intelligence tools to LSGDM. For instance, how to apply Natural Language Processing methods to improve the model of DMs' preferences when they are obtained from social networks in which millions of users take part or developing Group Recommendation Systems for managing millions of users which provide recommendations by taking into account a certain consensus degree when fusing the preferences of other users with similar profiles.

VII. Conclusions

The main aim of this review is to become a turning point for researchers to better understand the concept of LSGDM and introduce proposals that explore new challenges in the area related to new technological developments such as Big Data or social media and pay more attention to the validness of their models under these contexts.

This contribution has performed a systematic review of the existing literature regarding LSGDM. To do so, we have followed the indications for developing bibliographic analysis in Software Engineering proposed by Kitchenham and Charters [43]. By using this methodology, the existing proposals have been reviewed from four different points of view, namely Preference Structure, Group Decision Rules, Evaluation of Quality and Applications, which contain the most relevant keywords in the LSGDM literature and represent the different steps to consider when proposing LSGDM models. Since the developed analysis has revealed several major drawbacks regarding the current research in the topic, this contribution also provides a deep critical analysis of these bad habits found in the literature and some indications about how to redirect future investigation towards the original purpose of LSGDM, which was related to propose frameworks to face decision situations involving an elevated number of DMs.

It should be highlighted that defining theoretical models and testing their performance in toy examples, in which 20–50 DMs are considered, may be a profitable source of content from the point of view of publishing interests, but they would be hard to be applied in practical situations if they do not explicitly specify the number of DMs that are able to manage and prove their good performance in these contexts. In a purely applied area like this, researchers should focus future studies on dealing with real world problems involving a large group of DMs (for instance, Netflix manages 209 million paid memberships) instead of proposing more "large-scale" models which work just with 20 DMs.

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References

[1] S. V. Kovalchuk, V. Krotov, A. Smirnov, D. A. Nasonov, and A. N. Yakovlev, "Distributed data-driven platform for urgent decision making in cardiological ambulance control," *Future Generation Computer Systems-the Int. Journal of Escience*, vol. 79, pp. 144–154, 2018.

[2] Q. Q. Long, "Data-driven decision making for supply chain networks with agent-based computational experiment," *Knowledge-Based Systems*, vol. 141, pp. 55–66, 2018.

[3] I. Palomares, R. M. Rodriguez, and L. Martinez, "An attitude-driven web consensus support system for heterogeneous group decision making," *Expert Systems with Applications*, vol. 40, pp. 139–149, 2013.

[4] Y. Sun, S. He, and J. Leu, "Syndicating web services: A QoS and userdriven approach," *Decision Support Systems*, vol. 43, 2007.

[5] Eklund, A. Rusinowska, and H. de Swart, "Consensus reaching in committees," *Decision Support*, vol. 178, no. 1, pp. 185–193, 2007.

[6] Eklund, A. Rusinowska, and H. de Swart, "A consensus model of political decision-making," *Annals of Operations Research*, vol. 158, no. 1, pp. 5–20, 2008.

[7] I. Palomares, L. Martinez, and F. Herrera, "A consensus model to detect and manage noncooperative behaviors in large-scale group decision making," *IEEE Trans. Fuzzy Systems*, vol. 22, no. 3, pp. 516–530, 2014.

[8] Z. Zhang, C. H. Guo, and L. Martinez, "Managing multigranular linguistic distribution assessments in large-scale multiattribute group decision making," *IEEE Trans. Systems Man Cybernetics Systems*, vol. 47, no. 11, pp. 3063–3076, 2017.

[9] J. Kim, "A model and case for supporting participatory public decision making in e-democracy," *Group Decision and Negotiation*, vol. 17, no. 3, pp. 179–193, 2008.

[10] A. Mateos, A. Jiménez-Martín, and S. Ríos-Insua, "A group

decisionmaking methodology with incomplete individual beliefs applied to e-democracy," *Group Decision and Negotiation*, vol. 24, no. 4, pp. 633–653, 2015.

[11] R. Karthik and S. Ganapathy, "A fuzzy recommendation system for predicting the customers interests using sentiment analysis and ontology in e-commerce," *Applied Soft Computing*, vol. 108, p. 107396, 2021. DOI: 10.1016/j.asoc.2021.107396.

[12] T. T. Tham and H. Nguyen, "An integrated approach of fuzzy-ahptopsis for e-commerce evaluation," *Industrial Engineering & Management Systems*, vol. 20, no. 2, pp. 82–95, 2021.

[13] C. Sueur, J. L. Deneubourg, and O. Petit, "From social network (centralized vs. decentralized) to collective decision-making (unshared vs. shared consensus)," *PLoS* one, vol. 7, no. 2, p. e32566, 2012. DOI: 10.1371/journal.pone.0032566.

[14] Y. J. Xu, X. W. Wen, and W. C. Zhang, "A two-stage consensus method for large-scale multi-attribute group decision making with an application to earthquake shelter selection," *Computers & Industrial Engineering*, vol. 116, pp. 113–129, 2018.

[15] H. J. Zhang, J. Xiao, I. Palomares, H. M. Liang, and Y. C. Dong, "Linguistic distribution-based optimization approach for large-scale gdm with comparative linguistic information: An application on the selection of wastewater disinfection technology," *IEEE Trans. Fuzzy Systems*, vol. 28, no. 2, pp. 376–389, 2020.

[16] X. J. Gou, Z. S. Xu, and F. Herrera, "Consensus reaching process for large-scale group decision making with double hierarchy hesitant fuzzy linguistic preference relations," *Knowledge-Based Systems*, vol. 157, pp. 20–33, 2018.

[17] R. X. Ding, I. Palomares, X. Q. Wang, G. R. Yang, B. S. Liu, Y. C. Dong, E. Herrera-Viedma, and F. Herrera, "Large-scale decision-making: Characterization, taxonomy, challenges and future directions from an artificial intelligence and applications perspective," *Information Fusion*, vol. 59, pp. 84–102, 2020.

[18] R. M. Rodriguez, A. Labella, G. De Tre, and L. Martinez, "A large scale consensus reaching process managing group hesitation," *Knowledge-Based Systems*, vol. 159, pp. 86–97, 2018.

[19] R. X. Ding, X. Q. Wang, K. Shang, B. S. Liu, and F. Herrera, "Sparse representation-based intuitionistic fuzzy clustering approach to find the group intra-relations and group leaders for large-scale decision making," *IEEE Trans. Fuzzy Systems*, vol. 27, no. 3, pp. 559–573, 2019.

[20] A. Labella, Y. Liu, R. M. Rodriguez, and L. Martinez, "Analyzing the performance of classical consensus models in large scale group decision making: A comparative study," *Applied Soft Computing*, vol. 67, pp. 677–690, 2018.

[21] R. X. Nie, Z. Tian, J. Q. Wang, and H. Y. Luo, "An objective and interactive-information-based feedback mechanism for the consensus-reaching process considering a non-support degree for minority opinions," *Expert Systems*, vol. 37, no. 5, 2020.

[22] F. Liu, J. W. Zhang, and T. Liu, "A PSO-algorithm-based consensus model with the application to large-scale group decision-making," *Complex & Intelligent Systems*, vol. 6, no. 2, pp. 287–298, 2020.

[23] K. Kabirifar and M. Mojtahedi, "The impact of engineering, procurement and construction (EPC) phases on project performance: A case of large-scale residential construction project," *Buildings*, vol. 9, no. 1, 2019.

[24] G. Carvalho, A. S. Vivacqua, J. M. Souza, and S. J. Medeiros, "LaSca: A large scale group decision support system," *Journal of Universal Computer Science*, vol. 17, no. 2, pp. 261–275, 2011.

[25] I. Palomares, L. Martinez, and F. Herrera, "Mentor: A graphical monitoring tool of preferences evolution in large-scale group decision making," *Knowledge-Based Systems*, vol. 58, pp. 66–74, 2014.

[26] H. J. Zhang, S. H. Zhao, G. Kou, C. C. Li, Y. C. Dong, and F. Herrera, "An overview on feedback mechanisms with minimum adjustment or cost in consensus reaching in group decision making: Research paradigms and challenges," *Information Fusion*, vol. 60, pp. 65–79, 2020.

[27] M. Tang and H. Liao, "From conventional group decision making to large-scale group decision making: What are the challenges and how to meet them in big data era? A state-of-the-art survey" *Omega*, vol. 100, 2021.

[28] J. Kacprzyk, "Group decision making with a fuzzy linguistic majority," *Fuzzy Sets and Systems*, vol. 18, no. 2, pp. 105–118, 1986.

[29] J. Lu and D. Ruan, "Multi-objective group decision making: Methods, software and applications with fuzzy set techniques," *Imperial College Press*, vol. 6, 2007.

[30] M. Roubens, "Fuzzy sets and decision analysis," *Fuzzy Sets and Systems*, vol. 90, no. 2, pp. 199–206, 1997.

[31] C. Butler and A. Rothstein, *On Conflict and Consensus: A Handbook on Formal Consensus Decision Making*. Portland: Food Not Bombs Publishing, 1987.

[32] I. Palomares, F. J. Estrella, L. Martínez, and F. Herrera, "Consensus under a fuzzy context: Taxonomy, analysis framework AFRYCA and experimental case of study," *Information Fusion*, vol. 20, pp. 252–271, 2014.

[33] E. Herrera-Viedma, J. L. García-Lapresta, J. Kacprzyk, M. Fedrizzi, H. Nurmi, and S. Zadrożny, *Consensual Processes*. Springer, 2011, vol. 267.

[34] S. Modgil, S. Gupta, U. Sivarajah, and B. Bhushan, "Big data-enabled large-scale group decision making for circular economy: An emerging market context," *Technological Forecasting and Social Change*, vol. 166, 2021.

[35] T. Wu, X. W. Liu, and J. D. Qin, "A linguistic solution for double largescale group decision-making in e-commerce," *Computers & Industrial Engineering*, vol. 116, pp. 97–112, 2018.

[36] Q. F. Wan, X. H. Xu, X. H. Chen, and J. Zhuang, "A two-stage optimization model for large-scale group decision-making in disaster management: Minimizing group conflict and maximizing individual satisfaction," *Group Decision and Negotiation*, vol. 29, no. 5, pp. 901–921, 2020.

[37] L. M. Liu, W. Z. Cao, B. Shi, and M. Tang, "Large-scale green supplier selection approach under a q-rung interval-valued orthopair fuzzy environment," *Processes*, vol. 7, no. 9, 2019.

[38] Y. C. Dong, S. H. Zhao, H. J. Zhang, F. Chiclana, and E. Herrera-Viedma, "A self-management mechanism for noncooperative behaviors in large-scale group consensus reaching processes," *IEEE Trans. Fuzzy Systems*, vol. 26, no. 6, pp. 3276–3288, 2018.

[39] Z. B. Wu and J. Xu, "A consensus model for large-scale group decision making with hesitant fuzzy information and changeable clusters," *Information Fusion*, vol. 41, pp. 217–231, 2018.

[40] B. S. Liu, Q. Zhou, R. X. Ding, I. Palomares, and F. Herrera, "Large-scale group decision making model based on social network analysis: Trust relationship-based conflict detection and elimination," *European Journal of Operational Research*, vol. 275, no. 2, pp. 737–754, 2019.

[41] Y. Lu, Y. Xu, E. Herrera-Viedma, and Y. Han, "Consensus of large-scale group decision making in social network: The minimum cost model based on robust optimization," *Information Sciences*, vol. 547, pp. 910–930, 2021.

[42] Z. J. Shi, X. Q. Wang, I. Palomares, S. J. Guo, and R. X. Ding, "A novel consensus model for multi-attribute large-scale group decision making based on comprehensive behavior classification and adaptive weight updating," *Knowledge-Based Systems*, vol. 158, pp. 196–208, 2018.

[43] B. A. Kitchenham and S. Charters, "Guidelines for performing systematic literature reviews in software engineering," Keele University, Tech. Rep. EBSE 2007-001, 2007.

[44] I. Palomares, "Consensus model for large-scale group decision support in IT services management," *Intelligent Decision Technologies Netherlands*, vol. 8, no. 2, pp. 81–94, 2014.

[45] S. Orlovsky, "Decision-making with a fuzzy preference relation," *Fuzzy Sets and Systems*, vol. 1, no. 3, pp. 155–167, 1978.

[46] T. L. Saaty, "The analytic hierarchy process (AHP)," *The Journal of the Operational Research Society*, vol. 41, no. 11, pp. 1073–1076, 1980.

[47] F. Seo and M. Sakawa, "Fuzzy multiattribute utility analysis for

collective choice," *IEEE Trans. Systems*, *Man*, *and Cybernetics*, vol. 1, pp. 45–53, 1985.

[48] T. Tanino, "On group decision making under fuzzy preferences," in *Multiperson Decision Making Models Using Fuzzy Sets and Possibility Theory*. Springer, 1990, pp. 172–185.

[49] J. Xiao, X. L. Wang, and H. J. Zhang, "Managing personalized individual semantics and consensus in linguistic distribution large-scale group decision making," *Information Fusion*, vol. 53, pp. 20–34, 2020.

[50] C. C. Li, Y. C. Dong, and F. Herrera, "A consensus model for largescale linguistic group decision making with a feedback recommendation based on clustered personalized individual semantics and opposing consensus groups," *IEEE Trans. Fuzzy Systems*, vol. 27, no. 2, pp. 221–233, 2019.

[51] T. Wu, X. W. Liu, and F. Liu, "An interval type-2 fuzzy topsis model for large scale group decision making problems with social network information," *Information Sciences*, vol. 432, pp. 392–410, 2018.

[52] W. J. Zuo, D. F. Li, G. F. Yu, and L. Zhang, "A large group decision-making method and its application to the evaluation of property perceived service quality," *Journal of Intelligent & Fuzzy Systems*, vol. 37, no. 1, pp. 1513–1527, 2019.

[53] Y. M. Song and J. Hu, "Large-scale group decision making with multiple stakeholders based on probabilistic linguistic preference relation," *Applied Soft Computing*, vol. 80, pp. 712–722, 2019.

[54] Y. M. Song and G. X. Li, "Consensus constructing in large-scale group decision making with multi-granular probabilistic 2-tuple fuzzy linguistic preference relations," *IEEE Access*, vol. 7, pp. 56 947–56 959, 2019.

[55] S. L. Li and C. Wei, "A two-stage dynamic influence model-achieving decision-making consensus within large scale groups operating with incomplete information," *Knowledge-Based Systems*, vol. 189, 2020.

[56] J. F. Chu, Y. M. Wang, X. W. Liu, and Y. C. Liu, "Social network community analysis based large-scale group decision making approach with incomplete fuzzy preference relations," *Information Fusion*, vol. 60, pp. 98–120, 2020.

[57] Y. M. Song and G. X. Li, "A large-scale group decision-making with incomplete multi-granular probabilistic linguistic term sets and its application in sustainable supplier selection," *Journal of the Operational Research Society*, vol. 70, no. 5, pp. 827–841, 2019.

[58] H. Liao, R. Tan, and M. Tang, "An overlap graph model for largescale group decision making with social trust information considering the multiple roles of experts," *Expert Systems*, vol. 38, no. 3, 2021.

[59] R. X. Ding, X. Q. Wang, K. Shang, and F. Herrera, "Social network analysis-based conflict relationship investigation and conflict degree-based consensus reaching process for large scale decision making using sparse representation," *Information Fusion*, vol. 50, pp. 251–272, 2019.

[60] T. Wu, K. Zhang, X. W. Liu, and C. Y. Cao, "A two-stage social trust network partition model for large-scale group decision-making problems," *Knowledge-Based Systems*, vol. 163, pp. 632–643, 2019.

[61] Z. J. Du, S. M. Yu, H. Y. Luo, and X. D. Lin, "Consensus convergence in large-group social network environment: Coordination between trust relationship and opinion similarity," *Knowledge-Based Systems*, vol. 217, p. 106828, 2021. DOI: 10.1016/j.knosys.2021.106828.

[62] M. Tang, H. Liao, E. Herrera-Viedma, C. L. Chen, and W. Pedrycz, "A dynamic adaptive subgroup-to-subgroup compatibility-based conflict detection and resolution model for multicriteria large-scale group decision making," *IEEE Trans. Cybernetics*, vol. 51, no. 10, pp. 4784–4795, 2021.

[63] Z. J. Du, S. M. Yu, and X. H. Xu, "Managing noncooperative behaviors in large-scale group decision-making: Integration of independent and supervised consensus-reaching models," *Information Sciences*, vol. 531, pp. 119–138, 2020.

[64] Z. J. Du, H. Y. Luo, X. D. Lin, and S. M. Yu, "A trust-similarity analysis-based clustering method for large-scale group decision-making under a social network," *Information Fusion*, vol. 63, pp. 13–29, 2020.

[65] X. H. Xu, Z. J. Du, X. H. Chen, and C. G. Cai, "Confidence consensus-based model for large-scale group decision making: A novel approach to managing non-cooperative behaviors," *Information Sciences*, vol. 477, pp. 410–427, 2019.

[66] C. X. Zhang, M. Zhao, L. C. Zhao, and Q. F. Yuan, "A consensus model for large-scale group decision-making based on the trust relationship considering leadership behaviors and non-cooperative behaviors," *Group Decision and Negotiation*, vol. 30, no. 3, pp. 553–586, 2021.

[67] T. Wu, X. W. Liu, J. D. Qin, and F. Herrera, "Balance dynamic clustering analysis and consensus reaching process with consensus evolution networks in large-scale group decision making," *IEEE Trans. Fuzzy Systems*, vol. 29, no. 2, pp. 357–371, 2021.

[68] Z. Y. Liu, X. He, and Y. Deng, "Network-based evidential three-way theoretic model for large-scale group decision analysis," *Information Sciences*, vol. 547, pp. 689–709, 2021.

[69] M. Tang, H. Liao, X. Mi, X. Xu, and F. Herrera, "Dynamic subgroup-quality-based consensus in managing consistency, nearness, and evenness quality indices for large-scale group decision making under hesitant environment," *Journal of the Operational Research Society*, vol. 72, no. 4, pp. 865–878, 2020.

[70] M. Tang, H. C. Liao, J. Xu, D. Streimikiene, and X. S. Zheng, "Adaptive consensus reaching process with hybrid strategies for large-scale group decision making," *European Journal of Operational Research*, vol. 282, no. 3, pp. 957–971, 2020.

[71] F. J. Quesada, I. Palomares, and L. Martinez, "Managing experts behavior in large-scale consensus reaching processes with uninorm aggregation operators," *Applied Soft Computing*, vol. 35, pp. 873–887, 2015.

[72] Q. Gao, J. Huang, and Y. J. Xu, "A k-core decomposition-based opinion leaders identifying method and clustering-based consensus model for large-scale group decision making," *Computers & Industrial Engineering*, vol. 150, 2020.

[73] W u, Q. Wu, L. G. Zhou, and H. Y. Chen, "Optimal group selection model for large-scale group decision making," *Information Fusion*, vol. 61, pp. 1–12, 2020.

[74] X. Liu, Y. J. Xu, R. Montes, R. X. Ding, and F. Herrera, "Alternative ranking-based clustering and reliability index-based consensus reaching process for hesitant fuzzy large scale group decision making," *IEEE Trans. Fuzzy Systems*, vol. 27, no. 1, pp. 159–171, 2019.

[75] D. L. Zhang, Y. B. Yang, W. C. Wang, and X. S. You, "A LSGDM method based on social network and IVIFN's geometric characteristics for evaluating the collaborative innovation problem," *Journal of Intelligent & Fuzzy Systems*, vol. 40, no. 3, pp. 5119–5138, 2021.

[76] Y. W. Du, N. Yang, and J. Ning, "IFS/ER-based large-scale multiattribute group decision-making method by considering expert knowledge structure," *Knowledge-Based Systems*, vol. 162, pp. 124–135, 2018.

[77] Y. W. Du and Y. K. Shan, "A dynamic intelligent recommendation method based on the analytical ER rule for evaluating product ideas in large-scale group decision-making," *Group Decision and Negotiation*, 2020.

[78] Y. W. Du, Q. Chen, Y. L. Sun, and C. H. Li, "Knowledge structure-based consensus-reaching method for large-scale multiattribute group decision-making," *Knowledge-Based Systems*, vol. 219, 2021.

[79] E. Koksalmis and O. Kabak, "Sensor fusion based on Dempster-Shafer theory of evidence using a large scale group decision making approach," *Int. Journal of Intelligent Systems*, vol. 35, no. 7, pp. 1126–1162, 2020.

[80] S. F. He, X. H. Pan, and Y. M. Wang, "A shadowed set-based todim method and its application to large-scale group decision making," *Information Sciences*, vol. 544, pp. 135–154, 2021.

[81] M. Zhao, M. Gao, and Z. C. Li, "A consensus model for large-scale multi-attribute group decision making with collaboration-reference network under uncertain linguistic environment," *Journal of Intelligent & Fuzzy Systems*, vol. 37, no. 3, pp. 4133–4156, 2019.

[82] Wa ng, X. H. Xu, and S. Huang, "An improved consensus-based

model for large group decision making problems considering experts with linguistic weighted information," *Group Decision and Negotiation*, vol. 28, no. 3, pp. 619–640, 2019.

[83] B. S. Liu, T. F. Huo, C. Liao, J. Gong, and B. Xue, "A group decision-making aggregation model for contractor selection in large scale construction projects based on two-stage partial least squares (PLS) path modeling," *Group Decision and Negotiation*, vol. 24, no. 5, pp. 855–883, 2015.

[84] X. Gou, Z. Xu, H. Liao, and F. Herrera, "Consensus model handling minority opinions and noncooperative behaviors in large-scale group decision-making under double hierarchy linguistic preference relations," *IEEE Trans. Cybernetics*, vol. 51, no. 1, pp. 283–296, 2021.

[85] Q. Zhan, C. Fu, and M. Xue, "Distance-based large-scale group decision-making method with group influence," *Int. Journal of Fuzzy Systems*, vol. 23, no. 2, pp. 535–554, 2021.

[86] R. X. Ren, M. Tang, and H. C. Liao, "Managing minority opinions in micro-grid planning by a social network analysis-based large scale group decision making method with hesitant fuzzy linguistic information," *Knowledge-Based Systems*, vol. 189, 2020.

[87] X. Y. Zhong and X. H. Xu, "Clustering-based method for large group decision making with hesitant fuzzy linguistic information: Integrating correlation and consensus," *Applied Soft Computing*, vol. 87, 2020.

[88] Z. Zhang, W. Y. Yu, L. Martinez, and Y. Gao, "Managing multigranular unbalanced hesitant fuzzy linguistic information in multiattribute largescale group decision making: A linguistic distribution-based approach," *IEEE Trans. Fuzzy Systems*, vol. 28, no. 11, pp. 2875–2889, 2020.

[89] L. Xiao, Z. S. Chen, X. Zhang, J. Chang, W. Pedrycz, and K. S. Chin, "Bid evaluation for major construction projects under large-scale group decision-making environment and characterized expertise levels," *Int. Journal of Computational Intelligence Systems*, vol. 13, no. 1, pp. 1227–1242, 2020.

[90] S. L. Li and C. Wei, "A large scale group decision making approach in healthcare service based on sub-group weighting model and hesitant fuzzy linguistic information," *Computers & Industrial Engineering*, vol. 144, 2020.

[91] Z. Z. Ma, J. J. Zhu, K. Ponnambalam, and S. T. Zhang, "A clustering method for large-scale group decision-making with multi-stage hesitant fuzzy linguistic terms," *Information Fusion*, vol. 50, pp. 231–250, 2019.

[92] X. L. Wu, S. Nie, H. C. Liao, and Gupta, "A large-scale group decision making method with a consensus reaching process under cognitive linguistic environment," *Int. Transactions in Operational Research*, 2020.

[93] Z. Tian, R. X. Nie, and J. Q. Wang, "Social network analysis-based consensus-supporting framework for large-scale group decision-making with incomplete interval type-2 fuzzy information," *Information Sciences*, vol. 502, pp. 446–471, 2019.

[94] T. Wu, X. W. Liu, and F. Liu, "The solution for fuzzy large-scale group decision making problems combining internal preference information and external social network structures," *Soft Computing*, vol. 23, no. 18, pp. 9025–9043, 2019.

[95] T. Wu and X. W. Liu, "An interval type-2 fuzzy clustering solution for large-scale multiple-criteria group decision-making problems," *Knowledge-Based Systems*, vol. 114, pp. 118–127, 2016.

[96] Z. Wen and H. C. Liao, "Capturing attitudinal characteristics of decision-makers in group decision making: Application to select policy recommendations to enhance supply chain resilience under COVID-19 outbreak," *Operations Management Research*, 2021.

[97] X. Tan, J. J. Zhu, F. J. Cabrerizo, and E. Herrera-Viedma, "A cyclic dynamic trust-based consensus model for large-scale group decision making with probabilistic linguistic information," *Applied Soft Computing*, vol. 100, 2021.

[98] Q. Liu, H. Y. Wu, and Z. S. Xu, "Consensus model based on probability k-means clustering algorithm for large scale group decision making," *Int. Journal of Machine Learning and Cybernetics*, vol. 12, no. 6, pp. 1609–1626, 2021.

[99] B. L. Wang and J. Y. Liang, "A novel preference measure for multigranularity probabilistic linguistic term sets and its applications in large-scale group decision-making," *Int. Journal of Fuzzy Systems*, vol. 22, no. 7, pp. 2350–2368, 2020.

[100] X. Li, H. Liao, and Z. Wen, "A consensus model to manage the noncooperative behaviors of individuals in uncertain group decision making problems during the COVID-19 outbreak," *Applied Soft Computing*, vol. 99, p. 106879, 2021. DOI: 10.1016/j.asoc.2020.106879.

[101] X. R. Chao, G. Kou, Y. Peng, and E. Herrera-Viedma, "Large-scale group decision-making with non-cooperative behaviors and heterogeneous preferences: An application in financial inclusion," *European Journal of Operational Research*, vol. 288, no. 1, pp. 271–293, 2021.

[102] M. Tang, X. Y. Zhou, H. C. Liao, J. Xu, H. Fujita, and F. Herrera, "Ordinal consensus measure with objective threshold for heterogeneous large-scale group decision making," *Knowledge-Based Systems*, vol. 180, pp. 62–74, 2019.

[103] X. Liu, Y. J. Xu, and F. Herrera, "Consensus model for largescale group decision making based on fuzzy preference relation with self-confidence: Detecting and managing overconfidence behaviors," *Information Fusion*, vol. 52, pp. 245–256, 2019.

[104] Y. Cornet, M. J. Barradale, H. Gudmundsson, and M. B. Barfod, "Engaging multiple actors in large-scale transport infrastructure project appraisal: An application of MAMCA to the case of HS2 high-speed rail," *Journal of Advanced Transportation*, 2018.

[105] G. Beliakov, H. Bustince, and T. Calvo, *A Practical Guide to Averaging Functions*. Springer, 01 2016.

[106] X. Y. Mo, Z. S. Xu, H. Zhao, Z. N. Hao, and S. K. Xiang, "Hesitant fuzzy multiple integrals for information aggregation," *Int. Journal of Fuzzy Systems*, vol. 22, no. 2, pp. 668–685, 2020.

[107] E. Kian, M. Sun, and F. Bosche, "Complexity for megaprojects in the energy sector," *Journal of Management in Engineering*, vol. 33, Article No. 4, 2017.

[108] X. K. Zhou, W. Liang, S. Z. Huang, and M. Fu, "Social recommendation with large-scale group decision-making for cyber-enabled online service," *IEEE Trans. Computational Social Systems*, vol. 6, no. 5, pp. 1073–1082, 2019.

[109] R. Urena, F. Chiclana, and E. Herrera-Viedma, "DeciTrustNET: A graph based trust and reputation framework for social networks," *Information Fusion*, vol. 61, pp. 101–112, 2020.

[110] L. Martínez and J. Montero, "Challenges for improving consensus reaching process in collective decisions," *New Mathematics and Natural Computation*, vol. 3, no. 02, pp. 203–217, 2007.

[111] Á. Labella, H. Liu, R. M. Rodríguez, and L. Martinez, "A cost consensus metric for consensus reaching processes based on a comprehensive minimum cost model," *European Journal of Operational Research*, vol. 281, no. 2, pp. 316–331, 2020.

[112] D. García-Zamora, Á. Labella, R. M. Rodríguez, and L. Martínez, "Nonlinear preferences in group decision-making. Extreme values amplifications and extreme values reductions," *Int. Journal of Intelligent Systems*, vol. 36, no. 11, pp. 6581–6612, 2021.

Diego García-Zamora received the bachelor degree in mathematics from the University of Granada in 2016. In addition, he received the master degree in physics and mathematics FisyMat in 2017 and the master degree in teachers of spanish compulsory secondary education and bachiller in 2018, both from the University of Granada. Currently, he is a Ph.D. candidate at the University of Jaén and focuses his research on group decision making and their consensus reaching processes. He also holds the predoctoral contract Formación del Profesorado Universitario, granted by the Spanish Ministry of Science, Innovation, and Universities. He has published works in high-quality journals such as *Information Sciences or International Journal of Intelligent Systems* and participated in several international conferences, obtaining the Best Student Paper Award in the 16th International Conference on Intelligent Systems and Knowledge Engineering (ISKE 2021), celebrated in Chengdu, China.

Álvaro Labella received the B.Sc. diploma from University of Jaén, Spain, in 2014 and holds the Ph.D. in computer science from the same university in 2021. Currently, he is a Postdoctoral Researcher in the SINBAD2 research group at the University of Jaén and works in the area of decision making, computing with words and software application development. During the development of the doctoral thesis, he has published 20 articles in international journals, as well as 17 contributions in international conferences and 4 in national conferences. During the doctoral studies he stayed at the National Technical University of Athens (NTUA) in Athens, Greece and at the Portsmouth Business School in Portsmouth, UK. He was awarded by the University of Jaén with the Ada Lovelace Award for the Best Application modality, in 2019 he received the Best Student Paper Award at the International Conference on Intelligent and Fuzzy Systems in Istanbul, Turkey, and the Best Paper Award at the International Conference on Intelligent Systems and Knowledge Engineering in Dalian, China.

Weiping Ding (Senior Member, IEEE) received the Ph.D. degree in computer science from Nanjing University of Aeronautics and Astronautics (NUAA) in 2013. From 2014 to 2015, he was a Postdoctoral Researcher at the Brain Research Center, National Chiao Tung University (NCTU), Hsinchu, Taiwan, China. In 2016, He was a Visiting Scholar at National University of Singapore (NUS), Singapore. From 2017 to 2018, he was a Visiting Professor at University of Technology Sydney (UTS), Australia. He is currently a Professor at the School of Information Science and Technology, Nantong University. His research interests include deep neural networks, multimodal machine learning, granular data mining, and medical images analysis. He has published more than 150 journal papers, including IEEE T-FS, T-NNLS, T-CYB, T-SMCS, T-BME, T-EVC, T-II, T-ETCI, T-CDS, T-ITS and T-AI. and he has held 20 approved invention patents in total over 35 issued patents. He has co-authored two books. His nine authored/co-authored papers have been selected as ESI Highly Cited Papers. Dr. Ding is vigorously involved in editorial activities. He served/serves on the Editorial Advisory Board of Knowledge-Based Systems and Editorial Board of Information Fusion, Engineering Applications of Artificial Intelligence and Applied Soft Computing. He served/serves as an Associate Editor of IEEE Transactions on Neural Networks and Learning Systems, IEEE Transactions on Fuzzy Systems, IEEE/CAA Journal of Automatica Sinica, Information Sciences, Neurocomputing, Swarm and Evolutionary Computation, IEEE Access and Journal of Intelligent & Fuzzy Systems, and Co-Editor-in-Chief of Journal of Artificial Intelligence and System. He is the Leading Guest Editor of Special Issues in several prestigious journals, including IEEE Transactions on Evolutionary Computation, IEEE Transactions on Fuzzy Systems.

Rosa M. Rodríguez (Member, IEEE) received the B.Sc, M.Sc., and Ph.D. degrees in computer science, all of them from the University of Jaén in 2006, 2008, and 2013, respectively. She got the extraordinary award of Ph.D. (bestowed in 2017). Currently, she has a Postdoctoral contract (Ramón y Cajal) at the University of Jaén, Spain. Her research interests include linguistic preference modelling, decision making, decision support systems and fuzzy logic-based systems. She has more than 40 papers in journals indexed by the SCI. She is Editorial Assistant in *International Journal of Computational Intelligence Systems* and Associate Editor in *IEEE Transactions on Fuzzy Systems*, *International Journal of Fuzzy Systems* and *Journal of Intelligent and Fuzzy Systems*. She received the IEEE Transactions on Fuzzy Systems Outstanding Paper Award 2012 (bestowed in 2015). Finally, highlight that It has been classified as Highly Cited Researcher 2019, 2020, and 2021 according to the Essential Science Indicators of the Web of Science.

Luis Martínez (Senior Member, IEEE) is currently a Full Professor with the Department of Computer Science, University of Jaén, Spain. He is also Visiting Professor in University of Technology Sydney, University of Portsmouth (Isambard Kingdom Brunel Fellowship Scheme), and in the Wuhan University of Technology (Chutian Scholar). He has been main Researcher in 16 R&D projects, and also has published more than 190 papers in journals indexed by the SCI and more than 200 contributions in Inter/national conferences related to his areas. His current research interests include multi-criterion decision making, fuzzy logic-based systems, computing with words, and recommender systems. He was a Recipient of the IEEE Transactions on Fuzzy Systems Outstanding Paper Award 2008 and 2012 (bestowed in 2011 and 2015, respectively). He is a Co-Editor-in-Chief of the *International Journal of Computational Intelligence Systems* and an Associate Editor of journals including the *Information Sciences*, *Knowledge Based Systems and Information Fusion*. He is IFSA Fellow 2021, Senior Member of IEEE and of the European Society for Fuzzy Logic and Technology. Eventually, he has been appointed as Highly Cited Researcher 2017–2021 in computer sciences.

4.2 PREFERENCIAS NO LINEALES EN TOMA DE DECISIÓN EN GRUPO.

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Nonlinear preferences in group decision-making. Extreme values amplifications and extreme values reductions

Diego García-Zamora | **Álvaro Labella** | **Rosa M. Rodríguez** | **Luis Martínez**

Department of Computer Science, University of Jaén, Jaén, Spain

Correspondence
Diego García-Zamora, Department of Computer Science, University of Jaén, Campus Las Lagunillas, Jaén 23071, Spain.
Email: dgzamora@ujaen.es

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Abstract
Consensus Reaching Processes (CRPs) deal with those group decision-making situations in which conflicts among experts' opinions make difficult the reaching of an agreed solution. This situation, worsens in large-scale group decision situations, in which opinions tend to be more polarized, because in problems with extreme opinions it is harder to reach an agreement. Several studies have shown that experts' preferences may not always follow a linear scale, as it has commonly been assumed in previous CRP. Therefore, the main aim of this paper is to study the effect of modeling this nonlinear behavior of experts' preferences (expressed by fuzzy preference relations) in CRPs. To do that, the experts' preferences will be remapped by using nonlinear deformations which amplify or reduce the distance between the extreme values. We introduce such automorphisms to remap the preferences as Extreme Values Amplifications (EVAs) and Extreme Values Reductions (EVRs), study their main properties and propose several families of these EVA and EVR functions. An analysis about the behavior of EVAs and EVRs when are implemented in a generic consensus

model is then developed. Finally, an illustrative experiment to study the performance of different families of EVAs in CRPs is provided.

KEYWORDS
consensus reaching process, extreme values amplification, extreme values reductions, group decision-making, nonlinear preferences

1 | INTRODUCTION

Group decision-making (GDM) problems are those situations in which several individuals or experts have to choose a solution for a given problem which consists of two or more possible solutions or alternatives.[1]

Butler and Rothstein[2] proposed several rules to guide the decision process in real-world problems, like, the majority rule, the minority rule, or unanimity. The main issue around these general rules is the fact that some individuals or experts may not agree with the *solution* chosen by the group because they could consider that their opinions have not been sufficiently taken into account for achieving the solution.

Even though classic GDM problems have been proposed by considering a few number of experts, current technological advances, such as e-democracy[3] or social networks,[4] have led to situations in which many experts can be required. Large-scale GDM (LSGDM) problems are defined as those GDM problems in which there are more than 20 experts involved in the problem.[5] This increment in the number of experts implies more conflicts and polarized opinions,[6] and thus to obtain agreed solutions that become much more complex.

Consensus Reaching Processes (CRPs)[2,7] emerge for those GDM situations which demand an agreement among experts about the chosen solution.

According to Reference [8] the most widely used approach to deal with CRPs may be the idea of *soft consensus* proposed by Kacprzyk.[1] Soft consensus that is based on the notion of fuzzy linguistic majority provides a measure to compute the consensus among experts.[9]

Several CRPs based on the notion of soft consensus have been developed,[10–13] all of them assume the use of linear scales for the preferences elicited from experts. However, recent studies have shown that, in certain situations, better decisions can be obtained by using nonlinear scales for representing users' preferences.[14,15] Masthoff[14] studied people's behavior when they rate their opinions on a numerical scale and concluded that the ratings did not follow a linear scale because the same differences between two values at different levels of the scale represent different differences in people's minds. Therefore, it was concluded that a quadratic scale was a better measure than a linear one. Meanwhile, Delic et al.[15] pointed out that by using polynomial remappings of individual preferences (under both ranking and rating conditions in group decision schemes) the results of the decision-making processes are improved.

Paying attention to previous results, this paper aims at studying the effect of modeling nonlinear behaviors in CRPs for LSGDM. Therefore, we raise the following research questions that stem from our goal:

- RQ1: How are nonlinear scales modeled in CRPs for managing polarization in (LS)GDM?

- RQ2: Does the nonlinear approach improve the CRPs in comparison with linear approach?

Without loss of generality,[16] we assume that the preferences will be elicited by fuzzy preference relations (FPRs) and nonlinear deformations will be applied to each value of the preference relation, to *adjust* the initial experts' preferences to a more realistic nonlinear scale when the (LS)GDM problem faces a situation in which such scales are needed. The impact of the nonlinear remapping procedure in CRPs will be evaluated by comparing its convergence and degree of consensus achieved.

Consequently, we will introduce Extreme Values Amplifications (EVAs) as those functions that increase in a nonlinear way the distance between extreme values of the FPRs. Additionally, Extreme Values Reductions (EVRs) will be defined as those nonlinear deformations which reduce the distance between the extreme values of the FPRs.

Several families of these EVAs and EVRs are proposed. EVAs will be then applied in different CRPs to LSGDM problems to show the effectiveness of this nonlinear preference modeling by using the software AFRYCA.[10]

Such EVAs (resp., EVRs) will act as:

1. They remap the original linear-scaled FPRs into nonlinear-scaled FPRs.
2. They amplify (resp., reduce) the distance between the extreme values, and reduce (resp., amplify) the distance between the intermediate ones.
3. They have a concrete geometrical pattern.
4. The amplification (resp., reduction) of distances is greater when preferences are close to the extremes.

Finally, we analyze the performance of EVAs and EVRs in CRPs for LSGDM problems evaluating if EVA/EVR approaches outperform the classic linear approach in CRPs.

The remaining of this paper is set up as follows: In Section 2, a brief review of GDM problems and CRPs is presented. Section 3 introduces an exhaustive study of the properties of those automorphisms on the interval [0, 1] which remaps linear-scaled FPRs into nonlinear-scaled FPRs by increasing or reducing distances between extreme preferences. Section 4 defines the main concepts of this contribution, namely, EVA and EVR, and presents its fundamental characteristics. Section 5 proposes a general method to construct EVAs and introduces several families of EVAs and EVRs. In Section 6 we will discuss the performance of EVAs and EVRs when applied in a generic consensus model. In Section 7, we simulate the performance of EVAs when they are applied in CRPs for LSGDM problems. Finally, Section 8 will conclude the contribution.

2 | PRELIMINARIES

This section revises some essential concepts about GDM and CRP to easily understand the proposal.

2.1 | Group decision-making

A GDM problem[1,2] is a situation in which two or more individuals have to choose a collective solution for a certain problem. Formally, the main elements in a GDM problem are:

- A set $X = \{X_1, X_2, ..., X_n\}$, $2 \leq n \in \mathbb{N}$, of alternatives or possible solutions to the given problem.
- A set $E = \{e_1, e_2, ..., e_m\}$, where $2 \leq m \in \mathbb{N}$, of experts who express their opinions about the alternatives in X throughout certain preference structure.

In this study, without loss of generality,[16] we will assume that experts elicit their preferences by using an FPR, which has been proved to be effective in managing the uncertainty.[16,17] To obtain these FPRs, each expert e_k, $k = 1, ..., m$ will elicit the degree to which she/he prefers the alternative X_i over the alternative X_j, which will be denoted by $p_{i,j}^k$. The FPR associated with the expert e_k will be the matrix $P_k \in \mathcal{M}_{n\times n}([0,1])$ whose items are the values $p_{ij}^k \in [0,1]$ which must satisfy the symmetry condition $p_{ij}^k + p_{ji}^k = 1 \quad \forall\, i, j \in \{1, 2, ..., n\}, k \in \{1, 2, ..., m\}$.

Nowadays, technological developments have led to GDM situations[3,4] which demand a large number of experts. LSGDM problems are defined as those decision situations in which 20 or more experts are required to solve the GDM problems.[5]

2.2 | Consensus reaching processes

GDM solving processes may fail when using classical GDM rules, like, the majority rule, since experts may feel unsatisfied with the solution and think that their opinions have not been sufficiently considered.[10,18] To avoid such disagreements, it is necessary to include in the GDM solving process a CRP to obtain agreed solutions that reflect the opinion of all the experts involved in the GDM problem.[11,19]

A CRP is an iterative discussion process[7] usually coordinated by a moderator whose main responsibilities are to evaluate the level of agreement achieved in each round of discussion (and if it is enough), identify those experts' opinions that are far away from the collective opinion and provide some feedback/recommendations to such experts to increase the consensus degree in the next round.[8]

A general scheme of a CRP (see Figure 1) is briefly summarized as follows:

- *Gathering preferences*: Each expert elicits her/his preferences through a certain preference structure.[16]
- *Determining the level of consensus*: The moderator computes the level of agreement throughout a certain consensus measure.[10]
- *Consensus control*: The consensus level is compared with a threshold level previously established as acceptable. If either this consensus threshold is reached or the maximum number of rounds is surpassed, the process finishes. Otherwise the consensus progress keeps going.
- *Consensus progress*: To increase the level of consensus, experts should change their preferences according to the moderator's recommendations.

When large-scale contexts are considered, CRPs become more complex[10,18] because there are usually more conflicts and the opinions are more polarized.[6] Additionally, new challenges emerge to deal with a large number of experts in CRPs and several proposals have been presented to cope with them[19–22] in recent years.

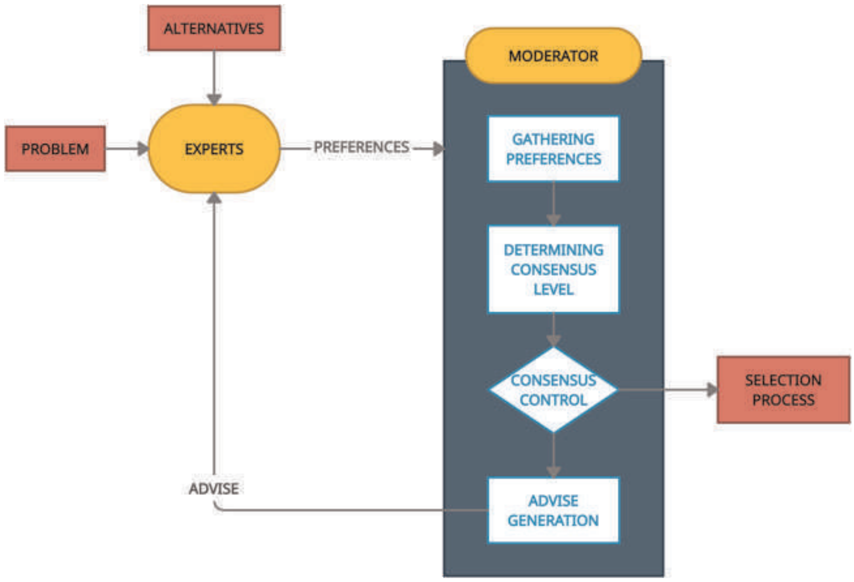

FIGURE 1 General scheme of consensus reaching process [Color figure can be viewed at wileyonlinelibrary.com]

3 | AMPLIFYING DISTANCES BETWEEN EXTREME PREFERENCES

Our main aim is to study the performance of CRPs when the experts' preferences are modeled by a nonlinear scale. To do this, the preferences elicited from experts with FPRs will be transformed by a nonlinear deformation to obtain more realistic FPRs in which extreme values are deformed so that the distances between them are increased or decreased.

To remap the experts' preferences by using a function D: $[0, 1] \to [0, 1]$ it is necessary to consider two different factors:

(i) The function D: $[0, 1] \to [0, 1]$ must remap FPRs into FPRs. This implies not only that the codomain of D must be the interval $[0, 1]$, but also that the nonlinear preferences keep the symmetry condition of an FPR $P \in \mathcal{M}_{n\times n}([0, 1])$, that is, $p_{ij} + p_{ji} = 1$,

(ii) The function D: $[0, 1] \to [0, 1]$ must transform the unit interval $[0, 1]$ such that the distance between the extreme values increases (or decreases) with respect to their original distance. In this section we will focus on those functions which deform the preferences by increasing the distance between extreme values and decreasing it between the intermediate ones, but similar arguments could be developed to describe those functions which increase the distance between the intermediate values and decrease it between the more extreme values.

For the sake of clarity, the description of these nonlinear deformations will be developed heuristically, by progressively adding requirements to a function D: $[0, 1] \to [0, 1]$. Therefore, some mathematical properties will be imposed to the function D due to their practical application and, in other cases, the mathematical properties will lead to useful features of these functions. In the following section all of these properties will be then compiled in the main definitions of our proposal, namely, EVAs and EVRs.

3.1 | Regularity

To obtain a proper deformation of the interval $[0, 1]$ the function D: $[0, 1] \to [0, 1]$, it must be a bijection. Otherwise, different values of the preferences would be mapped into the same value, which is not reasonable if we want to compare how different the preferences are.

In this context, both the strictly increasing character of D and the values $D(0) = 0$ and $D(1) = 1$ are mandatory.

Property 1. D: $[0, 1] \to [0, 1]$ is a strictly increasing bijection which satisfies the boundary conditions $D(0) = 0$ and $D(1) = 1$; that is, D is an automorphism on the interval $[0, 1]$.

The following well-known result will assure that a function satisfying this property is also a continuous function.

Proposition 1. *Let f: $[a, b] \to [c, d]$ be a bijection. Then f is strictly monotonous if and only if f is continuous.*

Proof. Let us prove first the sufficiency. Suppose f is strictly increasing (the decreasing case is similar) and pick $x_0 \in [a, b]$ and $\epsilon > 0$. Since f is a bijection we can find $A \subset [a, b]$ such that $f(A) = [f(x_0) - \epsilon, f(x_0) + \epsilon] \cap [c, d]$. Now, if we pick $x, y \in A$ such that $x < y$, because of the monotoniticity of f, we obtain

$$\max\{f(x_0) - \epsilon, c\} < f(x) < f(z) < f(y) < \min\{f(x_0) + \epsilon, d\}$$

for every $z \in]x, y[$. In such a case A is an interval containing x_0 and we can find $\delta > 0$ such that $|f(z) - f(x_0)| < \epsilon \quad \forall z \in [a, b]: |z - x_0| < \delta$, that is, f is continuous in x_0.

To prove the necessary condition pick $x, y, z \in [0, 1]$ such that $x < y < z$. Suppose $f(x) < f(y)$ and $f(y) > f(z)$. In that case, by using the Intermediate Value Theorem, we obtain $f(]x, y[) \cap f(]y, z[) \neq \emptyset$, which is impossible because of the bijectivity of f. We can apply the same reasoning to the remaining case (i.e., $f(x) > f(y)$ and $f(y) < f(z)$) and conclude it must be either $f(x) < f(y) < f(z)$ or $f(x) > f(y) > f(z)$. □

Since a function satisfying Property 1 is continuous, small changes on the original preferences are mapped into small changes of the deformed values.

As we will see, we will need some extra regularity on D to characterize the amplification of the distances between extreme values according to the value of D', so we impose now some additional *smoothness*:

Property 2. $D : [0, 1] \to [0, 1]$ is a differentiable function whose derivative D': $[0, 1] \to [0, 1]$ is continuous, that is, D is a $\mathcal{C}^1$ function.

Note that if D: $[0, 1] \to [0, 1]$ is a function satisfying Properties 1 and 2, then D also satisfies $D'(x) \geq 0 \quad \forall x \in [0, 1]$.

3.2 | Symmetry

When using an FPR to represent expert's preferences, it is usual to make the calculations only in the superior triangle due to that triangle and the inferior one are related by the standard negation $N: [0, 1] \to [0, 1]$ defined by $N(x) = 1 - x \quad \forall x \in [0, 1]$.

We have to translate this symmetry into an equivalent property for our function D, that is, the modified distance from 0.8 to 0.85 should be the same that the modified distance from 0.2 to 0.15. This kind of symmetry around the value $x = \frac{1}{2}$ will be imposed by the following property.

Property 3. $D: [0, 1] \to [0, 1]$ must be a symmetric function in the sense that $D(x) = 1 - D(1 - x) \quad \forall x \in [0, 1]$. In particular, $D(\frac{1}{2}) = \frac{1}{2}$.

In other words, this property guarantees that D remaps FPRs into FPRs. Furthermore, this property has a clear practical purpose since it allows us to construct these nonlinear deformations by only focusing on one half of the interval $[0, 1]$: if we manage to obtain $D_1: \left[\frac{1}{2}, 1\right] \to \left[\frac{1}{2}, 1\right]$ we can define $D_2: \left[0, \frac{1}{2}\right] \to \left[0, \frac{1}{2}\right]$ by $D_2(x) := 1 - D_1(1 - x) \quad \forall x \in \left[0, \frac{1}{2}\right]$ and construct $D: [0, 1] \to [0, 1]$ as a piecewise function. Note that, when D_1 is a differentiable function, D_2 is also differentiable and its derivative satisfies $D_2'(x) = D_1'(1 - x) \quad \forall x \in [0, 1]$, so $D_2'(\frac{1}{2}) = D_1'(\frac{1}{2})$ and D will be a differentiable function such that $D'(0) = D'(1)$. Furthermore, if D_1' is continuous, D' will be also continuous.

It should be highlighted that a function $D: [0, 1] \to [0, 1]$ satisfying these three properties induces the restricted dissimilarity $d_D: [0, 1] \times [0, 1] \to [0, 1]$[23] given by

$$d_D(x, y) = |D(x) - D(y)| \quad \forall x, y \in [0, 1],$$

and the Restricted Equivalence Function[23] $S_D: [0, 1] \times [0, 1] \to [0, 1]$ defined by

$$S_D(x, y) = 1 - |D(x) - D(y)| \quad \forall x, y \in [0, 1].$$

These tools allow one to compare *how similar* are the preferences taking into account the nonlinear approach.

3.3 | Distance amplification and derivatives

Here it is studied the relation between the first derivative of an arbitrary automorphism defined in $[0, 1]$ and the modification of the distances between elements that it will produce.

First, a theorem that characterizes those functions which amplify the distance between the elements of their domain is proposed.

Theorem 1. *Let $f: [a, b] \to \mathbb{R}$ be a $\mathcal{C}^1$ function defined on the interval $[a, b] \subset \mathbb{R}$. Then the following statements are equivalent:*

1. *f is an increasing function satisfying $|f(y) - f(x)| \geq |y - x| \quad \forall x, y \in [a, b]$,*
2. *$f'(z) \geq 1 \quad \forall z \in [a, b]$.*

Proof. (1) → (2) Suppose first that $|f(y) - f(x)| \geq |y - x| \quad \forall x, y \in [a, b]$ and pick $z \in [a, b[$. Let us choose $h > 0$ such that $z + h < b$. In that case the auxiliary function $g\colon]0, h[\to\mathbb{R}$ defined by $g(t) = \frac{f(z+t) - f(z)}{t} \quad \forall\, t \in]0, h[$ is a continuous function such that $g(t) \geq 1 \quad \forall\, t \in]0, h[$ and therefore $f'(z) = \lim_{t\to 0} g(t) \geq 1$. On the other hand, when $z = b$, we can consider $h > 0$ such that $b - a > h$ and use the analogue reasoning for the function $g\colon]0, h[\to\mathbb{R}$ defined by $g(t) = \frac{f(b) - f(b-t)}{t} \quad \forall\, t \in]0, h[$.

(2) → (1) Since $f'(z) \geq 1 \quad \forall\, z \in [a, b]$, f is increasing. To show the inequality pick $x, y \in [a, b]$ such that $x < y$. We can use the Mean Value Theorem to obtain $\xi \in]x, y[$ such that $f(y) - f(x) = f'(\xi)(y - x)$ and therefore it must be $|f(y) - f(x)| \geq |y - x|$. □

Remark 1. Note that if the derivative of D is between 0 and 1 in some subinterval, as it may occur on the intermediate values of $[0, 1]$, then the distance between the deformations of those elements will be lower than the distance between the original ones.

In the following proposition we will use the idea of Theorem 1 to describe the amplification of distances between values close to 1 when we are deforming the interval $[0, 1]$. The key is to ask for the derivative in $x = 1$ to be higher than 1 and use the continuity of the derivative to obtain a neighborhood of 1 wherein the derivative is greater or equal than 1.

Proposition 2. *Let $f\colon [1 - r, 1] \to [0, 1]$ be a $\mathcal{C}^1$ function defined on $[1 - r, 1]$ for some $r \in]0, 1[$. Then,*

- *If $f'(1) > 1$, there exists $r' \in]0, r[$ such that $|f(y) - f(x)| > |y - x| \quad \forall x, y \in [1 - r', 1]\colon x \neq y$,*
- *f is an increasing function satisfying $|f(y) - f(x)| \geq |y - x| \quad \forall x, y \in [1 - r, 1]$, if and only if $f'(z) \geq 1 \quad \forall\, z \in [1 - r, 1]$.*

Proof. To prove the first assumption let us use the continuity of f' around 1 to get $r' \in]0, r[$ such that $f'(z) > 1 \quad \forall\, z \in [1 - r', 1]$. Then, for every $x < y$, $x, y \in [1 - r', 1]$, we can use the Mean Value Theorem to obtain $\xi \in]x, y[$ such that $f(y) - f(x) = f'(\xi)(y - x)$. Since $f'(\xi) > 1$ and, in particular, f is strictly increasing we have $|f(y) - f(x)| = f(y) - f(x) = f'(\xi)(y - x) > |y - x|$. To show the second statement we just have to use Theorem 1. □

Using a similar argument we can show the analogous result for distance amplification in values close to 0:

Proposition 3. *Let $f\colon [0, r] \to [0, 1]$ be a $\mathcal{C}^1$ function defined on $[0, r]$ for some $r \in]0, 1[$. Then:*

- *If $f'(0) > 1$, there exists $r' \in]0, r[$ such that $|f(y) - f(x)| > |y - x| \quad \forall x, y \in [0, r'], x \neq y$,*
- *f is an increasing function satisfying $|f(y) - f(x)| \geq |y - x| \quad \forall x, y \in [0, r]$, if and only if $f'(z) \geq 1 \quad \forall\, z \in [0, r]$.*

The practical interpretation of these results is simple: to amplify distances close to the extremes we need that the derivative of D is greater than 1 in the extremes. Since D is asked to be a $\mathcal{C}^1$ function, the continuity of D' will give us two neighborhoods (one around 0 and the other around 1) where D' is always greater than 1 and thus the distance between two values which are inside one of these neighborhoods will increase when we apply D.

Property 4. D must be a $\mathcal{C}^1$ function satisfying $D'(0) > 1$ and $D'(1) > 1$.

3.4 | Distance amplification and convexity

Finally, we study the relation between the convexity and the distance amplification on extreme values. First, we compare the deformations D with the identity function on the interval $[0, 1]$.

Proposition 4. *Let* $f: [0, 1] \to [0, 1]$ *be a* $\mathcal{C}^1$ *increasing function such that* $f(0) = 0, f(1) = 1, f'(0) > 1,$ *and* $f'(1) > 1$. *Then we can find* $r \in \,]0, \frac{1}{2}[$ *such that* $f(x) > x \quad \forall x \in \,]0, r]$ *and* $f(x) < x \quad \forall x \in [1 - r, 1[$.

Proof. Due to $f'(0) > 1$ and $f'(1) > 1$, the function $g: [0, 1] \to [0, 1]$ given by $g(x) = f(x) - x \quad \forall x \in [0, 1]$ satisfies $g'(0) > 0$ and $g'(1) > 0$ and we can find $r \in \,]0, \frac{1}{2}[$ such that g is strictly increasing in both $[0, r]$ and $[1 - r, 1]$.

In that case $g(0) < g(x) \quad \forall x \in \,]0, r]$ and therefore $f(0) - 0 < f(x) - x \quad \forall x \in [0, r] \Leftrightarrow x < f(x) \quad \forall x \in \,]0, r]$ and $g(x) < g(1) \quad \forall x \in [1 - r, 1[\Leftrightarrow f(x) - x < f(1) - 1$ which means $f(x) < x \quad \forall x \in [1 - r, 1[$ □

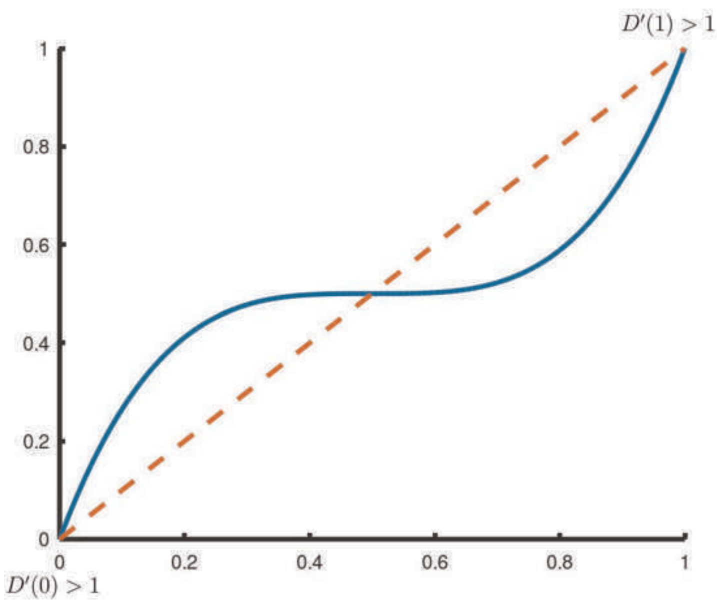

FIGURE 2 A function D satisfying Proposition 4 $\left(r = \frac{1}{2}\right)$ [Color figure can be viewed at wileyonlinelibrary.com]

This result provides a clear geometrical interpretation: the graph of D is over the diagonal of the square $[0, 1] \times [0, 1]$ for values close enough to 0 and it is under the same diagonal for those values close enough to 1 (see Figure 2).

We have already shown that the derivative of D must be greater than 1 close to the extremes. The following proposition, which is an immediate consequence of the previous one, will prove that D' must also be under 1 in some subinterval of $[0, 1]$ and thus the distances decrease between the values in such subinterval. Additionally, we will obtain that D cannot be convex nor concave on its full domain.

Proposition 5. *Let* $f: [0, 1] \to [0, 1]$ *be a* $\mathcal{C}^1$ *increasing function such that* $f(0) = 0, f(1) = 1, f'(0) > 1$*, and* $f'(1) > 1$*. Then*

- *There exists an interval* $I \subset [0, 1]$ *such that* $0 \leq f'(x) < 1 \quad \forall x \in I$*,*
- f' *cannot be increasing nor decreasing on the full domain* $[0, 1]$*.*

Proof. Let us define $g: [0, 1] \to \mathbb{R}$ by $g(x) = f(x) - x \quad \forall x \in [0, 1]$ as before. On the one hand, suppose that $f'(x) \geq 1 \quad \forall x \in [0, 1]$. In that case, $g' \geq 0$ and g is strictly increasing, which is a contradiction due to $g(0) = g(1)$. That means we can find some $x \in [0, 1]$ where $f'(x) < 1$ and the continuity of f' will give us the interval we are looking for.

On the other hand, if f' is increasing then g' is also increasing and due to $g'(0) > 0$ we obtain $g'(x) > 0 \quad \forall x \in [0, 1]$. Then g is strictly increasing and $g(0) < g(1)$, which is impossible. □

Propositions 6 and 7 provide a characterization of those functions which amplify the difference between nearby elements when we approach extreme values (0 or 1).

Proposition 6. *Let* $f: [a, b] \to \mathbb{R}$ *be a* $\mathcal{C}^1$ *increasing function. The following statements are equivalent:*

1. f *is a convex function,*
2. *For each* $x < y, \quad x, y \in \,]a, b]$ *the inequality*

$$|f(x) - f(x - t)| \leq |f(y) - f(y - t)|$$

holds for any $t \in [0, h]$*, where* $h = \min\{x - a, y - x\}$*.*

Proof. (1) → (2) If f is convex, f' is an increasing function in $[a, b]$. On the other hand, the Mean Value Theorem gives us $\xi_1 \in \,]x - t, x[$ and $\xi_2 \in \,]y - t, y[$ such that

$$\begin{aligned} |f(x) - f(x - t)| &= f(x) - f(x - t) = f'(\xi_1)t \leq f'(\xi_2)t = f(y) - f(y - t) \\ &= |f(y) - f(y - t)|, \end{aligned}$$

where we have used that f' is increasing and therefore $f'(\xi_1) \leq f'(\xi_2)$.

(2) → (1) Let us fix $x < y, \quad x, y \in \,]a, b]$ and define $g: [0, h] \to \mathbb{R}$ by

$$g(t) = \frac{f(y) - f(y-t) - (f(x) - f(x-t))}{t} \quad \forall\, t \in [0, h].$$

Since f is increasing $g \geq 0$. In addition

$$\lim_{t \to 0} g(t) = f'(y) - f'(x)$$

and the continuity of g leads to $f'(y) \geq f'(x)$, which is the convexity of f. □

Using a similar proof, we obtain:

Proposition 7. *Let f: $[a, b] \to \mathbb{R}$ be a $\mathcal{C}^1$ increasing function. The following statements are then equivalent:*

1. *f is a concave function,*
2. *For each $x < y, \quad x, y \in\,]a, b]$ the inequality*

$$|f(x) - f(x-t)| \geq |f(y) - f(y-t)|$$

holds for any $t \in [0, h]$, where $h = \min\{x - a, y - x\}$.

These propositions show that the convexity is related to an increment of the distances between *consecutive* values of the preferences when we approach the extremes of the interval [0, 1]. We summarize this in Property 5:

Property 5. D: [0, 1] → [0, 1] should be concave in a neighborhood of 0 and convex in a neighborhood of 1.

4 | EXTREME VALUES AMPLIFICATIONS AND EXTREME VALUES REDUCTIONS

This section introduces the concept of EVA and its dual concept EVR. First, we present the definition of EVAs as those functions satisfying the properties stated in Section 3.

Definition 1 (Extreme Values Amplification). Let D: [0, 1] → [0, 1] be a function satisfying:

1. D is an automorphism on the interval [0, 1],
2. D is a $\mathcal{C}^1$ function,
3. D satisfies $D(x) = 1 - D(1 - x) \quad \forall x \in [0, 1]$,
4. $D'(0) > 1$ and $D'(1) > 1$,
5. D is concave in a neighborhood of 0 and convex in a neighborhood of 1.*D will be called then an EVA on the interval* [0, 1].

This notation reminds that the main purpose of D is to remap FPRs of a GDM problem in a nonlinear way by amplifying the distance between the extreme values. So the new preferences show a larger distance between extreme elements and a smaller distance between elements close to $\frac{1}{2}$.

The following theorem, which compiles the main properties of the EVAs, is obtained by using the results discussed in Section 3:

Theorem 2. *Let* $D: [0, 1] \to [0, 1]$ *be an EVA on* $[0, 1]$. *Then,*

1. *The function* $d_D: [0, 1] \times [0, 1] \to [0, 1]$ *given by*
$$d_D(x, y) = |D(x) - D(y)| \quad \forall x, y \in [0, 1]$$
is a restricted dissimilarity[23] *and the function* $S_D: [0, 1] \times [0, 1] \to [0, 1]$ *defined by*
$$S_D(x, y) = 1 - |D(x) - D(y)| \quad \forall x, y \in [0, 1].$$
is a Restricted Equivalence Function.[23]
2. *We can find three intervals* $I_1, I_2, I_3 \subset [0, 1]$ *such that* $0 \in I_1, 1 \in I_3$, *and* $I_1 < I_2 < I_3$ *satisfying that*
$$\begin{aligned} |D(y) - D(x)| &> |y - x| \quad \forall\, x, y \in I_1: x \neq y, \\ |D(y) - D(x)| &< |y - x| \quad \forall\, x, y \in I_2: x \neq y, \\ |D(y) - D(x)| &> |y - x| \quad \forall\, x, y \in I_3: x \neq y. \end{aligned}$$
3. *The graph of D is over the diagonal of the square* $[0, 1] \times [0, 1]$ *for values close enough to 0 and it is under the same diagonal for those values close enough to* 1,
4. *There exist a neighborhood* U_0 *containing* 0 *and a neighborhood* U_1 *containing* 1 *such that for every* $x, y \in U_0^\circ, \quad x < y$, *there exists* $h_0 > 0$ *such that the inequality* $|D(x) - D(x - t)| \geq |D(y) - D(y - t)|$ *holds for any* $t \in [0, h_0]$ *and for every* $x, y \in U_1^\circ, x < y$, *there exists* $h_1 > 0$ *such that the inequality* $|D(x - t) - D(x)| \leq |D(y - t) - D(y)|$ *holds for any* $t \in [0, h_1]$.

Therefore, EVAs behave exactly as we aimed at:

1. They remap the original linear-scaled FPRs into nonlinear-scaled FPRs.
2. They amplify the distance between the extreme values, and reduce the distance between the intermediate ones.
3. They have a concrete geometrical pattern (see Figure 3A).
4. The amplification of distances is greater close to the extremes.

We can also consider the analogous notion of EVR:

Definition 2 (Extreme Values Reduction). Let $D: [0, 1] \to [0, 1]$ be a function satisfying:

1. D is an automorphism on the interval $[0, 1]$,
2. D is a $\mathcal{C}^1$ function,
3. D satisfies $D(x) = 1 - D(1 - x) \quad \forall x \in [0, 1]$,
4. $D'(0) < 1$ and $D'(1) < 1$,
5. D is convex in a neighborhood 0 and concave in a neighborhood of 1.

Then D will be called an EVR on the interval $[0, 1]$.

In this case, we obtain:

Theorem 3. *Let* D: $[0, 1] \to [0, 1]$ *be an EVR on* $[0, 1]$. *Then:*

1. *The function* d_D: $[0, 1] \times [0, 1] \to [0, 1]$ *given by*
$$d_D(x, y) = |D(x) - D(y)| \quad \forall x, y \in [0, 1]$$
is a restricted dissimilarity[23] *and the function* S_D: $[0, 1] \times [0, 1] \to [0, 1]$ *defined by*
$$S_D(x, y) = 1 - |D(x) - D(y)| \quad \forall x, y \in [0, 1]$$
is a Restricted Equivalence Function.[23]
2. *We can find three intervals* $I_1, I_2, I_3 \subset [0, 1]$ *such that* $0 \in I_1, 1 \in I_3$, *and* $I_1 < I_2 < I_3$ *satisfying that*
$$\begin{aligned} |D(y) - D(x)| &< |y - x| \quad \forall\, x, y \in I_1: x \neq y,\\ |D(y) - D(x)| &> |y - x| \quad \forall\, x, y \in I_2: x \neq y,\\ |D(y) - D(x)| &< |y - x| \quad \forall\, x, y \in I_3: x \neq y. \end{aligned}$$
3. *The graph of D is under the diagonal of the square* $[0, 1] \times [0, 1]$ *for values close enough to 0 and it is over the same diagonal for those values close enough to 1,*
4. *There exist a neighborhood* U_0 *containing 0 and a neighborhood* U_1 *containing 1 such that for every* $x, y \in U_0^\circ$, $x < y$, *there exists* $h_0 > 0$ *such that the inequality* $|D(x) - D(x - t)| \leq |D(y) - D(y - t)|$ *holds for any* $t \in [0, h_0]$ *and for every* $x, y \in U_1^\circ, x < y$, *there exists* $h_1 > 0$ *such that the inequality* $|D(x - t) - D(x)| \geq |D(y - t) - D(y)|$ *holds for any* $t \in [0, h_1]$.

The behavior of EVRs is dual to the EVAs as can be seen below:

1. They remap the original linear-scaled FPRs into nonlinear-scaled FPRs.
2. They reduce the distance between the extreme values, and amplify the distance between the intermediate ones.

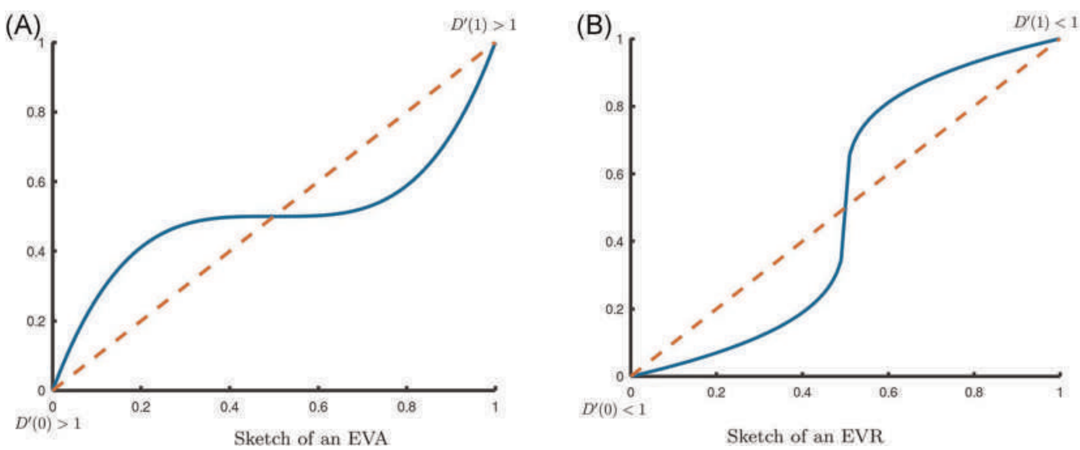

FIGURE 3 Comparing the shapes of EVAs and EVRs. EVA, Extreme Values Amplification; EVR, Extreme Values Reduction [Color figure can be viewed at wileyonlinelibrary.com]

3. They have a concrete geometrical pattern (see Figure 3B).
4. The reduction of distances is greater close to the extremes.

Both EVAs and EVRs are valid approaches to model nonlinear preferences, which gives an answer to the first research question.

5 | GENERATING EVAS AND EVRS

In this section several examples of EVAs and EVRs are provided. First, a generic method to construct EVAs is developed.

Let h: $[\alpha, \beta] \to [a, b]$ be the standard affine transformation given by

$$h(x) = \left(\frac{b-a}{\beta-\alpha}\right) \cdot (x-\alpha) + a \qquad \forall x \in [\alpha, \beta]$$

and let us consider the special cases h_1: $\left[\frac{1}{2}, 1\right] \to [0, 1]$ and h_2: $[0, 1] \to \left[\frac{1}{2}, 1\right]$.

Proposition 8. *Let f: $[0, 1] \to [0, 1]$ be a $\mathcal{C}^1$ convex automorphism on the interval $[0, 1]$ such that $f'(0) < 1$ and $f'(1) > 1$. Then the mapping D: $[0, 1] \to [0, 1]$ given by*

$$D(x) = \begin{cases} 1 - h_2 \circ f \circ h_1(1-x), & 0 \le x < \frac{1}{2}, \\ h_2 \circ f \circ h_1(x), & \frac{1}{2} \le x \le 1 \end{cases}$$

is an EVA.

Remark 2. The previous result could be adapted for EVRs by requiring concavity instead of convexity and changing the direction of the inequalities, that is, $f'(0) > 1$ and $f'(1) < 1$.

In the following we will show several families of EVAs and EVRs and study their properties.

5.1 | Sin-based EVAs and EVRs

Let $\alpha \in \left[0, \frac{1}{2\pi}\right]$. Then the $\mathcal{C}^\infty$ function s_α: $[0, 1] \to [0, 1]$ given by

$$s_\alpha(x) = x - \alpha{\cdot}\sin(2\pi x - \pi) \qquad \forall x \in [0, 1] \tag{1}$$

is an EVA (see Figure 4). Note that

$$s'_\alpha(x) = 1 - \alpha 2\pi{\cdot}\cos(2\pi x - \pi) \qquad \forall\, x \in [0, 1],$$
$$s''_\alpha(x) = \alpha(2\pi)^2{\cdot}\sin(2\pi x - \pi) \qquad \forall\, x \in [0, 1],$$

so s_α is strictly increasing, concave in $\left[0, \frac{1}{2}\right]$ and convex in $\left[\frac{1}{2}, 1\right]$. In addition $s'_\alpha(1) = s'_\alpha(0) > 1$ and $s'_\alpha(\frac{1}{2}) < 1$.

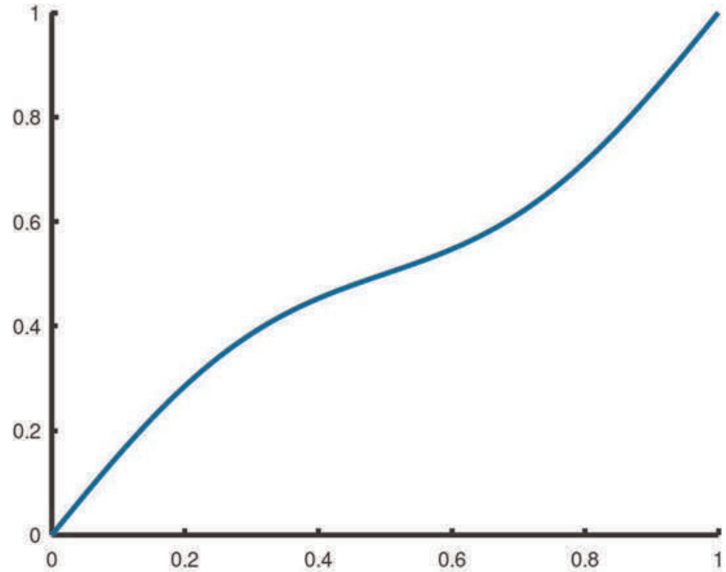

FIGURE 4 Extreme Values Amplification $s_{0.09}$ [Color figure can be viewed at wileyonlinelibrary.com]

Some interesting values are those where $s'_\alpha(x) = 1$. The solutions to this trigonometric equation are $x_1 = \frac{1}{4}$ and $x_2 = \frac{3}{4}$ (they do not depend on α), which are a kind of threshold for the distance amplification/reduction in the sense of

$$|s_\alpha(x) - s_\alpha(y)| \geq |x - y| \quad \forall\, x, y \in \left[0, \tfrac{1}{4}\right],$$
$$|s_\alpha(x) - s_\alpha(y)| \leq |x - y| \quad \forall\, x, y \in \left[\tfrac{1}{4}, \tfrac{3}{4}\right],$$
$$|s_\alpha(x) - s_\alpha(y)| \geq |x - y| \quad \forall\, x, y \in \left[\tfrac{3}{4}, 1\right].$$

Note that by defining $\hat{s}_\alpha\colon [0, 1] \to [0, 1]$ by

$$\hat{s}_\alpha(x) = x + \alpha \cdot \sin(2\pi x - \pi) \qquad \forall x \in [0, 1]$$

for $\alpha \in \left[0, \frac{1}{2\pi}\right]$ we obtain a family of EVRs.

5.2 | Polynomial EVAs and EVRs

By applying Proposition 8 to the automorphism $f_\alpha\colon [0, 1] \to [0, 1]$ given by

$$f_\alpha(x) = x^\alpha \qquad \forall x \in [0, 1],$$

where $\alpha > 1$, we obtain

$$m_\alpha(x) = \begin{cases} \frac{1}{2} - \frac{1}{2}(1 - 2x)^\alpha, & 0 \leq x < \frac{1}{2}, \\ \frac{1}{2} + \frac{1}{2}(2x - 1)^\alpha, & \frac{1}{2} \leq x \leq 1 \end{cases} \tag{2}$$

(see Figure 5), whose derivatives are

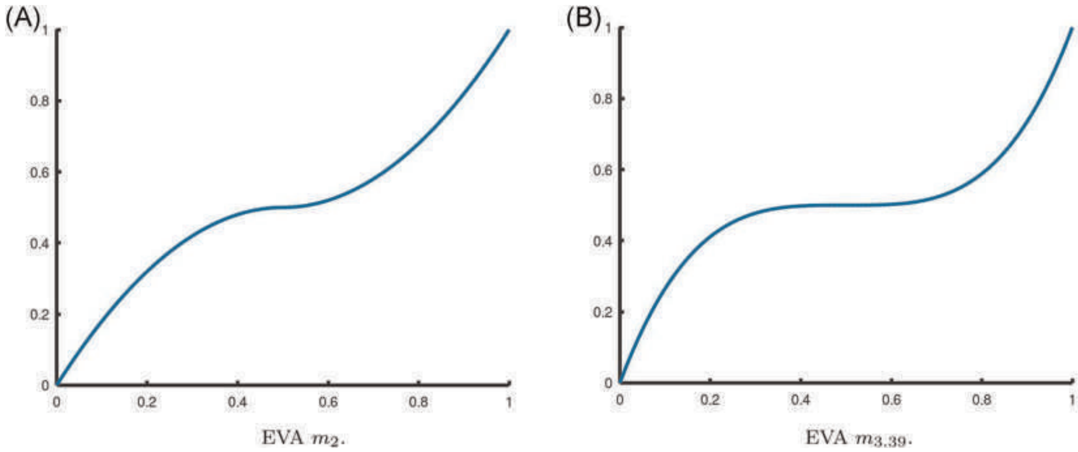

FIGURE 5 Graphs of the EVAs m_α: (A) EVA $m_2$and (B) EVA $m_{3.39}$. EVA, Extreme Values Amplification; EVR, Extreme Values Reduction [Color figure can be viewed at wileyonlinelibrary.com]

$$m'_\alpha(x) = \begin{cases} \alpha(1-2x)^{\alpha-1}, & 0 \le x < \frac{1}{2}, \\ \alpha(2x-1)^{\alpha-1}, & \frac{1}{2} \le x \le 1, \end{cases}$$

$$m''_\alpha(x) = \begin{cases} -2\alpha(\alpha-1)(1-2x)^{\alpha-2}, & 0 \le x < \frac{1}{2}, \\ 2\alpha(\alpha-1)(2x-1)^{\alpha-2}, & \frac{1}{2} \le x \le 1. \end{cases}$$

So m_α is strictly increasing, concave in $\left[0, \frac{1}{2}\right]$ and convex in $\left[\frac{1}{2}, 1\right]$. In addition $m'_\alpha(1) = m'_\alpha(0) = \alpha > 1$ and $m'_\alpha\left(\frac{1}{2}\right) = 0$.

In this case the calculation for the amplification/reduction threshold values is not as easy as before, but we can compute a numeric approximation. For $\alpha = 2$, we obtain that $m'_2\left(\frac{3}{4}\right) = 1$, and for $\alpha = 3.39$, $m'_{3.39}(0.8) \approx 1$.

Note that for $0 < \alpha < 1$, the functions $\hat{m}_\alpha$: $[0, 1] \to [0, 1]$ given by

$$\hat{m}_\alpha(x) = \begin{cases} \frac{1}{2} - \frac{1}{2}(1-2x)^\alpha, & 0 \le x < \frac{1}{2}, \\ \frac{1}{2} + \frac{1}{2}(2x-1)^\alpha, & \frac{1}{2} \le x \le 1 \end{cases}$$

behave like an EVR. Although they are not differentiable functions in $x = \frac{1}{2}$, this family satisfies $\lim_{x \to \frac{1}{2}-} f'(x) = \lim_{x \to \frac{1}{2}+} f'(x) = +\infty$. Therefore these functions can be used in almost any situation in which a proper EVR can be applied.

5.3 | Piecewise polynomial-based EVAs

Let us consider again the automorphism f_α: $[0, 1] \to [0, 1]$ given by

$$f_\alpha(x) = x^\alpha \qquad \forall x \in [0, 1],$$

where $\alpha > 1$.

For $r, s \in \left]\frac{1}{2}, 1\right[$ such that $r > s$ and $\epsilon \in]0, 1[$, the following standard affine transformations are considered:

$$h_a: \left[\frac{1}{2}, r\right] \to \left[\frac{1}{2}, s\right],$$
$$h_b: [r, 1] \to [\epsilon, 1],$$
$$h_c: [\epsilon^\alpha, 1] \to [s, 1].$$

We aim to use these affine transformations to construct a parametric EVA from the function $b_1: \left[\frac{1}{2}, 1\right] \to \left[\frac{1}{2}, 1\right]$ given by

$$b_1(x) = \begin{cases} h_a(x), & \frac{1}{2} \le x \le r, \\ h_c \circ f_\alpha \circ h_b(x), & r < x \le 1 \end{cases}$$

by defining $b_\alpha^{r,s}: [0, 1] \to [0, 1]$ by

$$b_\alpha^{r,s}(x) = \begin{cases} 1 - b_1(1 - x), & 0 \le x < \frac{1}{2}, \\ b_1(x), & \frac{1}{2} \le x \le 1. \end{cases}$$

Since

$$h'(x) = \frac{s - \frac{1}{2}}{r - \frac{1}{2}},$$

the parameters r and s allow us to control where and how much the differences between intermediate values are decreased. For instance, if we want the derivative of the EVA around $[1 - r, r]$ be equals to some $\lambda \in]0, 1[$ we just need to fix $s := \frac{1}{2} + \lambda(r - \frac{1}{2})$. The parameter $\alpha > 1$ controls *how faster* the distances between extreme values are amplified.

It is clear that the function b_1 is not C^1 for any combination of the parameters α, r, s. The parameter ϵ has been introduced to solve this issue. Note that

$$(h_c \circ f_\alpha \circ h_b)'(x) = \frac{1 - s}{1 - r}\frac{1 - \epsilon}{1 - \epsilon^\alpha}\alpha h_b^{\alpha-1}(x) = \frac{1 - s}{1 - r}\frac{1 - \epsilon}{1 - \epsilon^\alpha}\alpha\left(\frac{x - r}{1 - r}(1 - \epsilon) + \epsilon\right)^{\alpha-1}.$$

Therefore, for fixed $1 > r > s > \frac{1}{2}$ and $\alpha > 1$ we need to find $\epsilon \in]0, 1[$ such that the equality $(h_c \circ f_\alpha \circ h_b)'(r) = h'(r)$ holds, that is,

$$\frac{s - \frac{1}{2}}{r - \frac{1}{2}} = \frac{1 - s}{1 - r}\frac{1 - \epsilon}{1 - \epsilon^\alpha}\alpha\epsilon^{\alpha-1}.$$

Proposition 9. *Consider $r, s \in]\frac{1}{2}, 1[$ such that $r > s$ and $\epsilon \in]0, 1[$ and the standard affine transformations*:

$$h_a: \left[\frac{1}{2}, r\right] \to \left[\frac{1}{2}, s\right],$$
$$h_b: [r, 1] \to [\epsilon, 1],$$
$$h_c: [\epsilon^\alpha, 1] \to [s, 1].$$

Then the function $b_\alpha^{r,s}: [0, 1] \to [0, 1]$ given by

$$b_\alpha^{r,s}(x) = \begin{cases} 1 - b_1(1-x), & 0 \le x < \frac{1}{2}, \\ b_1(x), & \frac{1}{2} \le x \le 1, \end{cases} \tag{3}$$

where $b_1\colon \left[\frac{1}{2}, 1\right] \to \left[\frac{1}{2}, 1\right]$ *is defined by*

$$b_1(x) = \begin{cases} h_a(x), & \frac{1}{2} \le x \le r, \\ h_c \circ f_\alpha \circ h_b(x), & r < x \le 1 \end{cases}$$

is an EVA if and only if the following equality holds

$$\lambda = \frac{s - \frac{1}{2}}{r - \frac{1}{2}} = \frac{1-s}{1-r}\frac{1-\epsilon}{1-\epsilon^\alpha}\alpha\epsilon^{\alpha-1},$$

where λ *is the derivative of* h_a *and indicates how much the intermediate values become closer.*

Some useful combinations of these parameters are shown in Table 1 and the respective graphs are included in Figure 6.

Two limit cases are considered below.

5.3.1 | Special case $r = s > 1/2$

Let us first consider the limit case $r = s > \frac{1}{2}$. We would need

TABLE 1 Useful combinations of parameters for the EVA b

r	s	λ	α	ϵ
0.6	0.55	0.5	2	0.28571
0.6	0.533	0.333	3	0.38

Abbreviation: EVA, Extreme Values Amplification.

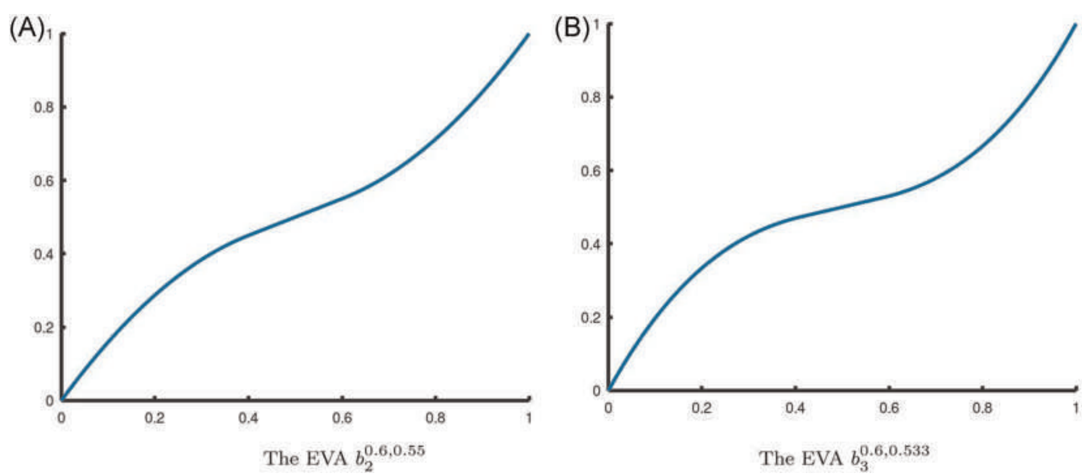

FIGURE 6 Graphs of the EVAs $b_\alpha^{r,s}$: (A) The EVA $b_2^{0.6,0.55}$and (B) The EVA $b_3^{0.6,0.533}$. EVA, Extreme Values Amplification [Color figure can be viewed at wileyonlinelibrary.com]

$$1 = \frac{1-\epsilon}{1-\epsilon^{\alpha}}\alpha\epsilon^{\alpha-1} \Leftrightarrow 1 = \frac{\epsilon^{\alpha-1}-\epsilon^{\alpha}}{1-\epsilon^{\alpha}}\alpha.$$

Consider the function g: $]0,1[\to\mathbb{R}$ defined by

$$g(\epsilon) = \alpha(\epsilon^{\alpha-1}-\epsilon^{\alpha}) - 1 + \epsilon^{\alpha} \quad \forall\, \epsilon \in [0,1].$$

Note that

$$\begin{aligned} g'(\epsilon) &= \alpha((\alpha-1)\epsilon^{\alpha-2} - \alpha\epsilon^{\alpha-1}) + \alpha\epsilon^{\alpha-1} \\ &= \alpha(\alpha-1)\epsilon^{\alpha-2} + \alpha\epsilon^{\alpha-1}(1-\alpha) \\ &= \alpha(\alpha-1)(\epsilon^{\alpha-2}-\epsilon^{\alpha-1}) \\ &= \alpha(\alpha-1)\epsilon^{\alpha-2}(1-\epsilon) > 0 \quad \forall\, \epsilon \in [0,1]. \end{aligned}$$

Since $\lim\limits_{\epsilon\to 0} g(\epsilon) = -\infty$, and $g' > 0$ the equation $g(\epsilon) = 0$ *has* the unique solution $\epsilon = 1$, which is not admissible in this problem.

5.3.2 | Special case $r = s = 1/2$

Note that this assumption implies that the affine transformation h_a disappears and $b_1(x) = h_c \circ f_\alpha \circ h_b(x) \quad \forall x \in [\frac{1}{2}, 1]$.

Define g: $]0,1[\to\mathbb{R}$ by

$$g(\epsilon) = \alpha\frac{\epsilon^{\alpha-1}-\epsilon^{\alpha}}{1-\epsilon^{\alpha}} \quad \forall\, \epsilon \in\,]0,1[.$$

Note that $\lim\limits_{\epsilon\to 0} g(\epsilon) = 0$ and

$$\lim_{\epsilon\to 1} g(\epsilon) = \lim_{\epsilon\to 1}\alpha\epsilon^{\alpha-1}\frac{1-\epsilon}{1-\epsilon^{\alpha}} = 1.$$

In addition

$$\begin{aligned} g'(\epsilon) &= \tfrac{\alpha}{(1-\epsilon^{\alpha})^2}\Big(((\alpha-1)\epsilon^{\alpha-2} - \alpha\epsilon^{\alpha-1})(1-\epsilon^{\alpha}) \\ &\quad + (\epsilon^{\alpha-1}-\epsilon^{\alpha})\alpha\epsilon^{\alpha-1}\Big) \\ &= \tfrac{\alpha\epsilon^{\alpha-2}}{(1-\epsilon^{\alpha})^2}((\alpha-1-\alpha\epsilon)(1-\epsilon^{\alpha}) + \alpha(\epsilon^{\alpha}-\epsilon^{\alpha+1})) \\ &= \tfrac{\alpha\epsilon^{\alpha-2}}{(1-\epsilon^{\alpha})^2}(\alpha(1-\epsilon)(1-\epsilon^{\alpha}) - (1-\epsilon^{\alpha}) + \alpha\epsilon^{\alpha}(1-\epsilon)) \\ &= \tfrac{\alpha\epsilon^{\alpha-2}}{(1-\epsilon^{\alpha})^2}(\alpha(1-\epsilon) - (1-\epsilon^{\alpha})) \\ &\quad \forall\, \epsilon \in [0,1]. \end{aligned}$$

To study the sign of g' let us consider h: $[0,1] \to \mathbb{R}$ defined by

$$h(\epsilon) = \alpha(1-\epsilon) - (1-\epsilon^{\alpha}) \quad \forall\, \epsilon \in\,]0,1[.$$

Since $h'(\epsilon) = \alpha(\epsilon^{\alpha-1} - 1) < 0 \quad \forall \epsilon \in]0,1[$ and $\lim_{\epsilon \to 0} h(\epsilon) = \alpha - 1$, $\lim_{\epsilon \to 1} h(\epsilon) = 0$ we can conclude that $h > 0$ in its domain and therefore $g'(\epsilon) > 0 \quad \forall \epsilon \in]0,1[$ In that case g is increasing and such that $\lim_{\epsilon \to 0} g(\epsilon) = 0$, $\lim_{\epsilon \to 1} g(\epsilon) = 1$. This fact allows one to state the following result.

Proposition 10. *Let* $\epsilon \in]0,1[$ *and consider the standard affine transformations* $h_b\colon \left[\frac{1}{2}, 1\right] \to [\epsilon, 1]$ *and* $h_c\colon [\epsilon^\alpha, 1] \to \left[\frac{1}{2}, 1\right]$ *and the function* $\hat{b}_1\colon \left[\frac{1}{2}, 1\right] \to \left[\frac{1}{2}, 1\right]$ *defined by* $\hat{b}_1(x) = h_c \circ f_\alpha \circ h_b(x) \quad \forall x \in \left[\frac{1}{2}, 1\right]$.

Then, for every $\lambda \in]0,1[$ *we can find* $\epsilon \in]0,1[$ *(i.e., the unique one which satisfies* $\lambda = \alpha \frac{\epsilon^{\alpha-1} - \epsilon^\alpha}{1 - \epsilon^\alpha}$*) such that the function* $\hat{b}\colon [0,1] \to [0,1]$ *defined by*

$$\hat{b}(x) = \begin{cases} \hat{b}_1(x), & \frac{1}{2} \le x \le 1, \\ 1 - \hat{b}_1(1-x), & 0 \le x \le \frac{1}{2} \end{cases}$$

is an EVA such that $\hat{b}'\left(\frac{1}{2}\right) = \lambda$.

Note that by taking $\epsilon = 0$ we would obtain the m_α family of EVAs.

5.4 | Comparing sin-based EVAs, polynomial EVAs, and piecewise polynomial EVAs

The main difference between these families of EVAs is the value of their derivative in $\frac{1}{2}$. Note that in all cases the derivative function reaches its minimum value at this point.

For *sin*-based EVAs, the derivative at $x = \frac{1}{2}$ is $s'_\alpha\left(\frac{1}{2}\right) = 1 - \alpha 2\pi$ and the derivative of the polynomial-based EVAs is zero. For piecewise polynomial EVAs, this derivative is a value $\lambda \in [0,1]$

In the first case, by choosing a proper $\alpha \in]0, \frac{1}{2\pi}[$, we can adjust *how much* the intermediate values will move closer. For example, for the intermediate parameters $\alpha = 0.08$ and 0.09 one obtains $s'_{0.08}\left(\frac{1}{2}\right) = 1 - 0.08 \cdot 2\pi \approx 0.49735$ and $s'_{0.09}\left(\frac{1}{2}\right) = 1 - 0.09 \cdot 2\pi \approx 0.43451$.

On the other hand, the main advantage of *polynomial*-based EVAs is the fact that we can decide where the values are going to start to move away by fixing $x_0 > \frac{1}{2}$ and solving (numerically) the equation $m'_\alpha(x_0) = 1$ for the variable $\alpha > 1$.

The *piecewise polynomial* EVA allows both to choose how much the intermediate values will become closer, by adjusting λ, and how much the extreme values will become distant, by choosing α. However, this family of EVAs is more complex and loses the regularity of the other two families.

6 | A THEORETIC DISCUSSION ABOUT THE PERFORMANCE OF EVAS AND EVRS IN CRPS FOR LSGDM

In Section 4 the EVA approach and the EVR approach have been introduced as models for nonlinear preferences. However, the strategy to deal with polarized opinions is totally different in EVA and EVR. According to References [24,25], the less extreme values have a more cohesive effect and greater success to reach an agreement. Therefore, this section is devoted to provide a sustained proof about why EVRs are not a good strategy to remap FPRs in CRPs meanwhile EVAs tend to improve the performance of the consensus models.

6.1 | EVAs and order of alternatives in CRPs

First a study of *how much* EVAs deform the original preferences is provided.

Proposition 11. *Let* $D: [0, 1] \to [0, 1]$ *be an EVA whose first derivative is strictly increasing in* $\left[\frac{1}{2}, 1\right]$. *Then, the equation*

$$D'(x) = 1$$

has an unique solution $x_0 \in \left]\frac{1}{2}, 1\right[$ *which satisfies*

$$|x - D(x)| \leq |x_0 - D(x_0)|$$

for every $x \in [0, 1]$.

Proof. The existence and the uniqueness of x_0 are given by the bijectivity of D'.

Now consider the function $g: \left[\frac{1}{2}, 1\right] \to \mathbb{R}$ given by

$$g(x) = x - D(x) \quad \forall x \in \left[\frac{1}{2}, 1\right].$$

Since $g'(x) = 1 - D'(x) \quad \forall x \in \left[\frac{1}{2}, 1\right]$ the candidates to be relative extremes for the function g are $\left(\frac{1}{2}, 0\right)$, $(1, 0)$ and $(x_0, g(x_0))$. Due to the continuity of D', g' is a continuous function which also satisfies that $g'\left(\frac{1}{2}\right) > 0$ and $g'(1) < 0$. In that case $(x_0, g(x_0))$ is a relative maximum for g and both $\left(\frac{1}{2}, 0\right)$ and $(1, 0)$ are minimums, which proves the inequality of the proposition. □

Note that we do not need that D' is strictly increasing in $\left[\frac{1}{2}, 1\right]$. It suffices to consider an EVA D whose first derivative is strictly increasing in a neighborhood of 1 which contains a value $r > \frac{1}{2}$ such that $D'(r) < 1$. The general version, whose proof is analogous, is stated:

Proposition 12. *Let* $D: [0, 1] \to [0, 1]$ *be an EVA whose first derivative is strictly increasing in* $[r, 1]$ *for some* $r > \frac{1}{2}$ *such that* $D'(r) < 1$. *If* D' *is monotonous in* $\left[\frac{1}{2}, 1\right]$ *then, the equation*

$$D'(x) = 1$$

has an unique solution $x_0 \in \,]r, 1[$ *which satisfies*

$$|x - D(x)| \leq |x_0 - D(x_0)|$$

for every $x \in [0, 1]$.

Remark 3. Note that the symmetry of D around $\frac{1}{2}$ would provide another $\hat{x}_0 \in \left[0, \frac{1}{2}\right]$ satisfying the same inequality.

Remark 4. The three families of EVAs introduced in this contribution satisfy the hypotheses of this proposition.

Suppose now that we have obtained the FPR $P = (p_{ij}) \in \mathcal{M}_{n\times n}([0,1])$ from a certain expert. We want to analyze how different the order of the alternatives chosen by that expert will be after applying an EVA.

To compute the order of the alternatives we just assign each alternative a score depending on the value of the preferences:

$$sc(x_i) = \frac{1}{n-1}\sum_{j=1,j\neq i}^{n} p_{ij}, \quad i \in \{1,2,\ldots,n\}.$$

Then we order the alternatives according to the score they have received.

We cannot prove that the order of the alternatives will not change after applying any EVA, but we can show that there exists a threshold which enables us to control the distance between the score obtained for the deformed preferences and the original score.

Proposition 13. *Let $D: [0,1] \to [0,1]$ be an EVA whose first derivative is monotonous in $\left[\frac{1}{2},1\right]$ and strictly increasing in $[r,1]$ for some $r > \frac{1}{2}$ such that $D'(r) < 1$. Consider an FPR $P = (p_{ij}) \in \mathcal{M}_{n\times n}([0,1])$ given for the alternatives $x_1, x_2, \ldots, x_n$. Then*

$$|sc(x_i) - sc(D(x_i))| \leq |x_0 - D(x_0)| \quad \forall\, i \in \{1,2,\ldots n\},$$

where $x_0 \in \left]\frac{1}{2},1\right[$ is the unique solution for the equation $D'(x) = 1$.

Proof. Note that for any alternative x_i we obtain

$$\begin{aligned}|sc(x_i) - sc(D(x_i))| &= \left|\tfrac{1}{n-1}\textstyle\sum_{j=1,j\neq i}^{n}(p_{ij} - D(p_{ij}))\right| \\ &\leq \tfrac{1}{n-1}\textstyle\sum_{j=1,j\neq i}^{n}|\,p_{ij} - D(p_{ij})| \leq \tfrac{1}{n-1}\textstyle\sum_{j=1,j\neq i}^{n}|\,x_0 - D(x_0)| \\ &\leq \tfrac{|x_0 - D(x_0)|}{n-1}\textstyle\sum_{j=1,j\neq i}^{n} 1 = |x_0 - D(x_0)|.\end{aligned}$$

□

Remark 5. Note that we would have obtained the same result if we had used any other aggregation operator based on weights to compute the score.

It should be highlighted that $x_0 \in \left]\frac{1}{2},1\right[$ such that $D'(x_0) = 1$ is not only a threshold for the amplification of the distances, but also provides a bound to study how similar is the order alternatives after applying the EVA with respect to the original order. Therefore, the value x_0 will receive the name of *amplification threshold*, and the value $R_0 := |x_0 - D(x_0)|$ will be called *maximum deformation*.

Let us study these quantities for different families of EVAs.

6.1.1 | Sin-based EVAs

Let $\alpha \in \left]0, \frac{1}{2\pi}\right]$ and consider the $\mathcal{C}^\infty$ function $s_\alpha: [0,1] \to [0,1]$ given by

$$s_\alpha(x) = x - \alpha \cdot \sin(2\pi x - \pi) \quad \forall x \in [0, 1].$$

Note that $s'(x) = 1, x > \frac{1}{2}$ satisfies if and only if

$$\alpha 2\pi \cdot \cos(2\pi x - \pi) = 0$$

and therefore $x_0 = \frac{3}{4}$ and

$$R_0 = g(x_0) = \frac{3}{4} - s_\alpha\left(\frac{3}{4}\right) = \alpha \cdot \sin\left(2\pi\frac{3}{4} - \pi\right) = \alpha.$$

6.1.2 | Polynomial-based EVAs

Consider the automorphism f_α: $[0, 1] \to [0, 1]$ given by

$$f_\alpha(x) = x^\alpha \quad \forall x \in [0, 1],$$

where $\alpha > 1$. We obtain the EVA

$$m_\alpha(x) = \begin{cases} \frac{1}{2} - \frac{1}{2}(1 - 2x)^\alpha, & 0 \leq x < \frac{1}{2}, \\ \frac{1}{2} + \frac{1}{2}(2x - 1)^\alpha, & \frac{1}{2} \leq x \leq 1. \end{cases}$$

In this case $m'(x) = 1, x > \frac{1}{2}$ satisfies if and only if

$$\alpha(2x - 1)^{\alpha-1} = 1,$$

thus the amplification threshold is $x_0 = \frac{1}{2}\left(\sqrt[\alpha-1]{\frac{1}{\alpha}} + 1\right)$ and the maximum deformation is

$$\begin{aligned} R_0 &= x_0 - m_\alpha(x_0) \\ &= \frac{1}{2}\left(\sqrt[\alpha-1]{\frac{1}{\alpha}} + 1\right) - \left(\frac{1}{2} + \frac{1}{2}\left(2\left(\frac{1}{2}\left(\sqrt[\alpha-1]{\frac{1}{\alpha}} + 1\right)\right) - 1\right)^\alpha\right) \\ &= \frac{1}{2}\left(\sqrt[\alpha-1]{\frac{1}{\alpha}} + 1\right) - \left(\frac{1}{2} + \frac{1}{2}\left(\sqrt[\alpha-1]{\frac{1}{\alpha}}\right)^\alpha\right) \\ &= \frac{1}{2}\left(\sqrt[\alpha-1]{\frac{1}{\alpha}} - \sqrt[\alpha-1]{\frac{1}{\alpha}^\alpha}\right) = \frac{1}{2}\left(\alpha^{-\frac{1}{\alpha-1}} - \alpha^{-\frac{\alpha}{\alpha-1}}\right) \\ &= \frac{1}{2}\left(\alpha^{-\frac{1}{\alpha-1}} - \alpha^{-\frac{\alpha+1-1}{\alpha-1}}\right) = \frac{1}{2}\left(\alpha^{-\frac{1}{\alpha-1}} - \alpha^{-\frac{1}{\alpha-1}}\alpha^{-1}\right) \\ &= \frac{1}{2\sqrt[\alpha-1]{\alpha}}\left(1 - \frac{1}{\alpha}\right). \end{aligned}$$

For $\alpha = 2$ we obtain $R_0 = \frac{1}{8}$ and for $\alpha = 3$ it is $R_0 \approx 0.2$.

6.1.3 | Piecewise polynomial-based EVA

Consider $r, s \in \left[\frac{1}{2}, 1\right]$ such that $r > s$ and $\epsilon \in [0, 1]$. Consider the standard affine transformations:

TABLE 2 Useful combinations of parameters for $b_\alpha^{r,s}$

r	s	λ	α	ϵ	R_0
0.6	0.55	0.5	2	0.28571	0.09
0.6	0.533	0.333	3	0.38	0.13

$$h_a\colon \left[\tfrac{1}{2}, r\right] \to \left[\tfrac{1}{2}, s\right],$$
$$h_b\colon [r, 1] \to [\epsilon, 1],$$
$$h_c\colon [\epsilon^\alpha, 1] \to [s, 1],$$

and the function $b_1\colon \left[\frac{1}{2}, 1\right] \to \left[\frac{1}{2}, 1\right]$

$$b_1(x) = \begin{cases} h_a(x), & \frac{1}{2} \le x \le r, \\ h_c \circ f_\alpha \circ h_b(x), & r < x \le 1. \end{cases}$$

Then the EVA $b_\alpha^{r,s}\colon [0, 1] \to [0, 1]$ defined, for some proper ϵ such that b is $\mathcal{C}^1$, by

$$b_\alpha^{r,s}(x) = \begin{cases} b_1(x), & \frac{1}{2} \le x \le 1, \\ 1 - b_1(1 - x), & 0 \le x \le \frac{1}{2} \end{cases}$$

has its amplification threshold at

$$x_0 = \left(\sqrt[\alpha-1]{\frac{(1-r)(1-\epsilon^\alpha)}{(1-s)(1-\epsilon)\alpha}} - \epsilon\right)\frac{1-r}{1-\epsilon} + r,$$

since this value satisfies $(h_c \circ f_\alpha \circ h_b)'(x_0) = 1$, that is,

$$\frac{1-s}{1-r}\frac{1-\epsilon}{1-\epsilon^\alpha}(h_b(x_0))^{\alpha-1}\alpha = 1.$$

In this case, the explicit formula for the amplification radius is complex and offers no advantage, so we can obtain that value by computing the numeric value of x_0 and then considering $R_0 = x_0 - b_\alpha^{r,s}(x_0)$. We show some values in Table 2.

6.2 | Why EVAs do work for improving CRPs for LSGDM and why EVRs do not

In Section 6.1 it has been proved that the impact of EVAs in the order of the alternatives is pretty low.

On the other hand, it has been shown in References [24,25] that the less extreme values have a more cohesive effect and make easier the reaching of an agreement whereas the more extreme values of the preferences tend to polarize situations, so any worthy CRP model should prioritize intermediate values of the preferences in its aggregations.

Let us consider all of these together: when the EVA approach is used on a consensus model which aggregates the preferences by prioritizing intermediate values, the model will ignore the extreme values and the intermediate ones will become closer because of the properties of the EVA function. In this case, the model will need a lower amount of rounds to reach the consensus, since the EVA has done part of the work by making the intermediate values closer. Since the order of the alternatives has not been changed *too much*, a consensus model which uses the EVA approach will choose a similar alternative faster than the original one.

This also explains why we are not using EVRs to model the nonlinear approach in CRPs. Although EVRs also modify the preferences in a nonlinear way, in this case the distances between extreme preferences are reduced and the distance between intermediate preferences is amplified. When EVRs are implemented in a consensus model, which usually prioritizes intermediate preferences to facilitate the consensus, the model will probably need more rounds to reach the consensus, since intermediate elements are less similar.

Both EVAs and EVRs are valid approaches to model nonlinear scales in CRPs (RQ1), but to improve classic models the EVA approach outperforms both the linear approach and the EVR approach (RQ2).

7 | EXTREME VALUES AMPLIFICATIONS IN GDM

This section aims at verifying, validating, and showing the better performance of EVA functions in LSGDM problems with respect to lineal preference modeling. Therefore, an illustrative LSGDM problem is solved by using the specialized CRP software AFRYCA[10] and comparing the performance of two widely used consensus models (see Remark 7) when they use linear and nonlinear preference scales.

Therefore, we have implemented the EVA families s_α (Equation 1) and m_α (Equation 2) into the CRPs introduced in References [16,26] (both included in AFRYCA) and then carried out several simulations to compare their performance by using EVAs and linear preferences.

Remark 6. In Section 6 it was pointed out that extreme values make more difficult the achievement of agreements.[24,25] Therefore, due to the fact that EVRs amplify the distances between less extreme values, we will only consider the study in further detail of EVAs for CRPs in LSGDM because EVRs are not suitable for smoothing the achievement of agreements.

Remark 7. The consensus model proposed by Herrera-Viedma et al.[16] has been selected since it has been widely used in the literature and several consensus models are based on its performance.[18] Quesada et al.'s proposal[26] has been chosen since it deals with GDM problems with a large number of experts and considers important aspects, such as the experts' behavior.

Unlike, most of the proposals about LSGDM in the specialized literature whose examples are based on 20 experts, we assume an LSGDM problem in which there are 100 experts who elicit their FPRs over four alternatives $X = \{X_1, X_2, X_3, X_4\}$. To accomplish the CRPs proposed in our example, several simulations have been run by using the default values of the parameters

TABLE 3 Average results on Herrera-Viedma et al.[16] (500 simulations)

EVA	Average rounds	Average consensus
Classical	5.158	0.8807
$s_{0.08}$	4.482	0.8835
$s_{0.09}$	4.384	0.8816
m_2	4.08	0.8805
$m_{3.39}$	3.606	0.8851

Abbreviation: EVA, Extreme Values Amplification.

TABLE 4 Average results on Quesada et al.[26] (500 simulations)

EVA	Average rounds	Average consensus
Classical	9.256	0.8579
$s_{0.08}$	7.532	0.8575
$s_{0.09}$	7.188	0.8592
m_2	6.074	0.8581
$m_{3.39}$	2.842	0.8598

Abbreviation: EVA, Extreme Values Amplification.

established in AFRYCA for such consensus models, by setting the consensus level at 0.85 and the maximum number of rounds at 15.

This section is divided into two subsections. In both, the performance of the classical models Herrera-Viedma et al.[16] and Quesada et al.[26] is compared with the EVA-modified models. To do so, for each consensus model five different scenarios are considered: the classical model (no EVA is used), the EVAs $s_{0.08}$ and $s_{0.09}$ (defined by Equation 1), and the EVAs m_2 and $m_{3.39}$ (defined by Equation 2).

In Section 7.1, 500 simulations are developed for each one of these scenarios. In all of them, 100 randomly defined FPRs are used to model experts' preferences. For these 500 simulations, both the average number of rounds required to obtain the consensus and the average degree of consensus are computed. In Section 7.2, the simulations are developed by using concrete values for the experts' FPRs[27] to be able to compare graphically the evolution of the experts' opinions through the different rounds of the CRPs.

7.1 | Average performance of EVAs

To validate the EVA approach, the average performance of EVA-modified models has been compared with the average performance of the classic models. To do so, 500 simulations with 100 randomly defined FPRs have been developed for each EVA in both Herrera-Viedma et al.[16] and Quesada et al.[26] models.

The obtained results, which are summarized in Tables 3 and 4, show that the EVA approach always outperforms the classic approach in terms of convergence, by keeping a similar average

TABLE 5 Results on Herrera-Viedma et al.[16] with and without EVA

EVA	Order of alternatives	Rounds	Consensus
Classical	$x1 \succ x2 \succ x4 \succ x3$	6	0.87
$s_{0.08}$	$x1 \succ x2 \succ x4 \succ x3$	6	0.89
$s_{0.09}$	$x1 \succ x2 \succ x4 \succ x3$	6	0.92
m_2	$x1 \succ x2 \succ x4 \succ x3$	5	0.86
$m_{3.39}$	$x1 \succ x2 \succ x4 \succ x3$	5	0.91

Abbreviation: EVA, Extreme Values Amplification.

TABLE 6 Results on Quesada et al.[26] with and without EVA

EVA	Order of alternatives	Rounds	Consensus
Classical	$x4 \succ x1 \sim x2 \sim x3$	10	0.85
$s_{0.08}$	$x4 \succ x1 \succ x2 \sim x3$	7	0.86
$s_{0.09}$	$x4 \succ x1 \succ x2 \sim x3$	7	0.87
m_2	$x4 \succ x1 \succ x2 \sim x3$	7	0.86
$m_{3.39}$	$x4 \succ x1 \succ x2 \sim x3$	5	0.87

Abbreviation: EVA, Extreme Values Amplification.

consensus. This fact provides a clear answer to the second research question: the classic consensus models improve on average when the nonlinear approach is modeled by an EVA.

7.2 | Performance of EVAs in a concrete example

To clarify the performance of EVAs in CRPs for LSGDM, we have chosen an individual simulation with the FPRs values provided in Reference [27] and then show the results obtained in Tables 4 and 5 together a graphical evolution of the consensus progress. The results obtained are summarized in Tables 5 and 6.

To facilitate the understanding of the simulation results, AFRYCA provides a visualization of the different CRP simulations based on the multidimensional scaling technique[28] (see Figures 7 and 8). This representation shows the collective opinion of the experts' group in the center of the plot. Around the collective opinion, the experts' preferences are represented. The closer the experts' preferences to the collective opinion, the greater the consensus reached. In this way, we can appreciate the state of the experts' preferences for each round and the evolution of the CRPs in the simulations. Furthermore, we have also shown the results obtained from AFRYCA in Tables 5 and 6.

The classical model[16] reached a consensus level of 0.87 in six rounds. For this model the EVAs $s_{0.08}$ and $s_{0.09}$ have not reduced the number of rounds required to reach the consensus, but have improved the consensus level reached. The latter can be appreciated in Figure 7, since in the last round (round 6) the experts are closer each other than with linear preferences. In addition, the polynomial-based EVA m_2 has reduced the amount of rounds required to reach a similar consensus level, whereas the EVA $m_{3.39}$ has improved both aspects by needing just five

FIGURE 7 Herrera Viedma et al.[16] simulations [Color figure can be viewed at wileyonlinelibrary.com]

FIGURE 8 Quesada et al.[26] simulations [Color figure can be viewed at wileyonlinelibrary.com]

rounds to reach a consensus level greater than the obtained in the original model. Again, the latter can be visualized in Figure 7.

On the other hand, the classical model[26] obtained a consensus level of 0.85 in 10 rounds (see Table 6). In this case, both families of EVAs have obtained significantly better performance than the original model (see Figure 8). The EVAs $s_{0.08}$, $s_{0.09}$, and m_2 have slightly improved the consensus level in just seven rounds. In this case, the EVA $m_{3.39}$ has performed surprisingly well by increasing the level of consensus reached in only five rounds.

The simulation has shown that the implemented EVAs improve the performance of both models. By keeping the same order for the alternatives, after using the EVAs either the number of rounds has been reduced or the consensus level is increased. These simulations clarify and reinforce the positive answer to the second research question when the nonlinear approach used is an EVA.

8 | CONCLUSIONS

Nowadays, CRPs are a prominent line of research in GDM. Several models have been proposed in the literature, but usually these models assume linear scales for experts' preferences.[16,26] This contribution has studied and proposed the use of nonlinear scales to obtain more realistic preference modeling from the original experts' preferences, even in large-scale contexts.

We have exhaustively studied the analytical properties of these nonlinear scales, obtaining the main mathematical characteristics of those functions which are good candidates to become a proper nonlinear deformation for the original preferences. These particular deformations of the preferences have received the name of EVAs. These EVA functions remap linear-scaled FPRs into nonlinear-scaled FPRs and deform the preferences in the way that the distances between extreme values are increased and the distances between intermediate values are decreased. In addition, we have stated the dual definition of EVRs, that is, those functions that reduce the distance between extreme values by amplifying the distance between the intermediate ones.

After introducing a general method to construct EVAs and EVRs, we have proposed several families of EVAs: s_α (Equation 1), m_α (Equation 2), and $b_\alpha^{r,s}$ (Equation 3). The first one is based on the sin function, the second one is constructed from a polynomial and the last one is obtained from a piecewise polynomial function. Finally, we have simulated the performance of some of these EVAs in two classical consensus models by using the software AFRYCA.[10]

The use of the nonlinear scales provided by the EVAs improves the performance of both classical models used in this study. The simulations with random FPRs showed that the EVA approach reduces the average number of rounds required to reach the consensus in both models. In addition, when using the same FPRs for the comparisons, the novel EVA-modified models either reach the consensus in a faster way or increase the level of consensus when we use the proper EVA.

Further studies should focus on either suggesting new EVAs or optimizing the parameters of the existing EVAs for concrete CRPs. In addition, a deeper study of the performance of EVAs in different consensus models should be developed. Additionally, since every EVA (resp., EVR) induces a similarity measure, it is also interesting to study the effects of using these proximity measures when comparing FPRs by moving closer (resp., bringing near) extreme values and bringing near (resp., moving closer) the intermediate ones. Another possible research work would be finding a concrete GDM problem adequate to the properties of EVRs. Furthermore, future works could be related to the application of the proposed framework to real-world problems, such as the high-speed rail passenger satisfaction and bid evaluation with LSGDM.[29,30]

ORCID

Diego García-Zamora https://orcid.org/0000-0002-0843-4714
Álvaro Labella https://orcid.org/0000-0003-2764-1011
Rosa M. Rodríguez https://orcid.org/0000-0002-1736-8915
Luis Martínez https://orcid.org/0000-0003-4245-8813

REFERENCES

1. Kacprzyk J. Group decision making with a fuzzy linguistic majority. *Fuzzy Sets Syst*. 1986;18(2):105-118.
2. Butler C, Rothstein A. *On Conflict and Consensus: a Handbook on Formal Consensus Decision Making*. Portland: Food Not Bombs Publishing; 1987.
3. Kim J. A model and case for supporting participatory public decision making in e-democracy. *Group Decis Negot*. 2008;17(05):179-193.
4. Sueur C, Louis Deneubourg J, Petit O. From social network (centralized vs. decentralized) to collective decision-making (unshared vs shared consensus). *PLOS One*. 2012;7(2):e32566.
5. Chen X-H, Liu R. Improved clustering algorithm and its application in complex huge group decision-making. *Xi Tong Gong Cheng Yu Dian Zi Ji Shu/Syst Eng Electron*. 2006;28(11):1695-1699.
6. Dong Q, Zhou X, Martínez L. A hybrid group decision making framework for achieving agreed solutions based on stable opinions. *Inf Sci*. 2019;490:227-243.
7. Saint S, Lawson J. *Rules for Reaching Consensus. A Modern Approach to Decision Making*. San Francisco: Jossey-Bass; 1994.
8. Martínez L, Montero J. Challenges for improving consensus reaching processes in collective decisions. *New Math Nat Comput*. 2007;03(02):203-217.
9. Kacprzyk J, Fedrizzi M. A human-consistent′ degree of consensus based on fuzzy login with linguistic quantifiers. *Math Soc Sci*. 1989;18(3):275-290.
10. Palomares I, Estrella F, Martinez L, Herrera F. Consensus under a fuzzy context: taxonomy, analysis framework AFRYCA and experimental case of study. *Inf Fusion*. 2014;20:11.
11. Labella Á, Liu H, Rodríguez RM, Martínez L. A cost consensus metric for consensus reaching processes based on a comprehensive minimum cost model. *Eur J Oper Res*. 2020;281:316-331.
12. Herrera-Viedma E, Cabrerizo FJ, Kacprzyk J, Pedrycz W. A review of soft consensus models in a fuzzy environment. *Inf Fusion*. 2014;17:4-13.Information fusion in consensus and decision making.
13. Zhang H, Kou G, Peng Y. Soft consensus cost models for group decision making and economic interpretations. *Eur J Oper Res*. 2019;277(3):964-980.
14. Masthoff J. Group modeling: selecting a sequence of television items to suit a group of viewers. *User Modeling User-Adapted Interact*. 2004;14:37-85.
15. Delic A, Ricci F, Neidhardt J. Preference networks and non-linear preferences in group recommendations. In: Barnaghi P, Gottlob G, Manolopoulos Y, Tzouramanis T, Vakali A, eds. *IEEE/WIC/ACM International Conference on Web Intelligence*, ser. WI '19. New York, NY, USA: Association for Computing Machinery; 2019:403-407.
16. Herrera-Viedma E, Herrera F, Chiclana F. A consensus model for multiperson decision making with different preference structures. *IEEE Trans Syst Man Cybern—Part A Syst Hum*. 2002;32(06):394-402.
17. Bryson N. Group decision-making and the analytic hierarchy process: exploring the consensus-relevant information content. *Comput Oper Res*. 1996;23(1):27-35.
18. Labella Á, Liu Y, Rodríguez RM, Martínez L. Analyzing the performance of classical consensus models in large scale group decision making: a comparative study. *Appl Soft Comput*. 2018;67:677-690.
19. Rodríguez RM, Labella Á, Tré GD, Martínez L. A large scale consensus reaching process managing group hesitation. *Knowl-Based Syst*. 2018;159(1):86-97.
20. Zhang Z, Guo C, Martínez L. Managing multigranular linguistic distribution assessments in large-scale multiattribute group decision making. *IEEE Trans Syst Man Cybern: Syst*. 2017;47(11):3063-3076.
21. Xiao L, Chen Z-S, Zhang X, Chang J-P, Pedrycz W, Chin K-S. Bid evaluation for major construction projects under large-scale group decision-making environment and characterized expertise levels. *Int J Comput Intell Syst*. 2020;(07):1227-1242.

22. Rodríguez RM, Labella Á, Dutta B, Martíinez L. Comprehensive minimum cost models for large scale group decision making with consistent fuzzy preference relations. *Knowl-Based Syst*. 2021;215:106780.
23. Bustince H, Barrenechea E, Pagola M. Relationship between restricted dissimilarity functions, restricted equivalence functions and normal en-functions: Image thresholding invariant. *Pattern Recognition Lett*. 2008; 29(4):525-536.
24. Sîrbu A, Loreto V, Servedio V, Tria F. Cohesion, consensus and extreme information in opinion dynamics. *Adv Complex Syst*. 2013;16(06):1350035.
25. Sîrbu A, Loreto V, Servedio V. Opinion dynamics with disagreement and modulated information. *J Stat Phys*. 2013;151:218-237.
26. Quesada FJ, Palomares I, Martínez, L. Managing experts behavior in large-scale consensus reaching processes with uninorm aggregation operators. *Appl Soft Comput*. 2015;35:873-887.
27. Data set with 100 experts. https://sinbad2.ujaen.es/afryca/sites/default/files/dataset/EVA100Preferences.pdf. Accessed January 4, 2021.
28. Blouvshtein L, Cohen-Or D. Outlier detection for robust multi-dimensional scaling. *IEEE Trans Pattern Analy Mach Intell*. 2018;41(9):2273-2279.
29. Chen ZS, Liu XL, Rodríguez RM, et al. Identifying and prioritizing factors affecting in-cabin passenger comfort on high-speed rail in china: a fuzzy-based linguistic approach. *Appl Soft Comput*. 2020;95:95.
30. Chen Z-S, Liu X, Chin K-S, Pedrycz W, Tsui K-L, Skibniewski M. Online-review analysis based large-scale group decision-making for determining passenger demands and evaluating passenger satisfaction: case study of high-speed rail system in china. *Inf Fusion*. 2021;69(05):22-39.

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4.3 PESOS SIMÉTRICOS PARA OPERADORES OWA. EL OPERADOR EVR-OWA

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Symmetric weights for OWA operators prioritizing intermediate values. The EVR-OWA operator

Diego García-Zamora [a,*], Álvaro Labella [a], Rosa M. Rodríguez [a], Luis Martínez [a]

[a] *Department of Computer Science, University of Jaén, Jaén, Spain*

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ABSTRACT

One of the most widely adopted approaches to define weights for Ordered Weighting Averaging (OWA) operators consists of using biparametric linear increasing fuzzy linguistic quantifiers. However, several shortcomings appear when using these quantifiers because depending on the values of these parameters, the aggregations could be biased or the extreme values might be completely ignored. In this contribution, the use of Extreme Values Reductions (EVRs) as fuzzy linguistic quantifiers is proposed to define weights for OWA operators in order to provide more realistic aggregations. First, the impact of the parameters of these linear fuzzy linguistic quantifiers in the OWA aggregations is studied. After that, EVR-OWA operators are introduced as those OWA operators whose weights are computed by using an EVR as fuzzy linguistic quantifier. It will be shown that when using EVR-OWA operators to fuse information, the aggregations are non-biased, take into account more information and the intermediate values are prioritized before the extreme ones. After proposing several families of EVRs, the generalising potential of the EVR-OWA operators is shown by proving that every family of symmetric weights for OWA operators that prioritize the intermediate information are the weights obtained from a certain EVR. Finally, an illustrative example is provided.

1. Introduction

Several real world problems demand the fusion of information or expert knowledge which might be fuzzy or imprecise. For instance, according to [13], Group Decision Making (GDM) problems require of an *aggregation phase* which combines the experts' preferences to obtain a collective opinion before carrying the *exploitation phase* out, in which the ranking of the alternatives to select the best one as solution of the decision problem is established. Even though lots of aggregation operators have been proposed in the literature [1,2], one of the most widely used is the Ordered Weighted Averaging (OWA) operator, which assigns weights to the input values according to their order [18,20,21]. In order to compute these weights, among other proposals [7,11,17], it is common to use the method proposed by Yager [21], which is based on the use of a biparametric family of linear fuzzy linguistic quantifiers [20] which assigns zero to the values that are close to zero and one to the values that are close to one.

This approach is simple and effective but it presents important drawbacks regarding the selection of necessary parameters. For instance, the aggregations could produce biased results (orness measure [2] not equal to 0.5) or even do not aggre-

* Corresponding author at: Universidad de Jaén, Computer Science Department, Campus Las Lagunillas, s/n Building A3, Office 102. 23071 - Jaén, Spain
E-mail addresses: dgzamora@ujaen.es (D. García-Zamora), alabella@ujaen.es (Á. Labella), rmrodrig@ujaen.es (R.M. Rodríguez), martin@ujaen.es (L. Martínez).

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gate enough information (low entropy measure [2]). In addition, the OWA operator constructed from these linear fuzzy linguistic quantifiers completely ignores the most extreme values in the aggregation process which could lead to non realistic aggregations. These biased or non realistic aggregations could be a major inconvenient in many real-world applications. For instance, in consensus processes for GDM [6,9], an OWA operator whose orness is greater than 0.5 tends to prioritize extreme values close to 1 regarding those close to 0, which is not reasonable because all of them are equally important. Furthermore, a theoretical consensus reached by completely ignoring the most extreme values would not be realistic. However, it has been proved that the less extreme information has a cohesive effect and facilitates the agreement among experts [14,15]. Therefore it seems reasonable to provide new ways of generating OWA weights which prioritize the intermediate information before the extreme data, as linear fuzzy linguistic quantifiers do, but taking into account more information in the aggregation process and avoiding biased aggregations in the results.

This work aims at solving three different problems.

i) First, it is necessary to clarify the relation between the biparametric family of linear fuzzy linguistic quantifiers [20] and the way that these quantifiers fuse the information.
ii) A new proposal is then required to deal with the limitations of these linear fuzzy linguistic quantifiers by keeping their simplicity and applicability [7] but allowing to generate weights which prioritize the intermediate information in a non biased way.
iii) Finally, the abstract conditions required to generate the aforementioned weights are discussed.

Therefore, we raise the following research questions:

- RQ1: How do the parameters of the linear fuzzy linguistic quantifiers impact the aggregation of information?
- RQ2: How to fuse information in a more realistic way than by using the linear fuzzy linguistic quantifier?
- RQ3: What properties share those linguistic quantifiers whose associated OWA weights allow to fuse information symmetrically by prioritizing the intermediate values?

Consequently, this proposal analyzes the impact in the aggregation of information of the parameters of these widely extended linear fuzzy linguistic quantifiers [20] and shows their main shortcomings. We then propose an abstract novel extension of OWA operators which uses an Extreme Values Reduction (EVR) [4] as a fuzzy linguistic quantifier instead of the traditional linear fuzzy linguistic quantifier. The resulting operator is characterized for assigning weights to information depending on its degree of polarization, such that the most important values are the intermediate ones and their importance (weight) progressively decreases for the most extreme values. In addition, this novel EVR-OWA operator provides a non biased way to fuse information which takes into account almost as much information as the arithmetic mean operator, which is the one with higher entropy measure [2]. Furthermore, the generalizing potential of this EVR approach is highlighted by showing that any family of positive symmetric weights which prioritize the intermediate information, like the ones studied in [17], are actually EVR-OWA weights.

The structure of this contribution is as follows: Section 2 is a brief review about OWA operators and fuzzy linguistic quantifiers. In Section 3 the impact of the parameters of linear fuzzy linguistic quantifier in the aggregation of information is analyzed and the main shortcomings of this approach are exposed. Section 4 introduces the main proposal of this contribution, the EVR-OWA operator, and studies some of its main properties. Section 5 completes this proposal by providing several families of EVRs. In Section 6, the relation between symmetric weights for OWA operators and EVRs is studied. In Section 7, an illustrative example of aggregations by using this EVR-OWA operator is developed. Section 8 summarizes the main contributions of this work. Finally, Section 9 concludes this contribution.

2. Preliminaries

This section provides a brief review about OWA operators [20] and Yager's method [21] to compute their weights with a fuzzy linguistic quantifier, which is the starting point of our proposal. Finally, the notion of EVR [4] is introduced.

2.1. Ordered weighted averaging operators

The Ordered Weighted Averaging (OWA) operators [20] are a family of aggregation functions which generalizes the notion of arithmetic mean.

Definition 1 (*OWA Operator*). Let $\omega \in [0,1]^m$ be a weighting vector $(\omega \in [0,1]^m, \sum_{i=1}^m \omega_i = 1)$. The OWA Operator $\Phi_\omega : [0,1]^m \to [0,1]$ associated to ω is given by:

$$\Phi_\omega\left(\vec{x}\right) = \sum_{i=1}^{m} \omega_i x_{\sigma(i)} \forall \vec{x} \in [0,1]^m$$

where σ is a permutation of the m-tuple $(1,2,\dots,m)$ which satisfies $x_{\sigma(1)} \geqslant x_{\sigma(2)} \geqslant \dots \geqslant x_{\sigma(m)}$.

Remark 1. If $\omega = (1, 0, \ldots, 0) \in [0,1]^m$, the corresponding OWA Operator is the maximum operator, whereas when $\omega = (0, 0, \ldots, 1) \in [0,1]^m$ the respective OWA Operator is the minimum operator. For $\omega = (\frac{1}{m}, \frac{1}{m}, \ldots, \frac{1}{m}) \in [0,1]^m$, the OWA Operator associated to ω is the arithmetic mean.

OWA operators have several remarkable properties such as the facts that they are idempotent non decreasing functions which are continuous, symmetric, homogeneous and shift-invariant [2].

There are different measures to study the behavior of an OWA Operator. Among the most extended measures are the arithmetic mean and the standard deviation of the weights. Other useful measures are the Orness and the entropy measures [2].

The orness measure of an OWA Operator quantifies the similarity between this OWA Operator and the maximum operator. It is given by

$$orness(\Phi_\omega) = \sum_{i=1}^{m} \omega_i \frac{m-i}{m-1}.$$

And, when m is large enough [10]:

$$orness(\Phi_\omega) \approx \int_0^1 Q(t)dt$$

When the coordinates of the weighting vector are increasing, i.e. $w_1 \leqslant w_2 \leqslant \ldots \leqslant w_m, orness(\Phi_\omega) \in \left[\frac{1}{2}, 1\right]$ whereas when these coordinates are decreasing, i.e. $w_1 \geqslant w_2 \geqslant \ldots \geqslant w_m, orness(\Phi_\omega) \in \left[0, \frac{1}{2}\right]$ [2]. In addition, it is known that the orness measure equals to 0.5 if and only if the weights are symmetric, i.e. $w_k = w_{m-k+1} \forall k = 1, 2, \ldots, m$ [2]. In particular, the orness measure for the arithmetic mean operator, in which all the weights are the same, is equal to $\frac{1}{2}$.

The entropy measure, or simply entropy, quantifies how much information is taken into account during the aggregation process. It is given by

$$Entropy(\Phi_\omega) = -\sum_{i=1}^{m} \omega_i \log \omega_i,$$

If no orness measure is specified, the weighting vector which maximizes the Entropy is the associated with the arithmetic mean operator [2].

2.2. Using fuzzy linguistic quantifiers to compute weights

Among others proposals [5] to compute OWA weights, Yager proposed the use of fuzzy linguistic quantifiers [22] to obtain the weights for OWA Operators [21].

Fuzzy linguistic quantifiers are fuzzy subsets $Q : [0,1] \to [0,1]$ of the unit interval $[0,1]$ and they were classified by Yager as follows [19]:

- Regular Increasing Monotone (RIM) quantifiers, i.e. $Q(0) = 0, Q(1) = 1, Q(x) \leqslant Q(y) \forall x \leqslant y$,
- Regular Decreasing Monotone (RDM) quantifiers, i.e. $Q(0) = 1, Q(1) = 0, Q(x) \geqslant Q(y) \forall x \leqslant y$,
- Regular UniModal (RUM) quantifiers, i.e. $Q(0) = 0, Q(1) = 0, Q(x) \leqslant Q(y) \forall x \leqslant y, y < a, Q(x) = 1 \forall x \in [a,b]$, and $Q(x) \geqslant Q(y) \forall x \leqslant y, x > b$ for some $a, b \in [0,1]$ such that $a < b$.

Yager introduced the use of RIM quantifiers for generating the weights [21] according to the following formula:

$$w_k = Q\left(\frac{k}{m}\right) - Q\left(\frac{k-1}{m}\right) \text{for } k = 1, 2, \ldots, m$$

The OWA operators based on linear fuzzy linguistic quantifiers have been widely used in the literature [12]. One of the most common approaches consists of using the linear RIM quantifier $Q_{\alpha,\beta} : [0,1] \to [0,1]$ given by:

$$Q_{\alpha,\beta}(x) = \begin{cases} 0 & 0 \leqslant x < \alpha \\ \frac{x-\alpha}{\beta-\alpha} & \alpha \leqslant x \leqslant \beta \\ 1 & x > \beta \end{cases},$$

where $\alpha < \beta$ are two parameters in the interval $[0,1]$. Several consensus models for GDM problems [6,10] have used this method to compute the weights of their aggregations since they allow to adjust the importance of the intermediate information by adjusting the value of α and β.

2.3. Extreme values reductions

García-Zamora et al. [4] studied the effect of remapping experts' preferences by using non linear scales in consensus models for GDM. To do that, the notion of EVR was introduced as those automorphisms on the interval $[0,1]$, i.e. strictly increasing bijections which satisfy the boundary conditions $D(0)=0$ and $D(1)=1$. These functions are characterized by reducing the distance between the values which are close to 0 and 1. Formally:

Definition 2 (*Extreme Values Reduction [4]*). Let $\widehat{D}:[0,1]\to[0,1]$ be a function satisfying:

1. $\widehat{D}$ is an automorphism on the interval $[0,1]$,
2. $\widehat{D}$ is a function of class $\mathscr{C}^1$, i.e. it is differentiable and its derivative is continuous,
3. $\widehat{D}$ satisfies $\widehat{D}(x)=1-\widehat{D}(1-x)\forall x\in[0,1]$,
4. $\widehat{D}'(0)<1$ and $\widehat{D}'(1)<1$,
5. $\widehat{D}$ is convex in a neighborhood of 0 and concave in a neighborhood of 1,

Then $\widehat{D}$ is called an Extreme Values Reduction (or EVR) on the interval $[0,1]$.

It was shown [4] that these EVR functions satisfy the following properties.

Theorem 1. *Let $\widehat{D}:[0,1]\to[0,1]$ be an EVR on $[0,1]$. Then:*

1. *The function $d_{\widehat{D}}:[0,1]\times[0,1]\to[0,1]$ given by*
$$d_{\widehat{D}}(x,y)=|\widehat{D}(x)-\widehat{D}(y)|\forall x,y\in[0,1],$$
is a Restricted dissimilarity [3] and the function $S_{\widehat{D}}:[0,1]\times[0,1]\to[0,1]$ defined by
$$S_{\widehat{D}}(x,y)=1-|\widehat{D}(x)-\widehat{D}(y)|\forall x,y\in[0,1].$$
is a Restricted Equivalence Function [3].
2. *We can find three intervals $I_1,I_2,I_3\subset[0,1]$ such that $0\in I_1, 1\in I_3$, and $I_1<I_2<I_3$ satisfying that*
$$|\widehat{D}(y)-\widehat{D}(x)|<|y-x|\forall x,y\in I_1:x\neq y,$$
$$|\widehat{D}(y)-\widehat{D}(x)|>|y-x|\forall x,y\in I_2:x\neq y,$$
$$|\widehat{D}(y)-\widehat{D}(x)|<|y-x|\forall x,y\in I_3:x\neq y.$$
3. *The graph of $\widehat{D}$ is under the diagonal of the square $[0,1]\times[0,1]$ for values close enough to 0 and it is over the same diagonal for those values close enough to 1,*
4. *There exist a neighborhood U_0 containing 0 and a neighborhood U_1 containing 1 such that for every $x,y\in U_0^\circ, x<y$, there exists $h_0>0$ such that the inequality $|\widehat{D}(x)-\widehat{D}(x-t)|\leqslant|\widehat{D}(y)-\widehat{D}(y-t)|$ holds for any $t\in[0,h_0]$ and for every $x,y\in U_1^\circ, x<y$, there exists $h_1>0$ such that the inequality $|\widehat{D}(x-t)-\widehat{D}(x)|\geqslant|\widehat{D}(y-t)-\widehat{D}(y)|$ holds for any $t\in[0,h_1]$.*

Note that EVRs deform the unit interval in a very particular way. According to the second thesis of the previous Theorem, when using an EVR the distances between extreme values (those which are close to 0 or 1) are decreased, whereas the distances between certain intermediate values are increased. In addition, the forth thesis of that result guarantees that the distances among extreme values are progressively reduced when the values become closer to 0 or 1.

3. On the drawbacks of the linear RIM quantifier for computing OWA weights

In this section, it is analyzed the implications of choosing the weighting vector by using the linear RIM quantifier defined in the previous section.

Suppose $0<\alpha<\beta<1$. Then for any $k\leqslant\alpha m$ we obtain $\omega_k=0$. When $k\geqslant m\beta+1$, we also get $w_k=0$. If $m\alpha>k$ but $k-1\leqslant m\alpha$ we obtain
$$w_k=\frac{k-m\alpha}{m(\beta-\alpha)}.$$
In the case $m\beta\leqslant k$ and $k-1<m\beta$:

$$w_k = 1 - \frac{(k-1) - m\alpha}{m(\beta - \alpha)} = \frac{m\beta - k + 1}{m(\beta - \alpha)}.$$

The remaining case $\alpha m + 1 < k < m\beta$ is reduced to:

$$w_k = \frac{k - m\alpha}{m(\beta - \alpha)} - \frac{k - 1 - m\alpha}{m(\beta - \alpha)} = \frac{1}{m(\beta - \alpha)}.$$

The previous discussion is summarized in the following result.

Proposition 1. The weights for the m-dimensional OWA operator obtained from the linear RIM quantifier $Q_{\alpha,\beta}$ are given by

$$w_k = \begin{cases} 0 & 1 \leqslant k \leqslant \alpha m \\ \frac{k - m\alpha}{m(\beta-\alpha)} & m\alpha < k \leqslant \alpha m + 1 \\ \frac{1}{m(\beta-\alpha)} & \alpha m + 1 < k < m\beta \\ \frac{m\beta + 1 - k}{m(\beta-\alpha)} & m\beta \leqslant k < m\beta + 1 \\ 0 & \beta m + 1 \leqslant k \leqslant m \end{cases}, \quad (1)$$

where $k = 1, 2, \ldots, m$.

Note that for any $k \leqslant m\alpha$, all the weights w_k are zero and the same occurs when $k \geqslant \beta m + 1$. In terms of the OWA operator associated to these weights, this fact means that the operator ignores the *first* $m\alpha$ values and the *last* $m - (\beta m + 1)$ values. In other words, the greater α, the less top ranked values are considered for the OWA aggregation. In the same way, the less value of β, the less bottom ranked values are considered for OWA aggregation. Depending on the choice of α and β, the ignored values could be high enough to declare *non realistic* any aggregation which is based in this linear RIM quantifier. Keep in mind, that OWA operators order the values to aggregate before applying the weights. So, the ignored information is the corresponding to the most extreme values (polarized, if polarization exist in such a set of values) among the elements to be aggregated.

In addition, if non biased aggregations are required, the orness measure of the corresponding OWA operator must be equal to 0.5. For instance, if the orness measure is greater than this value, the aggregation would swing towards the maximum, giving more importance to the values greater than the median value of the elements which are being aggregated. The following result provides a relation between the parameters α and β which characterizes non biased aggregations.

Proposition 2. The orness measure for the OWA operator associated with the linear RIM quantifier $Q_{\alpha,\beta}$ equals to 0.5 if and only if $\alpha + \beta = 1$.

Proof. According to [2], the orness measure of an OWA operator equals to 0.5 if and only if the associated weights are symmetric. It is clear that if $\alpha + \beta = 1$, the weights provided by the linear RIM quantifier $Q_{\alpha,\beta}$ are symmetric and consequently the orness measure equals to 0.5.

To prove the reciprocal assumption, fix $m \in \mathbb{N}$ and $0 \leqslant \alpha < \beta \leq 1$ fixed and consider

$$k_1 := E^+[m\alpha] = \min_{k=1,2,\ldots,m} \{k : w_k \neq 0\}$$
$$k_2 := E^+[m\beta] = \max_{k=1,2,\ldots,m} \{k : w_k \neq 0\},$$

where $E^+ : \mathbb{R} \to \mathbb{Z}$ denotes the ceiling function. Note that Proposition 1 allows to study the symmetry of the weights $w_1, w_2, \ldots, w_m$ by just looking at w_{k_1} and w_{k_2}. Therefore, if the weights are symmetric, the following chain of equalities holds:

$$w_{k_1} = w_{k_2} = w_{m-k_2+1}.$$

From the first one, the constrain $k_1 - m\alpha = m\beta + 1 - k_2$ is obtained, whereas comparing the first term and the last one leads to $k_1 + k_2 = m + 1$. If these constraints are combined, $\alpha + \beta = 1$ is obtained.

This fact induces a constraint for α and β. If they are not chosen in a symmetric way, i.e. $\beta = 1 - \alpha$, the aggregation gives more importance to certain extreme values, and this may have no sense when applied in some real world problems like consensus models for GDM.

Example 1. Consider a problem in which five experts express their preferences through the vector $P = (P_1, P_2, P_3, P_4, P_5) = (1, 1, 0.75, 0.5, 0.5)$ on how much they prefer the alternative X_1 to the alternative X_2. When using the linear RIM quantifier $Q_{\alpha,\beta}$ for $\alpha = 0.4$ and $\beta = 0.8$, the obtained weights are $w = (w_1, w_2, w_3, w_4, w_5) = (0, 0, 0.5, 0.5, 0)$.

Note that the orness of the corresponding OWA operator is $1 - (0.4 + 0.8)/2 = 0.4$. In practice, this means that the fusion of information does not prioritize the most intermediate values, i.e. the ones which are around the median value of the preferences. On the contrary, the aggregations made by this OWA operator prioritize the values which are slightly deviated to the lower values of the preferences.

For instance, when fusing the preferences in the previous problem, the aggregation of these preferences is $0.5 \cdot 0.75 + 0.5 \cdot 0.5 = 0.625$, which is deviated from the median value of preferences, i.e. 0.75. Note that this deviation implies introducing a bias in the computations because equally extreme values of the preferences are not weighted in the same way: the distance between the median value and both P_2 and P_4 are the same, but this is not reflected in the aggregations made by this operator, which prioritize the value P_4 because it is lower than the median value.

In addition, note that we have just three possibilities for the value of w_k:

- $w_k = 0$, for the most extreme values,
- $\frac{1}{m(\beta-\alpha)}$, which is much higher than $\frac{1}{m}$, for most of the intermediate values,
- $\frac{k-m\alpha}{m(\beta-\alpha)}$ or $\frac{m\beta+1-k}{m(\beta-\alpha)}$, which are lower than $\frac{1}{m(\beta-\alpha)}$, for at most two of the possible values of k.

The fact that there are just a few possible values for the weights is somehow against the fuzzy logic view. It should be convenient that the values of the weights change smoothly from the minimum possible value to the maximum one, instead of changing drastically from zero to $\frac{1}{m(\beta-\alpha)}$ as it occurs with the weights associated to the linear RIM quantifier $Q_{\alpha,\beta}$.

To summarize, the main shortcomings of the linear RIM quantifier (see Fig. 1) are:

- If α is too high or β is too low, the aggregations are non realistic because we are ignoring too many extreme values,
- If $\alpha + \beta \neq 1$ the results of the aggregations are biased,
- The obtained weights are against the fuzzy logic philosophy.

Therefore, we propose an alternative method to select such OWA weights which guarantees not only to take into account the more extreme values, but also allows the user to control the relevance given to these values. Our aim is to aggregate elements in a more realistic non biased way.

4. An OWA operator based on Extreme Values Reductions

This section presents the main novelty of this contribution, namely EVR-OWA operator. This operator is based in the notion of Extreme Values Reduction, detailed in SubSection 2.3. The properties of EVRs are applied to construct an OWA Operator which has similar measures to the arithmetic mean but giving more importance to the intermediate values and less importance to the more extreme values to smooth out the importance of polarized opinions in GDM, but taking them into account instead of ignoring them.

Let us start by analysing the fourth thesis of Theorem 1. Consider an EVR $\widehat{D} : [0,1] \to [0,1]$ which is convex in $[0,0.5]$ and concave in $[0.5,1]$. Suppose we have a partition of the interval $[0,1]$. For instance, we can take $m \in \mathbb{N}$ and define

$$x_k = \frac{k}{m}, \forall k \in \{1,2,\ldots,m\}$$

Since $\widehat{D}$ is convex in $[0,0.5]$, for any $k_1 \leqslant k_2$ such that $x_{k_1}, x_{k_2} \in \left[\frac{1}{m}, \frac{1}{2}\right]$ we obtain

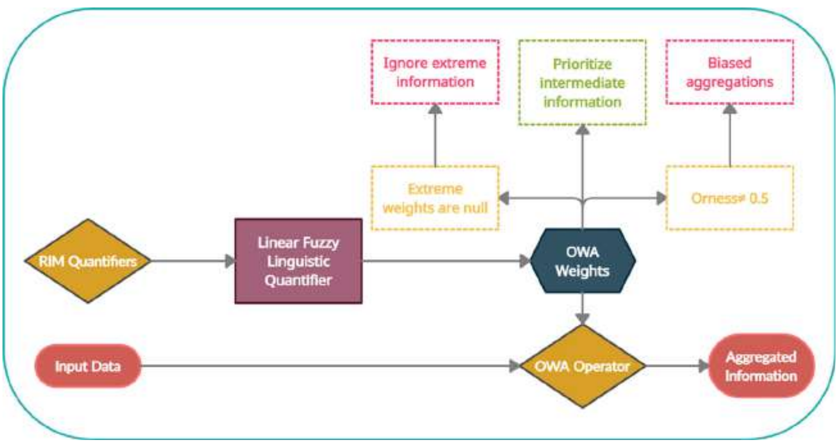

Fig. 1. Flowchart summarizing drawbacks of linear RIM quantifiers when used in OWA operators.

$$|\widehat{D}(x_{k_1}) - \widehat{D}\left(x_{k_1} - \frac{1}{m}\right)| \leqslant |\widehat{D}(x_{k_2}) - \widehat{D}\left(x_{k_2} - \frac{1}{m}\right)|.$$

In other words, the more closer x_k is to 0, the smaller the difference $|\widehat{D}(x_k) - \widehat{D}(x_k - \frac{1}{m})|$. A similar reasoning leads us to the concave counterpart of this: the more closer x_k is to 1, the smaller the difference $|\widehat{D}(x_k) - \widehat{D}(x_k - \frac{1}{m})|$.

This reasoning and the fact that EVRs are RIM quantifiers allow to define the weights of an OWA operator by using the general scheme introduced by Yager [21].

Definition 3. [EVR-OWA Operator] Let $\widehat{D}$ be an Extreme Values Reduction and consider $m \in \mathbb{N}$. We define

$$w_k = \widehat{D}\left(\frac{k}{m}\right) - \widehat{D}\left(\frac{k-1}{m}\right) \forall k \in \{1, 2, \ldots, m\}.$$

The family $W = \{w_1, w_2, \ldots, w_m\}$ receives the name of order m weights associated with the EVR $\widehat{D}$, and the OWA- operator given by $\Phi_{\widehat{D}} : [0,1]^m \to [0,1]$ defined by

$$\Phi_{\widehat{D}}(x_1, x_2, \ldots, x_m) = \sum_{k=1}^{m} w_k x_{\sigma(k)},$$
$$\forall (x_1, x_2, \ldots, x_m) \in [0,1]^m,$$

where σ is a permutation of the m-tuple $(1, 2, \ldots, m)$ which satisfies $x_{\sigma(1)} \geqslant x_{\sigma(2)} \geqslant \ldots \geqslant x_{\sigma(m)}$.

We highlight the philosophy behind this operator. As any OWA operator, Φ_W starts by ordering the values $x_1, x_2, \ldots, x_m$ from the largest one, to the smallest one. When we use these weights, those values of x which are closer to extremes, i.e. the largest ones, and the smallest ones, are matched with the smallest weights while the intermediate values of x are matched with the highest w_k's. Hence, this operator aggregates elements by assigning more relevance to the intermediate values of the elements which aggregates, and giving less importance to the more extreme elements, but taking them into account since $\widehat{D}$ is strictly increasing and therefore those weights can not be zero.

Let us analyze some properties of these weights. First note that since $\widehat{D}$ is strictly increasing, all of these weights are greater than zero. In addition,

$$\sum_{k=1}^{m} w_k = \sum_{k=1}^{m} \widehat{D}\left(\tfrac{k}{m}\right) - \widehat{D}\left(\tfrac{k-1}{m}\right) =$$
$$= \widehat{D}(1) - \widehat{D}(0) = 1,$$

and they are properly defined.

By using the third condition of the EVR definition, i.e. $\widehat{D}(x) = 1 - \widehat{D}(1-x) \forall x \in [0,1]$, we get:

$$\widehat{D}\left(\tfrac{k}{m}\right) = 1 - \widehat{D}\left(1 - \tfrac{k}{m}\right) = 1 - \widehat{D}\left(\tfrac{m-k}{m}\right),$$
$$\widehat{D}\left(\tfrac{k-1}{m}\right) = 1 - \widehat{D}\left(1 - \tfrac{k-1}{m}\right) = 1 - \widehat{D}\left(\tfrac{m-k+1}{m}\right),$$

therefore

$$w_k = \widehat{D}\left(\tfrac{k}{m}\right) - \widehat{D}\left(\tfrac{k-1}{m}\right) =$$
$$= \widehat{D}\left(\tfrac{m-k+1}{m}\right) - \widehat{D}\left(\tfrac{m-k}{m}\right) = w_{m-k+1}.$$

This symmetry and the fifth property of EVRs, used as we have explained before, give us an idea of the distribution of these weights. On the one hand, the smallest values of w_k are always located at $k = 1$ and $k = m$, i.e.

$$w_{min} = w_1 = D\left(\frac{1}{m}\right) = 1 - D\left(1 - \frac{1}{m}\right) = w_m.$$

We know that these weights are matched by pairs. So there is a minimum value at w_1 and the values of the weights strictly increase until a certain maximum value w_{max} and then, because of the symmetry $w_k = w_{m-k+1}$, the values of the weights start to decrease towards the value $w_m = w_1$. The maximum value for w_k depends on the parity of m. When m is even, the maximum value is at $k = \frac{m}{2}$ due to $w_{\frac{m}{2}} = w_{m-\frac{m}{2}+1} = w_{\frac{m+2}{2}}$. When m is odd, the maximum value is at $k = \frac{m+1}{2}$:

$$w_{max} = \begin{cases} w_{\frac{m}{2}} & \text{if } m \text{ is even} \\ w_{\frac{m+1}{2}} & \text{if } m \text{ is odd} \end{cases},$$

Let us remark here that w_{max} is a kind of median value for the weights.

Note that the arithmetic mean of these weights is

$$\overline{w} = \frac{1}{m}\sum_{k=1}^{m} w_k = \frac{1}{m}.$$

and the orness measure of the corresponding OWA operator must be equal to 0.5 because the weights are symmetric [1]. So $orness\left(\Phi_{\widehat{D}}\right) = 0.5$ for any EVR $\widehat{D}$ and any $m \in \mathbb{N}$.

To conclude this section, the main advantages of EVR-OWA operators are summarized (see Fig. 2). First, since their orness measure is equal to 0.5, the weights are symmetrically distributed and the aggregations give equal importance to the extreme values. In addition, all the weights are positive and the most extreme values are always considered in the aggregation process. Finally, they are simple to compute and the corresponding EVR can be used as RIM Quantifier in any scenario in which weights are required to prioritize intermediate values.

5. Examples of EVR-OWA operators

In this section several families of EVRs are introduced, complementing the examples of EVAs proposed in [4]. For the EVR-OWA operator associated to these families, their main measures, namely the arithmetic mean, the orness measure, the standard deviation and the entropy measure, are studied.

5.1. The EVR-OWA associated to $\hat{s}_\alpha$

Let $\alpha \in]0, \frac{1}{2\pi}[$ and consider the EVR $s_\alpha : [0,1] \to [0,1]$ given by

$$\hat{s}_\alpha(x) = x + \alpha \cdot \sin(2\pi x - \pi) \forall x \in [0,1].$$

The following result summarizes the performance of the EVR-OWA operator associated to $\hat{s}_\alpha$ regarding its main measures:

Proposition 3. Let $\alpha \in]0, \frac{1}{2\pi}[$ and $m \in \mathbb{N}, m > 1$ and consider the EVR-OWA operator of order m $\Phi : [0,1]^m \to [0,1]$ associated to the EVR $\hat{s}_\alpha$ and the respective weighting vector $w = (w_1, w_2, \ldots, w_m)$. Then

- The arithmetic mean of the weights is given by $\overline{w} = \sum_{k=1}^{m} \frac{w_k}{m} = \frac{1}{m}$,
- The orness measure of the OWA operator is $orness(\Phi) = 0.5$,
- The standard deviation of the weights is bounded by $\sigma_w \leqslant \alpha \frac{2\pi}{m} \in]0, 1/m]$.

Proof. The two first items are consequence of the discussion made in the previous section. In order to show the third one, let us study the difference $|w_k - \overline{w}|$:

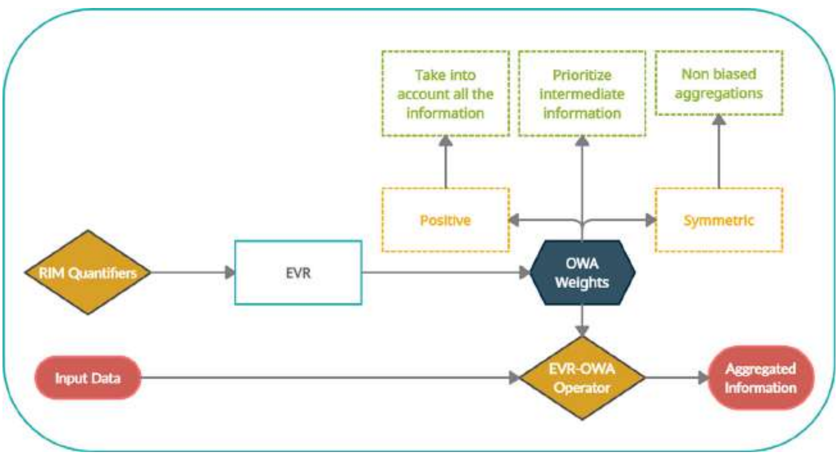

Fig. 2. Flowchart summarizing the main advantages of the EVR-OWA operator.

$$\begin{aligned}&|w_k-\overline{w}|=|\hat{s}_\alpha\left(\tfrac{k}{m}\right)-\hat{s}_\alpha\left(\tfrac{k-1}{m}\right)-\tfrac{1}{m}|=\\&|\hat{s}_\alpha\left(\tfrac{k}{m}\right)-\hat{s}_\alpha\left(\tfrac{k-1}{m}\right)-\tfrac{k-k+1}{m}|=\\&|\hat{s}_\alpha\left(\tfrac{k}{m}\right)-\tfrac{k}{m}+\tfrac{k-1}{m}+\hat{s}_\alpha\left(\tfrac{k-1}{m}\right)|=\\&\alpha|\sin(2\pi k/m-\pi)-\sin(2\pi(k-1)/m-\pi)|=\\&\alpha|cos(\xi)|(2\pi)/m\end{aligned}$$

where ξ is given by the Mean Value Theorem. Therefore:

$$|w_k-\overline{w}|\leqslant\alpha\frac{2\pi}{m}\forall k\in\{1,2,\ldots,m\}$$

and therefore the standard deviation of the w_k's is bounded by

$$\sigma_\alpha=\sqrt{\frac{1}{m}\sum_{k=1}^{m}(w_k-\bar{w})^2}\leqslant\alpha\frac{2\pi}{m}$$

which is a small value for m highly enough and $\alpha\in]0,\frac{1}{2\pi}[$.

For $m=100$, Table 1 shows the calculations of the most standard measures for the weights obtained for different values of α (keep in mind that for $\alpha=0$ we get the weights associated to the arithmetic mean operator). In Fig. 3 we show the comparison between values of the weights obtained for several values of α.

In order to analyze the behavior of the entropy measure for this EVR-OWA operator, we provide the plot in Fig. 4. For each fuzzy linguistic quantifier, namely the identity function (arithmetic mean), $\hat{s}_{0.08},\hat{s}_{0.15}$ and $Q_{0.2,0.8}$, the values of the entropy measure of the corresponding OWA operator have been computed for $m=2,3,\ldots,1000$. The graph shows that the OWA operators constructed from these EVRs present higher entropy than the linear RIM quantifier $Q_{0.2,0.8}$.

5.2. The EVR-OWA associated to $\hat{m}_\alpha$

Let $\alpha>1$ and consider the EVR $\hat{m}_\alpha:[0,1]\to[0,1]$ given by

$$\hat{m}_\alpha(x)=\begin{cases}\frac{1}{2}-\frac{1}{2}(1-2x)^{\frac{1}{\alpha}} & 0\leqslant x<\frac{1}{2}\\ \frac{1}{2}+\frac{1}{2}(2x-1)^{\frac{1}{\alpha}} & \frac{1}{2}\leqslant x\leqslant 1\end{cases}.$$

Remark 2. Note that $\hat{m}_\alpha$ is not an EVR strictly speaking since it is not differentiable in $x=\frac{1}{2}$. However, $\lim_{x\to\frac{1}{2}^-}f'(x)=\lim_{x\to\frac{1}{2}^+}f'(x)=+\infty$ and therefore there is no problem with considering it as an EVR.

The following result summarizes the performance of the EVR-OWA operator associated to $\hat{m}_\alpha$ regarding its main measures:

Proposition 4. Let $\alpha>1$ and $m\in\mathbb{N},m>1$ and consider the EVR-OWA operator of order m $\Phi:[0,1]^m\to[0,1]$ associated to the EVR $\hat{m}_\alpha$ and the respective weighting vector $w=(w_1,w_2,\ldots,w_m)$. Then

- The arithmetic mean of the weights is given by $\overline{w}=\sum_{k=1}^{m}\frac{w_k}{m}=\frac{1}{m}$,
- The orness measure of the OWA operator is $orness(\Phi)=0.5$,
- The standard deviation of the weights is bounded by $\sigma_w\leqslant\frac{1}{\alpha\sqrt[\alpha]{\alpha}}\left(1-\frac{1}{\alpha}\right)\in]0,1]$.

Table 1
Measures for $\hat{s}_\alpha$, $m=100$.

EVR	Orness	Entropy	Mean	SD	Min	Max
$\hat{s}_0$	0.5	4.6051	0.01	0	0.01	0.01
$\hat{s}_{0.04}$	0.5	4.5892	0.01	0.0017	0.0074	0.0125
$\hat{s}_{0.08}$	0.5	4.5398	0.01	0.0035	0.0049	0.0150
$\hat{s}_{0.09}$	0.5	4.5216	0.01	0.0039	0.0043	0.0156
$\hat{s}_{0.15}$	0.5	4.3443	0.01	0.0066	0.0005	0.0194

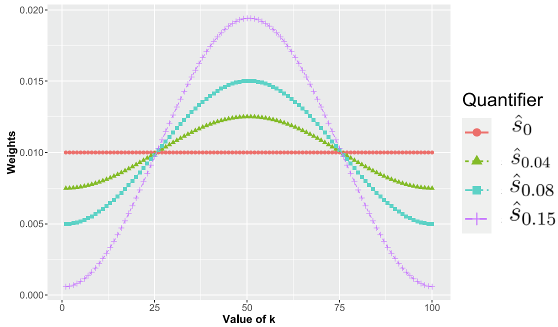

Fig. 3. Comparison of w_k for different $\hat{s}_\alpha, m = 100$.

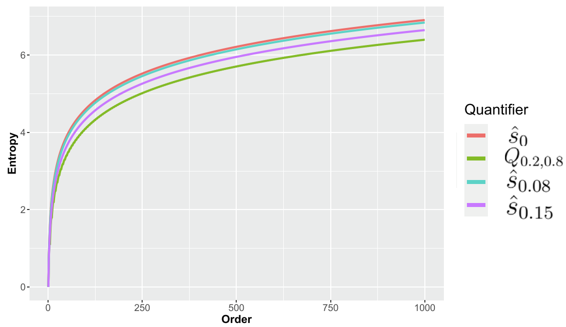

Fig. 4. Graph of the entropy function for several fuzzy linguistic quantifiers.

Proof. The two first items are consequence of the discussion made in the previous section. In order to show the third one, let us study the difference $|w_k - \overline{w}|$. To do that, let us consider the function $g : [0.5, 1] \to [0.5, 1]$ given by $g(x) = \hat{m}_\alpha(x) - x \forall x \in [0.5, 1]$, which reach its maximum value at $x_0 = \frac{1}{2}\left(\sqrt[\alpha-1]{\frac{1}{\alpha}} + 1\right)$. In that case:

$$|\hat{m}_\alpha(x) - x| \leqslant g(x_0) = \frac{1}{2\sqrt[\alpha-1]{\alpha}}\left(1 - \frac{1}{\alpha}\right).$$

Now we can compute:

$$\begin{aligned}
&|w_k - \overline{w}| = |\hat{m}_\alpha\left(\tfrac{k}{m}\right) - \hat{m}_\alpha\left(\tfrac{k-1}{m}\right) - \tfrac{1}{m}| = \\
&|\hat{m}_\alpha\left(\tfrac{k}{m}\right) - \hat{m}_\alpha\left(\tfrac{k-1}{m}\right) - \tfrac{k-k+1}{m}| = \\
&|\hat{m}_\alpha\left(\tfrac{k}{m}\right) - \tfrac{k}{m} + \tfrac{k-1}{m} - \hat{m}_\alpha\left(\tfrac{k-1}{m}\right)| = \\
&\leqslant \tfrac{2}{2\sqrt[\alpha-1]{\alpha}}\left(1 - \tfrac{1}{\alpha}\right).
\end{aligned}$$

So

$$|w_k - \overline{w}| \leqslant \frac{1}{\sqrt[\alpha-1]{\alpha}}\left(1 - \frac{1}{\alpha}\right)$$

and

$$\sigma_\alpha = \sqrt{\frac{1}{m}\sum_{k=1}^{m}(w_k - \overline{w})^2} \leqslant \frac{1}{\sqrt[\alpha-1]{\alpha}}\left(1 - \frac{1}{\alpha}\right).$$

Let us study this bound. Consider the function $h :]1, \infty[\to \mathbb{R}$, defined by

$$h(\alpha) = \frac{1}{\sqrt[\alpha-1]{\alpha}}\left(1 - \frac{1}{\alpha}\right) = \alpha^{\frac{-1}{\alpha-1}}\left(1 - \frac{1}{\alpha}\right), \forall \alpha \in]1, \infty[$$

Note that $\lim_{\alpha\to 1} h(\alpha) = 0$ and

$$\begin{aligned}&\lim_{\alpha\to\infty} h(\alpha) = \lim_{\alpha\to\infty} \alpha^{\frac{-1}{\alpha-1}}\left(1 - \tfrac{1}{\alpha}\right) = \\ &\lim_{\alpha\to\infty} \exp\left(\tfrac{-1}{\alpha-1}\log(\alpha)\right)\left(1 - \tfrac{1}{\alpha}\right) = 1\end{aligned}$$

In order to study h', note that for any $A \subset \mathbb{R}^+$ and $f : A \to \mathbb{R}$, the derivative of the function $g : A \to \mathbb{R}$ defined by $g(x) = x^{f(x)} = \exp(f(x)\log(x)) \forall x \in A$ is given by

$$g'(x) = x^{f(x)}\left(f'(x)\log(x) + \frac{f(x)}{x}\right) \forall x \in A.$$

In that case

$$\begin{aligned}&h'(\alpha) = \alpha^{\frac{-1}{\alpha-1}} + \\ &(\alpha - 1)\alpha^{\frac{-1}{\alpha-1}}\left(\tfrac{-(\alpha-1)+\alpha}{(\alpha-1)^2}\log(\alpha) - \tfrac{1}{\alpha-1}\right) = \\ &\alpha^{\frac{-1}{\alpha-1}}\left(1 + \tfrac{\log(\alpha)}{\alpha-1} - 1\right) = \\ &\qquad \alpha^{\frac{-1}{\alpha-1}}\tfrac{\log(\alpha)}{\alpha-1} \geqslant 0 \forall \alpha \in]1, \infty[\end{aligned}$$

So h is increasing and $\sigma_\alpha \in]0, 1[$ for any value of α.

For $m = 100$ we show the results of the calculations for most standard measures for the weights in Table 2 (in this case, the weights associated to the arithmetic mean operator are given by $\alpha = 1$). In Fig. 5 we show the comparison between values of the weights obtained for the different values of α. In order to analyze the behavior of the entropy measure for this EVR-OWA operator, we provide the plot in Fig. 6. For each fuzzy linguistic quantifier, namely the identity function (arithmetic mean), $\hat{m}_{1.5}, \hat{m}_2, \hat{m}_3$ and $Q_{0.2,0.8}$, the values of the entropy measure of the corresponding OWA operator have been computed for $m = 2, 3, \ldots, 1000$. The graph shows that the OWA operators constructed from these EVRs present higher entropy that the linear RIM quantifier $Q_{0.3,0.7}$.

Table 2
Measures for $\hat{m}_\alpha$, $m = 100$.

EVR	Orness	Entropy	Mean	SD	Min	Max
$\hat{m}_1$	0.5	4.605	0.01	0	0.01	0.01
$\hat{m}_{1.35}$	0.5	4.5581	0.01	0.0034	0.0074	0.0275
$\hat{m}_2$	0.5	4.3421	0.01	0.0098	0.0050	0.0707
$\hat{m}_{3.39}$	0.5	3.8057	0.01	0.0219	0.0029	0.1576
$\hat{m}_5$	0.5	3.3090	0.01	0.0317	0.0020	0.2286
$\hat{m}_{10}$	0.5	2.4379	0.01	0.0470	0.0010	0.3381

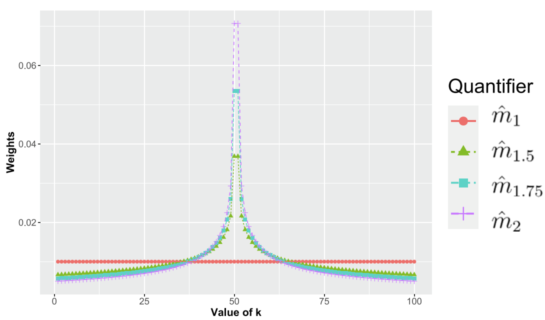

Fig. 5. Comparison of w_k for different $\hat{m}_\alpha, m = 100$.

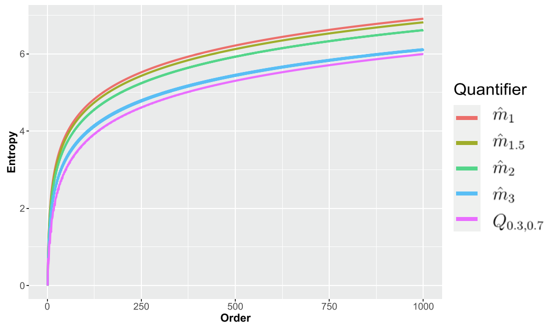

Fig. 6. Graph of the entropy function for several fuzzy linguistic quantifiers.

5.3. The EVR-OWA associated to $\hat{b}_\alpha^{r,s}$

Let us consider now the automorphism $f_\alpha : [0,1] \to [0,1]$ given by

$$f_\alpha(x) = x^\alpha \forall x \in [0,1],$$

where $0 < \alpha < 1$ and let $h : [\alpha, \beta] \to [a, b]$ be the standard affine transformation given by

$$h(x) = \left(\frac{b-a}{\beta-\alpha}\right) \cdot (x-\alpha) + a, \forall x \in [\alpha, \beta].$$

By using similar arguments that the ones applied in [4] the following result is clear.

Proposition 5. Consider $r, s \in]\frac{1}{2}, 1[$ such that $r < s, \alpha \in]0,1[$ and $\epsilon \in]0,1[$ and the standard affine transformations:

$$\begin{aligned} h_a &: \left[\tfrac{1}{2}, r\right] \to \left[\tfrac{1}{2}, s\right] \\ h_b &: [r, 1] \to [\epsilon, 1] \\ h_c &: [\epsilon^\alpha, 1] \to [s, 1]. \end{aligned}$$

Then the function $\hat{b}_\alpha^{r,s} : [0,1] \to [0,1]$ given by

$$\hat{b}_\alpha^{r,s}(x) = \begin{cases} 1 - b_*(1-x) & 0 \leqslant x < \frac{1}{2} \\ b_*(x) & \frac{1}{2} \leqslant x \leqslant 1 \end{cases}. \tag{2}$$

where $b_* : \left[\frac{1}{2}, 1\right] \to \left[\frac{1}{2}, 1\right]$ is defined by

$$b_*(x) = \begin{cases} h_a(x) & \frac{1}{2} \leqslant x \leqslant r \\ h_c \circ f_\alpha \circ h_b(x) & r < x \leqslant 1 \end{cases}$$

is an EVR if and only if the following equality holds

$$\lambda = \frac{s - \frac{1}{2}}{r - \frac{1}{2}} = \frac{1-s}{1-r} \frac{1-\epsilon}{1-\epsilon^\alpha} \alpha \epsilon^{\alpha-1},$$

where λ is the derivative of h_a and indicates how much the intermediate values further apart.

Some useful combinations of these parameters are shown in Table 3.

Let us analyse the special case $r = s = 1/2$. Note that this assumption implies that the affine transformation h_a disappears

Table 3
Useful combinations of parameters for the EVR $\hat{b}_\alpha^{r,s}$.

r	s	λ	α	ϵ
0.6	0.75	2.5	$\frac{1}{3}$	0.043034
0.5	0.5	2	$\frac{1}{2}$	0.111177
0.55	0.6	2	$\frac{1}{2}$	0.081757

and $b_1(x) = h_c \circ f_\alpha \circ h_b(x) \forall x \in \left[\frac{1}{2}, 1\right]$.

Define $g :]0,1[\to \mathbb{R}$ by

$$g(\epsilon) = \alpha \frac{\epsilon^{\alpha-1} - \epsilon^\alpha}{1 - \epsilon^\alpha} \forall \epsilon \in]0,1[$$

Note that $\lim_{\epsilon \to 0} g(\epsilon) = +\infty$ and

$$\lim_{\epsilon \to 1} g(\epsilon) = \lim_{\epsilon \to 1} \alpha \epsilon^{\alpha-1} \frac{1-\epsilon}{1-\epsilon^\alpha} = 1$$

In addition

$$\begin{aligned} g'(\epsilon) &= \tfrac{\alpha}{(1-\epsilon^\alpha)^2}\left(\left((\alpha-1)\epsilon^{\alpha-2} - \alpha\epsilon^{\alpha-1}\right)(1-\epsilon^\alpha) + \left(\epsilon^{\alpha-1} - \epsilon^\alpha\right)\alpha\epsilon^{\alpha-1}\right) = \\ &= \tfrac{\alpha\epsilon^{\alpha-2}}{(1-\epsilon^\alpha)^2}\left((\alpha - 1 - \alpha\epsilon)(1-\epsilon^\alpha) + \alpha\left(\epsilon^\alpha - \epsilon^{\alpha+1}\right)\right) = \\ &= \tfrac{\alpha\epsilon^{\alpha-2}}{(1-\epsilon^\alpha)^2}(\alpha(1-\epsilon)(1-\epsilon^\alpha) - (1-\epsilon^\alpha) + \alpha\epsilon^\alpha(1-\epsilon)) = \\ &= \tfrac{\alpha\epsilon^{\alpha-2}}{(1-\epsilon^\alpha)^2}(\alpha(1-\epsilon) - (1-\epsilon^\alpha)) \\ &\qquad \forall \epsilon \in]0,1[\end{aligned}$$

To study the sign of g' let us consider $h :]0,1[\to \mathbb{R}$ defined by

$$h(\epsilon) = \alpha(1-\epsilon) - (1-\epsilon^\alpha) \forall \epsilon \in]0,1[.$$

Since $h'(\epsilon) = \alpha(\epsilon^{\alpha-1} - 1) > 0 \forall \epsilon \in]0,1[$ and $\lim_{\epsilon \to 0} h(\epsilon) = \alpha - 1, \lim_{\epsilon \to 1} h(\epsilon) = 0$ we can conclude that $h < 0$ in its domain and therefore $g'(\epsilon) < 0 \forall \epsilon \in]0,1[$ In that case g is strictly decreasing and its codomain is $]1, +\infty[$. This fact allows to state the following result.

Proposition 6. Let $\epsilon \in]0,1[, \alpha \in]0,1[$ and consider the standard affine transformations $h_b : \left[\frac{1}{2}, 1\right] \to [\epsilon, 1]$ and $h_c : [\epsilon^\alpha, 1] \to \left[\frac{1}{2}, 1\right]$ and the function $\hat{b}_1 : \left[\frac{1}{2}, 1\right] \to \left[\frac{1}{2}, 1\right]$ defined by $\hat{b}_*(x) = h_c \circ f_\alpha \circ h_b(x) \forall x \in \left[\frac{1}{2}, 1\right]$.

Then, for every $\lambda \in]1, \infty[$ we can find $\epsilon \in]0,1[$ (i.e. the unique one which satisfies $\lambda = \alpha \frac{\epsilon^{\alpha-1} - \epsilon^\alpha}{1-\epsilon^\alpha}$) such that the function $\hat{b}^\lambda_\alpha : [0,1] \to [0,1]$ defined by

$$\hat{b}^\lambda_\alpha(x) = \begin{cases} \hat{b}_*(x) & \frac{1}{2} \leqslant x \leqslant 1 \\ 1 - \hat{b}_*(1-x) & 0 \leqslant x \leqslant \frac{1}{2} \end{cases},$$

is an EVR such that $\left(\hat{b}^\lambda_\alpha\right)'\left(\frac{1}{2}\right) = \lambda$.

Note that by taking $\epsilon = 0$ we would obtain the $\hat{m}_\alpha$ family of EVRs.

It should be highlighted that the previous reasoning allows to deduce the existence and unicity of a value ϵ which guarantees that the EVRs $b^{r,s}_\alpha$ is well defined for $\frac{1}{2} < r < s < 1$ and $\alpha \in]0,1[$.

Proposition 7. Consider $r, s \in]\frac{1}{2}, 1[$ such that $r < s, \alpha \in]0,1[$ and $\epsilon \in]0,1[$. Then there exist a unique $\epsilon \in [0,1]$ such that

$$\frac{s - \frac{1}{2}}{r - \frac{1}{2}} = \frac{1-s}{1-r} \frac{1-\epsilon}{1-\epsilon^\alpha} \alpha \epsilon^{\alpha-1}.$$

In the following, we use the notation $b^{r,s}_\alpha$ for the unique EVR which could be defined for the parameters $r < s$ and α.

The quality measures for the OWA weights generated for some of these EVRs for the order 100 case are sumarised in Table 4 and the respectively obtained weights are sketched in Fig. 7. Note that when using the family $\hat{b}^{r,s}_\alpha$ all the weights in the interval $[0.5 - r, 0.5 + r]$ remain the same and their value depend on the value of $\frac{s-0.5}{r-0.5}$. On the other hand, the family

Table 4
Measures for $\hat{b}^{r,s}_\alpha$, $m = 100$.

EVR	*Orness*	*Entropy*	*Mean*	*SD*	*Min*	*Max*
$\hat{b}^1_1$	0.5	4.605	0.01	0	0.01	0.01
$\hat{b}^{0.55,0.6}_{1/2}$	0.5	4.515	0.01	0.004	0.0057	0.020
$\hat{b}^{0.55,0.6}_{1/3}$	0.5	4.505	0.01	0.005	0.0052	0.020
$\hat{b}^2_{1/2}$	0.5	4.560	0.01	0.003	0.006	0.019

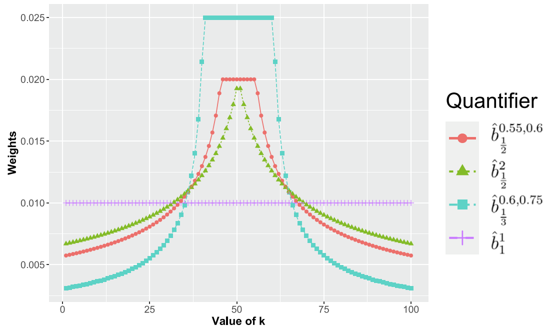

Fig. 7. Comparison of w_k for different $\hat{b}_{\alpha}^{r,s}, m = 100$.

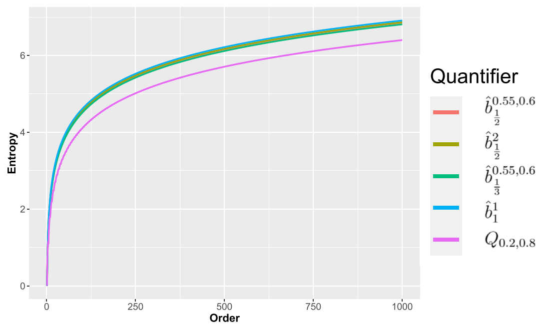

Fig. 8. Graph of the entropy function for several fuzzy linguistic quantifiers.

$\hat{b}_{\alpha}^{\lambda}$ allows to control the weight of the median value by modifying the value of λ. The comparative graph of the entropies for the obtained EVR weights is shown in Fig. 8.

5.4. Comparison of the families $\hat{s}_{\alpha}, \hat{m}_{\alpha}$ and $\hat{b}_{\alpha}^{r,s}$

Even though, there exist plenty of examples of EVRs, such as the cumulative distribution function for the Gaussian distribution [17], all the EVR families which have been proposed in this section are parametric families which allow to control the relevance of the intermediate values in the aggregations by adjusting the value of the parameters. The family $\hat{s}_{\alpha}$ consists of functions of class $\mathscr{C}^{\infty}$ which always assign the same weight to the values corresponding to the second and third quartiles. On the other hand, the family m_{α} resulted to be a particular case of the family $\hat{b}_{\alpha}^{r,s}$ and provided a simple way of generating OWA weights which give much more importance to the intermediate values than to the extreme ones. Finally, the family $\hat{b}_{\alpha}$ allows to control the amount of intermediate values which receive the higher relevance in the aggregations and the exact weights for these values.

6. Symmetric weights and EVRs

In Section 4 it was proved that any EVR is able to produce a family of symmetric weights which give more importance to the intermediate values in the OWA aggregations. This section is devoted to show the reciprocal statement, i.e., to some extent, any family of symmetric weights which prioritizes intermediate values is obtained from an EVR function. First, it is proposed a result which highlights an interesting property relating the weights associated to a fuzzy linguistic quantifier Q and its derivative.

Proposition 8. Let $Q : [0, 1] \rightarrow [0, 1]$ be a RIM quantifier of class $\mathscr{C}^1$. Then:

$$w_k = Q\left(\frac{k}{m}\right) - Q\left(\frac{k-1}{m}\right) = Q'(\xi_k)\frac{1}{m} = \int_{\frac{k-1}{m}}^{\frac{k}{m}} Q'(t)dt, k = 1,2,\dots,m$$

where $\xi_k \in]\frac{k-1}{m}, \frac{k}{m}[$. In addition,

$$|w_k - \overline{w}| = |Q'(\xi_k)\frac{1}{m} - \frac{1}{m}| = \frac{1}{m}|1 - Q'(\xi_k)|, \xi_k \in]\frac{k-1}{m}, \frac{k}{m}[$$

for any $k = 1,2,\dots,m$.

Proof. It is consequence of the Mean Value Theorem and the Fundamental Theorem of Calculus.

Remark 3. For m large enough $w_k \approx Q'\left(\frac{k}{m}\right)\frac{1}{m}$.

Remark 4. The previous proposition allows to measure how far the weights produced by a RIM quantifier are from their mean value. For instance, when the EVR $\hat{s}_\alpha, \alpha \in]0, \frac{1}{2\pi}[$ is considered, we obtain

$$|w_k - \overline{w}| = \frac{1}{m}2\pi\alpha|\cos(2\pi\xi_k - \pi)| \leqslant \frac{2\pi\alpha}{m}.$$

The following Theorem provides the reciprocal statement, i.e., given $w_1,\dots,w_m$, under certain conditions Theorem 2 assures that we can find an EVR such that, when Yager's method is applied to compute the associated OWA weights, the obtained values for these weights are precisely $w_1,\dots,w_m$.

Theorem 2. *Given a positive symmetric* $(w_k = w_{m-k+1} > 0, k = 1,2,\dots,m)$ *weighting vector* $w = (w_1, w_2, \dots, w_m)$ *consider a continuous function* $q : [0,1] \to \mathbb{R}_0^+$ *such that:*

1. *For any* $k = 1,2,\dots,m$

$$w_k = \int_{\frac{k-1}{m}}^{\frac{k}{m}} q(t)dt,$$

2. *q satisfies the symmetry condition* $q(x) = q(1-x) \forall x \in [0,1]$,
3. $q(1) < 1$ *and* $q(0) < 1$,
4. *q is strictly increasing in* $\left[0, \frac{1}{2}\right]$ *and strictly decreasing in* $\left[\frac{1}{2}, 1\right]$.

Then, the function $Q : [0,1] \to [0,1]$ *given by* $Q(x) := \int_0^x q(t)dt$ *is an EVR whose associated EVR-OWA operator is determined by the weighting vector w.*

Proof. Let us check the properties which characterizes EVRs.

1. Q is a function of class $\mathscr{C}^1$. Clear by using the Fundamental Theorem of Calculus.
2. Q is an increasing automorphism. Since $q \geqslant 0, Q$ must be an increasing function. The 4th hypothesis guarantees that in fact, Q is strictly increasing. In addition,

$$0 = \int_0^0 q(t)dt = Q(0)$$
$$1 = \sum_{k=1}^m w_k = \sum_{k=1}^m \int_{\frac{k-1}{m}}^{\frac{k}{m}} q(t)dt = \int_0^1 q(t)dt = Q(1).$$

3. Involution with respect to the standard negation. Note that

$$Q(x) + Q(1-x) = \int_0^x q(t)dt + \int_0^{1-x} q(t)dt =$$
$$= \int_0^x q(t)dt + \int_{1-x}^1 q(1-t)dt = \int_0^1 q(t)dt = 1,$$

4. $Q'(0) = q(0) < 1$ and $Q'(1) = q(1) < 1$,
5. The convexity of Q in $\left[0, \frac{1}{2}\right]$ is consequence of the fact that q is increasing in that interval. A similar argument can be provided to show that Q is concave in $\left[\frac{1}{2}, 1\right]$

By using the first hypothesis and the Fundamental Theorem of Calculus it is obtained that

$$Q\left(\frac{k}{m}\right) - Q\left(\frac{k-1}{m}\right) = \int_{\frac{k-1}{m}}^{\frac{k}{m}} q(t)dt = w_k.$$

Therefore, when using Yager's method to compute the weights for the RIM quantifier Q, the obtained family is the former one.

Let us analyze the hypotheses of the previous result. The first one is just about relating the derivative of the EVR Q with the given weights such that when using Proposition 8 to compute the corresponding weights, the obtained values are $w_1, \dots, w_m$. The second one is a symmetry condition. The third condition is easy to obtain because the minimum values for a family of weights which prioritizes intermediate values must be lower than the arithmetic mean of these weights, i.e. $w_1 < \frac{1}{m}$ and $w_m < \frac{1}{m}$. The last hypothesis is related with both the symmetry condition and the fact that the weights for the intermediate values should be higher than the weights for the more extreme values.

In order to construct the function $q : [0,1] \to \mathbb{R}_0^+$ required in the previous result, we suggest using a continuous linear spline such that its restriction to the interval $\left[\frac{k-1}{m}, \frac{k}{m}\right]$, $q_k := q|_{\left[\frac{k-1}{m}, \frac{k}{m}\right]}$ $k = 1, 2, \dots, m$, satisfies

$$w_k = \int_{\frac{k-1}{m}}^{\frac{k}{m}} q_k(t)dt, k = 1, 2, \dots, m.$$

If m is even we can consider the functions defined by $q_k(t) = \alpha_k + \beta_k t \forall t \in \left[\frac{k-1}{m}, \frac{k}{m}\right], k = 1, 2, \dots, \frac{m}{2}$, which provide a total of m parameters. By considering the equations

$$w_k = \int_{\frac{k-1}{m}}^{\frac{k}{m}} q_k(t)dt = \frac{\alpha_k}{m} + \frac{\beta_k(2k-1)}{2m^2}, k = 1, 2, \dots, \frac{m}{2},$$

the boundary condition $q_1(0) = \alpha_1 = 0$, and the continuity conditions

$$q_k\left(\frac{k}{m}\right) = \alpha_k + \frac{k}{m}\beta_k = \alpha_{k+1} + \beta_{k+1}\frac{k}{m} = q_{k+1}\left(\frac{k}{m}\right), k = 1, 2, \dots \frac{m}{2} - 1$$

We obtain a total of m linear constraints. The matrix representation of this linear system is

$$\begin{pmatrix}
1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
1 & -1 & 0 & 0 & \cdots & 0 & \frac{1}{m} & \frac{-1}{m} & 0 & 0 & \cdots & 0 \\
0 & 1 & -1 & 0 & \cdots & 0 & 0 & \frac{2}{m} & \frac{-2}{m} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & -1 & 0 & 0 & \cdots & 0 & \frac{m-2}{2m} & -\frac{m-2}{2m} \\
1 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{2m} & 0 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & 0 & \cdots & 0 & 0 & \frac{3}{2m} & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & 0 & \cdots & \frac{m-1}{2m}
\end{pmatrix}
\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{\frac{m}{2}} \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{\frac{m}{2}} \end{pmatrix}
=
\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ mw_1 \\ mw_2 \\ \vdots \\ mw_{\frac{m}{2}} \end{pmatrix}$$

The determinant of this $m \times m$ matrix is given by

$$\begin{vmatrix}
1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & -1 & 0 & 0 & \cdots & 0 & \frac{1}{m} & \frac{-1}{m} & 0 & 0 & \cdots & 0 \\
0 & 0 & -1 & 0 & \cdots & 0 & \frac{1}{m} & \frac{1}{m} & \frac{-2}{m} & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & -1 & \frac{1}{m} & \frac{1}{m} & \frac{1}{m} & \cdots & \frac{1}{m} & \frac{-(m-2)}{2m} \\
0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{2m} & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & \frac{1}{2m} & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{2m}
\end{vmatrix}$$

which is non zero. Therefore the linear system of equations has a unique solution. By using this solution to determine the functions $q_1, q_2, \dots, q_{\frac{m}{2}}$ a piecewise linear function $\acute{q} : \left[0, \frac{1}{2}\right] \to \mathbb{R}_0^+$ is obtained. This piecewise function can be extended to the entire interval $[0,1]$ by defining $\grave{q} : \left[\frac{1}{2}, 1\right] \to \mathbb{R}_0^+$ by $\grave{q}(t) = \acute{q}(1-t) \forall t \in \left[\frac{1}{2}, 1\right]$.

If m is odd, a similar reasoning can be developed by considering the system of equations consisting on

$$w_k = \int_{\frac{k-1}{m}}^{\frac{k}{m}} q_k(t)dt = \frac{\alpha_k}{m} + \frac{\beta_k(2k-1)}{2m^2}, k = 1, 2, \dots, \frac{m-1}{2},$$
$$\frac{w_{\frac{m+1}{2}}}{2} = \int_{\frac{m-1}{2m}}^{\frac{1}{2}} q_{\frac{m+1}{2}}(t)dt = \frac{\alpha_{\frac{m+1}{2}}}{2m} + \frac{\beta_{\frac{m+1}{2}}(2m-1)}{8m^2},$$

the boundary condition $q_1(0) = \alpha_1 = 0$, and the continuity conditions

$$q_k\left(\frac{k}{m}\right) = \alpha_k + \frac{k}{m}\beta_k = \alpha_{k+1} + \beta_{k+1}\frac{k}{m} = q_{k+1}\left(\frac{k}{m}\right), k = 1, 2, \dots \frac{m-1}{2}$$

which consists of $m+1$ parameters and $m+1$ linear constraints. The matrix representation of this linear system is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 1 & -1 & 0 & 0 & \cdots & 0 & \frac{1}{m} & \frac{-1}{m} & 0 & 0 & \cdots & 0 \\ 0 & 1 & -1 & 0 & \cdots & 0 & 0 & \frac{2}{m} & \frac{-2}{m} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 & 0 & 0 & \cdots & 0 & \frac{m-1}{2m} & -\frac{m-1}{2m} \\ 1 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{2m} & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & \frac{3}{2m} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 & \frac{m-2}{2m} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & 0 & \cdots & \frac{2m-1}{4m} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{\frac{m-1}{2}} \\ \alpha_{\frac{m+1}{2}} \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{\frac{m-1}{2}} \\ \beta_{\frac{m+1}{2}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ mw_1 \\ mw_2 \\ \vdots \\ mw_{\frac{m-1}{2}} \\ mw_{\frac{m+1}{2}} \end{pmatrix}$$

The determinant of this $m \times m$ matrix is given by

$$\begin{vmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & 0 & \cdots & 0 & \frac{1}{m} & \frac{-1}{m} & 0 & 0 & \cdots & 0 \\ 0 & 0 & -1 & 0 & \cdots & 0 & \frac{1}{m} & \frac{1}{m} & \frac{-2}{m} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & \frac{1}{m} & \frac{1}{m} & \frac{1}{m} & \cdots & \frac{1}{m} & \frac{-(m-1)}{2m} \\ 0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{2m} & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \frac{1}{2m} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \frac{1}{2m} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{4m} \end{vmatrix}$$

which is non zero too.

To summarise, every family of symmetric positive weights for OWA operators which prioritize the intermediate values in the aggregations is the family of weights obtained by using Yager's method with a certain EVR function.

7. An illustrative case of aggregation

In this section an illustrative example is provided in order to show in a practical environment the theoretical contents of this work.

Suppose a group consisting on 10 experts who give their opinions on how much they prefer a certain alternative x_1 to the alternative x_2. These preferences are denoted as $p_k := p^k_{1,2} \in [0,1], k = 1, 2, \dots, 10$ and are given as follows:

$$p_1 = 0.1118857 p_2 = 0.2152431 \; p_3 = 0.7635773 \; p_4 = 0.8914762$$
$$p_5 = 0.3605274 p_6 = 0.8650980 \; p_7 = 0.6478957 \; p_8 = 0.2588964$$
$$p_9 = 0.1553312 \; p_{10} = 0.1962114$$

For $m = 10$ and the EVR $\hat{s}_{0.08}$ the obtained list of weights is:

$$w_1 = 0.05297718 \; w_2 = 0.07093830 \; w_3 = 0.10000000 \; w_4 = 0.12906170$$
$$w_5 = 0.14702282 \; w_6 = 0.14702282 \; w_7 = 0.12906170 \; w_8 = 0.10000000$$
$$w_9 = 0.07093830 \; w_{10} = 0.05297718$$

and the collective preference by using these weights is given by

Table 5
Measures for several fuzzy linguistic quantifiers, $m = 10$.

EVR	*Orness*	*Entropy*	*Mean*	*SD*	*Min*	*Max*	Collective Preference
$\hat{s}_0 = Q_{0,1}$	0.5	2.302585	0.1	0	0.1	0.1	0.4466142
$\hat{s}_{0.04}$	0.5	2.287187	0.1	0.01748064	0.07648859	0.1235114	0.4353018
$\hat{s}_{0.08}$	0.5	2.239431	0.1	0.03496128	0.05297718	0.1470228	0.4239893
$\hat{s}_{0.15}$	0.5	2.052446	0.1	0.0655524	0.01183221	0.1881678	0.4041924
$Q_{0.1,0.9}$	0.5	2.079442	0.1	0.05	0	0.125	0.4328476
$Q_{0.2,0.8}$	0.5	1.791759	0.1	0.08164966	0	0.1666667	0.4070586
$Q_{0.3,0.7}$	0.5	1.386294	0.1	0.1224745	0	0.25	0.3706407

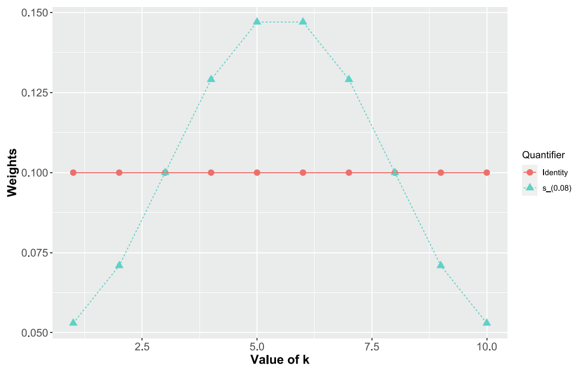

Fig. 9. Weights generated by the EVR $\hat{s}_{0.08}$ when $m = 10$.

$$p_c = < (p_1, p_2, \ldots, p_{10}) \searrow, (w_1, w_2, \ldots, w_{10}) >= 0.4239893.$$

Table 5 compiles the standard measures for the weighting vectors obtained for different fuzzy linguistic quantifiers. Fig. 9 shows the distribution of the weighting vector of order 10 associated with the EVR $\hat{s}_{0.08}$ around its mean value.

Table 5 shows that the OWA operator associated with the EVR $\hat{s}_\alpha$ provides, in general, values for the entropy measure which are greater than the obtained for the OWA operator associated with the linear RIM quantifier $Q_{\alpha,\beta}$. Consequently, the aggregations made by using the EVR-OWA operator take into account more information than the aggregations based on the linear RIM quantifiers $Q_{0.2,0.8}$ or $Q_{0.3,0.7}$, which are commonly used in the literature. In addition, the Standard Deviations obtained for the EVR-OWA operators are lower than the obtained for the linear RIM quantifiers $Q_{0.2,0.8}$ and $Q_{0.3,0.7}$. Furthermore, whereas the weights associated with linear RIM quantifiers just take the minimum value (0) or the maximum value ($0.125, 0.17$ or 0.25, depending on the case), the weights which are computed by using the EVR $\hat{s}_\alpha$ take different values in the interval defined by the respective minimum and maximum.

Note that the median for the preferences is 0.3097119. When using the linear RIM quantifier $Q_{\alpha,\beta}$, a reduction in the quantity $\beta - \alpha$ is translated in a value for the collective preference which is closer to the median preference and farther from the collective preference given by the arithmetic mean operator. However, when using the EVR-OWA operator associated with the EVR $\hat{s}_\alpha$, changes in the parameter α allows to control the importance given to the more extreme values without losing too much information in the aggregation process (higher entropy measure) and thus obtaining values for the collective preference similar to the obtained with the arithmetic mean operator.

8. Results and discussion

Currently, RIM quantifiers are widely used in order to compute the weights for OWA operators due to their simplicity and applications in several contexts such as poset environments [7]. Even though, several families of RIM quantifiers have been proposed in the specialized literature [17], one of the most extended approaches consists of using linear RIM quantifiers $Q_{\alpha,\beta} : [0,1] \to [0,1]$, where $0 \leqslant \alpha < \beta \leqslant 1$, to define OWA operators.

First, this study has analyzed the limitations of using these linear RIM quantifiers to generate OWA weights (RQ1). To do so, the impact of the parameters α and β which define this quantifier has been measured by checking out the corresponding standard quality measures. The obtained results are:

- If α is too high or β is too low the aggregations are non realistic because too many extreme values are ignored, i.e. the entropy of the obtained OWA operator is too low.

- If $\alpha + \beta \neq 1$ the aggregations are biased, i.e., the orness measure of the obtained operator is not 0.5, and therefore they are not suitable for those real world applications which require symmetry around the median value.
- With just two possible exceptions, the value w_k is either zero or the constant $\frac{1}{m(\beta-\alpha)}$. Therefore, the progressive behavior of the weighting vector calculated by using the EVR approach is more related to the fuzzy environment we are dealing with: whereas the classical linear RIM quantifier $Q_{\alpha,\beta}$ completely ignores the most extreme elements, the use of EVR based RIM quantifiers that is proposed here provides a progressive reduction of the importance we are giving to the more extreme values, which is closer to the *fuzzy philosophy*.

Second, the EVR-OWA operator, which is focused on overcoming the limitations of the OWA operator, has been introduced. Several families of EVRs have been proposed and it has been proved that their corresponding EVR-OWA operators satisfy:

- Symmetry: $w_1 = w_m, w_2 = w_{m-1}, w_3 = w_{m-2}, \ldots,$
- No w_k is zero,
- The smallest values of the w_k's are the *first* and, by symmetry, the *last* ones. The largest values are the intermediate ones,
- Their arithmetic mean is $1/m$,
- Their standard deviation is close to zero,
- Their orness measure is 0.5,
- Their entropy is similar to the entropy of the arithmetic mean operator, which is the maximum possible value.

Therefore, OWA operators based in EVRs show a considerably higher performance than those based on linear RIM quantifiers, but keeping their simplicity and applicability (RQ2).

Finally, it has been proved (Theorem 2) that any family of positive symmetric weights which prioritize the intermediate information is actually the family of OWA weights obtained from a certain EVR (RQ3).

To sum up, this study has highlighted the main disadvantages of the use of linear RIM quantifiers in OWA aggregations, concluding that these kinds of weights do not show a high quality when non biased aggregations which prioritize intermediate information without ignoring the most extreme values are required. The EVR-OWA operator, which is based on using EVRs as fuzzy linguistic quantifiers, allows to overcome this disadvantages by providing families of positive symmetric weights which prioritize the intermediate information. Furthermore, it has been proved that any other family of weights which satisfies these requirements can be obtained from a certain EVR, which guarantees that EVRs are actually a very complete proposal to generate weights with such properties.

9. Conclusions and future research

Linear RIM quantifiers are widely accepted in specialized literature [7,17], but they present several shortcomings. The novel EVR-OWA operator proposes using EVRs as fuzzy linguistic quantifiers in order to overcome these limitations by obtaining an OWA operator which takes into account the more extreme values, but gives more importance to the intermediate ones.

The aggregations made by EVR-OWA operators are not only more related with the fuzzy logic view than those done by using the linear RIM quantifier $Q_{\alpha,\beta}$, but are also better for certain real world applications such as consensus models for GDM [6,9], since they aggregate preferences in a non biased way and allow to take into account more information in the aggregation process.

In addition, the abstract nature of this proposal not only provides a simple and general method to obtain OWA weights with such properties, but also gives a characterization relating those families of symmetric positive OWA weights which prioritize intermediate values and EVRs: for every weighting vector w with these properties it can be found an EVR function such that the weighing vector for this EVR is the weighting vector w.

Further studies should be related with the behavior of these weights in large scale GDM problems and consensus models in which preferences tends to be polarized. In addition, it would be interesting to compute the optimal parameters for the proposed EVR families which maximize the entropy measure of the aggregations under certain constrains. Other research line could be to extend the proposed methods to other contexts such as Pythagorean fuzzy uncertain environments [16] or ELICIT information [8].

CRediT authorship contribution statement

Diego García-Zamora: Validation, Formal analysis, Writing - original draft, Writing - review & editing, Visualization. **Álvaro Labella:** Conceptualization, Methodology, Validation, Formal analysis, Writing - original draft, Writing - review & editing. **Rosa M. Rodríguez:** Conceptualization, Methodology, Supervision, Writing - original draft, Writing - review & editing. **Luis Martínez:** Conceptualization, Methodology, Formal analysis, Validation, Writing - original draft, Writing - review & editing, Supervision, Project administration, Funding acquisition.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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References

[1] G. Beliakov, A. Pradera, T. Calvo, Aggregation Functions: A Guide for Practitioners, Springer, 2007.
[2] G. Beliakov, H. Bustince, T. Calvo, A practical guide to averaging functions, 01, Springer, 2016.
[3] H. Bustince, E. Barrenechea, M. Pagola, Relationship between restricted dissimilarity functions, restricted equivalence functions and normal en-functions: Image thresholding invariant, Pattern Recogn. Lett. 29 (4) (2008) 525–536.
[4] D. García-Zamora, Á. Labella, R.M. Rodríguez, and L. Martínez, Non Linear Preferences in Group Decision Making. Extreme Values Amplifications and Extreme Values Reductions, Int. J. Intell. Syst., 2021. [Online]. Available: doi: 10.1002/int.22561..
[5] M. Grabisch, S.A. Orlovski, R. Yager, Fuzzy Aggregation of Numerical Preferences. Boston, MA: Springer, US, 1998, pp. 31–68..
[6] E. Herrera-Viedma, F. Herrera, and F. Chiclana, A consensus model for multiperson decision making with different preference structures, Syst., Man Cybern., Part A: Syst. Humans, IEEE Trans. 32 (2002) 394–402..
[7] L. Jin, R. Mesiar, R. Yager, Ordered weighted averaging aggregation on convex poset, IEEE Trans. Fuzzy Syst. 27 (3) (2019) 612–617.
[8] Á. Labella, R.M. Rodríguez, L. Martínez, Computing with comparative linguistic expressions and symbolic translation for decision making: Elicit information, IEEE Trans. Fuzzy Syst. 28 (10) (2020) 2510–2522.
[9] I. Palomares, F.J. Estrella, L. Martínez, F. Herrera, Consensus under a fuzzy context: Taxonomy, analysis framework afryca and experimental case of study, Inform. Fusion 20 (2014) 252–271.
[10] I. Palomares, J. Liu, Y. Xu, L. Martínez, Modelling experts' attitudes in group decision making, Soft Comput., 16 (2012) 1755–1766..
[11] X. Pu, Z. Wang, R. Mesiar, R. Yager, J. Qin, X. Feng, L. Jin, Extended and infinite ordered weighted averaging and sum operators with numerical examples, Iran. J. Fuzzy Syst. 17 (3) (2020) 33–41.
[12] M. Reformat, R. Yager, Z. Li, N. Alajlan, Human-inspired identification of high-level concepts using owa and linguistic quantifiers, Int. J. Comput. Commun. Control 6 (2011) 473–502.
[13] M. Roubens, Fuzzy sets and decision analysis, Fuzzy Sets Syst., 90(2) (1997) 199–206. fuzzy Sets: Where Do We Stand? Where Do We Go?..
[14] A. Sı̂rbu, V. Loreto, V. Servedio, Opinion dynamics with disagreement and modulated information, J. Stat. Phys. 151 (2013) 218–237.
[15] A. Sı̂rbu, V. Loreto, V. Servedio, F. Tria, Cohesion, consensus and extreme inormation in opinion dynamics, Adv. Complex Syst. 16 (06) (2013) 1350035.
[16] L. Wang, H. Garg, Algorithm for multiple attribute decision-making with interactive archimedean norm operations under pythagorean fuzzy uncertainty, Int. J. Comput. Intell. Syst. 14 (2020) 503–527.
[17] Z. Xu, An overview of methods for determining OWA weights, Int. J. Intell. Syst. 20 (8) (2005) 843–865.
[18] R. Yager, On ordered weighted averaging aggregation operators in multicriteria decisionmaking, IEEE Trans. Syst., Man, Cybern. 18 (1) (1988) 183–190.
[19] R. Yager, Connectives and quantifiers in fuzzy sets, Fuzzy Sets Syst. 40 (1) (1991) 39–75.
[20] R. Yager, Families of OWA operators, Fuzzy Sets Syst. 59 (2) (1993) 125–148.
[21] R. Yager, Quantifier guided aggregation using OWA operators, Int. J. Intell. Syst. 11 (1) (1996) 49–73.
[22] L. Zadeh, A computational approach to fuzzy quantifiers in natural languages, Comput. Math. Appl., 9 (1983) 149–184..

4.4 DISTANCIA Y GRADO DE CONSENSO EN MODELOS CCM INTEGRALES

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Decision Support

Relationship between the distance consensus and the consensus degree in comprehensive minimum cost consensus models: A polytope-based analysis

Diego García-Zamora[a,*], Bapi Dutta[a], Sebastia Massanet[b,c], Juan Vicente Riera[b,c], Luis Martínez[a]

[a] Department of Computer Science, University of Jaén, Jaén, Spain
[b] Department of Mathematics and Computer Science, University of the Balearic Islands, Palma, Spain
[c] Health Research Institute of the Balearic Islands (IdISBa), Palma, Spain

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ABSTRACT

Agreement in Group Decision-Making problems has recently been tackled through the use of Minimum Cost Consensus (MCC) models, which are associated with solving convex optimization problems. Such models minimize the cost of changing experts' preferences towards reaching a mutual consensus, and establish that the distance between the modified individual preferences and the collective opinion must be bounded by the threshold $\varepsilon > 0$. A recent MCC-based model, called the Comprehensive Minimum Cost Consensus (CMCC) model, adds another constraint related to a parameter $\gamma \in [0, 1]$ to the above constraint related to the parameter ε to enforce modified expert preferences in order to achieve a minimum level of agreement dictated by the consensus threshold $1 - \gamma \in [0, 1]$. This paper attempts to analyze the relationship between the aforementioned constraints in the CMCC models from two different perspectives. The first is based on inequalities and allows simple bounds to be determined to relate the parameters ε and γ. The second one is based on Convex Polytope Theory and provides algorithms that compute more precise bounds to relate these parameters, and could also be applied to other similar optimization problems. Finally, several examples are provided to illustrate the proposal.

1. Introduction

Group Decision-Making (GDM) problems are those situations in which a group of individuals or experts should decide, from a collective point of view, which alternative is the most suitable to solve a problem. Even though different rules such as majority, unanimity, or Borda count, among others, have been proposed in the classic literature to model these situations (Butler & Rothstein, 2006), the use of these rules in the formation of group opinions could leave some Decision Makers (DMs) feeling dissatisfied by not taking their opinions sufficiently into account in group opinion formation (Palomares, Estrella, Martinez, & Herrera, 2014). Therefore, it is of utmost interest to resolve conflicts among decision makers before forming a collective opinion to ensure that everyone is satisfied with unanimous acceptance.

Consensus Reaching Processes (CRPs) have been proposed (Labella, Liu, Rodríguez, & Martínez, 2018; Zhang, Dong, Chiclana, & Yu, 2019) to deal with conflicts between DMs' opinions in GDM. They consist of iterative discussion processes, usually coordinated by a human figure, called a moderator, which aim to smooth out conflicts in a GDM situation (Palomares et al., 2014). This iterative process is controlled by the measure of the level of agreement among decision makers, which we refer to as *consensus measure*. If the value obtained from this measure in a given round exceeds the consensus threshold set $\mu \in [0, 1]$, or the number of iterations exceeds the maximum number of rounds allowed $MaxRounds \in \mathbb{N}$ (the set of natural numbers), the CRP ends (Palomares et al., 2014). Often, these CRPs (we also refer to them as *consensus models*) are time-consuming as they require several discussion rounds among experts. Consequently, other consensus models have been proposed that aim to achieve agreement among experts quickly and automatically (Gong, Zhang, Forrest, Li, & Xu, 2015; Zhang, Dong, & Xu, 2012).

* Corresponding author.
E-mail address: dgzamora@ujaen.es (D. García-Zamora).

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Minimum Cost Consensus (MCC) models (Ben-Arieh & Easton, 2007; Zhang, Dong, Xu, & Li, 2011) stand out from other automatic CRPs because they express GDM problems in terms of an optimization model based on minimizing a cost function in the space of preferences (Ben-Arieh & Easton, 2007; Zhang et al., 2011). These models consider that the constraints defining the feasible region of the minimization problem are given by a maximum distance $\varepsilon > 0$ between the DMs and the collective opinion (Gong et al., 2015; Zhang, Dong, & Xu, 2013), but neglect the minimum level of agreement that characterizes classical CRPs (Zhang et al., 2012; Zhang, Gong, & Chiclana, 2017). To overcome this drawback, Comprehensive MCC (CMCC) models were introduced (Labella, Liu, Rodríguez, & Martínez, 2020; Rodríguez, Labella, Dutta, & Martínez, 2021) to generalize previous MCC approaches by including a constraint involving a consensus measure that enforces adjusted preferences to ensure a consensus degree $1 - \gamma \in [0, 1]$. However, the inclusion of such an additional inequality presents a major drawback in terms of redundancy, in some situations, of both types of constraints.

Rodríguez et al. (2021) observed that for a given fixed value of ε some constraints of the γ values become redundant and vice versa. Furthermore, the calculations shown by Labella et al. (2020) indicate that the parameters ε and γ could be related to one another. Concretely, one of the proposed examples shows that, for a fixed value of the parameter γ, the value of the cost of modifying the original preferences remains the same for several values of ε. Similarly, for certain fixed values of ε, the value of the cost function also remains invariant for some specific values of γ.

Although there are some initial observations on the dynamics of the relationship of the parameters ε and γ and their corresponding inequalities, a proper understanding of the behavior of the parameters with respect to the optimal solutions and specific theoretical results are yet to be established. From a practical application point of view, it is of utmost interest to understand the proper relationship between these parameters, since if the redundant constraints can be identified a priori, they can be suppressed in the computational resolution of the CRP. This simplification can be especially relevant when the number of DMs involved in the GDM is high because CMCC models are based on solving mathematical programming problems that can be especially slow in these situations.

Therefore, this paper is devoted to analyzing the relationship between the two parameters of the CMCC models. In particular, we attempt to explore the structure of such a relationship in light of the following research questions.

- RQ1: For a fixed value of γ (resp. ε), which values of ε (resp. γ) imply that the ε (resp. γ) constraint is redundant in CMCC?
- RQ2: For a fixed value of γ (resp. ε), which values of ε (resp. γ) imply that the γ (resp. ε) constraint is redundant in CMCC?

To answer these questions, this proposal studies the influence of the parameters γ and ε on the CMCC model using two different approaches. We will start with an inequality-based approach, which allows us to derive approximate bounds to relate the parameters that are very simple to calculate. The second approach relies on Convex Polytope Theory (Henk, Richter-Gebert, & Ziegler, 2018; Ziegler, 1995) to determine more precise bounds for these parameters. In addition, this polytope-based approach also provides a generic solution to the abstract problem of establishing a relationship between any linear constraints that define the feasible region of a convex minimization problem. In summary, the main novelties of this contribution are:

- The interactions between the consensus measure and the maximum distance between experts and the group in CMCC are formally analyzed from two different perspectives, one based on inequalities and the other on Polytope Theory, to explore the full potential of CMCC models in practice by providing a comprehensive view of parameter dynamics.
- A generic polytope-based algorithm is proposed to analyze the relationship between linear constraints in convex minimization problems, i.e., whether some of them are redundant and, consequently, determine the same feasible region.

A deeper understanding of the relationship between parameters in CMCC has the following main implications:

- It provides an explanation of certain peculiarities of the behavior of the cost function in the CMCC models that have been pointed out in the literature (Labella et al., 2020; Rodríguez et al., 2021).
- It simplifies CMCC models by eliminating redundant constraints.
- It helps the moderator to conduct the CRP more efficiently by explicitly detecting unnecessary consensus conditions and understanding possible changes in cost and solutions for different parameters configurations.
- It can significantly reduce the computational cost of generating experts' modified preferences in CMCC, which implies an immediate improvement in the total time required for the CRP.

The remainder of this contribution is set out as follows. Section 2 provides the necessary background on GDM, CRPs, MCC models, CMCC models, and Polytope Theory to easily understand this proposal. In Section 3, the minimization problem is reformulated by using a novel notation to simplify this proposal. Section 4 provides several relationships between the parameters γ and ε which have been obtained by chaining inequalities, and in Section 5 a novel approach based on Polytope Theory is developed to obtain more precise relationships between these parameters. In Section 6 several examples are proposed to illustrate this research. Finally, Section 7 concludes the paper.

2. Preliminaries

This section provides the background required to fully understand this proposal and introduces basic concepts about GDM and CRPs. In addition, a brief review of the historical evolution of MCC models is developed to emphasize the link between the various formulations. Finally, we provide a brief introduction to the fundamental concepts of convex polytopes.

2.1. Group decision-making

GDM problems are those situations in which several individuals or experts have to decide which alternative, out of a given set of possible solutions to a given problem, is the most appropriate (Butler & Rothstein, 2006; Kacprzyk, 1986). Formally, a GDM problem consists of:

- A set $X = \{x_1, x_2, \ldots, x_n\}$ of possible solutions to the problem.
- A set $E = \{e_1, e_2, \ldots, e_m\}$ of experts who express their preferences about the alternatives in X through a certain preference structure.

For the sake of simplicity, in this work, we restrict our investigation to two preference elicitation approaches, namely, numerical scale and Fuzzy Preference Relation (FPR). In numerical scale settings, experts evaluate the alternatives by using a number from $[0, 1]$. On the other hand, in the FPR setting, it is assumed that preferences are elicited from experts by using Fuzzy Preference Relations (FPRs), a widely used structure

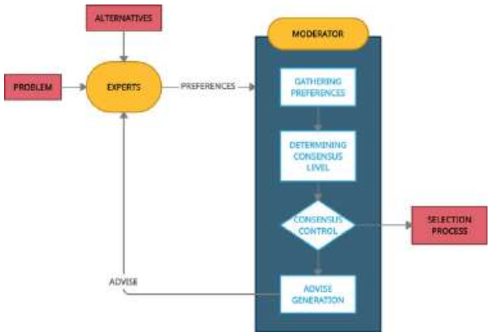

Fig. 1. Scheme of a CRP.

that has been shown to be effective in dealing with uncertainty (Bryson, 1996; Herrera-Viedma, Herrera, & Chiclana, 2002). These FPRs are obtained by asking the expert e_k to assess how much they prefer the alternative x_i to the alternative x_j using a value $p^k_{i,j}$ in the interval $[0, 1]$. The FPR associated with the expert e_k is the matrix $P_k = (p^k_{i,j}) \in \mathcal{M}_{n\times n}([0, 1])$, whose elements must satisfy the symmetry condition $p^k_{i,j} + p^k_{j,i} = 1$.

The classical scheme for a GDM problem consists of two phases:

- Aggregation. An aggregation operator is used to fuse the preferences elicited from the experts.
- Exploitation. The best alternative is selected taking into account the results of the previous phase.

Based on this scenario for a GDM problem, we will illustrate the key elements for carrying out the GDM process in the following subsection.

2.2. Consensus reaching processes and consensus measures

Classically, several rules have been used to select the best alternative in a GDM problem, such as the majority rule, the minority rule, or unanimity (Butler & Rothstein, 2006) but, when using these classic rules in the GDM solving process, some experts may disagree with the solution chosen by the group (Labella et al., 2018; Palomares et al., 2014).

A Consensus Reaching Process (CRP) is an iterative discussion process in which experts must modify their initial opinions to reach a collective agreement. CRPs have been developed to avoid disagreements and reach a collective opinion that satisfies all individuals who participate in the GDM problem.

Several consensus models have been proposed in the literature (Palomares et al., 2014). The general scheme for these models (see Fig. 1) is:

- *Consensus Measurement*. The preferences elicited from the experts are gathered and the level of agreement is computed using consensus measures (Beliakov, Calvo, & James, 2014).
- *Consensus control*. The obtained level of agreement is compared with a fixed consensus threshold $\mu \in [0, 1]$. If the level of agreement is greater than this threshold, a selection process is applied. Otherwise, another round of discussions is conducted. In order to avoid an endless process, a maximum number of rounds, $MaxRounds \in \mathbb{N}$, must be established beforehand.
- *Consensus Progress*. A moderator identifies experts' preferences that are difficult for the agreement process and generates recommendations for the experts to consider.

According to the taxonomy developed in Palomares et al. (2014), the consensus measures can be classified into two groups:

- Consensus measures based on the distance between each expert and the collective opinion,
- Consensus measures based on distances between experts.

Based on this background on the elements of the GDM process, in the following we will describe the automatic cost-based consensus models, which are the type of models for which we will study the relationships between parameters in this proposal.

2.3. Minimum cost consensus models

To study the cost of modifying experts' preferences, Ben-Arieh & Easton (2007) proposed the notion of MCC and introduced a model that considers consensus as being the minimum distance between each expert and the collective opinion, which is calculated using a weighted mean. This model aims to minimize the cost of moving preferences using a linear function. Specifically, for a set of experts $E = \{e_1, e_2, \dots, e_m\}$ who express the preferences $o = (o_1, o_2, \dots, o_m)$ over a certain alternative, the proposed optimization model is as follows:

$$\begin{aligned} &\min_{(x_1,\dots,x_m)} \sum_{i=1}^{m} c_i |x_i - o_i| \\ &s.t. \begin{cases} \bar{x} = \sum\limits_{i=1}^{m} w_i x_i \\ |x_i - \bar{x}| \le \varepsilon, i = 1, 2, \dots, m \end{cases} \end{aligned} \tag{M-1}$$

where the parameter $c_i \in \mathbb{R}_+$ models the cost of moving the opinion of the expert e_i one unit and $w_i \in [0, 1]$, $\sum_{i=1}^m w_i = 1$, is the importance of the expert e_i when aggregating the preferences.

By solving the non-linear programming problem defined in (M-1), a vector of optimal preferences $\hat{o} = (\hat{o}_1, \hat{o}_2, \dots, \hat{o}_m)$ is obtained that satisfies that the distance between its coordinates and the collective opinion $\bar{\hat{o}} = \sum_{i=1}^m w_i \hat{o}_i$ is bounded by ε.

Zhang et al. (2011) improved this previous proposal by considering that collective opinion could be calculated using different aggregation operators. To do so, the previous model was modified as follows:

$$\begin{aligned} &\min_{(x_1,\dots,x_m)} \sum_{i=1}^{m} c_i |x_i - o_i| \\ &s.t. \begin{cases} \bar{x} = F(x_1, x_2, \dots, x_m) \\ |x_i - \bar{x}| \le \varepsilon, i = 1, 2, \dots, m \end{cases} \end{aligned} \tag{M-2}$$

where F is an aggregation operator.

2.4. Comprehensive minimum cost consensus

Recent studies (Gong et al., 2015; Zhang et al., 2013; 2012; Zhang et al., 2017) have introduced new MCC approaches based on the original model proposed in Ben-Arieh & Easton (2007), but they all consider the distance of each expert from the collective opinion, ignoring a minimum level of agreement among experts, which is a milestone for CRPs (Chiclana, Mata, Martinez, Herrera-Viedma, & Alonso, 2008; Kacprzyk & Zadrożny, 2010). In order to deal with this shortcoming, Comprehensive MCC (CMCC) models were developed (Labella et al., 2020).

A CMCC model is a modification of the model (M-2) which includes a new constraint related to preferences holding a minimum consensus level:

$$\min_{(x_1,\dots,x_m)\in[0,1]^m} \sum_{i=1}^{m} c_i|x_i - o_i|$$
$$s.t.\begin{cases}\bar{x} = F(x_1,\dots,x_m)\\ |x_i - \bar{x}| \le \varepsilon, i = 1,2,\dots,m\\ consensus(x_1,\dots,x_m) \ge \mu,\end{cases} \tag{M-3}$$

where the function $consensus : [0,1]^m \to [0,1]$ measures the level of consensus reached by experts, $\mu \in [0,1]$ is a consensus threshold that is fixed a priori, $F : [0,1]^m \to [0,1]$ is an averaging aggregation operator, and ε is a parameter that measures the distance between each expert's adjusted opinion and the collective opinion.

2.4.1. MCC models dealing with numerical values

The model (M-3) was first adapted to two possible types of consensus measures (Palomares et al., 2014): those based on the distance between experts and collective opinion are modeled using (M-4) and those based on the distance between experts are modeled using (M-5), both of which are detailed below:

$$\min_{(x_1,\dots,x_m)\in[0,1]^m} \sum_{i=1}^{m} c_i|x_i - o_i|$$
$$s.t.\begin{cases}\bar{x} = \sum_{i=1}^{m} w_i x_i\\ |x_i - \bar{x}| \le \varepsilon, i = 1,2,\dots,m\\ \sum_{i=1}^{m} w_i|x_i - \bar{x}| \le \gamma,\end{cases} \tag{M-4}$$

$$\min_{(x_1,\dots,x_m)\in[0,1]^m} \sum_{i=1}^{m} c_i|x_i - o_i|$$
$$s.t.\begin{cases}\bar{x} = \sum_{i=1}^{m} w_i x_i\\ |x_i - \bar{x}| \le \varepsilon, i = 1,2,\dots,m\\ \sum_{i=1}^{m-1}\sum_{j=i+1}^{m-1} \frac{w_i+w_j}{m-1}|x_i - x_j| \le \gamma.\end{cases} \tag{M-5}$$

where $\gamma = 1-\mu$ and $w_i \in [0,1]$ ($\sum_{i=1}^{m} w_i = 1$) are the importance values of the expert e_i.

2.4.2. MCC models dealing with FPRs

The models provided in the previous section were also adapted for FPRs. Model (M-4) was rewritten as (M-6), while model (M-5) becomes (M-7). Given the fuzzy preference relations $P_k = (p_{ij}^k) \in \mathcal{M}([0,1])_{n\times n}$, $(k = 1,\dots,m)$, the model (M-6) is defined as follows:

$$\min_{(x_{ij}^k)\in\mathcal{M}_{n\times n}([0,1])} \sum_{k=1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} c_k|x_{ij}^k - p_{ij}^k|$$
$$s.t.\begin{cases}\bar{x}_{ij} = \sum_{k=1}^{m} w_k x_{ij}^k\\ |x_{ij}^k - \bar{x}_{ij}| \le \varepsilon, k = 1,\dots,m, i = 1,\dots,n-1, j = i+1,\dots,n\\ \frac{2}{n(n-1)}\sum_{k=1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} w_k|x_{ij}^k - \bar{x}_{ij}| \le \gamma,\end{cases} \tag{M-6}$$

where p_{ij}^k is the original preference of the expert e_k for the pair of alternatives x_i and x_j. Following the same scheme, the model (M-7) was defined as:

$$\min_{(x_{ij}^k)\in\mathcal{M}_{n\times n}([0,1])} \sum_{k=1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} c_k|x_{ij}^k - p_{ij}^k|$$
$$s.t.\begin{cases}\bar{x}_{ij} = \sum_{k=1}^{m} w_k x_{ij}^k\\ |x_{ij}^k - \bar{x}_{ij}| \le \varepsilon, k = 1,\dots,m, i = 1,\dots,n-1, j = i+1,\dots,n\\ \frac{2}{n(n-1)}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{k=1}^{m-1}\sum_{l=k+1}^{m} \frac{w_k+w_l}{m-1}|x_{ij}^k - x_{ij}^l| \le \gamma.\end{cases} \tag{M-7}$$

2.5. Polytopes

In this subsection, some basic notions about the Convex Polytope Theory are introduced. Convex Polytopes are the generalization of the 2-dimensional notion of convex polygon or the 3-dimensional concept of polyhedron. After introducing two different definitions for the concept of Polytope that are present in the literature (Henk et al., 2018; Ziegler, 1995), the Main Theorem of Polytope Theory, which unifies these two definitions, is stated.

Definition 1 (V-Polytope)**.** $\mathcal{R} \subset \mathbb{R}^m$ is said to be a V-polytope if $\mathcal{R}$ can be expressed as the convex hull of a finite set $V = \{v_1, v_2,\dots,v_n\} \subset \mathbb{R}^m$, i.e.:

$$\mathcal{R} = \left\{\sum_{k=1}^{n}\lambda_k v_k \ : \ \lambda_k \ge 0\ \forall\ k = 1,2,\dots,n \text{ and } \sum_{k=1}^{n}\lambda_k = 1\right\}$$

The set V is called the set of vertices of $\mathcal{R}$.

Definition 2 (H-Polytope)**.** $\mathcal{R} \subset \mathbb{R}^m$ is said to be a H-polytope if $\mathcal{R}$ can be expressed as the bounded solution set of a finite system of q linear inequalities, i.e., we can find $A \in \mathcal{M}_{q\times m}(\mathbb{R})$ and $B \in \mathcal{M}_{q\times 1}(\mathbb{R})$:

$\mathcal{R} = \{x \in \mathbb{R}^m \ : \ Ax \le B\} \subset B(0,r)$ for a certain $r > 0$,

where $B(0,r)$ denotes the ball of center $0 \in \mathbb{R}^m$ and radius $r > 0$.

The following fundamental theorem on the representation of polytopes (Henk et al., 2018; Ziegler, 1995), which relates the notions of V-polytope and H-polytope representations, forms the basis of our polytope-based analysis of the relationship between the inequalities involving the parameters μ and γ and the finding of tight bounds.

Theorem 1 (Main Theorem of Polytope Theory)**.** *The definitions of V-polytopes and of H-polytopes are equivalent. That is, every V-polytope has a description by a finite system of inequalities, and every H-polytope can be obtained as the convex hull of a finite set of points (its vertices).*

Definition 3 (Convex Polytope)**.** $\mathcal{R} \subset \mathbb{R}^m$ is said to be a convex polytope if $\mathcal{R} \subset \mathbb{R}^m$ is an H-polytope or a V-polytope.

From the computation point of view, it is important to understand how to switch between polytope representations. It was pointed out in Henk et al. (2018), Ziegler (1995) that there are three types of algorithms that allow us to transform one representation of a convex polytope into the other, namely, inductive algorithms (inserting vertices), projection algorithms and reverse search methods. Irrespective of the merits and demerits of the algorithms for numeric computations, this work will use the popular package `vertexenum` Robere (2018) developed for the software R (R Core Team, 2017), which allows us to transform the inequality-based representation of a convex polytope (H-polytope) into the vertex-based representation (V-polytope).

3. Problem description

This section is devoted to establishing a common notation to simplify the study of the relationships between the constraint induced by the consensus measure and the constraints involving the distances between preferences and collective opinion in CMCC models.

First, note that all models proposed in Section 2.4 can be described as follows.

$$\min \sum_{k=1}^{m} c_k|x_k - o_k| \\ s.t. \begin{cases} (x_1, x_2, \dots, x_m) \in \mathcal{R}_\varepsilon, \\ (x_1, x_2, \dots, x_m) \in \mathcal{R}_\gamma, \end{cases} \tag{M-G}$$

where

- $\mathcal{R}_\varepsilon$ and $\mathcal{R}_\gamma$ are the sets of points in $[0,1]^m$ that satisfy certain constraints related to the parameters ε and γ, respectively, that is,

  $$\mathcal{R}_\varepsilon := \{x \in [0,1]^m \ : \ g_k(x) \le \varepsilon \ \forall \ k = 1, 2, ..., m\},$$
  $$\mathcal{R}_\gamma := \{x \in [0,1]^m \ : \ g_0(x) \le \gamma\},$$

  where $g_0, g_1, \dots, g_m : [0,1]^m \to \mathbb{R}_+$ are defined as the composition of a linear function with the absolute values of some other linear combination of variables, that is,

  $$g_k(x_1, x_2, \dots, x_m) = \sum_{i=1}^{m} w_i^k |L_i(x_1, \dots, x_m)|, \quad \forall (x_1, x_2, \dots, x_m) \in \mathbb{R}^m, \tag{1}$$

  for all $k = 0, 1, \dots, m$, where $L_i : \mathbb{R}^m \to \mathbb{R}$ is a linear function for all $i = 1, 2, ..., m$ and $w_i^k \ge 0$.
- $(c_1, c_2, \dots, c_m) \in \mathbb{R}_+^m$ are the constant values for the cost of moving the opinion of each expert one unit,
- $(o_1, o_2, \dots, o_m) \in [0,1]^m$ are the initial values for experts' preferences.

As mentioned earlier, the experimental results conducted in Labella et al. (2020) suggest that the parameters γ and ε appearing in the models (M-4), (M-5), (M-6) and (M-7) could be related. Here, we are interested in exploring the relationships of these parameters by analyzing the containment relationship between the regions $\mathcal{R}_\gamma$ and $\mathcal{R}_\varepsilon$ that involve the inequalities corresponding to the parameters γ and ε, which attempt to restrict the adjusted preferences.

In light of the inclusion relationships of these regions, we first attempt to study if, given a value for $\gamma \in [0,1]$, certain values of ε exist for which the region $\mathcal{R}_\gamma$ is encapsulated in $\mathcal{R}_\varepsilon$. In other words, we want to find a value $\varepsilon_1 \in [0,1]$ such that $\mathcal{R}_\gamma \subseteq \mathcal{R}_\varepsilon \ \forall \ \varepsilon \ge \varepsilon_1$. This setup leads to some interesting consequences in the geometry of the regions that we describe in the following.

Denote $\mathcal{R}_{\gamma,\varepsilon} := \mathcal{R}_\gamma \cap \mathcal{R}_\varepsilon \ne \emptyset$ ($\bar{0} \in \mathcal{R}_{\gamma,\varepsilon}$). Note that for a fixed value of γ, the following statements are equivalent:

- $\mathcal{R}_\gamma \subseteq \mathcal{R}_\varepsilon \ \forall \ \varepsilon \ge \varepsilon_1$.
- $\mathcal{R}_{\gamma,\varepsilon} = \mathcal{R}_\gamma \ \forall \ \varepsilon \ge \varepsilon_1$.
- The constraints associated with ε, that is, $|x_k - \bar{x}| \le \varepsilon, k = 1, 2, ..., m$, do not affect the shape of $\mathcal{R}_{\gamma,\varepsilon} \ \forall \ \varepsilon \ge \varepsilon_1$.
- The values of the preferences that satisfy the constraint of $\mathcal{R}_\gamma$ also satisfy the constraints of $\mathcal{R}_\varepsilon$ for all $\varepsilon \ge \varepsilon_1$.

Remark 1. The notation $\mathcal{R}_\gamma^i, \mathcal{R}_\varepsilon^i$ and $\mathcal{R}_{\gamma,\varepsilon}^i$, $i = 4, 5, 6, 7$ will be used to relate these regions with the models (M-4), (M-5), (M-6) and (M-7). When referring to the generic model (M-G) the notation $\mathcal{R}_\gamma, \mathcal{R}_\varepsilon$ and $\mathcal{R}_{\gamma,\varepsilon}$ is kept for the sake of simplicity.

Keeping these consequences in mind, one can observe that for a fixed γ, studying the shape of the region $\mathcal{R}_{\gamma,\varepsilon}$ is equivalent to finding if a maximum value for the distance between the experts' opinions and the respective collective opinion ε is guaranteed. This fact leads to the following definitions:

Definition 4. ($\varepsilon_1(\gamma)$) For a fixed $\gamma \in [0,1]$ we will denote by $\varepsilon_1(\gamma)$, or simply ε_1 if no confusion is possible, the infimum value of ε such that $\mathcal{R}_\gamma \subseteq \mathcal{R}_\varepsilon$.

Definition 5. ($\gamma_1(\varepsilon)$) For a fixed $\varepsilon \in [0,1]$, we will denote by $\gamma_1(\varepsilon)$, or simply γ_1 if no confusion is possible, the infimum value of γ such that $\mathcal{R}_\varepsilon \subseteq \mathcal{R}_\gamma$.

The purpose of $\varepsilon_1(\gamma)$ and $\gamma_1(\varepsilon)$ is to detect when the region $\mathcal{R}_{\gamma,\varepsilon}$ starts to differ from the region $\mathcal{R}_\gamma$ or $\mathcal{R}_\varepsilon$, respectively. Note that finding the values $\varepsilon_1(\gamma)$ and $\gamma_1(\varepsilon)$ is equivalent to providing an answer to RQ1. However, a change in the region does not necessarily imply a change in the solution of the minimization problem. In order to control this change, we introduce the following alternative definition for the notion of boundary, which is more suitable for the problem that we are dealing with in this study:

Definition 6. Let $\mathcal{R} := \{x \in [0,1]^m \ : \ g(x) \le r\}$, for some $g : \mathbb{R}^m \to \mathbb{R}$ and $r > 0$, be a region in the unit hypercube. Then, the modified boundary of $\mathcal{R}$, $Bound^*(\mathcal{R})$, is defined as follows:

$$Bound^*(\mathcal{R}) := ((\mathcal{R}')^c \setminus (\mathcal{R}')^\circ) \cap [0,1]^m$$

where $\mathcal{R}' := \{x \in \mathbb{R}^m \ : \ g(x) \le r\}$ and c and $\circ$ denote, respectively, the standard closure and interior of a set.

The point of this definition is to avoid the interference of values such as $(0, 0, \dots, 0)$ or $(1, 1, \dots, 1)$ in the computation of the boundaries of the regions $\mathcal{R}_\gamma$ and $\mathcal{R}_\varepsilon$.

Based on this notion of a modified boundary, we propose the following definitions in order to find the values of the parameters ε and γ for which a change in the solution is ensured when the other parameter is fixed.

Definition 7. ($\varepsilon_2(\gamma)$) For a fixed $\gamma \in [0,1]$, we will denote by $\varepsilon_2(\gamma)$, or simply ε_2 if no confusion is possible, the supremum of the values of ε such that $Bound^*(\mathcal{R}_\gamma) \cap Bound^*(\mathcal{R}_{\gamma,\varepsilon}) = \emptyset$, or equivalently $\mathcal{R}_\varepsilon \subset \mathcal{R}_\gamma$.

Definition 8. ($\gamma_2(\varepsilon)$) For a fixed $\varepsilon \in [0,1]$, we will denote by $\gamma_2(\varepsilon)$, or simply γ_2 if no confusion is possible, the supremum of the values of γ such that $Bound^*(\mathcal{R}_\varepsilon) \cap Bound^*(\mathcal{R}_{\gamma,\varepsilon}) = \emptyset$, or equivalently $\mathcal{R}_\gamma \subset \mathcal{R}_\varepsilon$.

Note that if these boundaries have no points in common, the solution will inevitably change. In addition, by finding these values, an answer to RQ2 would be provided. To clarify this, consider a fixed value $\gamma \in [0,1]$ and the sequence $\{\varepsilon_n\} = \{\frac{1}{n}\}, n \in \mathbb{N}$. Note that for $\varepsilon_1 = 1$, the constraints related to ε_1 in the region $\mathcal{R}_{\gamma,\varepsilon_1}$ are always satisfied and consequently $\mathcal{R}_{\gamma,\varepsilon_1} = \mathcal{R}_\gamma$. If we increase the value of n, we will find some n_1 such that $\varepsilon_{n_1} < \varepsilon_1(\gamma)$, which means that for $n \ge n_1$, there is at least one point in $\mathcal{R}_\gamma$ which does not belong to $\mathcal{R}_{\varepsilon_n}$. However, the solution will not necessarily change until a value $n_2 \ge n_1$ such that $\varepsilon_{n_2} < \varepsilon_2(\gamma)$ is found and consequently $Bound^*(\mathcal{R}_\gamma) \cap Bound^*(\mathcal{R}_{\gamma,\varepsilon_n}) = \emptyset$ for any value $n \ge n_2$ (see Fig. 2).

Based on this formalization of our research questions in the form of the definitions mentioned above, we attempt to find rough bounds for the parameters in the next section.

4. An approach based on inequalities

In this section, we provide rough bounds for the values of $\varepsilon_1(\gamma)$ and $\gamma_1(\varepsilon)$ for each CMCC model. Specifically, we obtain the values ε_1' and γ_1' such that $\varepsilon_1' \ge \varepsilon_1(\gamma)$ and $\gamma_1' \ge \gamma_1(\varepsilon)$ for every model.

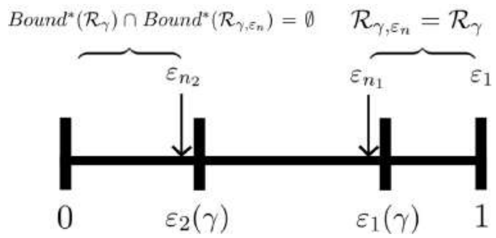

Fig. 2. A sketch of $\varepsilon_1(\gamma)$ and $\varepsilon_2(\gamma)$ for a fixed value of $\gamma \in [0, 1]$.

4.1. The model (M-4)

Adopting our earlier introduced notations, the regions associated with the consensus model (M-4) can be written as follows:

$$\mathcal{R}^4_\varepsilon := \{x \in [0,1]^m \ : \ |x_i - \bar{x}| \le \varepsilon, \ \forall \ i = 1, 2, \ldots, m\},$$
$$\mathcal{R}^4_\gamma := \left\{ x \in [0,1]^m \ : \ \sum_{i=1}^{m} w_i |x_i - \bar{x}| \le \gamma \right\},$$
$$\mathcal{R}^4_{\gamma,\varepsilon} := \mathcal{R}^4_\varepsilon \cap \mathcal{R}^4_\gamma.$$

The rough bound of γ for a given $\varepsilon > 0$ in terms of Definition 5 is illustrated in the following proposition.

Proposition 1. *For a given $\varepsilon > 0$ in the consensus model (M-4), the value of γ that ensures the satisfaction of the constraints of $\mathcal{R}^4_\gamma$ is $\gamma'_1(\varepsilon) = \varepsilon$, that is, $\mathcal{R}^4_\varepsilon = \mathcal{R}^4_{\gamma,\varepsilon} \ \forall \ \gamma \ge \varepsilon$.*

Proof. Suppose that $x \in \mathcal{R}^4_\varepsilon$. Then $|x_i - \bar{x}| \le \varepsilon$ and $\sum_{k=1}^m w_k |x_i - \bar{x}| \le \varepsilon$. Therefore, if $\gamma \ge \varepsilon$, $\mathcal{R}^4_\varepsilon \subseteq \mathcal{R}^4_{\gamma,\varepsilon}$ and consequently $\mathcal{R}^4_\varepsilon = \mathcal{R}^4_{\gamma,\varepsilon}$. $\square$

The opposite case is described in the following proposition.

Proposition 2. *For a given $\gamma > 0$ in the consensus model (M-4), the values of ε that ensure the satisfaction of the constraints in $\mathcal{R}^4_\varepsilon$ are $\varepsilon'_1(\gamma) = \frac{\gamma}{\min_{k=1,2,\ldots,m}\{w_k\}}$, that is,*

$$\mathcal{R}^4_\gamma = \mathcal{R}^4_{\gamma,\varepsilon} \ \forall \ \varepsilon \ge \frac{\gamma}{\min_{k=1,2,\ldots,m}\{w_k\}}$$

Proof. Let us suppose that $x \in \mathcal{R}^4_\gamma$. Then $\sum_{i=1}^m w_i|x_i - \bar{x}| \le \gamma$ and $w_i|x_i - \bar{x}| \le \gamma \ \forall \ i = 1, 2, ..., m$. Therefore,

$$|x_i - \bar{x}| \le \frac{\gamma}{\min_{k=1,2,\ldots,m}\{w_k\}} \ \forall \ i = 1, 2, ..., m.$$

So, if $\varepsilon \ge \varepsilon'_1 := \frac{\gamma}{\min_{k=1,2,\ldots,m}\{w_k\}}$, $x \in \mathcal{R}^4_{\gamma,\varepsilon}$ and consequently, $\mathcal{R}^4_\gamma = \mathcal{R}^4_{\gamma,\varepsilon}$. $\square$

4.2. The model (M-5)

The regions associated with the consensus model (M-5) can be cast as follows:

$$\mathcal{R}^5_\varepsilon := \{x \in [0,1]^m \ : \ |x_i - \bar{x}| \le \varepsilon, \ \forall \ i = 1, 2, \ldots, m\},$$
$$\mathcal{R}^5_\gamma := \left\{ x \in [0,1]^m \ : \ \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \frac{w_i + w_j}{m-1} |x_i - x_j| \le \gamma \right\},$$
$$\mathcal{R}^5_{\gamma,\varepsilon} := \mathcal{R}^5_\varepsilon \cap \mathcal{R}^5_\gamma.$$

Similarly, rough bounds for the parameters ε and γ when one of them is given can be obtained for (M-5) and are given in the following propositions.

Proposition 3. *For a given $\varepsilon > 0$ in the consensus model (M-5), the values of γ that ensure the satisfaction of the constraint of $\mathcal{R}^5_\gamma$ is $\gamma'_1(\varepsilon) = 2\varepsilon$, that is, $\mathcal{R}^5_\varepsilon = \mathcal{R}^5_{\gamma,\varepsilon} \ \forall \ \gamma \ge 2\varepsilon$.*

Proof. First, note that

$$\begin{aligned}
&\sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \frac{w_i + w_j}{m-1} \\
&= \frac{1}{m-1} \left(\sum_{i=1}^{m-1} (m-i) w_i + \sum_{i=2}^{m} (i-1) w_i \right) \\
&= \frac{1}{m-1} \left((m-1) w_1 + \sum_{i=2}^{m-1} (m-i) w_i \right. \\
&\left. + (m-1) w_m + \sum_{i=2}^{m} (i-1) w_i \right) \\
&= \frac{1}{m-1} \left((m-1)(w_1 + w_m) + \sum_{i=2}^{m-1} (m-1) w_i \right) = 1.
\end{aligned}$$

For an arbitrary $x \in \mathcal{R}^5_\varepsilon$, we have

$$\begin{aligned}
\sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \frac{w_i + w_j}{m-1} |x_i - x_j| &\le \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \frac{w_i + w_j}{m-1} \big(|x_i - \bar{x}| + |\bar{x} - x_j| \big) \\
&\le 2\varepsilon \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \frac{w_i + w_j}{m-1} = 2\varepsilon
\end{aligned}$$

Hence, $x \in \mathcal{R}^5_{\gamma,\varepsilon} \ \forall \ \gamma > 2\varepsilon$. $\square$

Proposition 4. *For a given $\gamma > 0$ in the consensus model (M-5), the values of ε that ensure the satisfaction of the constraints in $\mathcal{R}^5_\varepsilon$ are $\varepsilon'_1(\gamma) = (m-1)\gamma$, that is,*

$$\mathcal{R}^5_\gamma = \mathcal{R}^5_{\gamma,\varepsilon} \ \forall \ \varepsilon \ge (m-1)\gamma$$

Proof. Let $x \in \mathcal{R}^5_\gamma$. Then

$$\begin{aligned}
|x_i - \bar{x}| &= |x_i \sum_{j=1}^{m} w_j - \sum_{j=1}^{m} w_j x_j| \le \sum_{j=1, j\neq i}^{m} w_j |x_i - x_j| \\
&\le \sum_{i=1}^{m-1} \sum_{i=j+1}^{m} (w_i + w_j) |x_i - x_j| \le \gamma (m-1).
\end{aligned}$$

So, if $\varepsilon \ge (m-1)\gamma$, $\mathcal{R}^5_\gamma = \mathcal{R}^5_{\gamma,\varepsilon}$. $\square$

4.3. The model (M-6)

Let us define $\mathcal{M}_{n\times n}([0,1])^m := \mathcal{M}_{n\times n}([0,1]) \times \overset{m \text{ times}}{\cdots} \times \mathcal{M}_{n\times n}([0,1])$, whose elements are vectors of n-dimensional square matrices. With this notation, the regions associated with (M-6) can be represented as follows:

$$\begin{aligned}
\mathcal{R}^6_\varepsilon : &= \Big\{ X \in \mathcal{M}_{n\times n}([0,1])^m \ : \ |x^k_{ij} - \bar{x}_{ij}| \\
&\le \varepsilon, \ k = 1, \ldots, m, i = 1, \ldots, n-1, \\
&j = i+1, \ldots, n\},\\
\mathcal{R}^6_\gamma : &= \{ X \in \mathcal{M}_{n\times n}([0,1])^m \ : \\
&\left. \frac{2}{n(n-1)} \sum_{k=1}^{m} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} w_k |x^k_{ij} - \bar{x}_{ij}| \le \gamma \right\},\\
\mathcal{R}^6_{\gamma,\varepsilon} : &= \mathcal{R}^6_\varepsilon \cap \mathcal{R}^6_\gamma.
\end{aligned}$$

Similar results for the rough bounds for the parameters can also be obtained for (M-6) and are given in the following propositions.

Proposition 5. *For a given $\varepsilon > 0$ in the consensus model (M-6), the values of γ that ensure the satisfaction of the constraint of $\mathcal{R}^6_\gamma$ is $\gamma_1'(\varepsilon) = \varepsilon$, that is, $\mathcal{R}^6_\varepsilon = \mathcal{R}^6_{\gamma,\varepsilon}\ \forall\ \gamma \geq \varepsilon$.*

Proposition 6. *For a given $\gamma > 0$ in the consensus model (M-6), the values of ε that ensure the satisfaction of the constraints in $\mathcal{R}^6_\varepsilon$ are $\varepsilon_1'(\gamma) = \frac{\gamma(n-1)n}{2\min_{k=1,2,\ldots,m}\{w_k\}}$, that is,*

$$\mathcal{R}^6_\gamma = \mathcal{R}^6_{\gamma,\varepsilon}\ \forall\ \varepsilon \geq \frac{\gamma(n-1)n}{2\min_{k=1,2,\ldots,m}\{w_k\}}.$$

4.4. The model (M-7)

Analogously, we can cast the attached regions to the model (M-7) and find the results of the rough bounds for the parameters as follows:

$$\mathcal{R}^7_\varepsilon := \Big\{X \in \mathcal{M}_{n\times n}([0,1])^m\ :\ |x^k_{ij} - \bar{x}_{ij}| \leq \varepsilon,\ k = 1,\ldots,m, i = 1,\ldots,n-1, j = i+1,\ldots,n\Big\},$$

$$\mathcal{R}^7_\gamma := \left\{X \in \mathcal{M}_{n\times n}([0,1])^m\ :\ \frac{2}{n(n-1)}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{k=1}^{m-1}\sum_{l=k+1}^{m}\frac{w_k+w_l}{m-1}|x^k_{ij} - x^l_{ij}| \leq \gamma\right\},$$

$$\mathcal{R}^7_{\gamma,\varepsilon} := \mathcal{R}^7_\varepsilon \cap \mathcal{R}^7_\gamma.$$

The proof for the following results is analogous to some of those given previously, and is therefore omitted.

Proposition 7. *For a given $\varepsilon > 0$ in the consensus model (M-7), the values of γ that ensure the satisfaction of the constraint of $\mathcal{R}^7_\gamma$ is $\gamma_1'(\varepsilon) = 2\varepsilon$, that is, $\mathcal{R}^7_\varepsilon = \mathcal{R}^7_{\gamma,\varepsilon}\ \forall\ \gamma \geq 2\varepsilon$.*

Proposition 8. *For a given $\gamma > 0$ in the consensus model (M-7), the values of ε that ensure the satisfaction of the constraints in $\mathcal{R}^7_\varepsilon$ are $\varepsilon_1'(\gamma) = (m-1)n(n-1)\frac{\gamma}{2}$, that is,*

$$\mathcal{R}^7_\gamma = \mathcal{R}^7_{\gamma,\varepsilon}\ \forall\ \varepsilon \geq (m-1)n(n-1)\frac{\gamma}{2}.$$

Therefore, we have obtained rough bounds for the parameters ε and γ that correspond to the different consensus models by treating regions $\mathcal{R}_\gamma$ and $\mathcal{R}_\varepsilon$ as abstract spaces. Although these rough bounds are very easy to obtain and provide an immediate idea of the variations and a partial answer to RQ1 and RQ2, they are not very precise. In the following, we attempt to find more precise bounds for these parameters.

5. Approach based on polytopes

In this section, we further explore the geometry of the regions to obtain more precise bounds of the parameters ε and γ. We start by characterizing the regions $\mathcal{R}_\gamma$ and $\mathcal{R}_\varepsilon$ as convex polytopes. Subsequently, the properties of convex polytopes are used to derive the numerical bounds of the parameters along with the connection to the optimal solution of the models.

5.1. Properties of the regions $\mathcal{R}_\gamma$ and $\mathcal{R}_\varepsilon$

In this subsection, we show that both regions $\mathcal{R}_\gamma$ and $\mathcal{R}_\varepsilon$ in the generic model (M-G) are convex polytopes.

The first result proves that any system of inequalities involving compositions of linear functions with absolute values can be reduced to a system of linear inequalities.

Proposition 9. *Let $w = (w_1, w_2, \ldots w_m) \in \mathbb{R}^m$ and $\mu \in \mathbb{R}$, and consider the function $g : \mathbb{R}^m \to \mathbb{R}$ defined as*

$$g(x_1, x_2, \ldots, x_m) = \sum_{k=1}^{m} w_k|x_k|\ \forall\ (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m.$$

Then, for any $x = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m$, the following relations are equivalent:

- *$g(x) \leq \mu$,*
- *$< w_\sigma, x > \leq \mu$ for every $\sigma = (\sigma_1, \ldots, \sigma_m) \in \{-1, 1\}^m$.*

where $< \cdot, \cdot >$ denotes the standard dot product and $w_\sigma = (w_1\sigma_1, w_2\sigma_2, \ldots, w_m\sigma_m)$.

Proof. For $m = 1$, the statement is an immediate consequence of the properties of the absolute value function. Consider the case $m = 2$. Thus, $w_1, w_2 \geq 0$ and $g : \mathbb{R}^2 \to \mathbb{R}$ is defined as $g(x_1, x_2) = w_1|x_1| + w_2|x_2|$, for all $(x_1, x_2) \in \mathbb{R}^2$. Note that

$$w_1|x_1| + w_2|x_2| \leq \mu \iff w_1|x_1| \leq \mu - w_2|x_2| \iff \begin{matrix} w_1x_1 \leq \mu - w_2|x_2| \\ -w_1x_1 \leq \mu - w_2|x_2| \end{matrix}$$

$$\iff \begin{matrix} w_2|x_2| \leq \mu - w_1x_1 \\ w_2|x_2| \leq \mu + w_1x_1 \end{matrix} \iff \begin{matrix} w_2x_2 \leq \mu - w_1x_1 \\ -w_2x_2 \leq \mu - w_1x_1 \\ w_2x_2 \leq \mu + w_1x_1 \\ -w_2x_2 \leq \mu + w_1x_1 \end{matrix} \iff \begin{matrix} w_1x_1 + w_2x_2 \leq \mu \\ w_1x_1 - w_2x_2 \leq \mu \\ -w_1x_1 + w_2x_2 \leq \mu \\ -w_1x_1 - w_2x_2 \leq \mu \end{matrix}.$$

Therefore, this is also true for $m = 2$. The rest of the proof is an obvious induction. □

Note that the resulting linear inequalities are those obtained by considering all possible combinations of the different signs for the weights $w_1, \ldots, w_m$, as shown in the proof of Proposition 9.

This representation of inequalities involving absolute values allows us to characterize regions $\mathcal{R}_\gamma$ and $\mathcal{R}_\varepsilon$ as polytopes as stated in the following corollary.

Corollary 1. *Let $n \in \mathbb{N}$ and consider a set of m-dimensional weighting vectors $\{w^1, w^2, \ldots, w^n\}$, i.e., $w^k = (w^k_1, w^k_2, \ldots w^k_m) \in [0,1]^m$ such that $\sum_{i=1}^m w^k_i = 1$, $\forall\ k = 1, 2, ..., m$, and $\mu_1, \ldots, \mu_m \in \mathbb{R}_+$. Consider the functions $g_k : \mathbb{R}^m \to \mathbb{R}$ defined as*

$$g_k(x_1, x_2, \ldots, x_m) = \sum_{i=1}^{m} w^k_i|x_i|,\ \forall\ (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m, k = 1, 2, ..., m.$$

Then, the region $\mathcal{R} = \{x \in [0,1]^m\ :\ g_k(x) \leq \mu_k\ \forall\ k = 1, 2, ..., m\}$ is a non-empty polytope.

Remark 2. Note that the point $(0, 0, \ldots, 0) \in \mathbb{R}^m$ always satisfies the conditions that define $\mathcal{R}$.

Furthermore, we can assure that the obtained polytope is convex, even if the arguments of the absolute values are replaced with linear combinations of the variables.

Corollary 2. *Let $n \in \mathbb{N}$ and consider a set of m-dimensional weighting vectors $\{w^1, w^2, \ldots, w^n\}$, i.e., $w^k = (w^k_1, w^k_2, \ldots w^k_m) \in [0,1]^m$, such that $\sum_{i=1}^m w^k_i = 1\ \ \forall\ k = 1, 2, \ldots, n$, and $\mu_1, \ldots \mu_m \in \mathbb{R}_+$. Consider the functions $g_k : \mathbb{R}^m \to \mathbb{R}$ defined as*

$$g_k(x_1, x_2, \ldots, x_m) = \sum_{i=1}^{m} w^k_i|L_i(x_1, \ldots, x_m)|,\ \forall\ (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m, k = 1, 2, \ldots, n.$$

where $L_i : \mathbb{R}^m \to \mathbb{R}$ is a linear function for every $i = 1, 2, ..., m$. Then, the region

$$\mathcal{R} = \{x \in [0,1]^m\ :\ g_k(x) \leq \mu_k\ \forall\ k = 1, 2, \ldots, n\}$$

is a convex non-empty polytope.

Proof. Since the linear function L_i does not alter the linearity of the equations obtained by using Corollary 1, it is clear that the region $\mathcal{R}$ is a nonempty polytope. To show that $\mathcal{R}$ is convex, let us pick any $x, y \in \mathcal{R}$. By definition of $\mathcal{R}$, we have $g_k(x) \le \mu_k$ and $g_k(y) \le \mu_k$ for all $k = 1, 2, ..., n$. Now, we observe that for any $t \in [0, 1]$

$$\begin{aligned} g_k((1-t)x+ty) &= \sum_{i=1}^{m} w_i^k |L_i((1-t)x+ty)| \\ &= \sum_{i=1}^{m} w_i^k |(1-t)L_i(x)+tL_i(y)| \\ &\le \sum_{i=1}^{m} w_i^k ((1-t)|L_i(x)|+t|L_i(y)| \\ &= (1-t)g_k(x)+tg_k(y) \le \mu_k. \end{aligned}$$

Therefore, $(1-t)x+ty \in \mathcal{R}$ and, consequently, $\mathcal{R}$ is convex. □

The following result provides the sufficient and necessary conditions for a convex polytope to be contained in a fixed convex set.

Proposition 10. *Let $\mathcal{S} \subset \mathbb{R}^m$ be a convex set, and consider a convex polytope $\mathcal{R} \subset \mathbb{R}^m$. Then $\mathcal{R} \subseteq \mathcal{S}$ if and only if all the vertices of $\mathcal{R}$ belong to $\mathcal{S}$.*

Proof. The proof of the sufficient condition is straightforward. To prove the necessary condition, suppose that all vertices, say $\{v_1, v_2, \dots, v_n\}$, of $\mathcal{R}$ are contained in $\mathcal{S}$. As $\mathcal{R}$ is a convex polytope, any point $x \in \mathcal{R}$ can be expressed as a convex combination of its vertices $x = \sum_{k=1}^{n} a_k v_k$ for certain scalars $a_1, a_2, \dots, a_n$, such that $a_k \ge 0 \ \forall \ k = 1, 2, ..., n$ and $\sum_{k=1}^{n} a_k = 1$. Since $\mathcal{S}$ is convex and $\{v_1, v_2, \dots, v_n\} \in \mathcal{S}$, then $x \in \mathcal{S}$. Hence, $\mathcal{R} \subseteq \mathcal{S}$. □

Note that the constraints which define the feasible region in the models (M-4), (M-5), (M-6), and (M-7) are similar to the hypothesis constraints in Corollary 2. Based on Corollary 2 and Proposition 10, we can establish the containment relationship between regions $\mathcal{R}_\gamma$ and $\mathcal{R}_\varepsilon$ in terms of vertex representation, and that result is outlined in the following theorem.

Theorem 2. *The regions $\mathcal{R}_\gamma^i$ and $\mathcal{R}_\varepsilon^i$ $(i = 4, 5, 6, 7)$ which appear in the models (M-4), (M-5), (M-6) and (M-7) are convex nonempty polytopes. Furthermore, for any of these models, the region $\mathcal{R}_\gamma^i$ is contained in $\mathcal{R}_\varepsilon^i$ if and only if all the vertices of $\mathcal{R}_\gamma^i$ belong to the region $\mathcal{R}_\varepsilon^i$. In the same way, the region $\mathcal{R}_\varepsilon^i$ is contained in $\mathcal{R}_\gamma^i$ if and only if all the vertices in $\mathcal{R}_\varepsilon^i$ belong to the region $\mathcal{R}_\gamma^i$.*

Although Theorem 2 sketches the condition for the containment between the regions $\mathcal{R}_\gamma$ and $\mathcal{R}_\varepsilon$, it does not make any comment on the behavior of the optimal solution of the consensus model. Below, we take a closer look at this issue.

5.2. Existence of solution to (M-G)

In this section, we investigate the existence of a solution to the general optimization model (M-G). To do so, the link between this optimal solution and the region $\mathcal{R}_{\gamma,\varepsilon}$ is analyzed from a theoretical point of view.

Proposition 11. *The regions $\mathcal{R}_{\gamma,\varepsilon}, \mathcal{R}_\gamma$ and $\mathcal{R}_\varepsilon$ are compact subsets of the Euclidean spaces in which they are defined for any values of $\gamma, \varepsilon \in [0, 1]$.*

Proof. Note that all of these regions are contained in the unit hypercube of their respective Euclidean spaces, and therefore they are bounded. In addition, due to the fact that all of them are convex polytopes, they are also closed subsets in their respective Euclidean spaces. □

Proposition 12. *The function $c : [0, 1]^m \to [0, 1]$ defined as $c(x_1, x_2, \dots, x_m) := \sum_{i=1}^{m} c_i |x_i - o_i|$, for all $(x_1, x_2, \dots, x_m) \in [0, 1]^m$, where $o = (o_1, o_2, \dots, o_m) \in [0, 1]^m$ are the original preferences, is convex.*

Proof. Let us consider $x, y \in [0, 1]^m$. Then, for any $\alpha \in [0, 1]$

$$\begin{aligned} c(\alpha x + (1-\alpha)y) &= \sum_{i=1}^{m} c_i |\alpha x_i + (1-\alpha)y_i - o_i| \\ &= \sum_{i=1}^{m} c_i |\alpha x_i + (1-\alpha)y_i - (\alpha o_i + (1-\alpha)o_i)| \\ &\le \sum_{i=1}^{m} c_i |\alpha (x_i - o_i) + (1-\alpha)(y_i - o_i)| \\ &\le \alpha c(x) + (1-\alpha)c(y) \end{aligned}$$

which proves the convexity of c. □

Proposition 13. *(Corollary 2.8.1 in Giorgi, Guerraggio, & (Eds) (2004)) Let $A \subset \mathbb{R}^m$ be a nonempty convex subset and consider a convex function $f : A \to \mathbb{R}$. Then, if f reaches a local minimum at the point $x_0 \in A$, x_0 is also a global minimum for f.*

Based on the above results, we establish a link between the optimal solution and the region $\mathcal{R}_{\gamma,\varepsilon}$ in the following theorem.

Theorem 3. *The model (M-G) always has a solution. If the original preferences $o = (o_1, \dots, o_m)$ satisfy $o \in \mathcal{R}_{\gamma,\varepsilon}$, the solution to the minimization problem is the original preference vector o. Otherwise, the solution for the minimization problem is always obtained in the boundary of the region $\mathcal{R}_{\gamma,\varepsilon}$.*

Proof. Since c is a continuous function defined in the compact subset $\mathcal{R}_{\gamma,\varepsilon}$, it will always reach its maximum and minimum value within the region $\mathcal{R}_{\gamma,\varepsilon}$, so a solution for the (M-G) model always exist.

Suppose that the solution x_0 for the (M-G) model is an interior point of the region $\mathcal{R}_{\gamma,\varepsilon}$. Then, x_0 is a local minimum for the function c and according to Proposition 13 it is a global minimum. But it is obvious that the global minimum of c is the original preferences $o = (o_1, o_2, \dots, o_m)$, so it must be $x_0 = o$. Otherwise, if x_0 is not an interior point, then it must belong to the boundary of $\mathcal{R}_{\gamma,\varepsilon}$. □

So far, we have theoretically established the proper geometry of the regions associated with consensus models and the link of the optimal solution to these regions in light of the convex polytope-based analysis. We will further explore this theoretical foundation to compute more precise bounds for the parameters ε and γ.

5.3. Algorithms used to establish the relationship between γ and ε

In this section, we develop several generic algorithms to find numeric approximations for the bounds of the parameters when one is determined based on the key results of Theorems 2 and 3.

5.3.1. Obtaining ε_1 and γ_1

In this section, we take advantage of the fact that the containment relationship between the $\mathcal{R}_\varepsilon$ and $\mathcal{R}_\gamma$ regions is based on the vertex representation of convex polytopes in order to decisively find the value of the parameters. The idea is to first generate the vertex representation of a region for which the parameter value is given via successive applications of Proposition 9 and the `vertexenum` algorithm. We then successively reduce the value of the parameter we are looking for from the initialized level with a constant step-size until the containment relationship between regions holds. This principle of finding the parameters has been

Algorithm 1: Find $\gamma_1(\varepsilon)$, when ε is given.

Input : ε-related constrains $\mathcal{R}_\gamma$, γ-related constraint $\mathcal{R}_\varepsilon$, threshold $\varepsilon > 0$, step-size $\delta > 0$.

find the linear inequality-based representation of $\mathcal{R}_\varepsilon$ of the form $Az \le \varepsilon$ using Proposition 9;
obtain vertices of $\mathcal{R}_\varepsilon$ as $V =$ `vertexenum`$(Az \le \varepsilon)$;
initialize: $\gamma_1 = 1$;
while $V \subseteq \mathcal{R}_\gamma$ **do**
    $\gamma := \gamma - \delta$;
end
Return $\gamma_1(\varepsilon) := \gamma$;

Output: $\gamma_1(\varepsilon)$ up to a precision equal to the step-size δ.

Algorithm 2: Find $\varepsilon_1(\gamma)$, when γ is given.

Input : ε-related constrains $\mathcal{R}_\gamma$, γ-related constraint $\mathcal{R}_\varepsilon$, threshold $\gamma > 0$, step-size $\delta > 0$.

find the linear inequality-based representation of $\mathcal{R}_\gamma$ of the form $Az \le \varepsilon$ using Proposition 9;
obtain vertices of $\mathcal{R}_\gamma$ as $V =$ `vertexenum`$(Az \le \gamma)$;
initialize: $\varepsilon_1 = 1$;
while $V \subseteq \mathcal{R}_\varepsilon$ **do**
    $\varepsilon := \varepsilon - \delta$;
end
Return $\varepsilon_1(\gamma) := \varepsilon$;

Output: $\varepsilon_1(\gamma)$ up to a precision equal to the step-size δ.

Algorithm 3: Find $\gamma_2(\varepsilon)$, when ε is given.

Input : ε-related constrains $\mathcal{R}_\gamma$, γ-related constraint $\mathcal{R}_\varepsilon$, threshold $\varepsilon > 0$, step-size $\delta > 0$.

find the linear inequality-based representation of $\mathcal{R}_\varepsilon$ of the form $Az \le \varepsilon$ using Proposition 9;
obtain vertices of $\mathcal{R}_\varepsilon$ as $V =$ `vertexenum`$(Az \le \varepsilon)$
construct refinement of V, say, V' by removing vertices of V which automatically satisfy γ constraints;
initialize: $\gamma = 1$;
while $V' \cap \mathcal{R}_\gamma \neq \emptyset$ **do**
    $\gamma := \gamma - \delta$
end
Return $\gamma_2(\varepsilon) := \gamma$;

Output: $\gamma_2(\varepsilon)$ up to a precision equal to the step-size δ.

Algorithm 4: Find $\varepsilon_2(\gamma)$, when γ is given.

Input : ε-related constrains $\mathcal{R}_\gamma$, γ-related constraint $\mathcal{R}_\varepsilon$, threshold $\gamma > 0$, step-size $\delta > 0$.

find the linear inequality-based representation of $\mathcal{R}_\gamma$ of the form $Az \le \gamma$ using Proposition 9;
obtain vertices of $\mathcal{R}_\gamma$ as $V =$ `vertexenum`$(Az \le \gamma)$
construct refinement of V, say, V' by removing vertices of V which automatically satisfy ε constraints;
initialize: $\varepsilon = 1$;
while $V' \cap \mathcal{R}_\varepsilon \neq \emptyset$ **do**
    $\varepsilon := \varepsilon - \delta$;
end
Return $\varepsilon_2(\gamma) := \varepsilon$;

Output: $\varepsilon_2(\gamma)$ up to a precision equal to the step-size δ.

summarized in Algorithms 1–2. Theorem 2 guarantees the proper functioning of the algorithms.

If these algorithms are applied, for each parameter ε and γ the guaranteed value of the other parameter, i.e., γ_1 and ε_1, is obtained. Note that the algorithms estimate ε_1 and γ_1 with the maximum error bounded by $\delta > 0$. Depending on the precision requirement, the value of $\delta > 0$ may be adjusted. This algorithm provides a clear answer to RQ1. As long as the containment relationship is maintained, the region associated with one parameter becomes redundant and there is no change to the optimal solution. Now the question is: What parameter values would make changes to the optimal solution? We attempt to answer this question below by developing algorithms to find such parameter values using Theorem 3.

5.3.2. Obtaining ε_2 and γ_2

To develop the algorithms of this subsection, we have taken into account that Theorem 3 guarantees that the solution of the minimization problem is always reached in the boundary of $\mathcal{R}_{\gamma,\varepsilon}$ whenever the solution is not trivial. For example, if γ is fixed, when ε is decreased so that none of the vertices of $\mathcal{R}_\varepsilon$ belong to the boundary of $\mathcal{R}_\gamma$ (with the exception of those vertices like $(0, 0, \ldots, 0)$, which always satisfy the constraints), we can ensure that the solution of the minimization problem will change. Based on this idea, we summarize the steps for computing ε_2 and γ_2 in Algorithms 3–4.

When using these algorithms, for each parameter ε or γ the guaranteed value of the other parameter, resp. γ_2 and ε_2, is obtained. Again, the error is given by the value $\delta > 0$. These algorithms give the answer to RQ2. We propose a generalization of these algorithms below for the more generic case of the (M-G) optimization model.

Remark 3. It must be highlighted that the output of these algorithms does not depend on the values of the preferences given by the experts, but on the weights that have been assigned to them. In other words, if such weights are fixed, we can determine the relationship between γ and ε and, consequently, whether it is possible to simplify the CMCC model for any of their preference values by only running the algorithm once.

5.3.3. Generalizing the algorithm

To end this section, a generalization of this algorithm is provided to compare the constraints that define the feasible region of a minimization problem.

Theorem 4. *Let $\mathcal{P} \subset \mathbb{R}^m$ be a convex polytope in $\mathbb{R}^m$ and consider a continuous convex function $c : \mathcal{P} \to \mathbb{R}$. Consider two compact intervals I_1 and I_2 in $\mathbb{R}$ and the linear functions $g_k^1 : \mathbb{R}^m \to \mathbb{R}$, for $k = 1, 2, \ldots, n_1$ and $g_k^2 : \mathbb{R}^m \to \mathbb{R}$ for $k = 1, 2, \ldots, n_2$. For $\alpha \in I_1$ and $\beta \in I_2$ define the polytopes*

$$\mathcal{P}_\alpha := \left\{x \in \mathcal{P} \ : \ g_k^1(x) \le \alpha \ \forall \ k = 1, 2, \ldots, n_1\right\},$$

$$\mathcal{P}_\beta := \left\{x \in \mathcal{P} \ : \ g_k^2(x) \le \beta \ \forall \ k = 1, 2, \ldots, n_2\right\}.$$

Then:

- *The optimization problem*

$$\min_{x \in \mathcal{P}} \{c(x)\}$$
$$\text{s.t.} \begin{cases} x \in \mathcal{P}_\alpha \\ x \in \mathcal{P}_\beta \end{cases}$$

  always has a solution.
- *For any values $\alpha \in I_1$ and $\beta \in I_2$, $\mathcal{P}_\alpha \subseteq \mathcal{P}_\beta$ if and only if the vertices set of the polytope $\mathcal{P}_\alpha$ is contained in the polytope $\mathcal{P}_\beta$*

Given the hypotheses of the previous theorem, Algorithms 5 and 6 determine respectively when the feasible region changes and when the solution changes.

Algorithm 5: Algorithm for obtaining $\beta_1(\alpha)$, the infimum value of β such that $\mathcal{P}_\alpha \subseteq \mathcal{P}_\beta$.

Input: Problem defined in Theorem 4, a fixed $\alpha \in I_1$ and step-size $\delta > 0$.
find the linear inequality-based representation of $\mathcal{P}_\alpha$ of the form $Az \leq \alpha$ using Proposition 9;
obtain vertices of $\mathcal{P}_\alpha$ as $V = \texttt{vertexenum}(Az \leq \alpha)$;
Initialize $\beta := \max I_2$;
while $V \subset \mathcal{P}_\beta$ **do**
  $\beta := \beta - \delta$
end
Return $\beta_1(\alpha) := \beta$;
Output: $\beta_1(\alpha)$ up to a precision equal to the step-size δ.

Algorithm 6: Algorithm for obtaining $\beta_2(\alpha)$, the supremum value of β such that the solution of the optimization problem has changed.

Input: Problem defined in Theorem 4, a fixed $\alpha \in I_1$ and step size $\delta > 0$.
find the linear of the form representation of $\mathcal{P}_\alpha$ of the form $Az \leq \alpha$ using Proposition 9;
obtain vertices of $\mathcal{P}_\alpha$ as $V = \texttt{vertexenum}(Az \leq \alpha)$
construct refinement of V, say, V' by removing vertices of V which automatically satisfy β constraints;
initialize: $\beta = 1$;
while $V' \cap Bound^*(\mathcal{P}_\beta) \neq \emptyset$ **do**
  $\beta := \beta - \delta$;
end
Return $\beta_2(\alpha) := \beta$;
Output: $\beta_2(\alpha)$ up to a precision equal to the step-size δ.

6. Illustrative examples

In this section, a couple of examples are proposed to show the performance of the polytope-based algorithms developed in the previous section.

6.1. Example 1: Labella et al. (2020) GDM problem

The first illustrative example is related to one of the GDM problems with five experts provided by Labella et al. (2020) when they defined CMCC models for the first time, which has motivated this study.

Labella et al. considered a set of experts $E = \{e_1, e_2, e_3, e_4, e_5\}$ whose weights were

$$W = (0.375, 0.1875, 0.25, 0.0625, 0.125),$$

and their preferences were $(o_1, o_2, o_3, o_4, o_5) = (0, 0.09, 0.36, 0.45, 1)$. The values of the costs of moving experts' opinions were established as $(c_1, c_2, c_3, c_4, c_5) = (6, 3, 4, 1, 2)$. The values obtained by Labella et al. (2020) for the cost function for different values of ε and γ when using the model (M-5) are summarized in Table 1.

Note that Table 1 clearly suggests that the relationship between the parameters γ and ε exists. For instance, if we compare the values of the cost function for $\gamma = 0.1$ and the different values of ε, we can see that all of them, except the one obtained when $\varepsilon = 0.05$, are the same.

When CMCC models were first proposed, the authors were unable to provide any explanation for this phenomenon. In fact, existing studies do not offer any insight into the relationship of these parameters, how it influences the minimum consensus cost, and

Table 1
The costs with different values of ε and γ of (M-5) in Example 1.

	$\gamma = 0.3$	$\gamma = 0.25$	$\gamma = 0.2$	$\gamma = 0.15$	$\gamma = 0.1$	$\gamma = 0.05$
$\varepsilon = 0.30$	1.01	1.20	1.60	2.14	2.69	3.24
$\varepsilon = 0.25$	1.13	1.20	1.60	2.14	2.69	3.24
$\varepsilon = 0.20$	1.32	1.32	1.60	2.14	2.69	3.24
$\varepsilon = 0.15$	1.81	1.81	1.81	2.14	2.69	3.24
$\varepsilon = 0.10$	2.47	2.47	2.47	2.47	2.69	3.24
$\varepsilon = 0.05$	3.13	3.13	3.13	3.13	3.13	3.24

Table 2
Different bounds for ε when γ is fixed in (M-5) in Example 1.

	$\gamma = 0.3$	$\gamma = 0.25$	$\gamma = 0.2$	$\gamma = 0.15$	$\gamma = 0.1$	$\gamma = 0.05$
$\varepsilon_1'(\gamma)$	1.2	1	0.8	0.6	0.4	0.2
$\varepsilon_1(\gamma)$	0.94	0.79	0.64	0.48	0.32	0.16
$\varepsilon_2(\gamma)$	0.23	0.19	0.15	0.11	0.07	0.03

how one parameter could be computed when the other is given. Without such knowledge of the proper dynamics of CMCC models, the moderator, in practice, is forced to choose the other parameter on a hit and trial basis and cannot take advantage of computational cost reduction, which results in inefficiency when conducting the CRP. However, using the results of our study and the developed algorithms, it is not only possible to explain the rationale behind the behavior of the cost function, but we are also able to get a complete picture of how different parameter configurations will impact the cost of reaching consensus, which can help the moderator manage the CRP more efficiently. To explore this fact, we assume here that the parameter γ is given at the beginning for the problem described above, and the moderator wants to know how the values of the parameter ε impact the cost of solving the CMCC model. Therefore, we are going to use the results of our proposal to obtain the different bounds of the parameter ε for different values of γ (see Table 2). Specifically, the following values are collected:

- $\varepsilon_1'(\gamma)$: The upper bound for ε_1 provided by the results presented in Section 4, which are based on inequalities (Proposition 4). This value represents an easy-to-compute rough bound for which the constraints provided by ε in the optimization problem are guaranteed by the γ condition.
- $\varepsilon_1(\gamma)$: An estimate for ε_1 (up to a precision $\delta = 0.01$) calculated using the algorithms in Section 5 (Algorithm 2). Although this threshold also stands for a bound for the values of ε whose constraints are redundant with the one related to γ, this value is more accurate than the previous one.
- $\varepsilon_2(\gamma)$: An estimation for ε_2 (up to a precision $\delta = 0.01$) computed using the algorithms in Section 5 (Algorithm 4). This threshold for ε indicates when the feasible region related to ε is strictly contained in the one related to γ. In other words, if the moderator chooses a value of ε lower than this bound, the γ constraint is always guaranteed and the DMs will move their opinions just to satisfy the ε conditions.

If we look at the column $\gamma = 0.1$, we can deduce that the feasible region will not change until $\varepsilon < \varepsilon_1(0.1) = 0.32$ and we cannot ensure that the solution of the optimization problem changes until $\varepsilon < \varepsilon_2(0.1) = 0.07$. In fact, Table 1 shows that the values of the cost function are the same for all $\varepsilon > 0.07$.

Furthermore, the inequality-based approach reveals that for $\varepsilon > \varepsilon_1'(0.1) = 0.4$ the constraints related to the distance between experts' opinions and collective opinions are guaranteed by the constraint corresponding to the parameter γ and therefore they could be omitted in the resolution of the mathematical programming model, resulting in an immediate reduction of the computational costs. In fact, if more precision was necessary, the polytope-based

Table 3
The costs with different values of ε and γ of (M-5) in Example 2.

	$\gamma = 0.3$	$\gamma = 0.25$	$\gamma = 0.2$	$\gamma = 0.15$	$\gamma = 0.1$	$\gamma = 0.05$
$\varepsilon = 0.30$	0.19	0.26	0.32	0.39	0.46	0.53
$\varepsilon = 0.25$	0.19	0.26	0.32	0.39	0.46	0.53
$\varepsilon = 0.20$	0.21	0.26	0.32	0.39	0.46	0.53
$\varepsilon = 0.15$	0.30	0.30	0.33	0.39	0.46	0.53
$\varepsilon = 0.10$	0.40	0.40	0.40	0.40	0.46	0.53
$\varepsilon = 0.05$	0.50	0.50	0.50	0.50	0.50	0.53

Table 4
Different bounds for ε when γ is fixed in (M-5) in Example 2.

	$\gamma = 0.3$	$\gamma = 0.25$	$\gamma = 0.2$	$\gamma = 0.15$	$\gamma = 0.1$	$\gamma = 0.05$
$\varepsilon'_1(\gamma)$	0.60	0.50	0.40	0.30	0.20	0.10
$\varepsilon_1(\gamma)$	0.55	0.46	0.37	0.28	0.19	0.10
$\varepsilon_2(\gamma)$	0.19	0.16	0.13	0.09	0.06	0.03

algorithm guarantees that it is possible to ignore all the constraints obtained when $\varepsilon > \varepsilon_1(0.1) = 0.32$. In other words, if the moderator considers that a maximum distance between DMs and the collective equal to 0.35 is sufficient when $\gamma = 0.1$, all the constraints related to ε could be erased from the optimization model when implementing it in a numerical solver because they are already granted by the consensus constraint. In fact, we have used the Clp solver with the Julia programming language (Bezanson, Edelman, Karpinski, & Shah, 2017) to implement the linearized version of (M-5) with and without ε constraints when $\varepsilon = 0.3$ and considering the threshold $\gamma = 0.1$. While the first model takes around 300 milliseconds to be solved, the second one only needs around 200 milliseconds, yet both provide the same solution as guaranteed by our algorithms.

Similarly, since for $\varepsilon < 0.07 = \varepsilon'_2(0.1)$ the feasible region related to ε is strictly contained in the one determined by $\gamma = 0.1$, the constraints associated with the latter parameter could be omitted in the resolution of the model. In this case, we have also solved the optimization model in both scenarios: considering both $\gamma = 0.1$ and $\varepsilon = 0.05$ constraints and only taking into account the $\varepsilon = 0.05$ constraints. As before, deleting the $\gamma = 0.1$ constraints has also implied accelerating the model by about 50%.

6.2. Example 2: The 3-dimensional GDM problem

This second GDM situation aims at explaining the relationship between the parameters ε and γ from a geometrical point of view. To do so, a consensus process involving three experts has been studied. Note that if three experts are considered, their preferences can be represented as a point in a 3-dimensional space (o_1, o_2, o_3). This makes it possible to provide a graphical visualization of the relationship between the parameters γ and ε.

In this case, we consider a set of experts $E = \{e_1, e_2, e_3\}$ whose weights are given by $W = (0.5, 0.05, 0.45)$. Their initial preferences are $(o_1, o_2, o_3) = (0.3, 0.6, 0.9)$ and the cost vector is $(c_1, c_2, c_3) = (1, 1, 1)$. The optimization problem (M-5) has been solved for different values of ε and γ, and the values obtained for the cost function have been compiled in Table 3.

Let us compare these results with the bounds obtained by the algorithms proposed in this paper. To do so, we have compiled in Table 4 the different bounds for ε obtained when γ is fixed and the step-size $\delta = 0.01$ is considered.

We provided a graphical analysis of these values by taking advantage of the 3-dimensional nature of this decision problem below. Let us fix $\gamma = 0.2$. According to Table 4, the region will not change for $\varepsilon \in [0.4, 1]$, and we are sure that it changes when $\varepsilon < 0.37$. Fig. 3, which has been plotted using the Mathematica software, shows the evolution of the corresponding feasible region

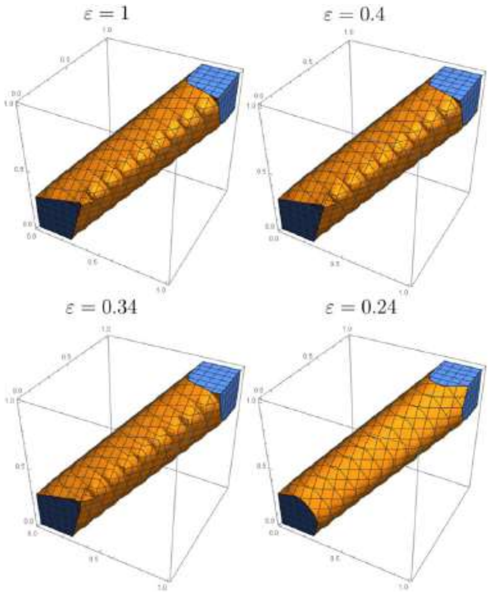

Fig. 3. The change of the region $\mathcal{R}_{\gamma,\varepsilon}$ for $\gamma = 0.2$ in Example 2.

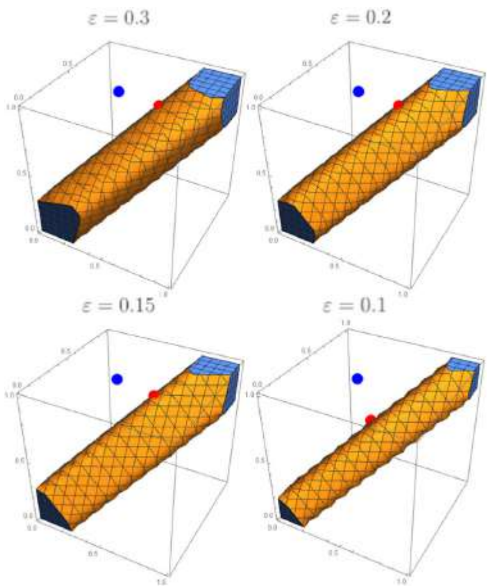

Fig. 4. The change of the region $\mathcal{R}_{\gamma,\varepsilon}$ and its solutions for $\gamma = 0.2$ in Example 2. The blue dot represents the original preferences and the red dot the solution for the given value of ε. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

$\mathcal{R}_{0.2,\varepsilon}$ for some values of ε. Note that the plots obtained for $\varepsilon = 1$ and $\varepsilon = 0.4$ are the same. However, the one obtained for $\varepsilon = 0.34$ is slightly different from the previous plots, and the plot corresponding to $\varepsilon = 0.24$ is completely different.

However, a change in the region does not necessarily imply a change in the solution. Fig. 4 shows the feasible region $\mathcal{R}_{0.2,\varepsilon}$, the position of the original preferences (blue dot), and the correspond-

Table 5
Computational time required for each GDM problem.

Decision-makers	5	10	50	100
$M-7$	10.0 ms	46.0 ms	16947.0 ms	480451.0 ms
$M-7'$	3.0 ms	7.0 ms	132.0 ms	531.0 ms

ing modified preferences obtained by solving the CMCC model (red dot). According to Table 4, we cannot guarantee that the solution will change until $\varepsilon < 0.13$, and this is exactly what the figure shows. Note that even though the values of ε are lower than $\varepsilon_1(0.2) = 0.37$ and we know that the feasible region is changing, the solution of the optimization problem is the same for $\varepsilon = 0.3$, $\varepsilon = 0.2$ and $\varepsilon = 0.15$. In contrast, the solution obtained for $\varepsilon = 0.1 < \varepsilon_2(\gamma) = 0.13$ changes.

6.3. Example 3: Simplifying CMCC in large-scale GDM

The following example is intended to show how the appropriate use of the relationship between the parameters ε and γ can be applied when solving a GDM problem to considerably improve computational time costs.

Let us consider a GDM problem which is going to be resolved by using the consensus model M-7. For a value $\varepsilon > 0$, Proposition 7 guarantees that, for any $\gamma \geq 2\varepsilon$, the model $M-7$ is equivalent to the following simplified version, which does not depend on the γ constraint:

$$\min_{(x_{ij}^k)\in\mathcal{M}_{n\times n}([0,1])} \sum_{k=1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} c_k|x_{ij}^k - p_{ij}^k|$$
$$s.t.\begin{cases} \bar{x}_{ij} = \sum_{k=1}^{m} w_k x_{ij}^k \\ |x_{ij}^k - \bar{x}_{ij}| \leq \varepsilon, k = 1,\dots,m, i = 1,\dots,n-1, j = i+1,\dots,n \end{cases} \tag{M-7$'$}$$

Note that, even though this model does not require the γ constraint, any solution for M-7′ ($\varepsilon > 0$) is also a solution for M-7 ($\varepsilon > 0$, $\gamma \geq 2\varepsilon$) and vice-versa.

To analyze the computational cost of using M-7 or M-7′, we implemented their respective linearized versions (Rodríguez et al., 2021) in the Julia 1.6 programming language (Bezanson et al., 2017) on the Google Colaboratory cloud service (Bisong, 2019) (2.20GHz Intel(R) Xeon(R) CPU and 13 GB RAM).

Four GDM problems were solved, each considering a different number of experts. For these scenarios, the experts' preferences were randomly generated, and the consensus parameters were set to $\varepsilon = 0.1, \gamma = 0.2$. We obtained the consensus solution to the different GDM situations for both M-7 and M-7′ by using the solver *GLPK* (GNU Linear Programming Kit) and we then measured the time required for the solver to compute the optimal solution in milliseconds (ms) (see Table 5).

The results show that the resolution of the M-7 model is much more time-consuming than the resolution of its modified version M-7′. While the former considerably increases the time cost when the number of experts also increases, the latter requires less than a second to provide a solution to optimization problems with the same characteristics as the former. In this case, understanding of the relationship between the parameters γ and ε has been key to shortening the computational time when dealing with many decision-makers.

7. Conclusions

CRPs, specifically MCC models, have been widely applied to GDM problems to achieve consensus solutions. Classical CRPs and MCC models fix different constraints using parameters specifically associated with the consensus level ($\gamma = 1 - \mu$) and the absolute distance (ε), to obtain a more general consensus solution. CMCC models include both types of constraints to obtain the agreed solution for the GDM problems. Nevertheless, the values of these parameters in CMCC have been traditionally fixed a priori by the moderator of the decision process according to the desired level of agreement among the stakeholders. However, our study shows that some configurations of such parameters may imply redundant constraints in the mathematical programming model, which, in practice, leads to higher computational costs. In this regard, our proposal defines a method to identify for which parameter pairs the optimization model can be simplified by removing such redundant inequalities. Understanding the proper dynamics of these parameters and corresponding constraints over the optimal consensus solution is key to applying these models in real-life large-scale group decision-making scenarios.

This paper has analyzed the relationship between both parameters γ and ε, and the associated constraints that determine the feasible region of the CMCC models. The use of inequalities for this analysis provides simple and straightforward relationships that are not very precise, but the analysis based on polytopes has provided a novel algorithm that relates the parameters of the linear constraints of any convex optimization problem, providing an accurate relationship between them. Based on the point of view of decision support in CMCC-based CRPs, the proposed algorithms could help the moderator set these parameters when one is given, and provide information on the impact of different configurations of these parameters in optimal solutions. Furthermore, understanding the relationship between these parameters also implies an immediate computational improvement because it allows redundant inequalities to be neglected and thus a better performance is achieved.

In the future, it would be interesting to analyze CMCC models that not only consider the minimization of the cost of changing opinions, but also attempt to minimize the number of opinion changes in light of the machinery developed in this work. Furthermore, investigations could be oriented to apply this algorithm to other optimization problems. For example, it would be interesting to develop a deeper analysis of the computational impact of removing the corresponding constraints when dealing with large-scale GDM problems where hundreds or thousands of DMs are considered. In this regard, such research could also highlight any differences when applying our algorithms to different preferences structures, such as utility vectors or FPRs.

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Supplementary material

Supplementary material associated with this article can be found, in the online version, at doi:10.1016/j.ejor.2022.08.015.

References

Beliakov, G., Calvo, T., & James, S. (2014). Consensus measures constructed from aggregation functions and fuzzy implications. *Knowledge-Based Systems, 55*, 1–8. https://doi.org/10.1016/j.knosys.2013.07.002.

Ben-Arieh, D., & Easton, T. (2007). Multi-criteria group consensus under linear cost opinion elasticity. *Decision Support Systems, 43*(3), 713–721. https://doi.org/10.1016/j.dss.2006.11.009. Integrated Decision Support

Bezanson, J., Edelman, A., Karpinski, S., & Shah, V. B. (2017). Julia: A fresh approach to numerical computing. *SIAM Review, 59*(1), 65–98. https://doi.org/10.1137/141000671.

Bisong, E. (2019). *Building machine learning and deep learning models on Google cloud platform*. Springer.

Bryson, N. (1996). Group decision-making and the analytic hierarchy process: Exploring the consensus-relevant information content. *Computers & Operations Research, 23*(1), 27–35. https://doi.org/10.1016/0305-0548(96)00002-H.

Butler, C., & Rothstein, A. (2006). *On conflict and consensus: A handbook on formal consensus decision making*. Takoma Park.

Chiclana, F., Mata, F., Martinez, L., Herrera-Viedma, E., & Alonso, S. (2008). Integration of a consistency control module within a consensus model. *International Journal of Uncertainty, Fuzziness and Knowlege-Based Systems, 16*(SUPPL. 1), 35–53. https://doi.org/10.1142/s0218488508005236.

Giorgi, G., Guerraggio, A., & (Eds), J. T. (2004). Chapter II - convex sets. convex and generalized convex functions. In *Mathematics of optimization* (pp. 23–200). Amsterdam: Elsevier Science. https://doi.org/10.1016/B978-044450550-7/50002-8.

Gong, Z., Zhang, H., Forrest, J., Li, L., & Xu, X. (2015). Two consensus models based on the minimum cost and maximum return regarding either all individuals or one individual. *European Journal of Operational Research, 240*(1), 183–192. https://doi.org/10.1016/j.ejor.2014.06.035.

Henk, M., Richter-Gebert, J., & Ziegler, G. M. (2018). Basic properties of convex polytopes. In J. E. Goodmanand, J. O'Rourke, & C. D. Tóth (Eds.), *Handbook of discrete and computational geometry*. In *CRC Press series on discrete mathematics and its applications* (3rd ed.). CRC Press.

Herrera-Viedma, E., Herrera, F., & Chiclana, F. (2002). A consensus model for multiperson decision making with different preference structures. *Systems, Man and Cybernetics, Part A: Systems and Humans, IEEE Transactions on, 32*, 394– 402. https://doi.org/10.1109/TSMCA.2002.802821.

Kacprzyk, J. (1986). Group decision making with a fuzzy linguistic majority. *Fuzzy Sets and Systems, 18*(2), 105– 118.

Kacprzyk, J., & Zadrożny, S. (2010). Soft computing and web intelligence for supporting consensus reaching. *Soft Computing, 14*, 833–846. https://doi.org/10.1007/s00500-009-0475-4.

Labella, Á., Liu, H., Rodríguez, R. M., & Martínez, L. (2020). A cost consensus metric for consensus reaching processes based on a comprehensive minimum cost model. *European Journal of Operational Research, 281*, 316–331. https://doi.org/10.1016/j.ejor.2019.08.030.

Labella, Á., Liu, Y., Rodríguez, R. M., & Martínez, L. (2018). Analyzing the performance of classical consensus models in large scale group decision making: A comparative study. *Applied Soft Computing, 67*, 677–690. https://doi.org/10.1016/j.asoc.2017.05.045.

Palomares, I., Estrella, F., Martinez, L., & Herrera, F. (2014). Consensus under a fuzzy context: Taxonomy, analysis framework AFRYCA and experimental case of study. *Information Fusion, 20*. https://doi.org/10.1016/j.inffus.2014.03.002.

R Core Team (2017). *R: A language and environment for statistical computing*. Vienna, Austria: R Foundation for Statistical Computing.

Robere, R. (2018). vertexenum: Vertex enumeration of polytopes. *R package version 1.0.2*. https://cran.r-project.org/package=vertexenum

Rodríguez, R. M., Labella, Á., Dutta, B., & Martínez, L. (2021). Comprehensive minimum cost models for large scale group decision making with consistent fuzzy preference relations. *Knowledge-Based Systems, 215*, 106780.

Zhang, B., Dong, Y., & Xu, Y. (2013). Maximum expert consensus models with linear cost function and aggregation operators. *Computers & Industrial Engineering, 66*(1), 147–157. https://doi.org/10.1016/j.cie.2013.06.001.

Zhang, G., Dong, Y., & Xu, Y. (2012). Linear optimization modeling of consistency issues in group decision making based on fuzzy preference relations. *Expert Systems with Applications, 39*(3), 2415–2420. https://doi.org/10.1016/j.eswa.2011.08.090.

Zhang, G., Dong, Y., Xu, Y., & Li, H. (2011). Minimum-cost consensus models under aggregation operators. *IEEE Transactions on Systems, Man, and Cybernetics - Part A: Systems and Humans, 41*(6), 1253–1261. https://doi.org/10.1109/TSMCA.2011.2113336.

Zhang, H., Dong, Y., Chiclana, F., & Yu, S. (2019). Consensus efficiency in group decision making: A comprehensive comparative study and its optimal design. *European Journal of Operational Research, 275*(2), 580–598. https://doi.org/10.1016/j.ejor.2018.11.052.

Zhang, N., Gong, Z., & Chiclana, F. (2017). Minimum cost consensus models based on random opinions. *Expert Systems with Applications, 89*, 149–159. https://doi.org/10.1016/j.eswa.2017.07.035.

Ziegler, G. M. (1995). Lectures on polytopes: Updated seventh printing of the first edition. *Graduate texts in mathematics 152* (1st ed.). New York: Springer-Verlag.

4.5 FORMULACIÓN PARA MODELOS CCM BASADA EN CONJUNTOS DIFUSOS.

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A Fuzzy-set based formulation for minimum cost consensus models☆

Diego García-Zamora *, Bapi Dutta, Álvaro Labella, Luis Martínez

University of Jaén, Department of Computer Science, Spain

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ABSTRACT

Minimum Cost Consensus (MCC) models are a popular approach to obtaining consensus in Group Decision-Making (GDM) problems, but previous extensions of these models have not been thoroughly analyzed in terms of their relationships and generalizability, which limits the practical application of these models. This paper presents a reformulation of MCC models for GDM problems using the Fuzzy Set Theory. The proposed fuzzy-based MCC (FZZ-MCC) framework offers a clearer understanding of MCC models and their extensions, as well as a rigorous and flexible methodology for addressing various types of GDM problems. The applicability of the FZZ-MCC framework is demonstrated through three practical examples related to e-democracy, personnel selection, and green supplier selection.

1. Introduction

In Group Decision-Making (GDM) problems, a group of DMs is asked to provide their preferences about the different alternative solutions for a certain problem (He et al., 2021; Song et al., 2023). However, the consideration of Decision-Makers (DMs) in decision problems usually implies the emergence of disagreements among their opinions. To manage such conflicts, it is necessary to develop a Consensus Reaching Process (CRPs) that ensures that the collective decision is accepted by the members of the group (Herrera-Viedma et al., 2002). Such CRPs are discussion processes whose aim is to increase the group consensus degree, which is usually computed using a consensus measure. The main goal of the CRP is to modify the DMs' initial opinions until the measured consensus degree surpasses a predefined consensus threshold (Herrera-Viedma et al., 2002). There are essentially two strategies to modify DMs' preferences (Palomares et al., 2014): (1) carrying out a feedback process, in which a moderator detects conflicting opinions and provides recommendations to the DMs, and (2) applying automatic changes according to a certain algorithm.

Minimum Cost Consensus (MCC) models are automatic CRPs for GDM that guarantee that DMs' opinions are modified as little as possible by satisfying a certain consensus constraint (Ben-Arieh & Easton, 2007). Due to their simplicity and unique vision of a CRP as a mathematical optimization problem, the original MCC models (Ben-Arieh & Easton, 2007) have been extended in order to deal with various demands in GDM (Zhang et al., 2020). For example, Zhang et al. (2011) analyzed the use of different aggregation operators when computing the collective opinion. On the other hand, Labella et al. (2020) proposed the so-called Comprehensive MCC (CMCC) models by arguing that the maximal deviation considered in classical MCC models does not ensure that classical consensus measures achieve a certain consensus threshold, and thus they included another constraint in the optimization model.

Since real-world group decision scenarios may have different characteristics, the underlying assumptions and constraints of the MCC models may not hold in all situations. For this reason, the popularity of extensions of MCC models has increased during recent years (Cheng et al., 2018; Ji et al., 2021). Although, these proposals have their own characteristics such as different preference structures (Wang et al., 2021) or some additional constraints to control the consensus level (Labella et al., 2020), all of them present several common elements whose relations and generalization have not been sufficiently analyzed yet. However, to increase the applicability and robustness of MCC models across various contexts, it is essential to generalize them. This can be achieved by introducing new constraints or objectives (Garcia-Zamora et al., 2022), or adapting the consensus process to account for the specific DMs' characteristics, such as weights (Zhang et al., 2011), or the environment, such as the required consensus level (Rodríguez et al., 2021). By doing so, we can streamline and unify context-specific MCC

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* Corresponding author.
E-mail addresses: dgzamora@ujaen.es (D. García-Zamora), bdutta@ujaen.es (B. Dutta), alabella@ujaen.es (Á. Labella), martin@ujaen.es (L. Martínez).

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models, making it possible to apply MCC models to a wide range of real-world GDM problems, while also gaining a deeper understanding of the underlying mechanisms involved in the consensus process (Xiao et al., 2022).

Therefore, in this paper, we aim at proposing a theoretical framework that allows generalizing MCC models and also adapting them for dealing with specific decision situations. Concretely, the research questions this contribution aims to answer are as follows.

- RQ1 How to propose a common rigorous framework to generalize MCC?
- RQ2 How to adapt such an abstract approach to address new GDM problems?

Fuzzy Set Theory allows both formalizing the idea of preference structure and expressing the different relevant functions required in MCC models, such as cost or consensus measure, in terms of mappings defined on Fuzzy Sets. Therefore, in this proposal, we apply Fuzzy Set Theory (Zadeh, 1996) to reformulate, from an abstract and rigorous point of view, the common parts of the different versions of MCC models. This fuzzy-set-based reformulation of MCC models (FZZ-MCC) not only provides a better understanding of the relations of the classic components in MCC but also yields a clear framework to propose new MCC-based models which are able to address different types of GDM situations. Additionally, we provide diverse application examples to illustrate the applicability of the FZZ-MCC framework to different decision-making problems. Such examples aim to demonstrate the versatility and effectiveness of our approach and highlight its potential applications in various decision-making contexts.

In summary, the main novelties introduced here are:

1. The key notions related to MCC models are redefined from the Fuzzy Set Theory point of view.
2. An abstract scheme for MCC models based on Fuzzy Set Theory is introduced.
3. This Fuzzy Set based formulation of MCC models, FZZ-MCC in short, is then utilized for solving three real-world GDM. Concretely, we adapt the FZZ-MCC framework for dealing with the following situations:
   - (a) Urban planning selection through e-democracy. FZZ-MCC models are adapted to manage thousands of preferences elicited through Multiplicative Preferences Relations (MPRs).
   - (b) Analyzing the cost of persuading a hiring committee about selecting a certain manager. FZZ-MCC models are applied to a decision problem involving the idea of driving DMs toward an agreement on a target solution.
   - (c) Combining expert's knowledge and data in green supplier selection. By adapting the FZZ-MCC framework, we propose a hybrid consensus model to combine ratings obtained from a database with the pairwise comparisons provided by DMs.

The remainder of this paper is broken down as follows: Section 2 introduces elementary concepts to facilitate the exposition of the proposal. Section 3 presents a novel framework for MCC models based on Fuzzy Set Theory. Afterward, this framework has been applied to the resolution of different real-world decision-making problems in Section 4. Finally, Section 5 draws some conclusions and future research.

2. Preliminaries

This section reviews some general concepts on GDM, its CRPs, and MCC proposals. In addition, some basic notions related to Fuzzy Sets Theory are also included.

2.1. GDM problems and CRPs

Formally, a GDM problem consists of a set of n, $(n \in \mathbb{N}, n \geq 2)$ alternatives $X = \{x_1, x_2, \dots, x_n\}$ and a set of m, $(m \in \mathbb{N}, m \geq 2)$ DMs $E = \{e_1, e_2, \dots, e_m\}$ who rate such alternatives by using a certain preference structure. In the following, GDM problems will be identified with their corresponding pairs (E, X) (García-Zamora et al., 2022).

In GDM, DMs have often different views about the solution to the problem, which implies the emergence of disagreements. If these conflicts are neglected in the resolution, some DMs may be unsatisfied with the solution achieved and do not support its implementation, leading to unsuccessful results (Butler & Rothstein, 2006; Saint & Lawson, 1994). To avoid this situation, a discussion process, so-called CRP (Herrera-Viedma et al., 2002), is included before the selection process. Such a CRP has been classically depicted as a process that aims to increase the level of agreement within the group by modifying DMs' initial opinions to bring their positions closer together (Yang et al., 2022).

Palomares et al. (2014) proposed a taxonomy for categorizing CRPs based on the type of advising process and the consensus measure used. The advising process can be either a feedback mechanism, in which DMs are asked if they want to change their preferences, or an automatic advising mechanism, in which changes are applied to DMs' preferences without asking DMs (García-Zamora et al., 2022). Consensus measures can be divided into two classes: (i) Measures of class 1, in which the consensus level is determined by comparing DMs' preferences with the collective opinion, and (ii) measures of class 2, in which the consensus level is determined by comparing DMs' preferences with each other (Palomares et al., 2014).

2.2. Minimum cost consensus models

MCC models are automatic CRPs that reformulate a GDM problem in terms of a mathematical programming problem. These models were firstly proposed by Ben-Arieh and Easton (2007) and they aim at minimizing the cost of moving DMs preferences to guarantee that the individual modified opinions are close enough (in terms of a predefined threshold, $\varepsilon \in]0,1]$) to the group opinion. Concretely, for the initial values of the preferences $(o_1, o_2, \dots, o_m) \in \mathbb{R}^n$ and a cost vector $(c_1, c_2, \dots, c_m) \in \mathbb{R}^n_+$, the MCC model was defined as follows:

$$\begin{aligned} &\min_{o'} \sum_{k=1}^{m} c_k |o'_k - o_k| \\ &s.t.\ |o'_k - \overline{o'}| \leq \varepsilon, k = 1, 2, \dots, m \end{aligned} \tag{MCC}$$

where $(o'_1, \dots, o'_m)$ are the adjusted opinions of the DMs, $\overline{o'}$ represents the collective opinion computed through a weighted mean operator and $\varepsilon \in]0,1]$ is the maximum absolute deviation of each DM and the collective opinion.

Lately, Zhang et al. (2011) studied the impact of the chosen aggregation operator on the resolution of the optimization model and proposed a more general reformulation of the former MCC model:

$$\begin{aligned} &\min_{o'} \sum_{k=1}^{m} c_k |o'_k - o_k| \\ &s.t. \begin{cases} \overline{o'} = F(o'_1, \dots, o'_m), \\ |o'_k - \overline{o'}| \leq \varepsilon, k = 1, 2, \dots, m, \end{cases} \end{aligned} \tag{MCC:AO}$$

where $\overline{o'}$ is now computed by using different aggregation operators $F : [0,1]^m \to [0,1]$.

Even though these models allow reformulating GDM situations as mathematical programming problems, the constraint defined by ε is quite simple and cannot assure that a certain consensus threshold $\mu_0 \in [0,1[$ is achieved by the group. To overcome such a limitation, the CMCC model was introduced by Labella et al. (2020). Such model includes the use of another constraint to control such consensus threshold:

$$\min_{o'} \sum_{k=1}^{m} c_k |o'_k - o_k|$$
$$s.t. \begin{cases} \overline{o'} = F(o'_1, \dots, o'_m), \\ |o'_k - \overline{o'}| \leq \varepsilon, k = 1, 2, \dots, m \\ consensus(o'_1, \dots, o'_n) \geq \mu_0, \end{cases} \tag{CMCC}$$

where $consensus(\cdot)$ represents the desired consensus measure. In addition to these models, there are other proposals in the literature. For instance, some of them are devoted to guaranteeing that the solution to the optimization model satisfies a minimum consistency degree (Rodríguez et al., 2021). Furthermore, there are some proposals aim at considering asymmetric cost functions as the objective of the mathematical programming problem (Ji et al., 2021) and others which adapt the MCC proposal to deal with linguistic information (Wang et al., 2021) or define performance metrics of CRPs (Labella et al., 2020).

2.3. Fuzzy sets and distance measures

In this subsection, some basics about the Fuzzy Sets Theory (Zadeh, 1996) are introduced in order to easily understand their use in our proposal. A fuzzy set on a certain universe of discourse X is usually understood as a generalization of the idea of membership function of a set. Note that each subset $A \subset X$ can be identified with the function $A : X \to \{0, 1\}$ defined as

$$A(x) = \begin{cases} 1 \text{ if } x \in A, \\ 0 \text{ if } x \notin A. \end{cases}$$

Fuzzy sets aim at considering membership functions whose possible values are not necessarily equal to 0 or 1, i.e., their image set can be any subset of the interval $[0, 1]$.

Definition 1 (*Fuzzy Sets Wang et al., 2009*)**.** Let X be the universe of discourse. A mapping $A : X \to [0, 1]$ is called a fuzzy set on X. The value $A(x)$ of A at $x \in X$ stands for the degree of membership of x in A. The set of all fuzzy sets on X will be denoted by $\mathcal{F}(X)$.

In this paper, fuzzy sets will be used as a model for representing the preferences elicited from the DMs participating in a GDM problem. Therefore, there are two essential operations required: fusing the information provided in several fuzzy sets and computing the similarities and distances between them. To address the fusion of information, we will use the notion of aggregation operator:

Definition 2 (*Aggregation Operator Bustince et al., 2008*)**.** $M : [0, 1]^m \to [0, 1]$ such that

- $M(x) = 0$ if and only if $x = (0, 0, \dots, 0)$,
- $M(x) = 1$ if and only if $x = (1, 1, \dots, 1)$,
- For any $x, y \in [0, 1]^m$ such that $x_k \leq y_k, k = 1, 2, \dots, n$, then $M(x) \leq M(y)$.

To make comparisons between two fuzzy sets, we will use the concept of distance measure:

Definition 3 (*Distance Measure Bustince et al., 2008*)**.** A function $D : \mathcal{F}(X) \times \mathcal{F}(X) \to [0, 1]$ is called a distance measure on $\mathcal{F}(X)$ if satisfies:

1. $D(A, B) = D(B, A)$,
2. $D(A, B) = 0$ if and only if $A = B$,
3. $D(A, B) = 1$ if and only if A and B are complementary non-fuzzy sets,
4. If $A \leq A' \leq B' \leq B$, then $D(A, B) \geq D(A', B')$.

These distance measures for fuzzy sets are based on Restricted Dissimilarity Functions, a sort of "distance" functions for real numbers.

Definition 4 (*Restricted Dissimilarity Function Bustince et al., 2008*)**.** A function $\delta : [0, 1] \times [0, 1] \to [0, 1]$ is called a restricted dissimilarity function if satisfies:

1. $\delta(x, y) = \delta(y, x)$,
2. $\delta(x, y) = 0$ if and only if $x = y$,
3. $\delta(x, y) = 1$ if and only if $\{x, y\} = \{0, 1\}$,
4. For all $x, y, z \in [0, 1]$ such that $x \leq y \leq z$, then $\delta(x, y) \leq \delta(x, z)$ and $\delta(y, z) \leq \delta(x, z)$.

For instance, the functions in the family $\delta_p(x, y) = |x^p - y^p|^{\frac{1}{p}} \ \forall\, x, y \in [0, 1]$, $p \in \mathbb{N}$ are restricted dissimilarity functions. Therefore, to propose distance measures for fuzzy sets, it is possible to extend their analogous counterpart for real numbers by using aggregation operators:

Theorem 1 (*Bustince et al., 2008*)**.** *Let $M : [0, 1]^m \to [0, 1]$ be an aggregation operator and consider $\delta : [0, 1] \times [0, 1] \to [0, 1]$ be a restricted dissimilarity function. Then, in any finite universe of discourse $X = \{x_1, x_2, \dots, x_m\}$, the function $D : \mathcal{F}(X) \times \mathcal{F}(X) \to [0, 1]$ defined as $D(A, B) = M_{k=1}^{m} \delta(A(x_k), B(x_k))$ is a distance measure.*

Even though it is also possible to use the notions of Similarity and Restricted Equivalence Function (Bustince et al., 2008) to deal with consensus, here we consider, for the sake of clarity, that they are essentially obtained by composing the standard negation ($N : [0, 1] \to [0, 1]$, $N(x) = 1 - x \ \forall\, x \in [0, 1]$) with Distance Measures and Restricted Dissimilarity Functions, respectively.

3. Generalized minimum cost consensus

MCC models are commonly used in various fields to reach a common decision or agreement among a group of DMs with conflicting preferences or objectives (Zhang et al., 2020). However, the involved DMs may have different characteristics that can affect their ability to participate in the consensus process. In addition, the decision scenario may require certain underlying assumptions and constraints. For this reason, it is necessary to generalize MCC models to make them more robust and applicable in diverse contexts. This can involve relaxing some assumptions, introducing new constraints or objectives, or adapting the consensus algorithms to the specific characteristics of the DMs or the decision problem environment.

Therefore, this section proposes a rigorous and deep reformulation of MCC models by redefining their generic concepts (Zadeh, 1996) based on Fuzzy Set Theory, which generalizes these consensus models. First, the main notions related to these models, namely preference structure, aggregation function, consensus measures, and cost functions, are redefined according to the fuzzy set nomenclature. Afterward, all these concepts are integrated into a generalized version of MCC models.

3.1. The main elements of MCC models

In order to make the MCC models applicable to a wider range of real-world situations and decision-making scenarios, it is essential to generalize the following key elements (see Fig. 1):

- Preference Structure. The preference structure refers to the way in which the DMs provide their opinions on the alternatives under consideration. This can take many forms, such as numerical ratings, rankings, or qualitative descriptions (Rodríguez et al., 2021).

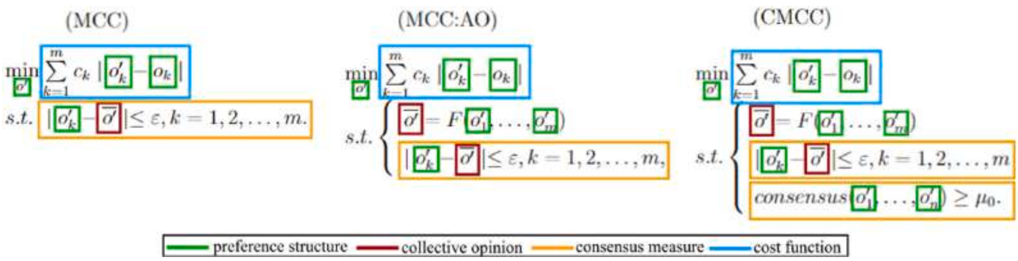

Fig. 1. MCC notions.

- Collective Opinion: Once DMs' individual preferences have been obtained, they need to be aggregated into a single group preference. There are many ways to aggregate individual preferences, such as taking the average or median of the ratings or using a weighted average, among others (Zhang et al., 2013).
- Consensus Measure: The consensus measure is the way in which the level of agreement within the group is calculated. This can be computed taking into account the distance between individual preferences or the distance between individual preferences and group opinion (García-Zamora et al., 2023).
- Cost Function: The cost function measures the cost of modifying the DMs' opinions to reach a consensus. This can take multiple forms that depend on how the distances between opinions are measured (Cheng et al., 2018).

By generalizing these key elements of MCC models, the models become more flexible and applicable to a wide range of decision-making situations. This enhances the usefulness and practicality of the models in real-world decision-making scenarios. In the following subsections, a brief review of the use of such key elements in the classic decision-making literature is provided. A generalized version of these components which relies on the Fuzzy Set Theory is then proposed.

3.1.1. *Preference structure*

A preference structure in a GDM problem is the format in which DMs are asked to give their opinions. In this contribution, since we aim at solving optimization problems at ease, we focus on those preference structures dealing with discrete numeric information, which are based on eliciting DMs' opinions by using utility vectors or pairwise comparison matrices (Herrera-Viedma et al., 2002). Note that, as it occurs with fuzzy sets, discrete numeric preference structures can be normalized to consider values in the unit interval without any loss of generality. In this context, the natural way to study preference structures from the fuzzy logic point of view is to establish the alternative set X as the universe of discourse. Therefore, a discrete numeric preference structure will be a collection of fuzzy sets defined on X or $X \times X$:

Definition 5 (*Discrete Numeric Preference Structure*)**.** Given a GDM problem (E, X), a numeric preference structure for the alternative set X is a subset $\mathbb{P} \subset \mathcal{F}(X^d)$, where $d = 1, 2$.

Remark 1. Note that $d = 1$ models the case in which the DMs provide numeric vectors and the index set of X^1 is $\{1, 2, \ldots, n\}$. For $d = 2$ the DMs provide pairwise comparison matrices and the index set of X^2 is $\{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\}$, which can be identified with the set $\{1, 2, \ldots, n^2\}$. For clarity, when no confusion is possible, we will use the notation $\{1, 2, \ldots, n^d\}$ to refer to the index set of X^d to simplify the reading of this contribution.

Usually, it is necessary to emphasize that a preference has been provided by a certain DM. Therefore, we also identify the set of preferences given by the DMs with a fuzzy set on $E \times X^d$.

Definition 6 (*Numeric Ratings Associated To* $\mathbb{P}$)**.** Given a GDM problem (E, X) and a discrete numeric preference structure $\mathbb{P}$, a numeric rating is a fuzzy set $P : (E, X^d) \to [0, 1]$ such that for every $k = 1, 2, \ldots, m$, $P(e_k, \cdot) \in \mathbb{P}$. The set containing all the possible numeric ratings for the GDM (E, X) and the preference structure $\mathbb{P}$, which is a subset of $\mathcal{F}(E, X^d)$, will be denoted by $\mathcal{P}$ and called the numeric ratings set associated to $\mathbb{P}$.

A classic example of preference structure based on pairwise comparisons is Fuzzy Preference Relations (FPRs) (Orlovsky, 1978). To facilitate the understanding of the proposal, let us adapt the ideas of Discrete Numeric Preference Structure and Numeric Ratings to the context of FPRs.

Example 1. According to the previous definitions, the Discrete Numeric Preference Structure corresponding to FPRs is:

$$\mathbb{P}_A := \left\{ P \in \mathcal{F}(X \times X) \;:\; P(x_i, x_j) + P(x_j, x_i) = 1, i, j = \{1, 2, \ldots, n\} \right\}$$

and the numeric ratings set is given by:

$$\mathcal{P}_A := \left\{ P \in \mathcal{F}(E, X \times X) : P(e_k, x_i, x_j) + P(e_k, x_j, x_i) = 1, \right.$$
$$\left. \forall\, i, j = \{1, 2, \ldots, n\} \;\; \forall\, k = \{1, 2, \ldots, m\} \right\}.$$

Concretely, the FPR

$$P = \begin{pmatrix} 0.5 & 0.6 & 0.3 \\ 0.4 & 0.5 & 0.8 \\ 0.7 & 0.2 & 0.5 \end{pmatrix}$$

can be identified, according to the previous definitions, with the fuzzy set $P : X \times X \to [0, 1]$ defined as $P(x_i, x_j) = P_{ij}\ \forall\, i, j = 1, 2, 3$. In that case $P(x_1, x_2) = 0.6$, $P(x_1, x_3) = 0.3$ and $P(x_2, x_3) = 0.8$. If we aim at making explicit that the FPR P corresponds with the opinion given by the DM e_k, then P is identified with the fuzzy set $P : \{e_k\} \times X \times X \to [0, 1]$ and some of their values are $P(e_k, x_1, x_2) = 0.6$, $P(e_k, x_1, x_3) = 0.3$ and $P(e_k, x_2, x_3) = 0.8$. In other words, $P(e_k, x_i, x_j)$ is the DM e_k's preference degree of the alternative x_i with respect to the alternative x_j.

It must be highlighted that, although the previous definitions are based on fuzzy sets theory, they are general enough to model decision-making scenarios in which preferences are not necessarily elicited as fuzzy sets. Further details are provided in Section 4.

3.1.2. *Aggregation of information*

The essence of any GDM problem is to obtain a single collective opinion that takes into account the information provided by all DMs' preferences using a suitable rule (Herrera-Viedma et al., 2002). In this paper, the collective opinion formation process will be modeled by aggregation operators (Beliakov et al., 2016).

Definition 7 (*Collective Opinion*)**.** Let us consider a GDM problem (E, X), a preference structure $\mathbb{P} \subset \mathcal{F}(X^d)$ and an aggregation operator $M : [0, 1]^m \to [0, 1]$. For $P \in \mathcal{P}$, the fuzzy set $\overline{P} : X^d \to [0, 1]$ defined as

$\overline{P}(x_i) := M_{k=1}^{m} P(e_k, x_i)\ \forall\ i = 1, 2, \ldots, n^d$ is called the collective opinion of P under M. When an aggregation operator $M : [0,1]^m \to [0,1]$ preserves the preference structure, i.e. $\overline{P} \in \mathbb{P}$, the respective induced collective opinion operator $M : \mathbb{P}^m \to \mathbb{P}$ is well-defined.

It should be highlighted that using an aggregation operator on preferences elicited via a certain preference structure does not guarantee that the obtained collective opinion adheres to that preference structure. In other words, not every aggregation operator is suitable for preserving the constraints required by a certain preference structure.

Example 2. A collective opinion for FPRs can be computed with the arithmetic mean $A : \mathbb{P}_A^m \to \mathbb{P}_A$ defined as:

$$A(P_1, \ldots, P_m)(x_i, x_j) = \frac{1}{m}\sum_{k=1}^{m} P_k(x_i, x_j)\ \forall\ i, j \in \{1, 2, \ldots, n\},\ \forall\ P_1, \ldots, P_m \in \mathbb{P}_A.$$

3.1.3. Consensus measurement

In some real-world GDM problems, it is essential to guarantee that the collective solution does not neglect the particular satisfaction of the DMs participating in the decision process. To provide these agreed solutions, several consensus models for GDM have been developed in the classic literature, which rely on the use of consensus measures to compute the degree of the agreement existing among DMs (Herrera-Viedma et al., 2002).

To reformulate the idea of consensus measure from the fuzzy set point of view, the main purpose must be computing a sort of average distance between DMs' opinions. Therefore, given a preference structure $\mathbb{P}$, a function to measure the consensus should be a mapping $\kappa : \mathcal{P} \to [0,1]$ which computes the degree of dissimilarity among the elements in the image set $P(E, X^d)$. Concretely, the consensus measures based on computing the distance between individual opinions and collective preference are defined as follows.

Definition 8 (*Consensus Metric of Class 1*)**.** Given a GDM problem (E, X) and a preference structure $\mathbb{P}$, the consensus metric of class 1 for $\mathcal{P}$ associated to an aggregation operator $\hat{M} : [0,1]^{mn^d} \to [0,1]$ and the restricted dissimilarity function δ is a function $\kappa : \mathcal{P} \to [0,1]$ defined as

$$\kappa(P) = \hat{M}_{k=1,\ldots,m;i=1,\ldots,n^d}\, \delta(P(e_k, x_i), \overline{P}(x_i))\ \forall\ P \in \mathcal{P},$$

where $\overline{P} \in \mathbb{P}$ is the collective opinion of P under an aggregation operator $M : [0,1]^m \to [0,1]$.

In addition, the consensus measures based on the distances between individual opinions are adapted to the fuzzy formulation in the following way.

Definition 9 (*Consensus Metric of Class 2*)**.** Given a GDM problem (E, X) and a preference structure $\mathbb{P}$, the consensus metric of class 2 for $\mathcal{P}$ associated to the aggregation operators $M : [0,1]^{m^2} \to [0,1]$ and $\hat{M} : [0,1]^{n^d} \to [0,1]$ and the restricted dissimilarity function δ is a function $\kappa : \mathcal{P} \to [0,1]$ defined as

$$\kappa(P) = M_{k,l=1}^{m} \hat{M}_{i=1}^{n^d}\, \delta(P(e_k, x_i), P(e_l, x_i))\ \forall\ P \in \mathcal{P}.$$

Example 3. As an example of these consensus metrics for FPRs, we can consider the function $\kappa : \mathcal{P}_A \to [0,1]$ defined as

$$\kappa(P) = \sum_{k=1}^{m} \frac{1}{m} \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{1}{n(n-1)} |P(e_k, x_i, x_j) - \overline{P}(x_i, x_j)|,\ \forall\ P \in \mathcal{P}_A$$

which is obtained from Definition 8 by taking $\hat{M} : [0,1]^{mn^2} \to [0,1]$ as an arithmetic average operator and $\delta(x, y) = |x - y|\ \forall\ x, y \in [0,1]$. Here, $\overline{P}$ denotes collective opinion for the rating $P \in \mathcal{P}_A$.

The reformulation of a consensus measure as a consensus metric $\kappa : \mathcal{P} \to [0,1]$ does not only allow providing a general vision of the existing idea of consensus measure, but also gives rise to other ways of defining consensus measures besides the identified by Palomares et al. (2014). For instance, we could think about consensus measures devoted to computing the distances between the collective preferences of clusters inside the global group, which would be especially useful to deal with large-scale GDM problems (García-Zamora et al., 2022).

Even though in classic literature the consensus measures are exclusively computed according to the similarity between the opinions (Palomares et al., 2014), some authors have studied the group's consensus using the dissimilarity between them (Ben-Arieh & Easton, 2007). However, these approaches have not been considered proper consensus measures until now. The general notion of *consensus metric*, here introduced, allows unifying both perspectives and, even though our definition is based on dissimilarities, using the similarity obtained by composing with the standard negation does not modify the results of the optimization problem (Bustince et al., 2008).

3.1.4. Cost function

A cost function measures a weighted difference between the DMs' original opinions and their modified ones. It could be interpreted as a measure of DMs' satisfaction because the less their preferences are changed, the more pleased they will be with the chosen solution. To adapt this idea to the fuzzy set framework, the cost function should provide a value that increases with the distance between DMs' original opinions $P_0 \in \mathcal{P}$ and the modified opinions $P \in \mathcal{P}$. In addition, such a distance should be weighted according to a certain relative cost that depends on the respective DM:

Definition 10 (*Cost Function*)**.** Given a GDM problem (E, X), a preference structure $\mathbb{P}$, an initial numeric rating $P_0 \in \mathcal{P}$, a relative cost vector $c \in [0,1]^m$ satisfying $\sum_{k=1}^{m} c_k = 1$ and a distance measure $D : \mathbb{P} \times \mathbb{P} \to [0,1]$. A cost function is a mapping $\xi_c : \mathcal{P} \to [0,1]$ defined as $\xi_c(P, P_0) = \sum_{k=1}^{m} c_k D(P(e_k, \cdot), P_0(e_k, \cdot))$, For simplicity, we will use the notation $\xi : \mathcal{P} \to [0,1]$ to denote a cost function in which all the relative costs are equal, i.e., $c_k = \frac{1}{m}\ \forall\ k = 1, 2, \ldots, m$.

Example 4. An example of a cost function for FPRs may be the function $\xi : \mathcal{P}_A \to [0,1]$ may be obtained from Definition 10 by considering $c_k = \frac{1}{m}\ \forall\ k = 1, 2, \ldots, m$ as follows:

$$\xi(P) = \sum_{k=1}^{m} \frac{1}{m} \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{1}{n(n-1)} |, P(e_k, x_i, x_j) - P_0(e_k, x_i, x_j)|,\ \forall\ P \in \mathcal{P}_A$$

where P_0 is an initial numeric rating.

The previous definition demands the sum of the costs to be equal to one. The purpose of this constraint is a normalization to guarantee that the corresponding cost function is a mapping valued in $[0,1]$. In practice, the cost values are usually defined as the cost of modifying one unit DMs' opinions and thus their sum is not necessarily equal to one. However, in such a situation, it is possible to compute the corresponding normalized costs by dividing each individual cost by the sum of all of them. In any case, minimizing a non-normalized cost function and the corresponding normalized version here proposed are equivalent from the optimization point of view and consequently, such a normalization does not have any impact on the solution of the mathematical programming model.

3.2. Fuzzy-set-based formulation for MCC models

Here, we propose a global model that integrates the Fuzzy-Set-based reformulations of the aforementioned classical elements of MCC models. This approach not only offers a generalization of previous proposals but also streamlines the adaptation of new MCC models to address specific real-world problems.

Definition 11 (*Fuzzy-Set-Based MCC Model*). Let (E, X) be a GDM problem and a preference structure $\mathbb{P}$. Given an initial numeric rating $P_0 \in \mathcal{P}$, a cost function $\xi_c : \mathcal{P} \to [0,1]$ for P_0 and a family of $q \in \mathbb{N}$ consensus metrics $\kappa = (\kappa_1, \ldots, \kappa_q) : \mathcal{P} \to [0,1]^q$, the corresponding Fuzzy-Set-based MCC model is given by:

$$\begin{aligned} &\min_{P\in\mathcal{P}} \xi_c(P, P_0) \\ &s.t. \quad \kappa(P) \le (\varepsilon_1, \ldots, \varepsilon_q), \end{aligned} \tag{FZZ-MCC}$$

where $(\varepsilon_1, \ldots, \varepsilon_q) \in [0,1]^q$ is the vector of desired parameters to control the q consensus metrics.

Please note that the definition of the FZZ-MCC model requires a preference structure, an aggregation operator, a set of consensus metrics, and a cost function. Moreover, additional constraints may be added to the model as required (see Appendix and Section 4). However, while this formulation provides a straightforward approach for defining MCC models, various factors can affect the resolution of the optimization model. For example, the impact of the aggregation operator was studied by Zhang et al. (2013), and Rodríguez et al. (2021) investigated how the consensus measure affects the performance of MCC models, while García-Zamora et al. (2023) analyzed the relationship between some consensus parameters. Determining the best configuration for these elements is not feasible in general, and they should be selected based on the specific requirements of the decision problem at hand. Therefore, it is essential to carefully consider the factors that influence the model's resolution and to choose the appropriate elements based on the specific needs of the decision-making process.

Furthermore, the feasibility of the FZZ-MCC model is guaranteed by the convexity of the constraints and the objective function. However, determining its computational complexity is not feasible in general because of the general nature of the constraints and the objective function. Nevertheless, if specific assumptions, such as linearity, are considered, similar analyses as those conducted by previous works (Garcia-Zamora et al., 2022; Rodríguez et al., 2021) can be applied.

Now, a couple of examples are introduced to illustrate how the FZZ-MCC modeling generalizes classical MCC models. First, we derive the classic model for real values introduced by Ben-Arieh and Easton (2007).

Example 5. Let us consider a GDM problem (E, X) where $E = \{e_1, \ldots, e_m\}$, $X = \{x\}$, with the preference structure $\mathbb{P} = \mathcal{F}(\{x\})$, the arithmetic mean operator $\overline{P} : \mathcal{P} \to [0,1]$ defined as $\overline{P}(x) = \frac{1}{m}\sum_{k=1}^{m} P(e_k, x)$, the consensus metric $\kappa_1 : \mathcal{P} \to [0,1]$ $(q = 1)$ given by $\kappa_1(P) = \max_{k=1,\ldots,m}\left\{|P(e_k, x) - \overline{P}(x)|\right\}$, and the cost function $\xi_c : \mathcal{P} \to [0,1]$ defined as $\xi_c(P, P_0) = \sum_{k=1}^{m} c_k|P(e_k, x) - P_0(e_k, x)|$. Then, under this configuration, the resulting (FZZ-MCC) model is the original (MCC) proposed by Ben-Arieh and Easton (2007).

Even though classic MCC models are defined for real values (Ben-Arieh & Easton, 2007), other authors have demonstrated the use of FPRs (Labella et al., 2020). Below, we show that the FZZ-MCC approach can be easily adapted to manage FPRs too.

Example 6. Let us consider the GDM problem (E, X) where $E = \{e_1, \ldots, e_m\}$, $X = \{x_1, \ldots, x_m\}$, in which the preferences are elicited using FPRs $\mathbb{P}_A$. Select the aggregation operator $A : \mathbb{P}_A^m \to \mathbb{P}_A$ defined as $A(P_1, \ldots, P_m)(x_i, x_j) = \frac{1}{m}\sum_{k=1}^{m} P_k(x_i, x_j)\ \forall\, i, j \in \{1, 2, \ldots, n\}$, the consensus measures $\kappa_1, \kappa_2 : \mathcal{P}_A \to [0,1]$ defined as $\kappa_1(P) = \max_{i,j,k} |P(e_k, x_i, x_j) - \overline{P}(x_i, x_j)|$, $\kappa_2(P) = \sum_{k=1}^{m}\frac{1}{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{n(n-1)} |P(e_k, x_i, x_j) - \overline{P}(x_i, x_j)|$, and the cost function $\xi : \mathcal{P}_A \to [0,1]$ given by $\xi(P) = \sum_{k=1}^{m}\frac{1}{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{n(n-1)}|P(e_k, x_i, x_j) - P_0(e_k, x_i, x_j)|$, where P_0 is an initial numeric rating. Then, the obtained (FZZ-MCC) model corresponds to (CMCC) for FPRs with a class 1 consensus measure (Labella et al., 2020).

This (FZZ-MCC) framework extends the classic MCC models, which are very specific examples of the generalized version here proposed. In addition, due to its general nature, it is possible to adapt the (FZZ-MCC) model to other decision-making problems, if the involved functions are properly selected. Therefore, the following section is devoted to illustrating how to effectively modify the FZZ-MCC approach to address various real-world decision scenarios.

4. Applications

The FZZ-MCC approach, introduced in the previous section, provides a flexible and comprehensive method for formulating MCC models that can be adapted to a variety of decision situations. In this section, we demonstrate the versatility of the FZZ-MCC framework by using it as a basis for defining several models for different scenarios never considered before in the specialized literature. Concretely, this section focuses on:

- Providing MCC models to manage preferences elicited using multiplicative scales,
- Examining the cost of bringing DMs to a consensus on a desired solution,
- Introducing a hybrid consensus model that combines data from a database with the preferences of DMs, gathered using various preference structures.

To show the practical application of the FZZ-MCC approach, we provide three real-world decision scenarios, namely, e-democracy (Xu et al., 2018), hiring personnel (Li et al., 2019), and green supplier selection (Zhang et al., 2022). For all these decision problems, the used original preferences have been randomly generated. In addition, the optimization problems considered in this section have been solved by considering their respective linearized versions (Rodríguez et al., 2021).

4.1. E-democracy for urban planning selection

In today's digital age, e-democracy has become an increasingly popular approach for facilitating group decision-making. E-democracy refers to the use of electronic platforms and tools to promote public engagement, transparency, and accountability in the decision-making process (Xu et al., 2018). The goal of e-democracy is to enhance citizen participation in decision-making and to ensure that the voices of all DMs are heard. In this regard, a major challenge for e-democracy is the difficulty of reaching a consensus among a large number of DMs (García-Zamora et al., 2022). For this reason, this subsection adapts the FZZ-MCC approach to solve an e-democracy situation regarding urban planning.

In urban planning, decisions must be made regarding the use of public spaces, the allocation of resources, and the development of infrastructure. These decisions have far-reaching consequences that can impact the quality of life of local residents.

Specifically, this subsection addresses the allocation of a new public park in a central location in a city. The council has identified four potential sites for the park, each with its own advantages and disadvantages. The council needs to make a decision on which site to choose. The four potential sites for the park are:

- Site A: This site is located in a residential area and would require the demolition of several homes to make way for the park. The site is relatively small but is easily accessible by public transport and is close to several schools and community centers.
- Site B: This site is located on the outskirts of the city, near a major highway. The site is large and would not require the demolition of any homes, but it is less accessible by public transport and is not close to any schools or community centers.

- Site C: This site is located in a commercial area and would require the demolition of several small businesses. The site is moderately sized and easily accessible by public transport, but it is not close to any schools or community centers.
- Site D: This site is located on a brownfield site, previously used for industrial purposes, and would require significant remediation before development. The site is large, but the remediation work is likely to be costly, and it is not easily accessible by public transport. However, it is close to a large residential area.

The city council has decided that it should be the citizens who choose where to build the public park. Consequently, they have made a questionnaire available online so that any citizen may provide his/her opinions by using Multiplicative Preferences Relations (MPRs) in a 1–9 Saaty's scale (Saaty, 1988). At the end of the survey, 4000 citizens $E = \{e_1, e_2, \dots, e_{4000}\}$ give their preferences over the 4 sites $X = \{x_1 =$ Site A, $x_2 =$ Site B, $x_3 =$ Site C, $x_4 =$ Site D$\}$, i.e., $m = 4000$ and $n = 4$.

Although FZZ-MCC has been designed to manage preferences in a 0–1 for the sake of simplicity, working with preference structures whose values are not in the interval $[0,1]$ is also possible within our FZZ-MCC framework by using appropriate transformations between the preference structures and customization of the compatible FZZ-MCC components. Therefore, we consider the previously defined GDM problem (E, X) whose preference structures are represented by MPRs defined as follows:

$$\mathbb{P}_M := \left\{ P' : X \times X \to [\tfrac{1}{9}, 9] : P'(x_i, x_j)P'(x_j, x_i) = 1, \ \forall\, i, j = \{1, 2, \dots, n\} \right\}.$$

Consequently, a numeric rating $P' \in \mathcal{P}_M$ can be decomposed in m matrices (one for each DM) $P'_1, P'_2, \dots, P'_m$ of dimension $n \times n$ whose values are within the interval $[\frac{1}{9}, 9]$ satisfying the reciprocal condition $P'(e_k, x_i, x_j)P'(e_k, x_j, x_i) = 1$.

In order to compute a collective opinion, the geometric mean can be used. Let us consider a family of weights $w = (w_1, w_2, \dots, w_m)$, $\sum_{k=1}^{m} w_k = 1, w_k \geq 0 \ \forall\ k = 1, 2, \dots, m$. Then, for each pairwise preference rating $P' \in \mathcal{P}_M$, the corresponding collective opinion will be given by

$$\overline{P'} : X \times X \to [\frac{1}{9}, 9],$$

$$\overline{P'}(x_i, x_j) = \prod_{k=1}^{m} P'(e_k, x_i, x_j)^{w_k} \ \forall i, j = 1, 2, \dots, n.$$

Note that the function $d : [\frac{1}{9}, 9] \times [\frac{1}{9}, 9] \to [0,1]$ defined as $d(x, y) = \frac{1}{2}|\log_9(x/y)| \ \forall\ x, y \in [\frac{1}{9}, 9]$ behaves as a dissimilarity function in these intervals. Therefore, this function can be used to define consensus metrics. For instance, by using the maximum operator, it is possible to obtain the consensus metric $\kappa'_1 : \mathcal{P}_M \to [0,1]$ as

$$\kappa'_1(P') = \frac{1}{2} \max_{k=1,\dots,m; i,j=1,\dots,n} \left\{ |\log_9(P'(e_k, x_i, x_j)/\overline{P'}(x_i, x_j))| \right\},$$

$\forall P' \in \mathcal{P}_M$. Another possible definition for a consensus metric could be $\kappa'_2 : \mathcal{P}_M \to [0,1]$ given as

$$\kappa'_2(P') = \frac{1}{mn(n-1)} \sum_{k=1}^{m} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} |\log_9(P'(e_k, x_i, x_j)/\overline{P'}(x_i, x_j))| \forall P' \in \mathcal{P}_M$$

which considers distances between DMs opinions.

Finally, given an initial preference $P'_0 \in \mathcal{P}_M$, the cost function $\xi'_c : \mathcal{P}_M \to [0,1]$ could be defined as

$$\xi'_c(P') = \frac{1}{2n^2} \sum_{k=1}^{m} c_k \sum_{i,j=1}^{n} |\log_9(P'(e_k, x_i, x_j)/P'_0(e_k, x_i, x_j))|, \forall P' \in \mathcal{P}'.$$

Therefore, the corresponding MCC model could be presented as:

$$\begin{aligned} &\min_{P' \in \mathcal{P}_M} \ \xi'_c(P') \\ &s.t. \begin{cases} \kappa'_1(P') \leq \varepsilon_1 \\ \kappa'_2(P') \leq \varepsilon_2 \end{cases} \end{aligned} \tag{M-FZZ-MCC}$$

It is clear that such a model is a nonlinear programming problem, and thus its resolution could be imprecise and time-consuming, especially when handling a large number of inputs as it occurs in a large-scale GDM problem (Rodríguez et al., 2021). Therefore, we linearize it by considering the functions $f : [\frac{1}{9}, 9] \to [0,1]$ defined as $f(x) = \frac{1}{2}(1 + \log_9(x)) \ \forall\ x \in [\frac{1}{9}, 9]$ and its inverse $f^{-1} : [0,1] \to [\frac{1}{9}, 9]$ defined as $f^{-1}(x) = 9^{2x-1} \ \forall\ x \in [0,1]$. In fact, $f(\mathbb{P}_M)$ is the set $\mathbb{P}_A$ of FPRs defined in $[0,1]$. In addition,

$$\begin{aligned} f(\overline{P'}(x_i, x_j)) &= \frac{1}{2}(1 + \log_9(\prod_{k=1}^{m} P'(e_k, x_i, x_j)^{w_k})) \\ &= \frac{1}{2}(1 + \sum_{k=1}^{m} w_k \log_9(P'(e_k, x_i, x_j))) \\ &= \frac{1}{2}(1 + \sum_{k=1}^{m} w_k \log_9(9^{2P(e_k, x_i, x_j)-1})) \\ &= \sum_{k=1}^{m} w_k P(e_k, x_i, x_j) = \overline{P}(x_i, x_j) \end{aligned}$$

where $P = f(P') \in \mathcal{P}_A \ \forall\ P' \in \mathcal{P}_M$. Furthermore, the functions $\kappa_1, \kappa_2, \xi : \mathcal{P}_A \to [0,1]$ defined as $\kappa_1 := \kappa'_1 \circ f^{-1}$, $\kappa_2 := \kappa'_2 \circ f^{-1}$ and $\xi := C' \circ f^{-1}$ can be expressed as:

$$\kappa_1(P) = \max_{k=1,\dots,m; i,j=1,\dots,n} \left\{ |P(e_k, x_i, x_j) - \overline{P}(x_i, x_j)| \right\},$$

$$\kappa_2(P) = \frac{2}{mn(n-1)} \sum_{k=1}^{m} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} |P(e_k, x_i, x_j) - \overline{P}(x_i, x_j)|,$$

$$\xi_c(P, P_0) = \frac{1}{n^2} \sum_{k=1}^{m} c_k \sum_{i,j=1}^{n} |P(e_k, x_i, x_j) - P_0(e_k, x_i, x_j)|,$$

$\forall P \in \mathcal{P}_M$, where $P_0 = f(P'_0) \in \mathcal{P}$. Note that the obtained linearization

$$\begin{aligned} &\min_{P \in \mathcal{P}_M} \ \xi_c(P, P_0) \\ &s.t. \begin{cases} \kappa_1(P) \leq \varepsilon_1, \\ \kappa_2(P) \leq \varepsilon_2, \end{cases} \end{aligned} \tag{L-M-FZZ-MCC}$$

corresponds to one of the CMCC models proposed by Labella et al. (2020).

At this point, the consensus parameters are set up as $\varepsilon_1 = \varepsilon_2 = 0.2$, because it would be not reasonable to demand a high level of agreement that may not be achieved in practice with so many DMs. In addition, the relative cost of moving each DM's opinion and their weights is $\frac{1}{4000}$ because we consider that all the citizens should present be equally important in e-democracy.

After the linearization process, it is possible to determine the collective opinion in just a few seconds. In fact, it shows a similar time complexity as reported by Rodríguez et al. (2021). Fig. 2 shows a multidimensional scaling (MDS) (Borg & Groenen, 2005) visualization of the DMs' preferences before and after applying the model L-M-FZZ-MCC. Note that the DMs' preferences are much closer to the corresponding collective opinions when consensus is considered to compute the group's preference. According to the L-M-FZZ-MCC model, the place preferred by the citizens is Site C, the commercial area.

This case study illustrates that the FZZ-MCC approach can be used to consider MPRs, which have not been used before in the MCC literature to the authors' knowledge. In addition, most of the CRPs in the literature, even in large-scale GDM, consider decision problems involving less than 50 DMs (García-Zamora et al., 2022). In this regard, the FZZ-MCC approach can be applied to solve actual e-democracy problems involving thousands of citizens.

Theoretically, this case study highlights the potential of the FZZ-MCC framework as a unifying tool to analyze MCC models with diverse preference structures. Instead of creating distinct models for each preference structure, the FZZ-MCC framework enables researchers to

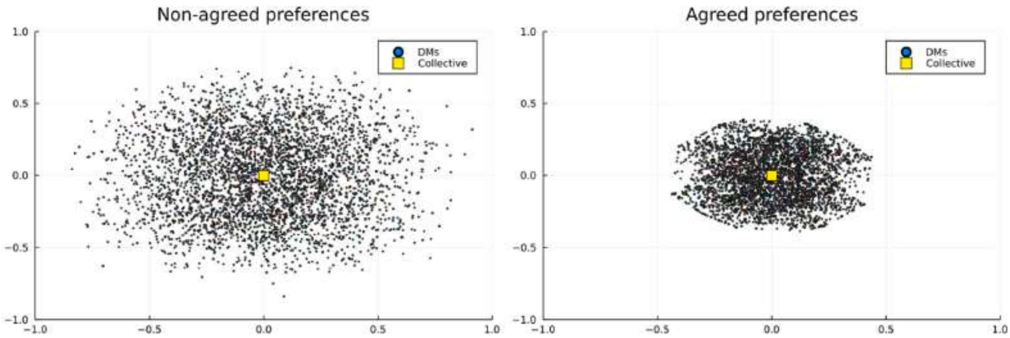

Fig. 2. MDS visualization of the DMs' preferences.

examine and compare these structures within a single framework. This approach promotes a more integrated and comprehensive understanding of MCC models, facilitating the development of new insights and hypotheses.

4.2. Persuading a committee to hire a manager

In any organization, selecting the right manager is crucial for its success (Li et al., 2019). The process of choosing a manager involves a committee that evaluates the candidates' qualifications, experience, and skills. However, when the council has a preference for a particular candidate, they may want to analyze if it is feasible to persuade the committee to select that candidate (Caillaud & Tirole, 2007; Chinn et al., 2018). In this subsection, we demonstrate the flexibility of our proposed generalized framework to design a new MCC model including the idea of persuasion. Specifically, we adapt our framework by designing a minimum-cost persuading model, which aims to shape the group's opinions as per a predefined target opinion.

Let us consider a corporation that operates in the manufacturing industry and is looking to hire a new manager for its production department. The committee responsible for hiring has received applications from four qualified candidates.

1. Candidate 1: John Smith — 10 years of experience in production management, with expertise in lean manufacturing and process optimization.
2. Candidate 2: Lisa Jones — 8 years of experience in production management with a successful track record of improving quality and reducing costs.
3. Candidate 3: Robert Brown — 12 years of experience in production management with a focus on safety and compliance.
4. Candidate 4: Amanda Green — 6 years of experience in production management with a strong background in project management.

The council of the corporation believes that Lisa Jones is the best candidate for the job, who had previously worked for the corporation and had an excellent track record, but the final decision must be made by the committee through a CRP. Such a CRP will be supervised by a moderator, who is selected by the council and will guide the DMs to reach an agreed solution. However, such a moderator, following the guidelines of the council, wants the committee to select Lisa Jones. For this reason, the moderator is interested in analyzing how much it would cost, in terms of time-consuming, effort, and resources, to convince the committee members about selecting each one of the candidates.

The committee that must make the decision is modeled by four DMs, $E = \{e_1, e_2, e_3, e_4\}$, who provide their preferences by using FPRs over the set of alternatives $\{x_1$: John Smith, x_2: Lisa Jones, x_3: Robert Brown, x_4: Amanda Green$\}$, i.e., $m = 4$ and $n = 4$.

To model this decision problem using the FZZ-MCC approach, let us consider the GDM problem (E, X) which uses FPRs

$$\mathbb{P}_A := \left\{ P : X \times X \to [0,1] : P(x_i, x_j) + P(x_j, x_i) = 1, i, j \in \{1, 2, \dots, n\} \right\}.$$

For $P \in \mathcal{P}_A$, the collective opinion may be computed by using the weighted arithmetic mean $\overline{P} : X \times X \to [0,1]$ defined as

$$\overline{P}(x_i, x_j) = \sum_{k=1}^{m} w_k P(e_k, x_i, x_j), \ \forall\ i, j = 1, 2, \dots, n.$$

The consensus metrics can be $\kappa_1, \kappa_2 : \mathcal{P}_A \to [0,1]$, defined as:

$$\kappa_1(P) = \max_{k=1\dots m} \max_{i,j=1\dots n} |P(e_k, x_i, x_j) - \overline{P}(x_i, x_j)|,$$

$$\kappa_2(P) = \frac{2}{mn(n-1)} \sum_{k=1}^{m} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} |P(e_k, x_i, x_j) - \overline{P}(x_i, x_j)|,$$

$\forall\ P \in \mathcal{P}_A$. In addition, given an initial preference $P_0 \in \mathcal{P}_A$, the cost function $\xi_c : \mathcal{P}_A \to [0,1]$ is defined as

$$\xi_c(P, P_0) = \sum_{k=1}^{m} c_k \frac{2}{n(n-1)} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} |P(e_k, x_i, x_j) - P_0(e_k, x_i, x_j)|,$$

$\forall\ P \in \mathcal{P}_A$, where c_k stands for the relative costs of moving DM e_k's preferences ($c_k \geq 0, k = 1, \dots, m, \sum_{k=1}^{m} c_k = 1$).

With this basic configuration of the FZZ-MCC framework, we are now ready to include the concept of persuasion. The goal is to analyze the cost of persuading DMs about reaching a consensus on a certain alternative x_{i_0}, $i_0 \in \{1, 2, \dots, n\}$, as the best choice among all the considered alternatives. This persuasion condition can be modeled by computing the group average preference of any alternative x_i regarding others via the function $\eta^i : \mathcal{P}_A \to [0,1]$ defined as

$$\eta^i(P) = \frac{1}{n} \sum_{j=1}^{n} \sum_{k=1}^{m} w_k P(e_k, x_i, x_j)\ \forall\ P \in \mathcal{P}_A,\ \forall\ i = 1, 2, \dots, n$$

and setting the conditions that the alternative x_{i_0} performs better than the rest of the alternatives as follows:

$$\eta^{i_0}(P) \geq \eta^i(P)\ \forall\ i = 1, 2, \dots, n\ (i \neq i_0).$$

These constraints in FZZ-MCC guarantee that the alternative x_{i_0} is preferred by the group and the corresponding optimization model is

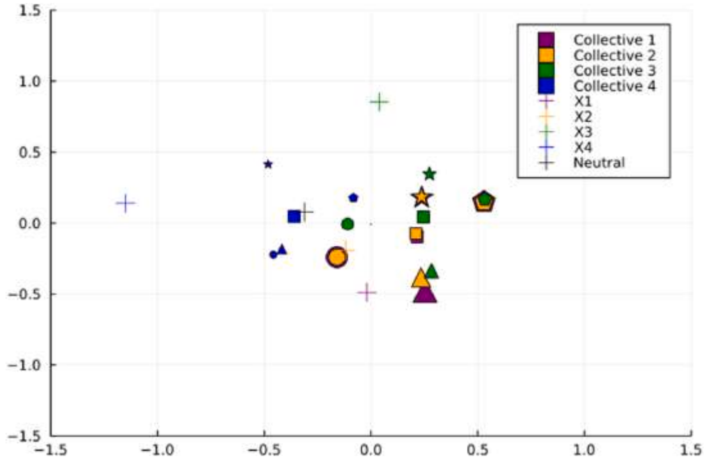

Fig. 3. Comparative visualization. Each shape stands for a different DM, except squares, which represent the collective opinion, and crosses, which represent the position of the extreme preference in which the corresponding alternative is completely preferred over the others and the others are equally preferred. Different colors correspond to different scenarios.

given as follows:

$$\min_{P\in\mathcal{P}} \xi_c(P,P_0)$$
$$s.t.\begin{cases}\kappa_1(P)\leq\varepsilon_1,\\ \kappa_2(P)\leq\varepsilon_2,\\ \eta^{i_0}(P)\geq\eta^{i}(P) \quad i\neq i_0.\end{cases} \tag{P-FZZ-MCC}$$

The committee's original preferences are given by FPRs:

$$P_1=\begin{pmatrix} - & 0.5 & 0.4 & 0.8\\ 0.5 & - & 0.4 & 0.8\\ 0.6 & 0.6 & - & 0.85\\ 0.2 & 0.2 & 0.15 & - \end{pmatrix} P_2=\begin{pmatrix} - & 0.95 & 1.0 & 1.0\\ 0.05 & - & 0.92 & 0.94\\ 0.0 & 0.08 & - & 0.58\\ 0.0 & 0.06 & 0.42 & - \end{pmatrix}$$
$$P_3=\begin{pmatrix} - & 0.6 & 0.33 & 1.0\\ 0.4 & - & 0.25 & 1.0\\ 0.67 & 0.75 & - & 1.0\\ 0.0 & 0.0 & 0.0 & - \end{pmatrix} P_4=\begin{pmatrix} - & 0.57 & 0.72 & 0.58\\ 0.43 & - & 0.67 & 0.51\\ 0.28 & 0.33 & - & 0.34\\ 0.42 & 0.49 & 0.66 & - \end{pmatrix}$$

The consensus conditions are, $\varepsilon_1=\varepsilon_2=0.2$, which guarantee a reasonable level of agreement, keeping in mind that the moderator have to try to convince DMs to select a specific candidate who, a priori, may not be their preferred one. The relative cost of modifying the opinions of each DM is equal, with a cost ratio of $\frac{1}{4}$, making it equally costly to adjust any of the DMs' opinions. In addition, we recall that the moderator wants to analyze the cost of persuading the committee to choose Candidate x_2, Lisa Jones. Therefore, to study the feasibility of persuading the committee about choosing her as the future manager, the moderator applies the P-FZZ-MCC model to analyze the relative cost of persuading the DMs about choosing x_2, i.e., $i_0=2$. In addition, the moderator also runs the P-FZZ-MCC for the remaining cases $i_0=1,3,4$ in order to also consider the relative cost of convincing the committee about all the others candidates.

At this point, let us apply the corresponding P-FZZ-MCC models to solve the GDM problem. Fig. 3 shows the evolution of the committee's preferences depending on the selected preferred candidate by means of an MDS representation (Borg & Groenen, 2005). Each color represents a different scenario: purple is for $i_0=1$ (candidate 1 is the most preferred by the council), orange for $i_0=2$ (candidate 2 is the most preferred by the council), green for $i_0=3$ (candidate 3 is the most preferred by the council), and blue for $i_0=4$ (candidate 4 is the most preferred by the council). The squares represent the collective opinion obtained in each scenario, whereas the crosses stand for the positions of the extreme preference in which the corresponding alternative is completely preferred over the others, which are equally preferred to each other. The preferences of each DM are represented by stars for e_1, triangles for e_2, pentagons for e_3 and circles for e_4. Note how every DMs' position moves toward the corresponding extreme preference in each scenario in order to obtain a collective opinion in which that candidate is preferred.

Table 1
Comparative results between Persuading and CMCC.

Consensus model	Consensus parameters	Desired alternative	Cost	Ranking of alternatives
P-FZZ-MCC	$\varepsilon_1=\varepsilon_2=0.2$	x_1	0.048	$x_1\succ x_2\succ x_3\succ x_4$
		x_2	0.072	$x_2=x_1\succ x_3\succ x_4$
		x_3	0.081	$x_3=x_1\succ x_2\succ x_4$
		x_4	0.173	$x_4=x_1=x_2\succ x_3$

Note that the collective opinions obtained in the scenarios $i_0=1$ and $i_0=2$ are pretty close, which means that these two are the most similar scenarios. On the contrary, the collective in the case $i_0=4$ is the one that is further from the other scenarios. In addition, such collective opinion is close to the neutral ideal preference (all alternatives are equally preferred).

Table 1 shows the numeric results of solving the P-FZZ-MCC model. The cost analysis reveals that convincing the committee to select the candidate x_2 : Lisa Jones implies a cost equal to 0.072, 50% extra relative cost regarding the committee's most preferred candidate, who is x_1, John Smith, because he is the one with the lowest cost (0.048). With this information, the council may evaluate if such 50% extra cost can be assumable or not to choose, in practice, the candidate x_2 instead of x_1.

This subsection has introduced the novel idea of persuading models as methods for GDM that aim at driving the DMs toward a target solution. The FZZ-MCC approach has been applied to propose a general

MCC model able to solve a decision problem related to influencing a staff recruitment process, which illustrates its flexibility to address new decision-making scenarios.

This case study highlights the practical value of the generalized FZZ-MCC model as a flexible and adaptable framework for designing new consensus models that meet specific decision-making requirements. By modifying key elements of the framework, researchers can easily tailor it to address the specific needs of a given decision context. This approach streamlines the development and analysis of new MCC models by emphasizing the crucial elements of change and facilitating connections with existing results. Thus, the generalized FZZ-MCC model offers a powerful tool for enhancing decision-making processes and advancing the field of consensus modeling.

4.3. Green supplier selection

Green supplier selection is the process of identifying and selecting suppliers who prioritize environmentally sustainable practices (Zhang et al., 2022). This involves evaluating suppliers based on their environmental performance and considering factors such as energy efficiency, waste management, and the use of hazardous materials. Green supplier selection can have a significant impact on a company's sustainability performance, as the environmental impact of a company's supply chain can be substantial. By selecting suppliers who prioritize sustainability, companies can reduce their environmental footprint and enhance their reputation among environmentally conscious consumers (Xing et al., 2022).

However, it can be difficult to assess suppliers' environmental performance and ensure that suppliers are adhering to sustainable practices. Additionally, some sustainable practices may be more costly for suppliers, which can affect the cost of goods and services for the purchasing company. For this reason, companies have to evaluate the performance of the suppliers over several aspects, and choose the one that best suits its needs.

In this section, we will explore how a company may successfully address green supplier selection through a hybrid model that integrates expert knowledge and data by adapting the FZZ-MCC approach.

Let us consider a manufacturing company that needs to select a new supplier for its raw materials. The company has identified four potential suppliers $X = \{x_1 = \text{Supplier A}, x_2 = \text{Supplier B}, x_3 = \text{Supplier C}, x_4 = \text{Supplier D}\}$, i.e., $n = 4$, each with their own strengths and weaknesses in terms of their environmental sustainability credentials:

- Supplier A: A local supplier with a strong track record of using renewable energy sources, but higher prices than other suppliers.
- Supplier B: A multinational supplier with an established sustainability program and a well-developed supply chain, but some concerns about the environmental impact of their transportation methods.
- Supplier C: A smaller supplier with a focus on sustainable production methods, but limited capacity to meet the company's needs.
- Supplier D: A relatively new supplier with a strong commitment to sustainability and innovative production methods, but some uncertainty about their long-term viability.

The company's procurement department has convened a group of stakeholders, including representatives from the company as well as external experts in sustainability and environmental issues, to evaluate each supplier and make a decision, $E_1 = \{e_1, e_2, e_3, e_4\}$, i.e., $m_1 = 4$, which will give their opinions using FPRs. In addition, the company also wants to consider the information within a database with ratings in a $[0, 1]$ scale for the suppliers obtained from other $m_2 = 2625$ companies and public institutions that have collaborated with them previously, $E_2 = \{e_5, e_6, \dots, e_{2629}\}$, i.e.,

$$E = E_1 \cup E_2 = \left\{ e_1, e_2, \dots, e_{m_1}, e_{m_1+1}, e_{m_1+2}, \dots, e_{m_1+m_2} \right\}.$$

Since the experts provide their opinions using FPRs it is necessary to consider the preference structure:

$$\mathbb{P}_1 := \mathbb{P}_A = \left\{ P : X \times X \to [0,1] : P(x_i, x_j) + P(x_j, x_i) = 1, \forall\, i, j \in \{1, 2, \dots, n\} \right\}.$$

In addition, since the ratings in the database are given in a $[0, 1]$ scale, we also need a preference structure in the form:

$$\mathbb{P}_2 := \left\{ P : X \to [0, \tfrac{1}{2}] \right\}.$$

Here, we have rescaled the ratings to the interval $[0, \frac{1}{2}]$ to be able to transform them into FPRs via the formula $\frac{1}{2} + P(x_i) - P(x_j)\ \forall\ P \in \mathbb{P}_2$. Note that, if we do not restrict the value $P(x_i)$ to be in the interval $[0, \frac{1}{2}]$, the result of such formula does not define an FPR (Herrera-Viedma et al., 2002).

Then, the collective opinion $\overline{(P_1, P_2)} : X \times X \to [0, 1]$ can be computed as:

$$\overline{(P_1, P_2)}(x_i, x_j) = \alpha \sum_{k=1}^{m_1} w_k P(e_k, x_i, x_j) + \frac{\beta}{m_2} \sum_{k=m_1+1}^{m_1+m_2} (\frac{1}{2} + P_2(e_k, x_i) - P_2(e_k, x_j)),$$

where $w_1, w_2, \dots, w_{m_1}$ are weights for the DMs in E_1, and $\alpha + \beta = 1$, ($\alpha > 0$ is the desired importance of DMs' opinions and $\beta > 0$ is the importance of the information in the database). Note that $\overline{(P_1, P_2)}$ is an FPRs too, i.e., $\overline{(P_1, P_2)} \in \mathcal{P}_1$:

$$\begin{aligned}
&\overline{(P_1, P_2)}(x_i, x_j) + \overline{(P_1, P_2)}(x_j, x_i) = \\
&\alpha \sum_{k=1}^{m_1} w_k P(e_k, x_i, x_j) + \frac{\beta}{m_2} \sum_{k=m_1+1}^{m_1+m_2} (\frac{1}{2} + P_2(e_k, x_i) - P_2(e_k, x_j)) + \\
&\alpha \sum_{k=1}^{m_1} w_k P(e_k, x_j, x_i) + \frac{\beta}{m_2} \sum_{k=m_1+1}^{m_1+m_2} (\frac{1}{2} + P_2(e_k, x_j) - P_2(e_k, x_i)) = \\
&\alpha \sum_{k=1}^{m_1} w_k (P(e_k, x_i, x_j) + P(e_k, x_j, x_i)) + \\
&\frac{\beta}{m_2} \sum_{k=m_1+1}^{m_1+m_2} (\frac{1}{2} + P_2(e_k, x_i) - P_2(e_k, x_j) + \frac{1}{2} + P_2(e_k, x_j) - P_2(e_k, x_i)) = \\
&\alpha \sum_{k=1}^{m_1} w_k + \frac{\beta}{m_2} \sum_{k=m_1+1}^{m_1+m_2} 1 = \alpha + \beta = 1,\ \forall\ i, j = 1, \dots, n.
\end{aligned}$$

Therefore, three compatible consensus metrics could be $\kappa_1, \kappa_2, \kappa_3 : \mathcal{P}_1 \times \mathcal{P}_2 \to [0, 1]$ defined as

$$\kappa_1(P_1, P_2) = \max_{\substack{i,j=1,\dots,n,\\ k=1,\dots,m_1}} |P_1(e_k, x_i, x_j) - \overline{(P_1, P_2)}(x_i, x_j)|,$$

$$\kappa_2(P_1, P_2) = \max_{\substack{i=1,\dots,n,\\ k=m_1+1,\\ \dots,m_1+m_2}} |\frac{1}{2} + P_2(e_k, x_i) - P_2(e_k, x_j) - \overline{(P_1, P_2)}(x_i)|,$$

$$\begin{aligned}
\kappa_3(P_1, P_2) = {} & \alpha \sum_{k=1}^{m_1} w_k | \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} P_1(e_k, x_i, x_j) - \overline{(P_1, P_2)}(x_i, x_j)| + \\
& \beta \frac{2}{m_2 n(n-1)} \sum_{k=m_1+1}^{m_1+m_2} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} |\frac{1}{2} + P_2(e_k, x_i) - P_2(e_k, x_j) - \overline{(P_1, P_2)}(x_i, x_j)|.
\end{aligned}$$

Finally, given an initial opinion $P^0 \equiv (P_1^0, P_2^0) \in \mathcal{P}_1 \times \mathcal{P}_2$, a feasible cost function may be $\xi_c : \mathcal{P}_1 \times \mathcal{P}_2 \to [0, 1]$ defined as

$$\begin{aligned}
\xi_c(P_1, P_2) = {} & \sum_{k=1}^{m_1} \frac{2c_k}{n(n-1)} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} |P_1(e_k, x_i, x_j) - P_1^0(e_k, x_i, x_j)| + \\
& \sum_{k=m_1+1}^{m_1+m_2} \frac{c_k}{n} \sum_{i=1}^{n} |P_2(e_k, x_i) - P_2^0(e_k, x_i)|,
\end{aligned}$$

where $\sum_{k=1}^{m1+m_2} c_k = 1, c_k \geq 0$.

With this basic configuration of the key elements of FZZ-MCC framework, the hybrid MCC model to derive consensual opinions can

Table 2
DMs' initial opinions.

e_1	x_1	x_2	x_3	x_4
x_1	0.5	0.1	0.33	0.85
x_2	0.9	0.5	0.36	0.4
x_3	0.67	0.64	0.5	0.0
x_4	0.15	0.6	1.0	0.5

e_2	x_1	x_2	x_3	x_4
x_1	0.5	0.96	0.92	0.74
x_2	0.04	0.5	0.5	0.55
x_3	0.08	0.5	0.5	0.49
x_4	0.26	0.45	0.51	0.5

e_3	x_1	x_2	x_3	x_4
x_1	0.5	0.5	0.32	0.49
x_2	0.5	0.5	0.51	0.85
x_3	0.68	0.49	0.5	0.01
x_4	0.51	0.15	0.99	0.5

e_4	x_1	x_2	x_3	x_4
x_1	0.5	0.16	0.57	0.15
x_2	0.84	0.5	0.02	0.85
x_3	0.43	0.98	0.5	0.62
x_4	0.85	0.15	0.38	0.5

Table 3
Examples of ratings in the database.

Ratings	x_1	x_2	x_3	x_4
e_5	0.28	0.19	0.16	0.36
e_6	0.31	0.17	0.22	0.28
e_7	0.14	0.55	0.28	0.00
e_8	0.29	0.29	0.02	0.37

be put into the following form:

$$\min_{(P_1,P_2)\in(\mathcal{P}_1,\mathcal{P}_2)} \xi_c(P_1,P_2)$$
$$s.t. \begin{cases} \kappa_1(P_1,P_2) \leq \varepsilon_1, \\ \kappa_2(P_1,P_2) \leq \varepsilon_2, \\ \kappa_3(P_1,P_2) \leq \varepsilon_3, \end{cases} \tag{H-FZZ-MCC}$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in]0,1]$ are the consensus parameters.

The stakeholders' preferences are expressed through the FPRs ($\mathbb{P}_1$) in Table 2, whereas some of the ratings in the database ($\mathbb{P}_2$) are shown in Table 3. In this case, the company needs to get an agreed solution that takes into account both the stakeholders' opinions according to the weights $\alpha = 0.7$ and $\beta = 0.3$ in the model (H-FZZ-MCC), which suggests that the company places more trust in the experts' opinions than in the information stored in the database. The consensus parameter $\varepsilon_1 = 0.1$ ensures that DMs' opinions, which are the most trusted in this decision-making problem, present a high consensus degree. In addition, the parameter $\varepsilon_2 = 0.2$ helps to reduce the variability of the ratings in the database, and the parameter $\varepsilon_3 = 0.15$, guarantees a moderate agreement between the experts' opinions and the data.

The results of the consensus model are represented in Fig. 4 by using MDS (Borg & Groenen, 2005). This figure shows how initially the ratings obtained from the database are quite far from each other (yellow crosses), even some stakeholders' preferences are also far from each other (purple circles). However, after the application of the H-FZZ-MCC model, all these preferences are closer and satisfy the consensus conditions previously defined. The agreed collective opinion, i.e., the solution of the (H-FZZ-MCC) model, is:

$$\begin{pmatrix} 0.5 & 0.45 & 0.47 & 0.55 \\ 0.55 & 0.5 & 0.45 & 0.59 \\ 0.53 & 0.55 & 0.5 & 0.49 \\ 0.45 & 0.41 & 0.51 & 0.5 \end{pmatrix}$$

Therefore, by computing the dominance of each alternative (Montgomery, 1983; Yakowitz et al., 1993), the consensual selection of the supplier is x_2 Supplier B.

This case study has shown that FZZ-MCC can manage several preference structures at the same time if the proper metrics are considered. In addition, we have shown that FZZ-MCC can be used to define hybrid models that integrate expert knowledge and data through an application for green supplier selection.

From a theoretical point of view, here it is demonstrated the potential of the FZZ-MCC framework to manage heterogeneous preference structures in MCC-driven consensus models. By modifying key elements such as consensus and distance metrics, researchers can effectively integrate diverse preference structures within a unified framework. This approach offers a flexible and adaptable tool for analyzing and synthesizing complex preference data, ultimately enabling more robust and informed decision-making processes. Thus, the FZZ-MCC framework offers a promising avenue for addressing the challenges posed by heterogeneous preference structures in consensus modeling.

5. Conclusions and future works

This contribution introduces the FZZ-MCC approach, a reformulation of classical MCC models in terms of fuzzy sets. This reformulation introduces three main advantages:

- A rigorous unified notation based on Fuzzy Sets which allows generalizing previous studies regarding MCC,
- Generalization of classical notions regarding GDM such as preference structure, consensus measure or cost function,
- Abstract nature which implies flexibility to adapt the FZZ-MCC scheme to address diverse decision situations.

In addition, this proposal has exploited such flexibility of FZZ-MCC to propose several novel MCC-based models:

- A FZZ-MCC model is defined to deal with an e-democracy scenario that involves urban planning by managing thousands of preferences through FZZ-MCC models and Multiplicative Preferences Relations.
- A FZZ-MCC model is used to persuade a hiring committee to select a particular manager by analyzing the associated cost and driving DMs toward agreement on a target solution.
- A hybrid FZZ-MCC model that merges database ratings with pairwise comparisons from DMs is proposed to combine expert knowledge and data in a green supplier selection problem.

Furthermore, all these models have been proposed in terms of linear and absolute-value-based objective functions and constraints, which facilitates their linearization to improve both their accuracy

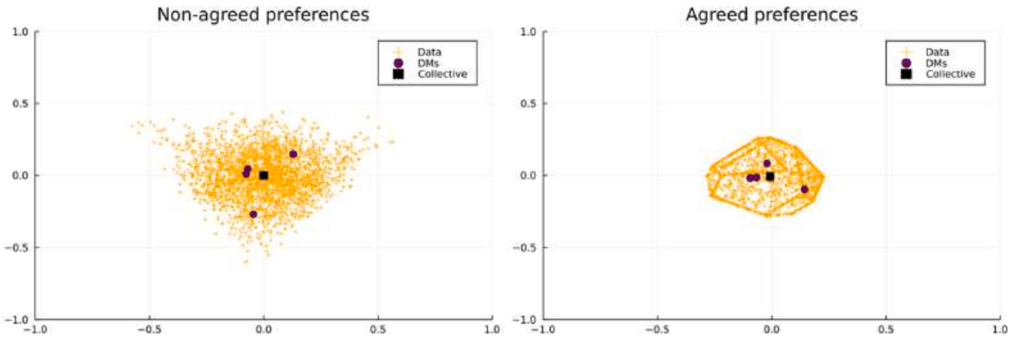

Fig. 4. MDS representation for (H-FZZ-MCC).

and computational efficiency aspects, which are essential to deal with large-scale GDM problems (Rodríguez et al., 2021).

The FZZ-MCC approach also gives raise to endless possibilities regarding future studies. For instance, it may be studied how to adapt MCC to deal with other preferences structures such as those related to linguistic information or include the use of nonlinear scales when modeling DMs' opinions. Furthermore, the FZZ-MCC may be applied to propose new optimization models to solve other real-world decision problems. Finally, the use of asymmetric cost may also be considered from the FZZ-MCC point of view to address those situations in which the cost of each individual depends on the adjustment direction of her/his opinion.

CRediT authorship contribution statement

Diego García-Zamora: Conceptualization, Formal analysis Visualization Writing – original draft. **Bapi Dutta:** Formal analysis, Validation, Writing – review & editing. **Álvaro Labella:** Validation, Writing – original draft, Visualization. **Luis Martínez:** Validation, Writing – review & editing, Funding acquisition.

Data availability

No data was used for the research described in the article.

Appendix. Consistency metric

Inconsistencies usually emerge when the preferences are elicited from DMs using pairwise comparison matrices (Saaty, 1990). In order to obtain more realistic results, some authors highlight the importance of considering consistency measures to ensure that the modified preferences are not contradictory (Rodríguez et al., 2021).

Even though there are several consistency measures in the literature (Rezaei, 2015; Rodríguez et al., 2021), all of them are based on a certain consistency formula that derives a degree of consistency for a preference structure. Such a consistency formula usually stands for the ideal situation in which the pairwise comparisons provided by the DMs do not contain contradictory information. In order to maintain that flexibility when defining the idea of consistency, we introduce the notion of g-consistency.

Definition A.1. Consider a GDM problem (E, X). Let $g : [0,1]\times[0,1] \to [a,b] \supseteq [0,1]$ be a surjective mapping and $\mathbb{P} \subset \mathcal{F}(X \times X)$ a preference structure. We say that $P \in \mathbb{P}$ is g-consistent if

$$P(x_i, x_j) = g(P(x_i, x_k), P(x_j, x_k)) \;\forall\; i, j, k \in \{1, 2, \dots, n\}.$$

Example A.1. For instance, when using FPRs it is possible to consider the function $g : [0,1] \times [0,1] \to [\frac{-1}{2}, \frac{3}{2}]$ defined as $g(x, y) = \frac{1}{2} + x - y \;\forall\; x, y \in [0,1]$. In that case, an FPR $P \in \mathbb{P}_A$ is g-consistent if and only if

$$P(x_i, x_j) = \frac{1}{2} + P(x_i, x_k) - P(x_j, x_k) \iff$$

$$P(x_i, x_j) + P(x_j, x_k) + P(x_k, x_i) = \frac{3}{2} \;\forall\; i, j, k \in \{1, 2, \dots, n\},$$

which are the classical definitions for additive consistency (Herrera-Viedma et al., 2004).

The previous definition allows expressing the idea of consistency according to the satisfaction of a certain formula, which is modeled by the function g. The surjectivity is required to emphasize that the result of computing the value $g(P(x_i, x_k), P(x_j, x_k))$ should be comparable to $P(x_i, x_j)$ for any $i, j, k = 1, \dots, n$. In other words, if g takes values in an interval that does not contain $[0,1]$, not a single $P \in \mathbb{P}$ could satisfy the g-consistency condition.

Obtaining fully consistent pairwise comparisons from DMs is challenging in practice, particularly when there are numerous elements to compare (Rezaei, 2015). To provide more flexibility, it is essential to establish a consistency metric that quantifies the level of consistency in a pairwise comparison matrix. Before stating the definition of such a consistency metric, we need to extend the idea of restricted dissimilarity function to be able to compare the output of the function g, which is valued in $[a,b]$ and the values of the preference, given in the interval $[0,1]$. Therefore, we will say that a mapping $\delta : [0,1] \times [a,b] \to [0,1]$ is a restricted dissimilarity function if it satisfies:

1. $\delta(x, y) = \delta(y, x) \;\; \forall\; x, y \in [0,1]$,
2. $\delta(x, y) = 0$ if and only if $x = y$,
3. $\delta(x, y) = 1$ if and only if either $x = 0$ and $y = b$ or $x = 1$ and $y = a$,
4. For all $x, y, z \in [0,1]$ such that $x \le y \le z$, then $\delta(x, y) \le \delta(x, z)$ and $\delta(y, z) \le \delta(x, z)$.

By using this notion of dissimilarity, we can introduce the notion of consistency metric for pairwise preference structures and their respective numeric ratings.

Definition A.2 (*g-Consistency Metric for Preference Structure*)**.** Consider a GDM problem (E, X). Let $g : [0,1]\times[0,1] \to [a,b] \supseteq [0,1]$ be a surjective mapping and $\mathbb{P} \subset \mathcal{F}(X \times X)$ a preference structure. Then a consistency metric $\hat{\eta} : \mathbb{P} \to [0,1]$ for $\mathbb{P}$ and g is given by

$$\hat{\eta}(P) = \hat{M}^{n}_{i,j,k=1} \delta(P(x_i, x_j), g(P(x_i, x_k), P(x_k, x_j)))$$

where $\hat{M} : [0,1]\times \overset{n^3}{\cdots} \times[0,1] \to [0,1]$ is an aggregation operator and $\delta : [0,1] \times [a,b] \to [0,1]$ is a restricted dissimilarity measure.

Following the lines described in Section 3, it is possible to measure the consistency of a group by extending the g-consistency metric for a preference structure to the corresponding set of numeric ratings.

Definition A.3 (*g-Consistency Metric for Numeric Ratings*)**.** Consider a GDM problem (E, X). Let $g : [0,1]\times[0,1] \to [a,b] \supset [0,1]$ be a surjective mapping and $\mathbb{P} \subset \mathcal{F}(X \times X)$ a preference structure. Then a consistency metric $\eta : \mathcal{P} \to [0,1]$ for $\mathcal{P}$ and g is a function $\eta : \mathcal{P} \to [0,1]$ given by

$$\eta(P) = M(\hat{\eta}(P(e_1,\cdot,\cdot)),\dots,\hat{\eta}(P(e_m,\cdot,\cdot))),$$

where $\hat{\eta}$ is a consistency metric for the preference structure associated to $\mathcal{P}$ and $M : [0,1]\times \overset{m}{\dots} \times[0,1] \to [0,1]$ is an aggregation operator.

Example A.2. Let us consider the restricted dissimilarity $\delta : [0,1] \times [\frac{-1}{2}, \frac{3}{2}] \to [0,1]$ given by $\delta(x,y) = \frac{2}{3}|x-y|\ \forall\, x,y \in [0,1]$. Then, by using the maximum as an aggregation operator, a consistency metric for FPRs $\hat{\eta} : \mathbb{P} \to [0,1]$ can be defined as

$$\hat{\eta}(P) = \frac{2}{3}\max_{i,j,k=1,\dots,n}\left\{|P(x_i,x_j)+P(x_j,x_k)+P(x_k,x_i)-\frac{3}{2}|\right\},$$

and the consistency metric for numeric ratings $\eta : \mathcal{P} \to [0,1]$ defined as

$$\eta(P) = \frac{2}{3}\max_{\substack{l=1,\dots,m,\\ i,j,k=1,\dots,n}}\left\{|P(e_l,x_i,x_j)+P(e_l,x_j,x_k)+P(e_l,x_k,x_i)-\frac{3}{2}|\right\}.$$

Note that for a preference matrix $P \in \mathbb{P} \subset \mathcal{F}(X\times X)$ and a surjective function $g : [0,1]\times[0,1] \to [a,b]$, $[0,1] \subset [a,b] \subset \mathbb{R}$, P is g-consistent if and only if $\hat{\eta}(P) = 0$. In the same way, for a numeric rating $P \in \mathcal{P}$, each fuzzy set $P(e_k,\cdot,\cdot)$ is g-consistent if and only if $\eta(P) = 0$. In addition, the greater the values of η, the further the preference structure or the numeric rating are from the ideal g-consistency.

To summarize, if a consistency metric $\eta : \mathcal{P} \to [0,1]$ is considered, the FZZ-MCC model could be rewritten as

$$\begin{aligned}&\min_{P\in\mathcal{P}} \xi_c(P,P_0)\\ &s.t.\begin{cases}\kappa(P) \le (\varepsilon_1,\dots,\varepsilon_q),\\ \eta(P) \le \theta,\end{cases}\end{aligned}$$

where $\theta \in [0,1]$, and ξ_c, κ and $\varepsilon_1,\dots,\varepsilon_q$ are defined as in Section 3.2.

To remark on the flexibility of the notion of g-consistency, please note that if we consider another scale that is not in $[0,1]$, such as the multiplicative 1–9 Saaty' scale (Saaty, 1990), it is possible to define the function $g : [\frac{1}{9},9]\times[\frac{1}{9},9] \to [\frac{1}{81},81]$ as $g(x,y) = \frac{x}{y}\ \forall\, x,y \in [\frac{1}{9},9]$ and conduct a similar discussion to derive the classical definitions for multiplicative consistency (Rezaei, 2015). We further analyze the adaptation of FZZ-MCC to multiplicative scales in Section 4.

References

Beliakov, G., Bustince, H., & Calvo, T. (2016). *A practical guide to averaging functions*. Springer.

Ben-Arieh, D., & Easton, T. (2007). Multi-criteria group consensus under linear cost opinion elasticity. *Decision Support Systems*, *43*(3), 713–721, Integrated Decision Support.

Borg, I., & Groenen, P. J. (2005). *Modern multidimensional scaling: Theory and applications*. Springer Science & Business Media.

Bustince, H., Barrenechea, E., & Pagola, M. (2008). Relationship between restricted dissimilarity functions, restricted equivalence functions and normal EN-functions: Image thresholding invariant. *Pattern Recognition Letters*, *29*(4), 525–536.

Butler, C., & Rothstein, A. (2006). *On conflict and consensus: A handbook on formal consensus decision making*. Takoma Park.

Caillaud, B., & Tirole, J. (2007). Consensus building: How to persuade a group. *American Economic Review*, *97*(5), 1877–1900.

Cheng, D., Zhou, Z., Cheng, F., Zhou, Y., & Xie, Y. (2018). Modeling the minimum cost consensus problem in an asymmetric costs context. *European Journal of Operational Research*, *270*(3), 1122–1137.

Chinn, S., Lane, D. S., & Hart, P. S. (2018). In consensus we trust? Persuasive effects of scientific consensus communication. *Public Understanding of Science*, *27*(7), 807–823.

García-Zamora, D., Dutta, B., Massanet, S., Riera, J. V., & Martínez, L. (2023). Relationship between the distance consensus and the consensus degree in comprehensive minimum cost consensus models: A polytope-based analysis. *European Journal of Operational Research*, *306*(2), 764–776. http://dx.doi.org/10.1016/j.ejor.2022.08.015, URL https://www.sciencedirect.com/science/article/pii/S0377221722006580.

García-Zamora, D., Labella, A., Ding, W., Rodríguez, R. M., & Martínez, L. (2022). Large-scale group decision making: A systematic review and a critical analysis. *IEEE/CAA Journal of Automatica Sinica*, *9*(6), 949–966.

Garcia-Zamora, D., Labella, A., Rodríguez, R. M., & Martínez, L. (2022). A linguistic metric for consensus reaching processes based on elicit comprehensive minimum cost consensus models. *IEEE Transactions on Fuzzy Systems*, 1–13. http://dx.doi.org/10.1109/TFUZZ.2022.3213943.

He, S., Pan, X., & Wang, Y. (2021). A shadowed set-based TODIM method and its application to large-scale group decision making. *Information Sciences*, *544*, 135–154.

Herrera-Viedma, E., Herrera, F., & Chiclana, F. (2002). A consensus model for multiperson decision making with different preference structures. *Systems, Man and Cybernetics, Part A: Systems and Humans, IEEE Transactions on*, *32*, 394–402.

Herrera-Viedma, E., Herrera, F., Chiclana, F., & Luque, M. (2004). Some issues on consistency of fuzzy preference relations. *European Journal of Operational Research*, *154*(1), 98–109.

Ji, Y., Li, H., & Zhang, H. (2021). Risk-averse two-stage stochastic minimum cost consensus models with asymmetric adjustment cost. *Group Decision and Negotiation*, 1–31.

Labella, A., Liu, H., Rodríguez, R. M., & Martínez, L. (2020). A cost consensus metric for consensus reaching processes based on a comprehensive minimum cost model. *European Journal of Operational Research*, *281*, 316–331.

Li, J., Wang, J.-q., & Hu, J.-h. (2019). Consensus building for hesitant fuzzy preference relations with multiplicative consistency. *Computers & Industrial Engineering*, *128*, 387–400. http://dx.doi.org/10.1016/j.cie.2018.12.051.

Montgomery, H. (1983). Decision rules and the search for a dominance structure: Towards a process model of decision making. In P. Humphreys, O. Svenson, & A. Vári (Eds.), *Advances in psychology*: *vol. 14*, *Analysing and aiding decision processes* (pp. 343–369). North-Holland.

Orlovsky, S. (1978). Decision-making with a fuzzy preference relation. *Fuzzy Sets and Systems*, *1*(3), 155–167.

Palomares, I., Estrella, F., Martinez, L., & Herrera, F. (2014). Consensus under a fuzzy context: Taxonomy, analysis framework AFRYCA and experimental case of study. *Information Fusion*, *20*.

Rezaei, J. (2015). Best-worst multi-criteria decision-making method. *Omega*, *53*, 49–57.

Rodríguez, R. M., Labella, Á., Dutta, B., & Martínez, L. (2021). Comprehensive minimum cost models for large scale group decision making with consistent fuzzy preference relations. *Knowledge-Based Systems*, *215*, Article 106780.

Saaty, T. L. (1988). What is the analytic hierarchy process? In *Mathematical models for decision support* (pp. 109–121). Springer.

Saaty, T. L. (1990). How to make a decision: the analytic hierarchy process. *European Journal of Operational Research*, *48*(1), 9–26.

Saint, S., & Lawson, J. (1994). *Rules for reaching consensus. A modern approach to decision making*. Jossey-Bass.

Song, H., Gong, Z., Forrest, J. Y.-L., Guo, W., & Herrera-Viedma, E. (2023). Social network utility consensus model with empathic and fuzzy interactions. *Computers & Industrial Engineering*, *175*, http://dx.doi.org/10.1016/j.cie.2022.108904.

Wang, Z., Rodríguez, R. M., Wang, Y.-m., & Martínez, L. (2021). A two-stage minimum adjustment consensus model for large scale decision making based on reliability modeled by two-dimension 2-tuple linguistic information. *Computers & Industrial Engineering*, *151*, Article 106973.

Wang, X., Ruan, D., & Kerre, E. (2009). *Mathematics of fuzziness—basic issues, Vol. 245*. Springer.

Xiao, J., Wang, X., & Zhang, H. (2022). Coping with diversity ratings in prioritizing design requirements in quality function deployment: A consensus-based approach with minimum-maximum adjustments. *Computers & Industrial Engineering*, *163*, http://dx.doi.org/10.1016/j.cie.2021.107799.

Xing, Y., Cao, M., Liu, Y., Zhou, M., & Wu, J. (2022). A choquet integral based interval type-2 trapezoidal fuzzy multiple attribute group decision making for sustainable supplier selection. *Computers & Industrial Engineering*, *165*, http://dx.doi.org/10.1016/j.cie.2022.107935.

Xu, Y., Wen, X., & Zhang, W. (2018). A two-stage consensus method for large-scale multi-attribute group decision making with an application to earthquake shelter selection. *Computers & Industrial Engineering*, *116*, 113–129. http://dx.doi.org/10.1016/j.cie.2017.11.025, URL https://www.sciencedirect.com/science/article/pii/S0360835217305582.

Yakowitz, D., Lane, L., & Szidarovszky, F. (1993). Multi-attribute decision making: dominance with respect to an importance order of the attributes. *Applied Mathematics and Computation*, *54*(2), 167–181.

Yang, Y., Fu, Y., Lin, J., & Huang, G. Q. (2022). An attitudinal consensus method under uncertainty in 3PRLP selection. *Computers & Industrial Engineering*, *172*(A), http://dx.doi.org/10.1016/j.cie.2022.108493.

Zadeh, L. A. (1996). Fuzzy sets. In *Fuzzy sets, fuzzy logic, and fuzzy systems: Selected papers by lotfi a zadeh* (pp. 394–432). World Scientific.

Zhang, B., Dong, Y., & Xu, Y. (2013). Maximum expert consensus models with linear cost function and aggregation operators. *Computers & Industrial Engineering*, *66*(1), 147–157. http://dx.doi.org/10.1016/j.cie.2013.06.001.

Zhang, G., Dong, Y., Xu, Y., & Li, H. (2011). Minimum-cost consensus models under aggregation operators. *IEEE Transactions on Systems, Man, and Cybernetics - Part A: Systems and Humans*, *41*(6), 1253–1261.

Zhang, H., Wang, F., & Song, X. (2022). From collective opinion dependence to personalization: The impacts of social trust relationship in consensus formation. *Computers & Industrial Engineering*, *172*(A), http://dx.doi.org/10.1016/j.cie.2022.108541.

Zhang, H., Zhao, S., Kou, G., Li, C.-C., Dong, Y., & Herrera, F. (2020). An overview on feedback mechanisms with minimum adjustment or cost in consensus reaching in group decision-making: Research paradigms and challenges. *Information Fusion*, *60*, 65–79.

4.6 UNA MÉTRICA PARA PROCESOS DE ALCANCE DE CONSENSO A GRAN ESCALA

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A Linguistic Metric for Consensus Reaching Processes Based on ELICIT Comprehensive Minimum Cost Consensus Models

Diego García-Zamora, Álvaro Labella, Rosa M. Rodríguez, *Member, IEEE*, and Luis Martínez, *Senior Member, IEEE*

***Abstract*—Linguistic group decision making (LiGDM) aims at solving decision situations involving human decision makers (DMs) whose opinions are modeled by using linguistic information. To achieve agreed solutions that increase DMs' satisfaction toward the collective solution, linguistic consensus reaching processes (LiCRPs) have been developed. These LiCRPs aim at suggesting DMs to change their original opinions to increase the group consensus degree, computed by a certain consensus measure. In recent years, these LiCRPs have been a prolific research line, and consequently, numerous proposals have been introduced in the specialized literature. However, we have pointed out the nonexistence of objective metrics to compare these models and decide which one presents the best performance for each LiGDM problem. Therefore, this article aims at introducing a metric to evaluate the performance of LiCRPs that takes into account the resulting consensus degree and the cost of modifying DMs' initial opinions. Such a metric is based on a linguistic comprehensive minimum cost consensus (CMCC) model based on Extended Comparative Linguistic Expressions with Symbolic Translation information that models DMs' hesitancy and provides accurate Computing with Words processes. In addition, the linguistic CMCC optimization model is linearized to speed up the computational model and improve its accuracy.**

***Index Terms*—Computing with Words (CW), extended comparative linguistic expressions with symbolic translation (ELICIT) information, fuzzy linguistic approach, linguistic cost metric, minimum cost consensus.**

I. Introduction

In GROUP decision making (GDM), a group of decision makers (DMs) faces a decision situation in which they provide their preferences to select the best alternative as a solution to the decision problem. Even though the participation of several DMs allows the consideration of several points of view in the decision process, it often implies the emergence of disagreements among them, which should be properly managed to avoid unsatisfactory results. Consensus reaching processes (CRPs) were designed to soften such discrepancies and drive the group toward an agreed solution [1], [2], [3]. Classically, a desired consensus threshold is fixed a priori; then, a discussion process is carried out in which a moderator suggests the DMs to modify their preferences in order to increase the group consensus degree. A CRP is usually an iterative process, which is repeated for several rounds until either the consensus degree surpasses the consensus threshold or the number of rounds exceeds a maximum limit [2].

Real-world GDM problems and their CRPs are generally presented in uncertain contexts characterized by the absence of objective information, which increases the complexity of the decision situation. Under these circumstances, the DMs may have difficulties in providing their opinions by using numerical assessments. To offer more realistic and suitable frameworks for DMs to express their preferences according to their natural way of thinking, the use of the fuzzy linguistic approach and linguistic variables [4], [5], [6] has increased its popularity in recent years. When DMs provide their opinions through linguistic assessments, we talk about linguistic group decision making (LiGDM) [7] and linguistic consensus reaching processes (LiCRPs) [8], [9], [10].

Since achieving linguistic agreed solutions is essential in many real-world decision situations [11], [12], the interest of researchers has been aroused, leading to many LiCRP proposals in the specialized literature [1]. Although a priori having many proposals could make the resolution of LiGDM problems easier, the bibliographic analysis developed by García-Zamora et al. [1] pointed out that there is an evident lack of objective metrics to compare the performance of different LiCRPs and discern which one presents a better performance to deal with a certain LiGDM problem. The main consequence of this situation is that the authors justify the alleged well performance of their proposals through the resolution of simple illustrative examples, which could easily be biased to obtain good results [1]. In this regard, the authors have used different measures to compare consensus proposals, such as the number of rounds necessary to reach the

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The authors are with the Department of Computer Science, University of Jaén, 23071 Jaén, Spain (e-mail: dgzamora@ujaen.es; alabella@ujaen.es; rmrodrig@ujaen.es; martin@ujaen.es).

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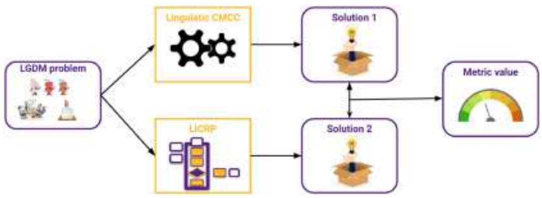

Fig. 1. Scheme of the proposed metric.

consensus threshold [13], [14], the trust among experts [13], or the consensus degree [2], [15]. However, these aspects could not be the representative of the quality of the models because they do not provide enough information about their global performance, and consequently, the authors could show them in the most convenient way. For instance, a fast consensus model in terms of the number of rounds may present several drawbacks related to the achieved consensus degree or the changes performed in the original preferences, which could have been excessively modified. In addition, these measures may allow comparing models in a specific case study, but they do not offer a global vision of the performance of the model when different DMs' opinions are used.

Therefore, the main goal of this article is proposing the first linguistic metric to objectively compare linguistic consensus models and show which one presents the best performance in the resolution of an LiGDM problem. The proposed metric compares the results of the LiCRP with an ideal scenario in which the consensus threshold is achieved by making as few changes as possible to DMs' original opinions (see Fig. 1). This article uses the comprehensive minimum cost consensus (CMCC) [16], [17] models, which are automatic CRPs, to determine such ideal results but extending them to deal with linguistic information. Consequently, we raise the following research questions.

- RQ1: How to define CMCC models in a linguistic environment?
- RQ2: How to evaluate objectively the performance of LiCRPs?

To answer these questions, we first propose a linguistic CMCC model for extended comparative linguistic expressions with symbolic translation (ELICIT) information [7], a recently proposed linguistic modeling approach that guarantees precise computations with linguistic information [4], [5], [6]. ELICIT information hybridizes the 2-tuple linguistic approach [18] and hesitant fuzzy linguistic term sets (HFLTS) [19] by introducing a Computing with Words (CW) [20], [21] framework that guarantees precise computations with hesitant expressions without losing interpretability during the operational process [7]. These ELICIT-CMCC models inherit the properties of classic CMCC models [16] for numeric assessments; thus, they provide modified DMs' preferences, which preserve as much as possible the initial opinions and, in turn, guarantee the predefined consensus threshold. In addition, ELICIT-CMCC models follow the CW methodology [20], [21], i.e., linguistic results are obtained from linguistic inputs. Since such optimization models do not only require the use of many variables, but also the use of nonlinear constraints involving the absolute value, this proposal also includes a linearized version of the proposed ELICIT-CMCC models to speed up the computational model and improve the accuracy of the solution for the decision situation. Finally, these novel linguistic CMCC models are used as the basis to define a linguistic cost metric to evaluate LiCRPs that is based on two indicators to determine the quality of a consensus model: 1) the consensus degree achieved and 2) the minimum changes necessary to obtain an agreed solution. The former is essential to ensure that the consensus process has been carried out successfully, i.e., it would be nonsense to score a consensus model that does not achieve the desired level of consensus with a high score [14], [15]. The latter guarantees that the original opinions of the DMs are not modified beyond the strictly necessary to reach the consensus threshold [16]. Therefore, an LiCRP that performs unnecessary changes on DMs' opinions to reach the consensus will receive a low mark.

To summarize, the main novelties of this proposal are as follows.

1) CMCC models for linguistic information are proposed following a CW approach.
2) Such models are then linearized to accelerate computational cost, even with dealing with hundreds or thousands of DMs, and improve the precision of the results.
3) From the linearized ELICIT-CMCC model, a linguistic cost metric is proposed to objectively evaluate the performance of LiCRPs.

The rest of this article is organized as follows. Section II includes some preliminary notions required to better understand this proposal related to LiGDM, 2-tuple, and ELICIT linguistic representation schemes and minimum cost consensus (MCC) models. In Section III, CMCC models for ELICIT information are proposed and then linearized. Here, we also provide a brief analysis regarding the feasibility of such linear models when dealing with decision situations in which hundreds or thousands of DMs take part. Afterward, Section IV introduces a linguistic cost metric based on the previous CMCC models, and a couple of CRPs are evaluated to illustrate its working. Section V shows the CW nature of the ELICIT-CMCC models through the resolution of an LiGDM problem, and Section V-C includes a comparative analysis between the novel linguistic CMCC model for ELICIT information and other proposals. Finally, Section VI concludes this article.

II. Background

This section introduces a revision of the basic notions related to the proposal. First, the basic concepts of LiGDM are revised. Afterward, the linguistic 2-tuple model and the ELICIT linguistic representation model are reviewed, and some notations are fixed to simplify their understanding. Finally, LiCRPs and MCC models are revised.

A. Linguistic Group Decision Making

Decision processes are inherent in human beings' daily life. These decision situations consist of making the best possible choice among several possible solutions to a certain problem.

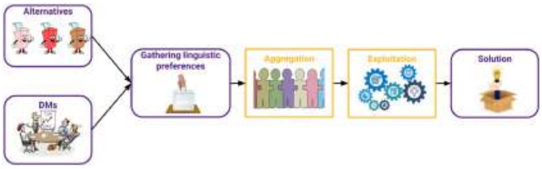

Fig. 2. LiGDM resolution scheme.

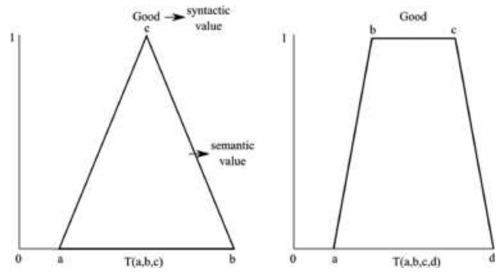

Fig. 3. Linguistic label.

Some decision problems are simple to solve and may involve just one individual. However, other decision problems are more complex and require several DMs, who may contribute with different points of view and knowledge. Formally, a GDM problem is modeled as a decision situation in which several individuals or DMs $E = \{e_1, e_2, \ldots, e_m\}$, $m \in \mathbb{N}$, have to decide which alternative from a set $X = \{x_1, x_2, \ldots, x_n\}$, $n \in N$, is the best solution to a problem [9], [22].

In addition, the complexity of GDM problems increases when the available information is not objective, but vague and imprecise. In such contexts, the stakeholders must address the decision situation from a subjective point of view by using qualitative assessments. In this regard, modeling DMs' opinions properly becomes crucial to managing the uncertainty inherent in these situations. Although some proposals translate qualitative information to a numerical scale, the goal of LiGDM is to model the uncertainty using linguistic expressions close to the natural way of human thinking (see Fig. 2).

This article uses the fuzzy linguistic approach [4], [5], [6] based on fuzzy sets theory [23], [24] to model uncertainty in LiGDM. This approach represents the linguistic information using linguistic variables [4], [5], [6], which usually model the information through parametric membership functions with triangular or trapezoidal graphical representation, among others (see Fig. 3).

The resolution of LiGDM problems implies to carry out computations with linguistic information. In this sense, the CW approach aims to provide linguistic solutions to problems formulated with linguistic expressions that emulate human thinking. There are several CW proposals in the literature [1] such as the linguistic model based on the fuzzy relation proposed by Tang and Zheng [25], linguistic distribution assessment proposed by Dong et al. [26], or the fuzzy set approach to treat determinacy and consistency of linguistic terms introduced by Ma et al. [27].

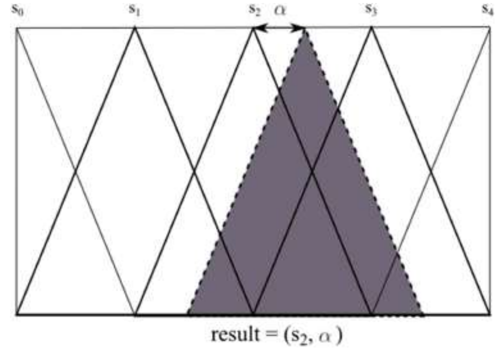

Fig. 4. Symbolic translation.

In particular, this article considers that the linguistic information is modeled by the 2-tuple linguistic model [18] and the ELICIT information [7], which highlight because they allow modeling uncertainty according to the fuzzy linguistic approach [24] without losing information or interpretability.

B. 2-Tuple Linguistic Model

The 2-tuple linguistic model [18] aimed to overcome the lack of precision in classical linguistic computational approaches through a continuous fuzzy representation of the linguistic information and a computational model capable of carrying out simple symbolic precise computations without approximations, obtaining accurate linguistic results according to the CW scheme.

A 2-tuple linguistic value is a tuple $(s_i, \alpha) \in \overline{S} := S \times [-0.5, 0.5[$, where s_i is a linguistic term that belongs to a certain linguistic term set $S = \{s_0, s_1, \ldots s_g\}$ (for a fixed even number $g \in \mathbb{N}$) and α is the so-called symbolic translation, i.e., a numerical value that represents the shifting of s_i fuzzy membership function (see Fig. 4). Note that for a linguistic 2-tuple value $(s_i, \alpha) \in \overline{S}$, the possible values for the symbolic translation α are

$$\alpha \in \begin{cases} [-0.5, 0.5), & \text{if } s_i \in \{s_1, s_2, \ldots, s_{g-1}\} \\ [0, 0.5), & \text{if } s_i = s_0 \\ [-0.5, 0], & \text{if } s_i = s_g \end{cases}.$$

The key characteristic of 2-tuple linguistic expressions is the fact that they can be translated into a numerical quantity $x \in [0, g]$, which simplifies the computations.

Proposition 1 (See [18]): Let $S = \{s_0, \ldots s_g\}$ be a linguistic term set. Then, the function $\Delta_S^{-1} : \overline{S} \to [0, g]$ defined by

$$\Delta_S^{-1}(s_i, \alpha) = i + \alpha, \forall (s_i, \alpha) \in \overline{S}$$

is a bijection whose inverse $\Delta_S : [0, g] \to \overline{S}$ is given by

$$\Delta_S(x) = (s_{\text{round}(x)}, x - \text{round}(x)) \forall x \in [0, g]$$

where round$(\cdot)$ is the function that assigns the closest integer number $i \in \{0, \ldots, g\}$.

Remark 1: Note that any linguistic term $s_i \in S$ can be represented as a 2-tuple linguistic value by considering $(s_i, 0) \in \overline{S}$.

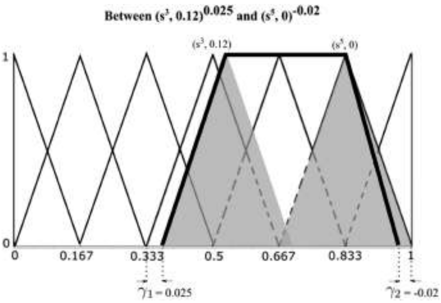

Fig. 5. Example of ELICIT linguistic expressions.

C. ELICIT Information

The 2-tuple linguistic framework follows a CW scheme to carry out computations, obtaining precise results that are easy to understand. However, it presents an important drawback regarding the lack of expressiveness, because the linguistic 2-tuple values are not able to model the DMs' hesitancy between several linguistic terms like HFLTS [19] do. Labella et al. [7] proposed the use of ELICIT information to address this limitation by introducing a linguistic approach that preserves the accuracy and understandability of the 2-tuple linguistic model and improves the expressiveness by hybridizing it with HFLTS.

Formally, ELICIT information is denoted here by an expression $[\overline{s}_i, \overline{s}_j]_{\gamma_1,\gamma_2}$, where $\overline{s}_i, \overline{s}_j \in \overline{S}, i \leq j$ are two 2-tuple linguistic values. In addition, ELICIT values also consider two parameters γ_1 and γ_2, which guarantee that no information is lost during the computations with these expressions. It should be noted that any trapezoidal fuzzy number (TrFN) [23], [24] can be unequivocally represented as an ELICIT value (see Fig. 5).

Remark 2: A TrFN is a function $T \equiv T(a,b,c,d) : [0,1] \to [0,1]$ of the form

$$T(x) = \begin{cases} 0, & \text{if } 0 \leq x \leq a \\ \frac{x-a}{b-a}, & \text{if } a < x < b \\ 1, & \text{if } b \leq x \leq c \quad \forall x \in [0,1] \\ \frac{d-x}{d-c}, & \text{if } c < x < d \\ 0, & \text{if } d \leq x \leq 1 \end{cases}$$

for certain $0 \leq a \leq b \leq c \leq d \leq 1$. For the sake of clarity, the set of all TrFNs on the interval $[0,1]$ will be denoted by

$$\mathcal{T} = \{T : [0,1] \to [0,1] : T \text{ is a TrFN}\}.$$

Proposition 2: Let $\overline{\overline{S}}$ be the set of all possible ELICIT values. Then, the mapping ζ given by

$$\zeta : \mathcal{T} \to \overline{\overline{S}}$$
$$T(a,b,c,d) \to [\overline{s}_1, \overline{s}_2]_{\gamma_1,\gamma_2}$$

where

$$\overline{s}_1 = \Delta_S(gb) \;\; \gamma_1 = a - \max\left\{b - \frac{1}{g^2}, 0\right\}$$
$$\overline{s}_2 = \Delta_S(gc) \;\; \gamma_2 = d - \min\left\{c + \frac{1}{g^2}, 1\right\}$$

is a bijection whose inverse ζ^{-1} is defined by

$$\zeta^{-1} : \overline{\overline{S}} \to \mathcal{T}$$
$$[\overline{s}_1, \overline{s}_2]_{\gamma_1,\gamma_2} \to T(a,b,c,d)$$

and allows computing the fuzzy representation of an ELICIT expression as follows:

$$a = \gamma_1 + \max\left\{\frac{\Delta_S^{-1}(\overline{s}_1) - \frac{1}{g}}{g}, 0\right\}, b = \frac{\Delta_S^{-1}(\overline{s}_1)}{g}$$
$$c = \frac{\Delta_S^{-1}(\overline{s}_m)}{g}, d = \gamma_2 + \min\left\{\frac{\Delta_S^{-1}(\overline{s}_m) + \frac{1}{g}}{g}, 1\right\}.$$

Remark 3: It must be highlighted that the notation $[\overline{s}_1, \overline{s}_2]_{\gamma_1,\gamma_2}$ is used for the sake of clarity, but the reader should keep in mind that, in spite of its formal nature, this notation resembles a linguistic expression. In other words, ELICIT information can be used to represent the hesitancy between several linguistic terms and perform precise computations on them by providing a linguistic result.

The ELICIT computational model follows a CW approach that computes the fuzzy representation of the respective linguistic expressions, whose results are lately retranslated to ELICIT information. From a theoretical point of view, ELICIT expressions are generated by a context-free grammar, which models comparative linguistic structures close to human language such as *at least bad*, *at most fast*, or *between expensive and rather expensive* [7]. Thus, this context-free grammar together with a linguistic term set, for instance,

$$S = \{\text{Much Worse (MW)}, \text{Worse (W)}, \text{Slightly Worse (SW)}$$
$$\text{Equal (E)}, \text{Slightly Better (SB)}, \text{Better (B)}, \text{Much Better (MB)}\}$$

can model linguistic expressions such as *at least* $(\mathrm{W}, 0.2)^{0.2}$, *at most* $(\mathrm{W}, 0.1)^{0.1}$, or *between* $(E, 0)^{-0.11}$ *and* $(SB, 0.32)^0$.

Remark 4: Note that any linguistic term $s_i \in S$ can be represented as the ELICIT expression $(s_i, 0)_0 \equiv [(s_i, 0), (s_i, 0)]_{00}$. In the same way, an HFLTS $\{s_i, s_{i+1}, \ldots, s_j\}$, $i < j$, can be translated to the ELICIT value $[(s_i, 0), (s_j, 0)]_{00}$.

To aggregate ELICIT values, Labella et al. [7] proposed the use of the fuzzy weighted average operator $A : \mathcal{T}^m \to \mathcal{T}$ defined by

$$A(T_1, T_2, \ldots, T_m) = \left(\sum_{k=1}^m \omega_k T_k^a, \sum_{k=1}^m \omega_k T_k^b, \sum_{k=1}^m \omega_k T_k^c, \sum_{k=1}^m \omega_k T_k^d\right)$$

where T_k^t denotes the tth $t \in \{a,b,c,d\}$ coordinate of the TrFN $T_k, k = 1, 2, \ldots, m$ and $\omega_1, \omega_2, \ldots, \omega_m \geq 0, \sum_{k=1}^m \omega_k = 1$ are the weights for the DMs.

A comparison measure to order ELICIT values based on the method presented by Abbasbandy and Hajjari [28] was also proposed. This method translates the fuzzy representation of the ELICIT values, given by a TrFN, into a numerical value called magnitude, which is defined by

$$\mathrm{Mag}([s_i, s_j]_{\gamma_1,\gamma_2}) = \mathrm{Mag}(T(a,b,c,d)) = \frac{a + 5b + 5c + d}{12}.$$

To compare two ELICIT values, it suffices to compute the respective magnitudes. According to Labella et al. [7], the higher the magnitude, the larger the ELICIT value.

Furthermore, to measure the distance between two ELICIT values, Labella et al. [22] proposed using the geometric distance [29] between their respective associated TrFNs, $\delta : \mathcal{T} \times \mathcal{T} \to [0,1]$ defined by

$$\delta(T_1, T_2) = \frac{1}{4}(|a_1 - a_2| + |b_1 - b_2| + |c_1 - c_2| + |d_1 - d_2|)$$

where $T_1 \equiv (a_1, b_1, c_1, d_1)$, and $T_2 \equiv (a_2, b_2, c_2, d_2)$. Note that, even though the geometric distances were originally proposed as a parametric family [29], here, we consider just the distance δ because it is defined in terms of absolute values rather than powers, and this facilitates the linearization of the optimization models we aim at proposing in the following section.

The use of ELICIT information can be adapted in classical linguistic preference structures. In the following, we consider that DMs' opinions are modeled by using ELICIT preference relations (EPRs), i.e., matrices of ELICIT values whose associated TrFNs are additive reciprocal matrices of TrFNs.

Remark 5: Let us define the set of matrices whose items are TrFN

$$\mathcal{M}_{n\times n}(\mathcal{T}) := \left\{(T^{ij})_{n\times n} : T^{ij} \in \mathcal{T}\ \forall 1 \le i \le n, 1 \le j \le n\right\}.$$

We will say that $T \in \mathcal{M}_{n\times n}(\mathcal{T})$ is additive reciprocal [30] if

$$\begin{aligned} T^{ij}[1] + T^{ji}[4] &= 1 \\ T^{ij}[2] + T^{ji}[3] &= 1 \\ T^{ij}[3] + T^{ji}[1] &= 1 \\ T^{ij}[4] + T^{ji}[1] &= 1 \end{aligned}$$

for any $i, j \in \{1, 2, \ldots, n\}$, where $T^{ij}[t]$, $t = 1, 2, 3, 4$ represents the tth coordinate of the TrFN T^{ij}. Furthermore, we will use the notation $\mathcal{M}_{n\times n}(\mathcal{T})^*$ to denote the set of TrFN matrices that are additive reciprocal.

Therefore, EPRs allow the generalization of other commonly used preference structures based on linguistic pairwise comparison matrices that rely on triangular or TrFNs such as linguistic preference relations [31] or hesitant fuzzy linguistic preference relations (HFLPRs) [32]. For example, the HFLPR on the linguistic term set S given by

$$\begin{pmatrix} E & W & \text{Bt SW and E} \\ B & E & SB \\ \text{Bt E and SB} & SW & E \end{pmatrix}$$

may be expressed as the EPR

$$\begin{pmatrix} (E,0)_0 & (W,0)_0 & [(SW,0),(E,0)]_{00} \\ (B,0)_0 & (E,0)_0 & (SB,0)_0 \\ [(E,0),(SB,0)]_{00} & (SW,0)_0 & (E,0)_0 \end{pmatrix}.$$

D. Linguistic Consensus Reaching Processes

In order to address GDM making problems, several rules have been proposed in the classical literature, such as the majority rule, the minority rule, unanimity, or the Borda count [33], [34]. However, even using these rules, some DMs may feel unsatisfied with the solution chosen by the group because their opinions have not been considered as much as they expected. This situation may especially be undesired in certain real-world problems that require a concrete level of agreement among the DMs.

To soften these disagreements, CRPs have been developed to guide DMs toward an agreed solution [9], [16], [22]. Usually in a CRP, a moderator or automatic moderator process suggests the DMs how to modify their opinions to lead the group to a greater agreement through different discussion rounds. Owing to the increasing necessity of LiGDM, CRPs have also been adapted to manage linguistic information, emerging LiCRPs. The general scheme of an LiCRP follows the scheme of CRPs but includes the management of linguistic information and presents the following phases [2].

1) *Aligning preferences:* DMs' opinions are elicited by using linguistic information.
2) *Determining consensus degree:* In each round of discussion, the current consensus degree $\mu \in [0,1]$ in the group is derived to evaluate the evolution of the consensus process.
3) *Consensus control:* After the discussion, the moderator computes if the group has reached a certain consensus threshold ($\mu_0 \in [0,1]$). If so, the CRP stops and the exploitation process starts. If not, the discussion process continues for another round. In any case, if a predefined maximum number of rounds $MaxRounds \in \mathbb{N}$ is exceeded, the CRP stops.
4) *Recommendation process:* In case the desired consensus threshold μ_0 is not achieved, those DMs whose opinions are furthest from the rest of the group are identified and modified if necessary.
5) *Exploitation:* After the desired consensus threshold is reached, the consensual modified opinions are aggregated in order to derive the group collective opinion.

Over the years, researchers have proposed many consensus models to support CRPs [17], [35]. For this reason, Palomares et al. [2] proposed a taxonomy to categorize them based on two characteristics related to consensus models.

1) Type of recommendation process to modify DMs' opinions.
a) *Feedback mechanism:* The moderator asks the DMs if they want to change or not their preferences [9], [22].
b) *Automatic changes:* DMs' opinions are automatically modified according to a certain algorithm without asking the DMs [17], [35].
2) Type of consensus measure to derive the consensus degree.
a) *Consensus measure of class 1:* The consensus degree among the DMs is computed by comparing the DMs' preferences with the collective opinion [17], [36], [37].
b) *Consensus measure of class 2:* The consensus degree among the DMs is computed by comparing the DMs' preferences with each other [17], [22], [38].

E. Comprehensive Minimum Cost Consensus

Ben-Arieh and Easton [35] proposed MCC models to study the cost of changing DMs' preferences in a consensus process.

These models are automatic CRPs (without feedback mechanism) that minimize the cost of changing DMs' original preferences by assuring that a maximum absolute deviation ($\varepsilon \in]0,1]$) between the individual assessments and the collective opinion is not surpassed. Formally, for the initial values of the preferences $(o_1, o_2, \ldots, o_m) \in \mathbb{R}$ and a cost vector $(c_1, c_2, \ldots, c_m) \in \mathbb{R}^+$, the proposed CRPs were defined by

$$\min \sum_{k=1}^{m} c_k |\overline{o}_k - o_k|$$
$$\text{s.t. } |\overline{o}_k - \overline{o}| \le \varepsilon, \quad k = 1, 2, \ldots, m \qquad \text{(MCC)}$$

where $(\overline{o}_1, \ldots, \overline{o}_m)$ are the adjusted opinions of the DMs, $\overline{o}$ represents the group collective opinion computed by using a weighted average operator, and ε is the maximum acceptable distance of each DM to the collective opinion.

Lately, Zhang et al. [39] studied the influence of the aggregation operator used to derive the collective opinion on the solution of the optimization problem. Consequently, they proposed a generalized version of MCC as follows:

$$\min \sum_{k=1}^{m} c_k |\overline{o}_k - o_k|$$
$$\text{s.t.} \begin{cases} \overline{o} = F(\overline{o}_1, \ldots, \overline{o}_m) \\ |\overline{o}_k - \overline{o}| \le \varepsilon, \quad k = 1, 2, \ldots, m \end{cases} \qquad \text{(MCC : AO)}$$

where $\overline{o}$ is now calculated using a different aggregation operator $F : \mathbb{R}^m \to \mathbb{R}$.

Even though these proposals allow translating a CRP situation into a mathematical programming problem, the constraint defined by ε is quite simple and does not guarantee that a certain consensus threshold $\mu_0 \in [0, 1[$ is achieved by the group. This drawback is solved by the CMCC models introduced by Labella et al. [16]. These models include the use of another constraint to control such a consensus threshold

$$\min \sum_{i=1}^{m} c_i |\overline{o}_i - o_i|$$
$$\text{s.t.} \begin{cases} \overline{o} = F(\overline{o}_1, \ldots, \overline{o}_m) \\ |\overline{o}_i - \overline{o}| \le \varepsilon, i = 1, 2, \ldots, m \\ \text{consensus}(\overline{o}_1, \ldots, \overline{o}_n) \ge \mu_0 \end{cases} \qquad \text{(CMCC)}$$

where consensus$(\cdot)$ represents the desired consensus measure.

III. ELICIT-CMCC Models for LiGDM

Keeping in mind that our main goal is to define an objective metric for measuring the performance of different LiCRPs, it is essential to compute some ideal values for the DMs' modified preferences. To obtain such optimal values, we follow the CMCC philosophy [16], which assumes that the best possible values for such modified opinions are those that, by satisfying the consensus threshold, are closest to their original preferences.

Even though MCC and CMCC models are focused on numerical assessments [16], [17], [35], [39], some proposals introduce extensions of the MCC models to a fuzzy environment. Nevertheless, the extended models either neglect the CW approach [40] or are not able to model hesitancy [18], [41]. Because ELICIT information allows carrying out computations with linguistic expressions that model hesitancy without loss of information, this section extends the numeric CMCC models [16] to deal with ELICIT information and obtain an optimal adjustment consensus model for the CW approach.

The general scheme of this article is as follows: let us consider an LiGDM problem in which $E = \{e_1, e_2, \ldots, e_m\}$ DMs have to decide in a consensual way which alternative $X = \{x_1, x_2, \ldots, x_n\}$ is the best solution for a concrete problem. To do so, each DM provides an HFLPR [32], which is expressed in terms of ELICIT information as an EPR. The ELICIT information contained in these matrices is then expressed as the corresponding TrFNs by using the mapping ζ^{-1} (see Proposition 2). Such TrFNs are used as inputs for the ELICIT-CMCC model, whose output provides the agreed preferences that are closest to the original opinions given by the DMs. Finally, the modified preferences obtained of solving the optimization problem, represented by TrFNs, are retranslated into ELICIT information by using the mapping ζ (see Fig. 6).

Let $O_1, O_2, \ldots, O_m \in \mathcal{M}_{n\times n}(\mathcal{T})^*$ be the additive reciprocal matrices of TrFNs corresponding to the translation via the mapping ζ^{-1} of DMs' original preferences expressed in form of EPRs, and let $T_1, T_2, \ldots, T_m \in \mathcal{M}_{n\times n}(\mathcal{T})^*$ be the respective modified DMs' opinions. The cost function and the consensus measures for these values are modeled by using the distance δ revised in Section II-C. Consequently, the classical distance measure between DMs' opinions and the collective opinion $(0 < \varepsilon \le 1)$ and the consensus threshold used in CMCC models $(0 \le \mu_0 < 1)$ are adapted to the ELICIT-CMCC models as follows.

1) *ELICIT-CMCC model considering a consensus measure of class 1*

$$\min_{T_1^{i,j}, \ldots, T_m^{i,j} \in \mathcal{T}} \sum_{k=1}^{m} \sum_{i<j} c_k^{ij} \delta(T_k^{i,j}, O_k^{i,j})$$
$$\text{s.t.} \begin{cases} \overline{T}^{i,j} = A(T_1^{i,j}, T_2^{i,j}, \ldots, T_m^{i,j}), 1 \le i < j \le n \\ \delta(T_k^{i,j}, \overline{T}^{i,j}) \le \varepsilon, 1 \le i < j \le n, k = 1, 2, \ldots, m \\ 1 - \frac{1}{N} \sum_{k=1}^{m} \sum_{i<j} w_k \delta(T_k^{i,j}, \overline{T}^{i,j}) \ge \mu_0 \end{cases}$$
$$\text{(ELICIT} - \text{CMCC : 1)}$$

2) *ELICIT-CMCC model considering a consensus measure of class 2*

$$\min_{T_1^{i,j}, \ldots, T_m^{i,j} \in \mathcal{T}} \sum_{k=1}^{m} \sum_{i<j} c_k^{ij} \delta(T_k^{i,j}, O_k^{i,j})$$
$$\text{s.t.} \begin{cases} \overline{T}^{i,j} = A(T_1^{i,j}, T_2^{i,j}, \ldots, T_m^{i,j}), 1 \le i < j \le n \\ \delta(T_k^{i,j}, \overline{T}^{i,j}) \le \varepsilon, 1 \le i < j \le n, k = 1, 2, \ldots, m \\ 1 - \frac{1}{N} \sum_{k<l} \sum_{i<j} \frac{w_k + w_l}{m-1} \delta(T_k^{i,j}, T_l^{i,j}) \ge \mu_0 \end{cases}$$
$$\text{(ELICIT} - \text{CMCC : 2)}$$

where $c_k^{ij} \in [0,1] (\sum_{k=1}^{m} \sum_{i<j} c_k^{ij} = 1)$ models the cost of moving the DM e_k's preference of the alternative x_i over x_j, $w_1, w_2, \ldots, w_m \in [0,1]$ $(\sum_{k=1}^{m} w_k = 1)$ are the weights for the DMs, $N = \frac{n(n-1)}{2}$, and $A : \mathcal{T}^m \to \mathcal{T}$ is a fuzzy weighted average operator.

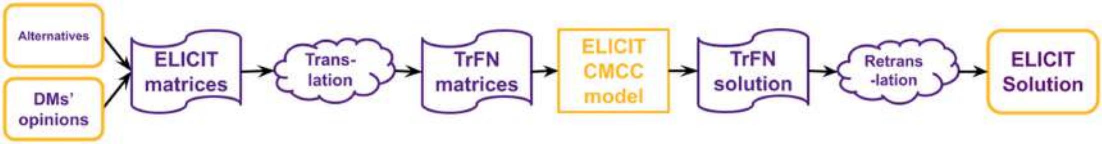

Fig. 6. ELICIT-CMCC scheme.

Remark 6: To adapt these linguistic models to return triangular fuzzy numbers, the condition $T_k^{ij}[1] \leq T_k^{ij}[2] \leq T_k^{ij}[3] \leq T_k^{ij}[4]$ should be replaced by $T_k^{ij}[1] \leq T_k^{ij}[2] = T_k^{ij}[3] \leq T_k^{ij}[4]$.

It should be highlighted that both the inputs and the outputs of these models are represented by using linguistic information (EPRs), following a CW scheme that facilitates the understandability of the results by the involved DMs (RQ1).

Note that the resolution of the previous consensus models requires numerous variables and constraints of a nonlinear optimization problem, which may lead to a high time consumption [9]. To overcome this drawback, we introduce below linearized versions of both (ELICIT-CMCC:1) and (ELICIT-CMCC:2). For the sake of clarity, the domains of the constraints in the models below use the notation $\mathcal{I}_a^b := [a,b] \cap \mathbb{N}$ for any pair $a < b \in \mathbb{N}$.

Theorem 1 (Linear ELICIT-CMCC:1): Let $O_k^{ij}[t]$ be the tth coordinate ($t = 1,2,3,4$) of the TrFN O_k^{ij}, which represents the initial rating about the alternative x_i over x_j provided by the DM e_k. In the same way, $T_k^{ij}[t]$ $t = 1,2,3,4$ denotes the corresponding modified opinions. Then, the model (ELICIT-CMCC:1) is linearized as follows:

$$\min_{T_k^{i,j}[t]\in[0,1]} \frac{1}{4}\sum_{k=1}^{m}\sum_{i<j} c_k^{ij} \sum_{t=1}^{4} v_k^{ij}[t]$$

$$\text{s.t.}\begin{cases} 0 \leq v_k^{ij}[t] \leq 1, k \in \mathcal{I}_1^m, i \in \mathcal{I}_1^{n-1}, j \in \mathcal{I}_{i+1}^n, t \in \mathcal{I}_1^4 \\ -1 \leq u_k^{ij}[t] \leq 1, k \in \mathcal{I}_1^m, i \in \mathcal{I}_1^{n-1}, j \in \mathcal{I}_{i+1}^n, t \in \mathcal{I}_1^4 \\ u_k^{ij}[t] = T_k^{ij}[t] - O_k^{ij}[t], k \in \mathcal{I}_1^m, i \in \mathcal{I}_1^{n-1}, \\ \quad j \in \mathcal{I}_{i+1}^n, t \in \mathcal{I}_1^4 \\ v_k^{ij}[t] \geq u_k^{ij}[t], k \in \mathcal{I}_1^m, i \in \mathcal{I}_1^{n-1}, j \in \mathcal{I}_{i+1}^n, t \in \mathcal{I}_1^4 \\ v_k^{ij}[t] \geq -u_k^{ij}[t], k \in \mathcal{I}_1^m, i \in \mathcal{I}_1^{n-1}, j \in \mathcal{I}_{i+1}^n, t \in \mathcal{I}_1^4 \\ \overline{T}^{ij}[t] = \sum_{k=1}^{m} \omega_k T_k^{ij}[t], i \in \mathcal{I}_1^{n-1}, j \in \mathcal{I}_{i+1}^n, t \in \mathcal{I}_1^4 \\ 0 \leq z_k^{ij}[t] \leq 1, k \in \mathcal{I}_1^m, i \in \mathcal{I}_1^{n-1}, j \in \mathcal{I}_{i+1}^n, t \in \mathcal{I}_1^4 \\ -1 \leq y_k^{ij}[t] \leq 1, k \in \mathcal{I}_1^m, i \in \mathcal{I}_1^{n-1}, j \in \mathcal{I}_{i+1}^n, t \in \mathcal{I}_1^4 \\ y_k^{ij}[t] = T_k^{ij}[t] - \overline{T}_k^{ij}[t], k \in \mathcal{I}_1^m, i \in \mathcal{I}_1^{n-1}, \\ \quad j \in \mathcal{I}_{i+1}^n, t \in \mathcal{I}_1^4 \\ z_k^{ij}[t] \geq y_k^{ij}[t], k \in \mathcal{I}_1^m, i \in \mathcal{I}_1^{n-1}, j \in \mathcal{I}_{i+1}^n, t \in \mathcal{I}_1^4 \\ z_k^{ij}[t] \geq -y_k^{ij}[t], k \in \mathcal{I}_1^m, i \in \mathcal{I}_1^{n-1}, j \in \mathcal{I}_{i+1}^n, t \in \mathcal{I}_1^4 \\ T_k^{ij}[1] \leq T_k^{ij}[2] \leq T_k^{ij}[3] \leq T_k^{ij}[4], k \in \mathcal{I}_1^m, \\ i \in \mathcal{I}_1^{n-1}, j \in \mathcal{I}_{i+1}^n \\ \frac{1}{4}\sum_{t=1}^{4} z_k^{ij}[t] \leq \varepsilon, k \in \mathcal{I}_1^m, i \in \mathcal{I}_1^{n-1}, j \in \mathcal{I}_{i+1}^n \\ 1 - \frac{1}{4N}\sum_{k=1}^{m}\sum_{i<j} w_k \sum_{t=1}^{4} z_k^{ij}[t] \geq \mu_0 \end{cases}.$$

$$\mathrm{L-ELICIT-CMCC:1}$$

where $c_k^{ij} \in [0,1] (\sum_{k=1}^{m}\sum_{i<j} c_k^{ij} = 1)$ model the cost of moving the DM e_k's preference of the alternative x_i over x_j, $w_1, w_2, \ldots, w_m \in [0,1]$ ($\sum_{k=1}^{m} w_k = 1$) are the weights for the DMs, $N = \frac{n(n-1)}{2}$, and $\omega_1, \omega_2, \ldots, \omega_m \in [0,1]$ ($\sum_{k=1}^{m} \omega_k = 1$) are the weights for a fuzzy weighted average operator.

Theorem 2 (Linear ELICIT-CMCC:2): Let $O_k^{ij}[t]$ be the tth coordinate ($t = 1,2,3,4$) of the TrFN O_k^{ij}, which represents the initial rating about the alternative x_i over x_j provided by the DM e_k. In the same way, $T_k^{ij}[t]$ $t = 1,2,3,4$ denotes the for the corresponding modified opinions. Then, the linearized version of the model (ELICIT-CMCC:2) is given by, unnumbered equation shown at the bottom of the next page.

where $c_k^{ij} \in [0,1] (\sum_{k=1}^{m}\sum_{i<j} c_k^{ij} = 1)$ model the cost of moving the DM e_k's preference of the alternative x_i over x_j, $w_1, w_2, \ldots, w_m \in [0,1]$ ($\sum_{k=1}^{m} w_k = 1$) are the weights for the DMs, $N = \frac{n(n-1)}{2}$, and $\omega_1, \omega_2, \ldots, \omega_m \in [0,1]$ ($\sum_{k=1}^{m} \omega_k = 1$) are the weights for a fuzzy weighted average operator.

Proof: The proof of these results are provided in Appendix A, available in the online supplementary material. ■

This linear formulation of the ELICIT-CMCC models allows us to considerably accelerate the resolution of the optimization problem and improve the accuracy of the results provided by computational solvers. Indeed, the linear formulation also allows applying these models in large-scale GDM problems [1], [42], namely, decision situations in which hundreds or thousands of DMs may take part. In this regard, we have tested the performance of the proposal in such contexts under randomly generated initial preferences. The simulations have considered $n = 4$, $\mu_0 = 0.8$, and $\varepsilon = 0.2$ and have been carried out by using the solver *Clp* for the programming language Julia 1.6 [43] on the cloud service Google Colaboratory [44] (2.20-GHz Intel(R) Xeon(R) CPU and 13-GB RAM). These simulations have shown that the model (ELICIT-CMCC:1) is able to deal with problems involving hundreds of DMs in a few seconds and just needs around 4 min to solve problems with 2000 DMs. However, since the volume of constraints and variables required to linearize (ELICIT-CMCC:2) is much higher, the latter requires around 26 min to solve problems in which 200 DMs are considered.

Remark 7: Note that, according to the literature review carried out by García-Zamora et al. [1], most of the existing large-scale CRPs are evaluated by using GDM problems involving just 20 or 50 DMs.

IV. LINGUISTIC COST METRIC BASED ON ELICIT-CMCC

The high prevalence of LiGDM problems in society has attracted the attention of researchers, who have proposed many

LiCRPs based on the fuzzy linguistic approach [9], [22]. However, this large number of proposals implies a considerable problematic related to choose the most suitable consensus model for solving a certain LiGDM problem. Even though several authors carry out a comparative analysis with other proposals in order to show their advantages, the lack of objective metrics prevents from categorically claiming that one model is better than another. In addition, this absence of metrics harms the research in the area, since there is no filter to evaluate the novel CRPs from a performance point of view [1].

Hereafter, we introduce a linguistic metric based on the ELICIT-CMCC models presented in the previous section. This linguistic metric aims at measuring the performance of those LiCRPs that model the linguistic information by means of linguistic variables with a triangular or trapezoidal membership function representation because they can be easily written in terms of ELICIT information. As in the previous section, here, we consider an LiGDM problem in which m DMs want to reach a consensus about which alternative, from a set of n, is the most suitable one with a consensus threshold $\mu_0 \in [0,1[$.

To do so, their judgments, which are elicited by using linguistic expressions and pairwise comparisons, are first translated into TrFNs. If two TrFN matrices T and T' that are additive reciprocal are given, the distance between them is computed by using the function $\nu : \mathcal{M}_{n\times n}(\mathcal{T})^* \times \mathcal{M}_{n\times n}(\mathcal{T})^* \to [0,1]$ defined by

$$\nu(T,T') = \frac{2}{n(n-1)} \sum_{i<j} \delta(T^{ij}, T'^{ij})$$
$$= \frac{2}{n(n-1)} \sum_{i<j} \frac{1}{4} \sum_{t=1}^{4} |T^{ij}[t] - T'^{ij}[t]| \forall (T,T')$$
$$\in \mathcal{M}_{n\times n}(\mathcal{T})^* \times \mathcal{M}_{n\times n}(\mathcal{T})^*$$

where δ is the geometric distance between TrFNs defined in Section II-C and $T^{ij}[t], T'^{ij}[t]$ $t = 1,2,3,4$ denote the tth coordinates of the TrFNs T^{ij} and T'^{ij}, respectively.

Let $O = \{O_1, O_2, \ldots, O_m\} \subset \mathcal{M}_{n\times n}(\mathcal{T})^*$ be the TrFN matrices corresponding to the initial values of DMs' preferences for the aforementioned LiGDM problem, and let $T = \{\hat{T}_1, \hat{T}_2, \ldots, \hat{T}_m\} \subset \mathcal{M}_{n\times n}(\mathcal{T})^*$ be the set of modified agreed preferences obtained as output from a certain LiCRP. In the same way, the set $T^0 = \{T_1^0, \ldots, T_m^0\} \in \mathcal{M}_{n\times n}(\mathcal{T})^*$ denotes the optimal solution obtained for the consensus threshold μ_0 by using either the model ELICIT-CMCC:1 if the LiCRP uses a consensus measure of class 1 or ELICIT-CMCC:2 if the LiCRP uses a consensus measure of class 2. From these TrFN matrices, the mean distance between the outputs of the corresponding

$$\min_{T_k^{i,j}[t]\in[0,1]} \frac{1}{4} \sum_{k=1}^{m} \sum_{i<j} c_k^{ij} \sum_{t=1}^{4} v_k^{ij}[t]$$

$$\text{s.t.} \begin{cases} 0 \le v_k^{ij}[t] \le 1, k \in \mathcal{I}_1^m, i \in \mathcal{I}_1^{n-1}, j \in \mathcal{I}_{i+1}^n, t \in \mathcal{I}_1^4 \\ -1 \le u_k^{ij}[t] \le 1, k \in \mathcal{I}_1^m, i \in \mathcal{I}_1^{n-1}, j \in \mathcal{I}_{i+1}^n, t \in \mathcal{I}_1^4 \\ u_k^{ij}[t] = T_k^{ij}[t] - O_k^{ij}[t], k \in \mathcal{I}_1^m, i \in \mathcal{I}_1^{n-1}, j \in \mathcal{I}_{i+1}^n, t \in \mathcal{I}_1^4 \\ v_k^{ij}[t] \ge u_k^{ij}[t], k \in \mathcal{I}_1^m, i \in \mathcal{I}_1^{n-1}, j \in \mathcal{I}_{i+1}^n, t \in \mathcal{I}_1^4 \\ v_k^{ij}[t] \ge -u_k^{ij}[t], k \in \mathcal{I}_1^m, i \in \mathcal{I}_1^{n-1}, j \in \mathcal{I}_{i+1}^n, t \in \mathcal{I}_1^4 \\ \overline{T}^{ij}[t] = \sum_{k=1}^m \omega_k T_k^{ij}[t] \\ 0 \le z_k^{ij}[t] \le 1, i \in \mathcal{I}_1^{n-1}, j \in \mathcal{I}_{i+1}^n, t \in \mathcal{I}_1^4 \\ -1 \le y_k^{ij}[t] \le 1, k \in \mathcal{I}_1^m, i \in \mathcal{I}_1^{n-1}, j \in \mathcal{I}_{i+1}^n, t \in \mathcal{I}_1^4 \\ y_k^{ij}[t] = T_k^{ij}[t] - \overline{T}_k^{ij}[t], k \in \mathcal{I}_1^m, i \in \mathcal{I}_1^{n-1}, j \in \mathcal{I}_{i+1}^n, t \in \mathcal{I}_1^4 \\ z_k^{ij}[t] \ge y_k^{ij}[t], k \in \mathcal{I}_1^m, i \in \mathcal{I}_1^{n-1}, j \in \mathcal{I}_{i+1}^n, t \in \mathcal{I}_1^4 \\ z_k^{ij}[t] \ge -y_k^{ij}[t], k \in \mathcal{I}_1^m, i \in \mathcal{I}_1^{n-1}, j \in \mathcal{I}_{i+1}^n, t \in \mathcal{I}_1^4 \\ T_k^{ij}[1] \le T_k^{ij}[2] \le T_k^{ij}[3] \le T_k^{ij}[4], k \in \mathcal{I}_1^m, i \in \mathcal{I}_1^{n-1}, j \in \mathcal{I}_{i+1}^n \\ \frac{1}{4}\sum_{t=1}^4 z_k^{ij}[t] \le \varepsilon, k \in \mathcal{I}_1^m, i \in \mathcal{I}_1^{n-1}, j \in \mathcal{I}_{i+1}^n \\ 0 \le q_{kl}^{ij}[t] \le 1, k \in \mathcal{I}_1^{m-1}, l \in \mathcal{I}_{k+1}^m i \in \mathcal{I}_1^{n-1}, j \in \mathcal{I}_{i+1}^n, t \in \mathcal{I}_1^4 \\ -1 \le p_{kl}^{ij}[t] \le 1, k \in \mathcal{I}_1^{m-1}, l \in \mathcal{I}_{k+1}^m i \in \mathcal{I}_1^{n-1}, j \in \mathcal{I}_{i+1}^n, t \in \mathcal{I}_1^4 \\ p_{kl}^{ij}[t] = T_k^{ij}[t] - T_l^{ij}[t], k \in \mathcal{I}_1^{m-1}, l \in \mathcal{I}_{k+1}^m i \in \mathcal{I}_1^{n-1}, j \in \mathcal{I}_{i+1}^n, t \in \mathcal{I}_1^4 \\ q_{kl}^{ij}[t] \ge p_{kl}^{ij}[t], k \in \mathcal{I}_1^{m-1}, l \in \mathcal{I}_{k+1}^m i \in \mathcal{I}_1^{n-1}, j \in \mathcal{I}_{i+1}^n, t \in \mathcal{I}_1^4 \\ q_{kl}^{ij}[t] \ge -p_{kl}^{ij}[t], k \in \mathcal{I}_1^{m-1}, l \in \mathcal{I}_{k+1}^m i \in \mathcal{I}_1^{n-1}, j \in \mathcal{I}_{i+1}^n, t \in \mathcal{I}_1^4 \\ 1 - \frac{1}{4\,N} \sum_{k=1}^m \sum_{i<j} \frac{w_k+w_l}{m-1} \sum_{t=1}^4 q_{kl}^{ij}[t] \ge \mu_0, \end{cases} \quad (\mathrm{L-ELICIT-CMCC:2})$$

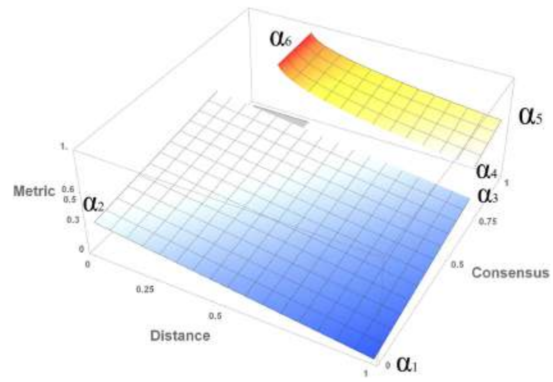

Fig. 7. Sketch of the graph of $\Phi_{0.25,0.75}$.

consensus models and the original preferences are computed as

$$d := \frac{1}{m}\sum_{k=1}^{m}\nu(T_k, O_k) \in [0,1]$$

$$d_0 := \frac{1}{m}\sum_{k=1}^{m}\nu(T_k^0, O_k) \in [0,1].$$

Note that these values strongly depend on the original values of the DMs' preferences, but such dependence is not reflected in the notation for the sake of simplicity.

To analyze the performance of the LiCRP, the distance d computed from the corresponding modified preferences is compared to the distance d_0 computed by using the ELICIT-CMCC model, which provides the preferences that require the lowest changes to reach the consensus threshold μ_0 (when $\varepsilon = 1$).

To compare these values, we use the metric $\Phi_{d_0,\mu_0} : [0,1] \times [0,1] \to [-1, \alpha_6]$ given by

$$\Phi_{d_0,\mu_0}(x,y) = \begin{cases} (\alpha_1-\alpha_2)x + \frac{\alpha_3-\alpha_2}{\mu_0}y + \alpha_2, & 0 \le y < \mu_0 \\ -1, & \begin{cases} \mu_0 \le y \le 1 \\ 0 \le x < d_0 \end{cases} \\ (\frac{x-d_0}{1-d_0})^{\frac{1}{3}}(\alpha_4 - \alpha_6 + \\ (\frac{y-\mu_0}{1-\mu_0})(\alpha_5-\alpha_4)) + \alpha_6, & \begin{cases} \mu_0 \le y \le 1 \\ d_0 \le x \le 1 \end{cases} \end{cases}$$

$\forall x, y \in [0,1]$, where $0 \le \alpha_1 < \alpha_2 < \alpha_3 \le \alpha_4 < \alpha_5 < \alpha_6$ are some parameters to configure the scale. In this regard, we propose the use of the default values $\alpha_1 = 0.0, \alpha_2 = 0.3, \alpha_3 = 0.5, \alpha_4 = 0.5, \alpha_5 = 0.6$, and $\alpha_6 = 1.0$, which guarantee that the function Φ_{d_0,μ_0} is valuated in the interval [0, 1]. For such values, the graph shown in Fig. 7 is obtained when the distance between the minimal solution to the ELICIT-CMCC optimization problem and the original preferences is $d_0 = 0.25$, and the consensus threshold is $\mu_0 = 0.75$.

Note that this metric provides a numeric rating in a [0, 1] scale, which is higher when the performance of the analyzed LiCRP is better. Consequently, to objectively evaluate the performance of an LiCRP in a certain LiGDM problem, it suffices to compute the value of $\Phi_{d_0,\mu_0}(d,\mu)$, where d is the distance between the original preferences and the modified opinions provided as output of the evaluated LiCRP and μ is the consensus degree of such modified preferences.

Remark 8: It should be highlighted that changing the values of the parameters $\alpha_1, \alpha_2, \ldots, \alpha_6$ implies a change of the scale in which the marks of the CRPs are given, but the better CRPs will still receive the higher marks.

Let us analyze the geometrical interpretation of the value $\Phi_{d_0,\mu_0}(d,\mu)$.

1) $0 \le \mu < \mu_0$: In this case, the consensus degree μ obtained by the LiCRP is worse than the consensus threshold μ_0. In this case, the worst scenario is $\Phi_{d_0,\mu_0}(1,0) = \alpha_1$ and the best ones are those close to the pair $(0,\mu_0)$, which receives a value close to α_3. α_2 is the value assigned to the pairs close to (0, 0).
2) $\mu_0 \le \mu \le 1$: In the case in which the LiCRP reaches the consensus threshold, it is necessary to differentiate two scenarios:
1) $0 \le d < d_0$: This case is unfeasible in practice because to achieve the consensus threshold μ_0, the minimum distance required is d_0. Therefore, the metric assigns -1 to the values in this region.
2) $d_0 \le d \le 1$: In this case, the LiCRP achieves the consensus threshold μ_0, but the distance d between the modified preferences and the original ones may not be close to the optimal distance d_0. The best pairs are those in which the distance d is equal to the optimal, and therefore, the metric receives the value α_6. If the LiCRP reaches the consensus threshold but makes unnecessary changes (d close to 1), the metric returns values close to α_4. The value α_5 is obtained when the distance is maximal, but the consensus level is close to 1.

The metric Φ_{d_0,μ_0} allows testing the performance of a model by comparing it with the optimal modified preferences obtained from the ELICIT-CMCC models (RQ2). However, the value of $\Phi^{\alpha}_{d_0,\mu_0}(d,\mu)$ highly depends on the original values of the preferences given by the DMs $O = \{O_1, O_2, \ldots, O_m\}$. To provide fair comparisons, the value of this metric should be computed for different LiGDM problems. To do that, the consensus model should be tested under several contexts $O^1, O^2, \ldots, O^r$ in order to better evaluate its performance, thus obtaining an average value $\Phi_{\mu_0} := \frac{1}{r}\sum_{s=1}^{r}\Phi_{d_0^r,\mu_0}(d^s,\mu^s)$, where d_0^r is the minimum value of the cost function for the initial preferences O^s, d_0^r is the value of the cost function for the preferences modified by the LiCRP, and μ^s is the corresponding consensus degree. Therefore, we propose solving the same LiGDM problem for several randomized preferences and computing the average value of the metric.

For instance, this metric has been used to evaluate the performance of two LiCRPs: the consensus model for ELICIT information introduced by Labella et al. [22] and the model proposed by Rodriguez et al. [9] for large-scale dealing with comparative linguistic expressions (CLEs). To do so, ten simulations with random preferences have been carried out in both the models. In each simulation, five DMs have to decide which alternative within a collection of four possible choices is the best one from

TABLE I
LABELLA ET AL. [22] SIMULATIONS RESULTS FOR $\mu_0 = 0.8$

Simulations	S1	S2	S3	S4	S5	S6	S7	S8	S9	S10	Average
d_0	0.07	0.09	0.08	0.04	0.09	0.09	0.1	0.09	0.08	0.06	0.08
d	0.09	0.13	0.09	0.06	0.13	0.12	0.16	0.15	0.1	0.07	0.11
μ	0.81	0.83	0.8	0.81	0.83	0.8	0.83	0.82	0.81	0.8	0.81
$\Phi_{d_0,\mu_0}(d,\mu)$	0.855	0.831	0.876	0.875	0.841	0.852	0.811	0.803	0.863	0.886	0.849

TABLE II
RODRÍGUEZ ET AL. [9] SIMULATIONS RESULTS FOR $\mu_0 = 0.8$

Simulations	S1	S2	S3	S4	S5	S6	S7	S8	S9	S10	Average
d_0	0.06	0.12	0.04	0.1	0.06	0.09	0.04	0.06	0.11	0.05	0.07
d	0.14	0.16	0.07	0.19	0.11	0.15	0.07	0.15	0.15	0.13	0.13
μ	0.85	0.81	0.82	0.82	0.82	0.82	0.82	0.88	0.8	0.86	0.82
$\Phi_{d_0,\mu_0}(d,\mu)$	0.788	0.815	0.846	0.77	0.813	0.802	0.848	0.79	0.816	0.795	0.808

a consensual point of view. The consensus threshold has been established in $\mu_0 = 0.8$ and the maximum number of allowed rounds is $MaxRounds = 5$.

The results of both the models are, respectively, shown in Tables I and II. Whereas the average value of our metric for the Labella et al. model is 0.849, the Rodríguez et al. model obtained an average mark of 0.808. Although both the models usually reach the consensus threshold $\mu_0 = 0.8$, the Rodríguez et al. model has shown a slightly worse performance because it changes DMs' initial opinions more than Labella et al. model, i.e., the average value $d - d_0$ is larger for the Rodríguez et al. model.

Finally, in order to perform a comparative analysis of this metric with other proposals, a search in Web of Science of the topics "metric" and "consensus reaching process" reveals that there is only one proper related paper proposed by Labella et al. [16]. Even though such work also considers as input the cost of modifying experts' opinions, the metric here proposed includes the following novelties regarding the one in [16].

1) The proposed metric in this article is capable to deal with flexible comparative linguistic information, which allows applying the metric in LiCRPs that require the modeling of DMs' hesitancy with expressions closer to their way of thinking.
2) It can be used to rate consensus models for large-scale LiGDM problems due to the linearization of the ELICIT-CMCC model.
3) Whereas the Labella et al. metric [16] assigns the same value to models with similar cost, the proposed metric assigns the metric value according to not only the cost but also the consensus degree reached by the consensus model. Consequently, the mathematical definition of the proposed metric is completely different to the one given in [16] (see Fig. 7) to ensure that the models are evaluated according to different scenarios that are determined by the consensus threshold and the minimum feasible cost.
4) The metric proposed in [16] is valuated in $[-1, 1]$, where 0 is the best scenario in terms of cost and 1 and -1 are bad scenarios with different meanings. On the contrary, the metric here introduced returns a value in a 0–1 scale that increases according to the quality of the evaluated model. This new metric, even though it is formally more complex, simplifies the comparison process because the higher the value of the metric, the better the quality of the model.

V. APPLYING THE LICRP METRIC TO LIGDM PROBLEMS

Here, the performance of both the ELICIT-CMCC models and the proposed linguistic cost metric is shown. First, in Section V-A, an illustrative LiGDM problem is introduced. Afterward, Section V-B solves such an LiGDM problem by using the CW ELICIT-CMCC:2 model. Finally, in Section V-C, two LiCRPs proposed in the literature [9], [22] are used to solve the same LiGDM problem in order to compare their performances through the linguistic cost metric. Since the purpose of this section is not solving a real-world problem, but showing how to use our proposals, we consider a toy problem with five DMs to simplify the process.

A. Illustrative LiGDM Problem Description

The LiGDM problem we aim at solving consists of a group of five friends $m = 5$ who want to decide in a consensual way (to avoid none of them feel unsatisfied with the chosen alternative), which movie franchise is the most preferred by the group to do a marathon. The possible alternatives are x_1 : Avengers, x_2 : Harry Potter, x_3 : Star Wars, and x_4 : The Lord of the Rings. In order to facilitate the decision process, they are asked to provide linguistic assessments by comparing the alternatives to each other. Since they may doubt in their preferences, we use HFLPRs to model their opinions. The linguistic expression domain is as follows:

$$S = \{\text{Much Worse (MW)}, \text{Worse (W)}, \text{Slightly Worse (SW)}$$
$$\text{Equal (E)}, \text{Slightly Better (SB)}, \text{Better (B)}, \text{Much Better (MB)}\}.$$

The initial values provided by the three DMs are compiled in Appendix B.A.

B. Solving the LiGDM Problem With ELICIT-CMCC Models

Here, the resolution of the illustrative LiGDM problem using the ELICIT-CMCC:2 model is carried out. First, the HFLPRs provided by the DMs (see Appendix B.A) are rewritten as EPRs

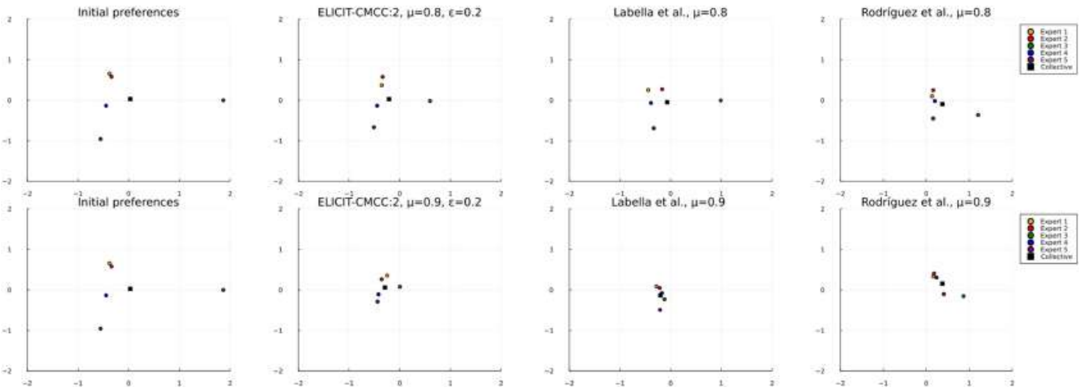

Fig. 8. Graphical visualization regarding the DMs' preferences in the different simulations and consensus models.

TABLE III
DOMINANCES AND MAGNITUDES FOR DETERMINING THE RANKING OF EACH ALTERNATIVE

Alternative	ELICIT expression	Magnitude
x_1	$Bt\ (SW,-0.33)^{0.002}$ and $(SW,0.01)^{-0.002}$	0.30620
x_2	$Bt\ (E,0.28)^{0.019}$ and $(SB,0.17)^{-0.032}$	0.61935
x_3	$Bt\ (SB,-0.34)^{0.003}$ and $(B,-0.13)^{-0.053}$	0.70694
x_4	$Bt\ (SW,-0.15)^{0.025}$ and $(E,-0.5)^{0.035}$	0.36750

(Appendix B.B) and then expressed as TrFNs by using the mapping ζ^{-1} (Appendix B.C).

To obtain the results of the linearized optimization problem, we have used the programming language Julia [43], concretely the package *Clp* which allows solving linear optimization problems. For a consensus threshold established as $\mu_0 = 0.8$ and a maximal distance between DMs and the collective opinion $\varepsilon = 0.2$, the optimal agreed preferences obtained for the ELICIT-CMCC:2 model are shown in Appendix B.D, and their translation into ELICIT values are in Appendix B.E.

From the collective values, the ELICIT expressions corresponding to the dominance degree [45], [46] of each alternative over the others are computed by using the fuzzy weighted average. For each one of such dominances, the respective value of its magnitude [7] (see Section II-C) is computed in order to determine the ranking of the alternatives. Both the dominances and their magnitudes are summarized in Table III.

Therefore, the ranking of the alternatives is $x_3 \succ x_2 \succ x_4 \succ x_1$. In other words, choosing the alternative x_3 : Star Wars is the best option from a consensual point of view, which requires the lowest cost.

C. Comparative Analysis

This section is devoted to compare the performance of two different LiCRPs to the ELICIT-CMCC approach when facing the problem described in the previous section. To do so, several aspects of these models are analyzed, such as the value of the metric Φ_{μ_0} or the number of rounds required to reach the desired consensus under different scenarios.

TABLE IV
COMPARATIVE RESULTS OF LABELLA ET AL. [22], RODRÍGUEZ ET AL. [9], AND ELICIT-CMCC:2 FOR $\mu_0 = 0.8$ AND $d_0 = 0.06$

Consensus model	Consensus degree (μ)	Distance to collective (ε)	Cost	Rounds Required	Metric (Φ_{d_0,μ_0})
ELICIT-CMCC:2	0.8	0.2	0.06	–	0.939
Labella et al. [20]	0.81	0.44	0.08	1	0.875
Rodríguez et al. [7]	0.85	0.4	0.14	2	0.801

TABLE V
COMPARATIVE RESULTS OF LABELLA ET AL. [22], RODRÍGUEZ ET AL. [9], AND ELICIT-CMCC:2 FOR $\mu_0 = 0.9$ AND $d_0 = 0.12$

Consensus model	Consensus degree (μ)	Distance to collective (ε)	Cost	Rounds Required	Metric (Φ_{d_0,μ_0})
ELICIT-CMCC:2	0.9	0.2	0.12	–	1.0
Labella et al. [20]	0.91	0.12	0.15	5	0.843
Rodríguez et al. [7]	0.92	0.37	0.17	3	0.815

The selected consensus models for this comparative analysis are the consensus model for ELICIT information introduced by Labella et al. [22] and the consensus model that deals with CLEs proposed by Rodríguez et al. [9]. Both the proposals have solved the problem previously introduced under two different scenarios.

1) Scenario 1: $\mu_0 = 0.8$ and $MaxRounds = 5$ (see Table IV).
2) Scenario 2: $\mu_0 = 0.9$ and $MaxRounds = 5$ (see Table V).

In addition, the value for the parameter ε used in the ELICIT-CMCC:2 is set as $\varepsilon = 0.2$. This model is also evaluated under the two aforementioned consensus situations.

In the first scenario, the Labella et al. model [22] achieves a consensus degree $\mu = 0.81$ in one discussion round, and the Rodríguez et al. [9] model achieves a consensus degree of $\mu = 0.85$ in two discussion rounds. Regarding the maximal distance between DMs and collective opinion, note that the condition $\varepsilon \leq 0.2$ guarantees such a maximal distance in ELICIT-CMCC:2 (see Table IV). However, such distance is much higher in both Labella et al. and Rodríguez et al. models, which can be appreciated in Fig. 8.

In the second scenario, the consensus degree obtained by Labella et al. model is $\mu = 0.91$ in five rounds and the obtained by Rodríguez et al. model is $\mu = 0.92$ in three rounds. In this scenario, the distance between modified preferences and the collective opinion is lower than before for the Labella et al. model (0.12), but still higher than $\varepsilon = 0.2$ for the Rodríguez et al. (see Table V and Fig. 8).

As expected, the costs obtained in ELICIT-CMCC:2 (0.06 and 0.12) are lower than the costs of both Labella et al. [22] (0.08 and 0.15) and Rodríguez et al. [9] (0.14 and 0.17) models. In this regard, the ELICIT:CMCC:2 stands out because of its efficiency.

Regarding the marks provided by our metric for these three approaches, in the $\mu_0 = 0.8$ scenario, Labella et al. CRP gets a score of 0.875, whereas Rodríguez et al. proposal obtains a score equal to 0.801. The performance of both the models to solve this specific LiGDM problem in terms of "extra cost" could be considered "good" but far from the optimal modified preferences provided by the ELICIT-CMCC model, whose mark is 0.939.

In the $\mu_0 = 0.9$ case, the Labella et al. model is still better than the Rodríguez et al. approach, but their marks are closer than in the previous scenario (0.843 and 0.815, respectively). Meanwhile, the ELICIT-CMCC:2 proposal gets an approximate mark of 1, which means that, for these values of the initial preferences, the solutions for the optimization problems corresponding to $\varepsilon = 1$, which provides the ideal modified preferences used in the metric, and $\varepsilon = 0.2$, which is the value used to derive the agreed solution in this illustrative example, are very close.

To sum up, the marks provided by the cost metric are quite simple and intuitive and allow evaluating properly the performance of LiCRPs, because it compares the output provided by the LiCRPs with the one provided by the ELICIT-CMCC model in terms of cost and consensus degree achieved.

VI. Conclusion

This article proposed a cost metric for LiCRPs, which takes into account both the cost of modifying the original DMs' preferences and the final consensus degree obtained by the group.

The definition of such a metric relies on ELICIT-CMCC models, a novel extension of CMCC models to manage linguistic information. The use of ELICIT information guarantees the manipulation of linguistic values without losing information in the process and assuring the interpretability of the results. Concretely, the output obtained from ELICIT-CMCC models present the following properties.

1) It is expressed in a linguistic domain.
2) It minimizes the cost of moving DMs' preferences.
3) It guarantees a maximal absolute deviation ε between the modified opinions and the collective one.
4) The obtained consensus degree is equal to or greater than a predefined consensus threshold μ_0.

In order to improve the computational performance of these ELICIT-CMCC models, we also proposed the corresponding linearized version, which additionally grants more precise solutions when it is implemented in a computer solver. Furthermore, the performance of these linear ELICIT-CMCC in GDM problems involving hundreds or thousands of DMs was briefly discussed.

The inherent features of the previous models have also allowed us to address one of the most recurrent limitations in the LiCRP literature: the lack of metrics capable to evaluate the performance of these processes. In this sense, the proposed linguistic cost metric compares the optimal cost necessary to reach the desired consensus threshold, which is obtained from solving an ELICIT-CMCC model (ELICIT-CMCC:1 or ELICIT-CMCC:2), with the changes made by the LiCRP. In addition, if the resulting consensus degree after the LiCRP is lower than the desired consensus threshold, the metric will rate such LiCRP with a low mark. This metric has also been used to evaluate the performance of two linguistic consensus models already defined in the specialized literature [9], [22] to show its implementation in practice.

Finally, we developed a comparative analysis that reveals that ELICIT-CMCC models are much better in terms of efficiency (lower cost and better values for μ and ε) than two LiCRPs [9], [22].

To summarize, the main contributions of this article are as follows:

1) linguistic CMCC models for LiGDM based on ELICIT information which follow a CW approach;
2) linearization of the ELICIT-CMCC models to improve their performance and expand their use to LiCRP with many DMs;
3) a linguistic cost metric to evaluate LiCRPs.

As future works, we will analyze some formal aspects such as the use of other linguistic preference structures to propose ELICIT-CMCC, instead of pairwise comparison matrices, such as utility linguistic vectors. Furthermore, we will study the impact of using different aggregation operators to compute the collective opinion to improve the scope of ELICIT-CMCC, as well as the use of different weighting mechanisms to determine experts' importance [47]. In addition, the influence of the parameters μ_0 and ε in the resolution of the GDM problem should be discussed. From the application point of view, ELICIT-CMCC will be used to solve real-world decision problems with hundreds or thousands of DMs. Last but not least, the proposed metric must be applied to the evaluation of novel proposed LiCRPs to draw conclusions about their capability.

References

[1] D. García-Zamora, Á. Labella, W. Ding, R. M. Rodríguez, and L. Martínez, "Large-scale group decision making: A systematic review and a critical analysis," *IEEE/CAA J. Autom. Sinica*, vol. 9, no. 6, pp. 949–966, Jun. 2022.

[2] I. Palomares, F. Estrella, L. Martínez, and F. Herrera, "Consensus under a fuzzy context: Taxonomy, analysis framework AFRYCA and experimental case of study," *Inf. Fusion*, vol. 20, pp. 252–271, 2014.

[3] X. Pan, Y. Wang, and K.-S. Chin, "Dynamic programming algorithm-based picture fuzzy clustering approach and its application to the large-scale group decision-making problem," *Comput. Ind. Eng.*, vol. 157, 2021, Art. no. 107330.

[4] L. A. Zadeh, "The concept of a linguistic variable and its application to approximate reasoning—I," *Inf. Sci.*, vol. 8, pp. 199–249, 1975.

[5] L. A. Zadeh, "The concept of a linguistic variable and its application to approximate reasoning—II," *Inf. Sci.*, vol. 8, pp. 301–357, 1975.

[6] L. A. Zadeh, "The concept of a linguistic variable and its application to approximate reasoning—III," *Inf. Sci.*, vol. 9, pp. 43–80, 1975.

[7] Á. Labella, R. M. Rodríguez, and L. Martínez, "Computing with comparative linguistic expressions and symbolic translation for decision making: ELICIT information," *IEEE Trans. Fuzzy Syst.*, vol. 28, no. 10, pp. 2510–2522, Oct. 2020.

[8] Y. Dong, Y. Xu, H. Li, and B. Feng, "The OWA-based consensus operator under linguistic representation models using position indexes," *Eur. J. Oper. Res.*, vol. 203, no. 2, pp. 455–463, 2010.

[9] R. M. Rodríguez, Á. Labella, M. Sesma-Sara, H. Bustince, and L. Martínez, "A cohesion-driven consensus reaching process for large scale group decision making under a hesitant fuzzy linguistic term sets environment," *Comput. Ind. Eng.*, vol. 155, 2021, Art. no. 107158.

[10] P. Liu, X. Zhang, and W. Pedrycz, "A consensus model for hesitant fuzzy linguistic group decision-making in the framework of Dempster–Shafer evidence theory," *Knowl.-Based Syst.*, vol. 212, 2021, Art. no. 106559.

[11] X. Pan and Y. Wang, "Evaluation of renewable energy sources in China using an interval type-2 fuzzy large-scale group risk evaluation method," *Appl. Soft Comput.*, vol. 108, 2021, Art. no. 107458.

[12] X. Pan, Y. Wang, S. He, and K.-S. Chin, "A dynamic programming algorithm based clustering model and its application to interval type-2 fuzzy large-scale group decision-making problem," *IEEE Trans. Fuzzy Syst.*, vol. 30, no. 1, pp. 108–120, Jan. 2022.

[13] Z.-J. Du, S.-M. Yu, H.-Y. Luo, and X.-D. Lin, "Consensus convergence in large-group social network environment: Coordination between trust relationship and opinion similarity," *Knowl.-Based Syst.*, vol. 217, 2021, Art. no. 106828.

[14] F. Mata, L. G. Pérez, S.-M. Zhou, and F. Chiclana, "Type-1 OWA methodology to consensus reaching processes in multi-granular linguistic contexts," *Knowl.-Based Syst.*, vol. 58, pp. 11–22, 2014.

[15] Y. Wu, Y. Dong, J. Qin, and W. Pedrycz, "Flexible linguistic expressions and consensus reaching with accurate constraints in group decision-making," *IEEE Trans. Cybern.*, vol. 50, no. 6, pp. 2488–2501, Jun. 2020.

[16] A. Labella, H. Liu, R. M. Rodríguez, and L. Martínez, "A cost consensus metric for consensus reaching processes based on a comprehensive minimum cost model," *Eur. J. Oper. Res.*, vol. 281, pp. 316–331, 2020.

[17] R. M. Rodríguez, Á. Labella, B. Dutta, and L. Martínez, "Comprehensive minimum cost models for large scale group decision making with consistent fuzzy preference relations," *Knowl.-Based Syst.*, vol. 215, 2021, Art. no. 106780.

[18] L. Martínez, R. M. Rodríguez, and F. Herrera, *The 2-Tuple Linguistic Model*. New York, NY, USA: Springer, 2015.

[19] R. M. Rodríguez, L. Martínez, and F. Herrera, "Hesitant fuzzy linguistic term sets for decision making," *IEEE Trans. Fuzzy Syst.*, vol. 20, no. 1, pp. 109–119, Feb. 2012.

[20] L. A. Zadeh, "Fuzzy logic = Computing with words," *IEEE Trans. Fuzzy Syst.*, vol. 4, no. 2, pp. 103–111, May 1996.

[21] P. K. Gupta and J. Andreu-Perez, "A gentle introduction and survey on computing with words (CWW) methodologies," *Neurocomputing*, vol. 500, pp. 921–937, 2022.

[22] Á. Labella, R. M. Rodríguez, A. A. Alzahrani, and L. Martínez, "A consensus model for extended comparative linguistic expressions with symbolic translation," *Mathematics*, vol. 8, no. 12, 2020, Art. no. 2198.

[23] L. A. Zadeh, "Fuzzy sets," *Inf. Control*, vol. 8, no. 3, pp. 338–353, 1965.

[24] L. A. Zadeh, *Fuzzy Sets, Fuzzy Logic, and Fuzzy Systems: Selected Papers*. Singapore: World Sci., 1996.

[25] Y. Tang and J. Zheng, "Linguistic modelling based on semantic similarity relation among linguistic labels," *Fuzzy Sets Syst.*, vol. 157, no. 12, pp. 1662–1673, 2006.

[26] Y. Dong, G. Zhang, W. Hong, and S. Yu, "Linguistic computational model based on 2-tuples and intervals," *IEEE Trans. Fuzzy Syst.*, vol. 21, no. 6, pp. 1006–1018, Dec. 2013.

[27] J. Ma, D. Ruan, Y. Xu, and G. Zhang, "A fuzzy-set approach to treat determinacy and consistency of linguistic terms in multi-criteria decision making," *Int. J. Approx. Reason.*, vol. 44, no. 2, pp. 165–181, 2007.

[28] S. Abbasbandy and T. Hajjari, "A new approach for ranking of trapezoidal fuzzy numbers," *Comput. Math. Appl.*, vol. 57, no. 3, pp. 413–419, 2009.

[29] S. Heilpern, "Representation and application of fuzzy numbers," *Fuzzy Sets Syst.*, vol. 91, no. 2, pp. 259–268, 1997.

[30] S. Orlovski, "Decision-making with fuzzy preference relations," *Fuzzy Sets Syst.*, vol. 1, pp. 155–167, 1978.

[31] F. Herrera, E. Herrera-Viedma, and J. L. Verdegay, "A sequential selection process in group decision making with a linguistic assessment approach," *Inf. Sci.*, vol. 85, no. 4, pp. 223–239, 1995.

[32] B. Zhu and Z. Xu, "Consistency measures for hesitant fuzzy linguistic preference relations," *IEEE Trans. Fuzzy Syst.*, vol. 22, no. 1, pp. 35–45, Feb. 2014.

[33] C. Butler and A. Rothstein, *On Conflict and Consensus: A Handbook on Formal Consensus Decision Making*. Takoma Park, MD, USA, 2006.

[34] I. Palomares, "Group decision making and consensual processes," in *Large Group Decision Making*. New York, NY, USA: Springer, 2018, pp. 5–36.

[35] D. Ben-Arieh and T. Easton, "Multi-criteria group consensus under linear cost opinion elasticity," *Decis. Support Syst.*, vol. 43, no. 3, pp. 713–721, 2007.

[36] F. Herrera, E. Herrera-Viedma, and J. L. Verdegay, "A model of consensus in group decision making under linguistic assessments," *Fuzzy Sets Syst.*, vol. 78, no. 1, pp. 73–87, 1996.

[37] B. Spillman, J. Bezdek, and R. Spillman, "Development of an instrument for the dynamic measurement of consensus," *Commun. Monographs*, vol. 46, pp. 1–12, 1979.

[38] J. Kacprzyk, M. Fedrizzi, and H. Nurmi, "Group decision making and consensus under fuzzy preferences and fuzzy majority," *Fuzzy Sets Syst.*, vol. 49, no. 1, pp. 21–31, 1992.

[39] G. Zhang, Y. Dong, Y. Xu, and H. Li, "Minimum-cost consensus models under aggregation operators," *IEEE Trans. Syst., Man, Cybern. A, Syst. Humans*, vol. 41, no. 6, pp. 1253–1261, Nov. 2011.

[40] J. Liu, F. Chan, Y. Li, Y. Zhang, and Y. Deng, "A new optimal consensus method with minimum cost in fuzzy group decision," *Knowl.-Based Syst.*, vol. 35, pp. 357–360, 2012.

[41] P. Ren, Z. Liu, W.-G. Zhang, and X. Wu, "Consistency and consensus driven for hesitant fuzzy linguistic decision making with pairwise comparisons," *Expert Syst. Appl.*, vol. 202, 2022, Art. no. 117307.

[42] S. He, X. Pan, and Y. Wang, "A shadowed set-based TODIM method and its application to large-scale group decision making," *Inf. Sci.*, vol. 544, pp. 135–154, 2021.

[43] J. Bezanson, A. Edelman, S. Karpinski, and V. B. Shah, "Julia: A fresh approach to numerical computing," *SIAM Rev.*, vol. 59, no. 1, pp. 65–98, 2017.

[44] E. Bisong, *Building Machine Learning and Deep Learning Models on Google Cloud Platform*. New York, NY, USA: Springer, 2019.

[45] H. Montgomery, "Decision rules and the search for a dominance structure: Towards a process model of decision making," in *Analysing and Aiding Decision Processes* (Advances in Psychology Series), vol. 14, P. Humphreys, O. Svenson, and A. Vári, Eds. Amsterdam, The Netherlands: North Holland, 1983, pp. 343–369.

[46] D. S. Yakowitz, L. J. Lane, and F. Szidarovszky, "Multi-attribute decision making: Dominance with respect to an importance order of the attributes," *Appl. Math. Comput.*, vol. 54, no. 2, pp. 167–181, 1993.

[47] S. Wang, J. Wu, F. Chiclana, Q. Sun, and E. Herrera-Viedma, "Two stage feedback mechanism with different power structures for consensus in large-scale group decision-making," *IEEE Trans. Fuzzy Syst.*, vol. 30, no. 10, pp. 4177–4189, Oct. 2022.

5 CONCLUSIONES Y TRABAJOS FUTUROS

Para finalizar esta memoria, presentaremos las conclusiones que hemos extraído durante nuestra investigación, así como los posibles proyectos futuros que podrían abordarse a partir de los resultados obtenidos.

5.1 CONCLUSIONES

La TDGGE surge para resolver problemas reales de decisión en contextos de incertidumbre que requieren considerar las opiniones de un gran número de decisores. Al tener en cuenta las opiniones de muchas personas se pueden lograr resultados más objetivos que cuando se utilizan grupos reducidos, pero a cambio aumenta la complejidad.

El primer objetivo de esta memoria consistía en proponer nuevos enfoques para modelar la naturaleza no lineal de las preferencias dadas por seres humanos. Para cumplirlo, hemos definido las funciones EVAs y EVRs, que permiten transformar las valoraciones de los decisores en problemas de TDG de forma no lineal. Hemos comprobado que utilizar EVAs en PAC, además de ser más realista desde el punto de vista psicológico, generalmente mejora los resultados del PAC. Por otro lado, se ha demostrado que los EVRs pueden ser utilizados para fusionar las opiniones de los decisores dando mayor prioridad a los valores intermedios, que son los de mayor relevancia para alcanzar el consenso.

El siguiente objetivo estaba relacionado con mejorar los modelos de TDGGE para abordar problemas que requieran de un gran número de decisores. Para ello, nos hemos centrado en los modelos de CCM, que, al ser PAC automáticos, son especialmente útiles para manejar problemas en TDGGE, pues no requieren de varias rondas de negociación para alcanzar una solución consensuada. En este aspecto, hemos analizado en detalle la estructura matemática de los modelos CCM integrales, concluyendo que es posible mejorar considerablemente su eficiencia cuando se eliminan restricciones que pueden ser redundantes. Además, hemos propuesto los modelos FZZ-MCC, que generalizan los modelos CCM existentes en la literatura de una forma flexible. Desde un punto de vista práctico, hemos definido diversos modelos de consenso que permiten abordar problemas de TDGGE tales como democracia electrónica y la integración de datos y conocimiento experto en la TD.

El último objetivo consistía en la definición de métricas que permitan evaluar objetivamente el rendimiento de los PACGE. Para lograr esto, hemos desarrollado una métrica que evalúa los PAC atendiendo simultáneamente al nivel de acuerdo alcanzado y a las modificaciones realizadas en las preferencias originales.

En conclusión, es importante resaltar que hemos logrado alcanzar todos los objetivos establecidos al inicio de esta investigación, lo que ha permitido desarrollar herramientas, modelos y resultados que superan el estado del arte previo y abren nuevas posibilidades de investigación, tal como se describe en la siguiente sección.

5.2 TRABAJOS FUTUROS

Los resultados de esta investigación abren la posibilidad de explorar nuevas áreas de estudio que pueden ser abordadas en trabajos futuros. Algunas de las posibles líneas de investigación que pueden continuar el trabajo realizado en esta tesis doctoral son:

- Aplicar la teoría de EVAs y EVRs para hacer frente a retos de la TDGGE como la polarización de opiniones o las opiniones minoritarias.
- Estudiar el uso de Operadores EVR-OWA en problemas de TDGGE.
- Desarrollar nuevos modelos de TDGGE escalables para resolver problemas de TD del mundo real que involucren a millones de personas.
- Extender los modelos FZZ-MCC para hacer frente a nuevas situaciones de decisión, por ejemplo, integrando análisis de redes sociales.
- Explorar nuevos tipos de problemas de decisión desde la perspectiva de la TDGGE, como los de sorting, que buscan clasificar alternativas en diferentes categorías.

5.3 PUBLICACIONES ADICIONALES

En el desarrollo de esta investigación se han presentado otras publicaciones que no han sido recogidas en esta memoria y que enumeramos a continuación:

- Publicaciones en otras revistas internacionales:
  - H. Song, D. García-Zamora, A. Labella Romero, X. Jia, Y. Wang, and L. Martínez. "Handling multi-granular hesitant information: A group decision-making method based on cross-efficiency with regret theory". In: *Expert Systems with Applications*, 2023, 120332. Impact factor 8.665, Q1.
  - Y. Wang, S. He, D. García Zamora, X. Pan, and L.Martínez, "A Large Scale Group Three-Way Decision-based consensus model for site selection of New Energy Vehicle charging stations" In: *Expert Systems with Applications*, 214, 119107 (2023). Impact factor 8.665, Q1.
  - S. Feng, Y. Xin, S. Xiong, Z. Cheng, M. Devecy, D. García-Zamora, and W. Pedrycz. "Safety Perception Evaluation of Civil Aviation Based on Weibo Posts in China: An Enhanced Large-Scale Group Decision-Making Framework". *Int. J. Fuzzy Syst.* (2023). Impact factor 4.085, Q2.
  - M. Zhou, Z. Chen, J. Jiang, G. Qian, D. García-Zamora, B. Dutta, Q. Zhan, and L. Jin. "Auto-generated Relative Importance for Multi-agent Inducing Variable in Uncertain and Preference Involved Evaluation". In: *International Journal of Computational Intelligence Systems* 15, 108 (2022). Impact factor 2.259, Q3.
  - Y. Wang, X. Pan, S. He, B. Dutta, D. García-Zamora, and L. Martínez, "A New Decision-Making Framework for Site Selection of Electric Vehicle Charging Station with Heterogeneous Information and Multi-Granular Linguistic Terms". In: *IEEE Transactions on Fuzzy Systems* (2022). Impact factor 12.253, Q1.
- Capítulos de libro:
  - D. García-Zamora, A. Labella, R. M. Rodríguez and L. Martínez. "Nonlinear Scaled Preferences in Linguistic

Multi-Criteria Group Decision Making". In: *Real Life Applications of Multiple Criteria Decision-Making Techniques in Fuzzy Domain*. Springer Nature Singapore Pte Ltd; 2022.

- Congresos internacionales:
  - Á. Labella, D. García-Zamora, W. He, R.M. Rodríguez, and L. Martínez (2022). "Grouping representative points in AHP-FuzzySort with agglomerative hierarchical clustering". In: The International Symposium on the Analytic Hierarchy Process (ISAHP2022), Online, December 15-18, 2022.
  - D. García-Zamora, A. Labella, R. M. Rodríguez and L. Martínez. "Comprehensive Minimum Cost Consensus for Analyzing the Cost of Different Agreed Solutions". In: *15th International FLINS Conference on Machine Learning, Multi agent and Cyber physical systems and the 17th International ISKE Conference* (FLINS/ISKE 2022), 26-28 August Tianjin (China) 2022.
  - D. García-Zamora, A. Labella, R. M. Rodríguez and L. Martínez. "Comprehensive Minimum Cost Consensus Models for ELICIT Information". In: *2022 Word Congress on Computational Intelligence* (IEEE WCCI2022), Padua, Italy 18-23 July 2022.
  - J. Baz, D. García-Zamora, I. Díaz, S. Montes, L. Martínez. "Flexible-Dimensional EVR-OWA as Mean Estimator for Symmetric Distributions". In: *The 19th International Conference On Information Processing And Management Of Uncertainty In Knowledge-Based Systems* (IPMU 2022), Milan (Italy), July 11-15, 2022.
  - Á. Labella, D. García-Zamora, R.M. Rodríguez, and L. Martínez (2022). "Fuzzy TODIM for ELICIT Information". In: *International Conference on Intelligent and Fuzzy Systems* (INFUS 2022), Istanbul (Turkey), July 19-21, 2022 .
  - A. Labella, D. García-Zamora, R. M. Rodríguez and L. Martínez. "A Consensus-based Best-Worst Method for Multi-criteria Group Decision-Making". In: *The Third International Workshop on Best-Worst Method* (BWM-2022), Delft, The Netherlands, 09-10 June 2022.

  - D. García-Zamora, A. Labella, R. M. Rodríguez and L. Martínez. "Modelling non linear preferences in Consensus Reaching Processes. A sustainability application". In: *16th International Conference on Intelligent Systems and Knowledge Engineering* (ISKE 2021), 26-28 November Chengdu (China) 2021.
  - D. García-Zamora, A. Labella, R. M. Rodríguez and L. Martínez. "An Ordered Weighted Averaging Operator based on Extreme Values Reductions´´. In: *The 19th World Congress of the International Fuzzy Systems Association. The 12th Conference of the European Society for Fuzzy Logic and Technology jointly with the AGOP, IJCRS, and FQAS conferences*. Bratislava (Slovakia), September 19-24, 2021.
  - D. García-Zamora, A. Labella, R. M. Rodríguez and L. Martínez. "Non Linear Scales in GDM: Extreme Values Amplifications". In: *International Virtual Workshop on Business Analytics Eureka*, 2-4 Junio, Ciudad Juarez (Mexico), 2021.
  - A. Labella, D. García-Zamora, R. M. Rodríguez and L. Martínez. "A Novel Linguistic Cohesion Measure based on Restricted Equivalence Functions for Weighting Experts' subgroups in Large-scale Group Decision Making Problems´´. In: *International Virtual Workshop on Business Analytics Eureka*, 2-4 Junio, Ciudad Juarez (Mexico), 2021.

- Congresos Nacionales:
  - D. García-Zamora, A. Labella, R. M. Rodríguez and L. Martínez. "Nonlinear Preferences in Consensus Reaching Processes. Extreme Values Amplifications". In: *XXI Congreso español sobre Tecnologías y Lógica Fuzzy* (ESTYLF 2022), celebrado en Toledo el 4 –7 de Septiembre 2022.
  - D. García-Zamora, A. Labella, P. Nuñez-Cacho, R. M. Rodríguez and L. Martínez. "Modelos de consenso basados en Mínimo Coste para expresiones ELICIT". In: *XXI Congreso español sobre Tecnologías y Lógica Fuzzy* (ESTYLF 2022), celebrado en Toledo el 4 –7 de Septiembre 2022.
  - D. García-Zamora, A. Labella, R. M. Rodríguez and L. Martínez. "Preferencias no lineales en Toma de Decisión

en Grupo. Amplificación de Valores Extremos”. In: *CEDI 20/21: VI Congreso Español de Informática*, celebrado el 22 – 24. Septiembre 2021.

5.4 RECONOCIMIENTOS

Es importante mencionar que algunos de los trabajos realizados durante esta tesis doctoral han sido reconocidos por la comunidad científica:

1. El trabajo “Modelling Non Linear Preferences in Consensus Reaching Processes: A Sustainability Application”, presentado en the 16th International Conference on Intelligent Systems and Knowledge Engineering (ISKE 2021), celebrado en Chengdu, China, en noviembre de 2021 recibió el premio al mejor artículo de estudiante.

2. El poster titulado “ Non-linear preferences in Group Decision Making: Extreme Values Amplifications” recibió la mención especial en las Jornadas Doctorales para jóvenes investigadores de la Universidad de Jaén, celebradas en noviembre de 2021.

3. El trabajo titulado “Comprehensive Minimum Cost Consensus for Analyzing the Cost of Different Agreed Solutions”, presentado en the 15th International FLINS Conference on Machine Learning, Multi agent and Cyber physical systems and the 17th International ISKE Conference (FLINS/ISKE 2022), celebrado en Tianjin, China, durante agosto de 2022, también recibió el premio al mejor artículo de estudiante.

Los diplomas asociados con los reconocimientos mencionados anteriormente se incluyen al final de este capítulo.

5.5 ESTANCIAS Y COLABORACIONES

Durante la tesis doctoral, se llevaron a cabo dos estancias de investigación con el fin de mejorar la formación investigadora del doctorando mediante el conocimiento y la experiencia de expertos en la materia. Gracias a las ayuda de movilidad para beneficiarios del contrato FPU EST22/00031, concedida por el Ministerio de Universidades, se pudo realizar una estancia de tres meses (desde el

15/04/2022 hasta el 14/07/2022) en la School of Computing, Ulster University, en el Reino Unido. Además, se realizó otra estancia de un mes (desde el 23/01/2023 hasta el 24/02/2023) en el Grupo de Investigación en Inteligencia Artificial y Razonamiento Aproximado de la Universidad Pública de Navarra.

CERTIFICATE

BEST STUDENT PAPER

Presented to

Diego Garcia-Zamora

Paper Title

Modelling Non Linear Preferences in Consensus Reaching Processes: A Sustainability Application

At the 16th International Conference on Intelligent Systems and Knowledge Engineering (ISKE 2021) held in Chengdu, China during November 26-28, 2021

General Chair	16th International Conference on Intelligent Systems and Knowledge Engineering	**Program Chair**
Jie Lu	ISKE 2021	Luis Martínez López
University of Technology Sydney, Australia		University of Jaén, Spain

Universidad de Jaén

Escuela de Doctorado

Vicerrectorado de Coordinación y Calidad de las Enseñanzas

D. Antonio Gálvez del Postigo Ruiz, Director de la Escuela de Doctorado de la Universidad de Jaén

Hace constar

Que D. **DIEGO GARCÍA ZAMORA**, con DNI **26054246F**, ha obtenido **MENCIÓN ESPECIAL** en las **Jornadas Doctorales para jóvenes investigadores de la Universidad de Jaén 2021** celebradas del 22 al 26 de noviembre de 2021 con la comunicación:

"Non-linear preferences in Group Decision Making: Extreme Values Amplifications"

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Fdo: D. Antonio Gálvez del Postigo Ruiz

Director de la Escuela de Doctorado

ESCUELA DE DOCTORADO

2022 FLINS / ISKE

CERTIFICATE

The 15th International FLINS Conference
On Machine learning, Multi agent and Cyber physical systems

August 26-28, 2022
Nankai University, Tianjin, China

This is to certify that (Given name, Family name) Diego García-Zamora

participated in the Conference and presented a paper entitled

Comprehensive Minimum Cost Consensus for Analyzing the Cost of Different Agreed Solutions

Best Student Paper Award

Qinglin Sun *Jie Lu*
Chairs of International Program Committee

Parte II

RESUMEN EN INGLÉS

A ENGLISH SUMMARY

As a partial requirement for obtaining the International Ph.D. Award, this appendix consists of an English summary of the thesis titled "Intelligent Decision-making under Uncertainty". The summary includes a brief introduction to the research topic and the motivation for the research conducted. It also outlines the objectives of this thesis and the structure of the chapters that shape it. Additionally, a summary of the research papers presented in the thesis is provided. Finally, the conclusions are discussed.

A.1 STATE OF THE ART

A Decision-making problem is usually understood as a cognitive process that involves various mental and reasoning processes to choose the most appropriate alternative among several possible solutions in a given situation [23]. A decision-making problem becomes more complex in situations of uncertainty that require consideration of various types of costs, conflicts, and large volumes of data [36]. To address problems of this complexity, intelligent decision techniques tailored specifically to the problem at hand are developed [5]. Generally speaking, decision-making processes have certain common phases (see Fig. A.1), such as [18]:

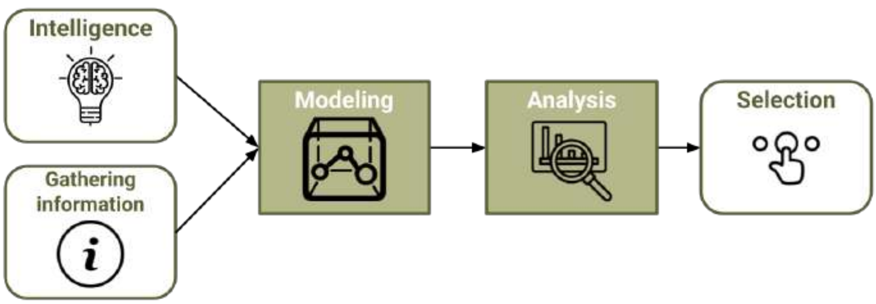

Fig. A.1: Decision-making scheme

INTELLIGENCE Observe the real world to identify the problem.

INFORMATION GATHERING Obtain data, knowledge, and preferences related to the problem.

MODELING Define a framework that establishes the structure of the problem, preferences, and uncertainty.

ANALYSIS Study and combine information according to the objectives, constraints, and results considered in the selection phase.

SELECTION Exploit the results of the analysis to select an alternative/solution to the problem.

Nowadays, it is increasingly common to rely on data-driven decision-making processes and quantitative models that may limit the participation of human experts, who usually develop their decision processes using qualitative information [1]. Despite this, in many fields that require intelligent, efficient, and effective decisions under uncertainty, the participation of human decision-makers remains essential [37]. Indeed, to address decision situations where no objective data or information is available about the problem at hand, expert-driven decision-making approaches remain indispensable [41]. However, in order to efficiently integrate expert knowledge into decision-making, it is necessary to take into account certain limitations associated with the human beings.

First, to ensure effective decision-making it is necessary to pay attention to the modeling of the experts' preferences. Unlike numerical data, using experts' opinions requires not only modeling the uncertainty associated with their preferences but also considering the psychological aspects involved in the preference elicitation process [14, 22]

Second, it is necessary to consider that when experts are required to elicit their preferences about the possible solutions to the decision-making problem, they may have a partial view of the problem based not only on their background and experience but also on their own interests [24]. To eliminate possible biases in the decision-making process caused by these factors, it is suitable to use Group Decision-Making (GDM) techniques, which simultaneously take into account the opinions of several experts [32].

Finally, when different experts participate in a GDM problem, it is expected that, in a certain sense, they all seek to see their particular opinions reflected in the final decision, which may lead to conflict situations among the members of the group [9]. Consensus Reaching Processes (CRPs) are intended to ensure a level of agreement on the decisions obtained in GDM problems [40]. Generally, a CRP is said to be an iterative process, usually coordinated by a human moderator, whose goal is to increase the degree of agreement among group members over multiple rounds of discussion.

GDM problems classically involved a small number of experts. However, recent technological advances make it possible to consider many decision-makers in the resolution of GDM problems [25]. In this context, a great interest has recently arisen in the so-called Large-Scale Group Decision-Making (LSGDM) problems, as well as in their Large-Scale Consensus Reaching Processes (LSCRPs), whose goal is to solve GDM problems that involve a large number of decision-makers [27]. Considering a large number of decision-makers in decision-making processes allows taking into account a greater number of experiences, backgrounds, and points of view, which can lead to a more comprehensive decision that considers a wider range of factors and possible outcomes. Consequently, large groups can make more diverse and better-informed decisions than decisions made by a single individual or a small group [25].

A.2 MOTIVATION

LSGDM offers a greater capacity to solve real-world GDM problems involving a large number of decision-makers, as opposed to the classical GDM in which the number is limited to a reduced set of participants. However, increasing the number of decision-makers involved also means an increase in the complexity of the decision-making process, which requires the development of new models, methods, and tools to improve and analyze these problems under the new assumptions. This is especially important to ensure the effectiveness of LSGDM processes and to address new emerging challenges such as the following ones:

1. Nonlinear preference modeling. Traditionally, it has been assumed that experts provide their opinions in a linear way

[10]. However, recent psychological studies argue that using nonlinear scales to represent their preferences could improve decision-making results [6].

2. LSGDM processes that involve hundreds or thousands of experts. In the specialized literature, LSGDM has been defined as a GDM problem involving more than twenty experts [5]. However, the philosophy of this type of problem, due to current technological advances and social demand, should be focused on problems in which hundreds, thousands, or even more decision-makers can take part. Therefore, LSGDM processes need to be improved to efficiently handle opinions in really large groups, such as users of a marketplace or a social network.

3. Metrics for LSCRPs. In the related literature, there are multiple proposals for LSCRP, but it is difficult to objectively discriminate which one is the most suitable to solve a particular LSGDM problem. Consequently, it is necessary to develop metrics capable of determining the performance of LSCRPs in an objective manner [15].

These limitations and challenges in the field of LSGDM under uncertainty led us to formulate the following hypotheses at the beginning of this research:

1. The use of nonlinear representation models of experts' opinions will allow improving the results of GDM problems and their CRPs.

2. The optimization of current LSGDM processes will facilitate addressing problems with a large number of decision-makers efficiently, improving the current results and fields of application.

3. The definition of metrics for LSCRPs will enable more effective performance evaluation of both existing CRPs and new proposals.

A.3 OBJECTIVES

Taking into account the motivations derived from the limitations in the current LSGDM literature and considering the initial hypotheses, the main purpose of this doctoral thesis is to improve LSGDM processes under uncertainty, and their LSCRPs, through the use of mathematical tools and models. The aim is to overcome the current methodological deficiencies and contribute to greater accuracy and robustness in the decision-making field. Consequently, the following objectives are proposed:

1. Define a methodological framework for modeling the nonlinear behavior of experts when providing their opinions that allows correcting deviations derived from human psychology.

2. Optimize LSGDM processes to address decision-making problems that require the participation of a large number of decision-makers (hundreds, thousands, ...).

3. Introduce objective metrics for LSCRPs that establish performance standards for consensus outreach.

A.4 STRUCTURE

In accordance with Article 25 point 2 of the current regulations for Doctoral Studies at the University of Jaén (RD. 99/2011), this doctoral thesis consists of a compilation of articles published by the doctoral student. The aim of this compilation is to achieve the objectives established in the previous section. Specifically, this memory consists of seis articles, published or accepted, in Q1 international journals indexed in the Journal Citation Reports (JCR) database.

The remainder of the report consists of the following chapters:

CHAPTER 2. The fundamental concepts related to the subject matter of the doctoral thesis are presented. GDM problems are described, paying special attention to LSGDM, and the advantages and limitations of the existing decision models are analyzed. Also, the need for LSCRPs to reach consensual solutions is presented.

CHAPTER 3 . The papers that make up this report are described, highlighting the results and conclusions obtained from each one of them.

CHAPTER 4. It consists of the above-mentioned papers.

CHAPTER 5. The main conclusions of the doctoral thesis are identified and possible fields of research for future work are suggested.

Finally, a bibliographic compilation of the most relevant articles related to this report has been included.

A.5 BACKGROUND

This section presents a brief summary of the theoretical concepts and background relevant to the research presented in this report. Specifically, it addresses the basic concepts of GDM and its CRPs, as well as a summary of the main proposals in the literature, and the existing limitations in LSGDM. The contents of this chapter are further developed in Section 4.1, which corresponds to a systematic review of LSGDM performed during my research and in which these concepts and related issues are shown in more detail.

A.5.1 *Group Decision Making and Consensus Reaching Processes*

A GDM problem arises when several individuals are required to choose the best among several alternatives to solve a given problem [10]. Formally, GDM problems are modeled as a pair (D, A) in which D is a finite set of m decision-makers $D = \{dm_1, dm_2, \dots, dm_m\}$, who are asked to evaluate n alternatives $A = \{a_1, a_2, \dots, a_n\}$ with the goal of selecting the best solution for the decision-making problem. In general, solving this type of problem consists of two main steps [28] (see Fig. A.2):

AGGREGATION To represent the overall opinion of the group, the preferences of the decision-makers are combined by an aggregation operator into a single collective preference.

EXPLOITATION. One or several alternatives are chosen as a solution to the problem.

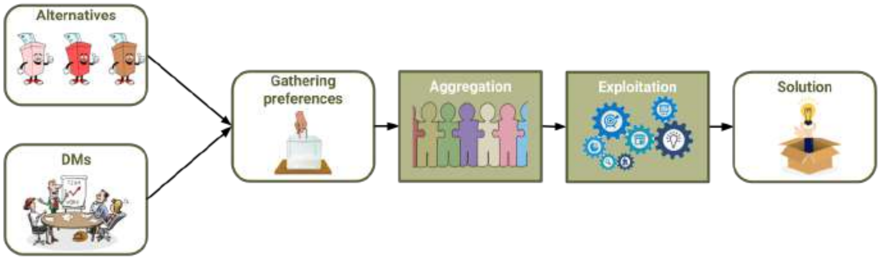

Fig. A.2: Scheme of a GDM problem

Butler and Rothstein [4] have proposed several rules to solve the GDM process, such as the majority or minority rules, or the Borda count. However, when employing these rules, some decision-makers could not be completely satisfied with the chosen solution since their views may not have been properly considered in the final collective choice. In order to address possible discrepancies among the decision makers' opinions, a CRP is often incorporated into the resolution of the GDM problem. A CRP is a dynamic, iterative process in which decision-makers discuss and adjust their initial opinions to achieve greater consensus within the group [23]. These processes are usually overseen by a moderator who is responsible for providing the decision-makers with the necessary information on the status of negotiations to smooth out conflicts. Generally speaking, a CRP is composed of four main steps, as described in the literature [24] (see Fig. A.3):

GATHERING PREFERENCES. Decision-makers provide their evaluations on the alternatives using a certain preference structure.

DETERMINING CONSENSUS LEVEL. Consensus measures are used to obtain the current level of agreement in the group.

CONSENSUS CONTROL. Such a level of agreement is compared to a previously established desired level of consensus. If the group succeeds in reaching the desired level, the CRP is terminated and the selection process is carried out to choose the best alternative. If not, another round is conducted. To prevent the process from dragging on indefinitely, a limit is set on the number of rounds allowed.

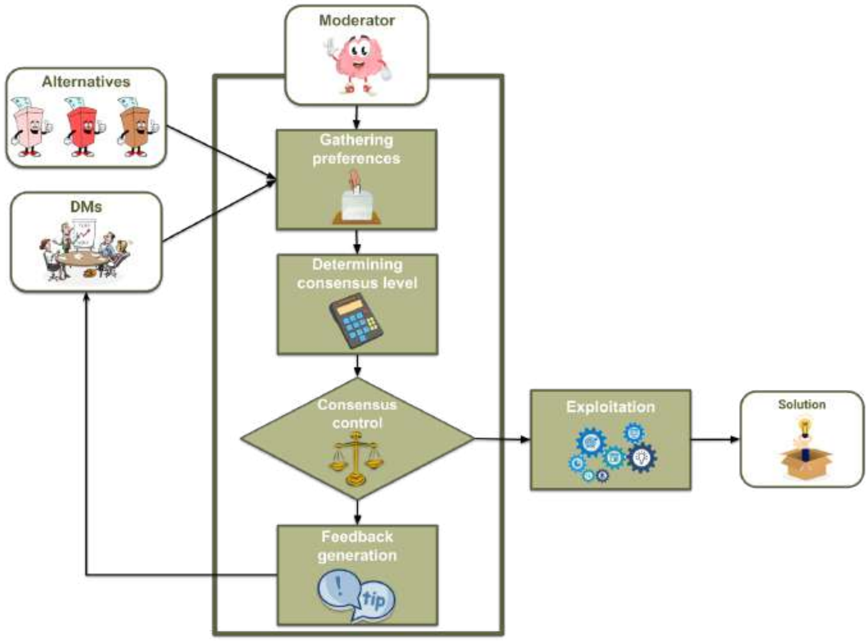

Fig. A.3: Scheme of a CRP

FEEDBACK GENERATION. The moderator identifies the decision-makers and opinions that cause the conflicts in the group and suggests ways to modify them to reduce the disagreements.

A.5.2 *Large-Scale Group Decision-Making*

Traditionally, GDM problems and their CRPs have involved only a small number of decision-makers. However, recent technological advances, such as Big Data [21] and e-commerce [35], together with the demands of modern society to address critical problems such as emergency situations [34] or sustainability [17], have given rise to new decision problems that require the involvement of a larger number of decision-makers. In this context, the so-called LSGDM arises, which, according to the classical definition, refers to GDM problems involving a large number of decision-makers (usually defined in the specialized literature as twenty or more decision-makers) [7].

Tang et al. [33] and Labella et al. [12] pointed out that the involvement of a large number of decision-makers with different views and

Fig. A.4: Scheme of an LSGDM problem

preferences requires considering new aspects in the overall LSGDM problem-solving process (see Fig. A.4):

DIMENSION REDUCTION. To handle the large amount of information involved in LSGDM models, some mechanisms are introduced to reduce the dimensionality of the data.

WEIGHTING AND AGGREGATION. These are related to the task of properly determining the importance of the decision-makers involved in the decision process and fusing their opinions effectively.

BEHAVIOR MANAGEMENT. Refers to the need to include mechanisms to detect and manage uncooperative decision-makers in the decision-making process, in order to prevent these decision-makers from negatively affecting the final outcome.

COST MANAGEMENT It is necessary to consider the human, economic, and time resources required to develop models that aim to manage hundreds, thousands, or even millions of decision-makers in the decision process.

Social network analysis. In large groups, it is necessary to consider how relationships between decision-makers (such as trust or reputation) influence the decision process.

CONSENSUS As the number of people involved in a decision increases, the likelihood of disagreement also increases. Therefore, it is essential to develop LSCRPs for large groups in order to reach agreed solutions.

LSGDM is based on the classical GDM scheme (see Fig. A.2), which consists of two phases: aggregation and exploitation. However, the aggregation phase in LSGDM is much more complex as several aspects are taken into account to merge the original values of the decision-makers' opinions. Thanks to this combination of techniques, it is possible to propose a wide variety of LSGDM schemes. Some of the most important ones are described below.

1. Palomares et al. [25] and Dong et al. [8] introduced consensus models that consider non-cooperative behavior management and dimension reduction for information weighting and aggregation.

2. In their work, Zhang et al. [42] used a linguistic aggregation process to address multi-attribute LSGDM problems under uncertainty.

3. Liu et al. [16] proposed a consensus model that integrates mechanisms to control the cost of modifying decision-makers' opinions and use social network analysis to determine their importance.

4. Lu et al. [19] introduced a consensus process that combines social network analysis and clustering techniques to perform dimension reduction and determine the influence of decision-makers in a decision process. This process also takes into account the cost associated with changing decision-maker preferences.

5. Shi et al. [29] used behavior and cost management techniques in their consensus model, which also incorporates a dimension reduction with adaptive weights.

A.5.3 *Main limitations of the literature on LSGDM*

Currently, LSGDM is a hot topic among researchers in various areas (operations research, politics, computer science, management, engineering, marketing, etc. [15, 25, 37]). However, the foundations of LSGDM are based on assumptions inherited from its widespread use, rather than on solid theoretical or practical foundations. To determine the challenges in the research area, this section aims to discuss the main shortcomings of the current research on LSGDM.

In general terms, the main limitations of the LSGDM stem from its own definition. The widespread definition in the specialized literature considers that LSGDM consists of “GDM problems with more than twenty experts”. This definition seems to be completely obsolete, especially considering that new technologies allow considering decision situations involving hundreds or thousands of decision-makers. It is necessary to revise this definition to ensure not only the applicability of the models in real-world problems but also to be able to compare the different processes in a fair way.

Moreover, in the specialized literature, it is common to find GDM techniques applied directly in LSGDM without studying their feasibility and performance [27]. Regarding the models reviewed in Section 4.1, an important critique is the lack of studies demonstrating the good performance of classical GDM techniques in large-scale contexts involving hundreds or thousands of decision-makers. Although these methods have been successfully applied in contexts with fifty or fewer decision-makers, there is no evidence that they are equally effective in decision situations with larger numbers of decision-makers. Rigorous studies are needed to assess the feasibility of these techniques in large-scale contexts, and if necessary, adapt existing approaches to address LSGDM problems.

Another consequence of defining LSGDM as "GDM with twenty or more experts" is its abuse in publications that are not aimed at solving real-world problems. The vast majority of the proposals reviewed in section 4.1 are limited to testing the validity of the corresponding models on examples that consider less than fifty experts, without giving an objective and verifiable analysis of their performance when handling thousands of decision makers (see Figure A.5). In this sense, it is necessary to demand a higher level of quality in terms of the conditions under which the validity of a method is tested.

In addition, it is a common practice to apply the more convenient measure to highlight the advantages of the proposed models when comparing them with others, but there are no objective metrics to fairly present both the positive and negative aspects of such models. A new approach with this capability has recently been proposed for consensus models in GDM [15], but there are no proposals for LSGDM. It is important to develop new metrics to address other problems to objectively analyze different features of LSGDM methods.

Regarding preference structures, it is worth noting the large number of them that can be found in the literature (see Section 4.1).

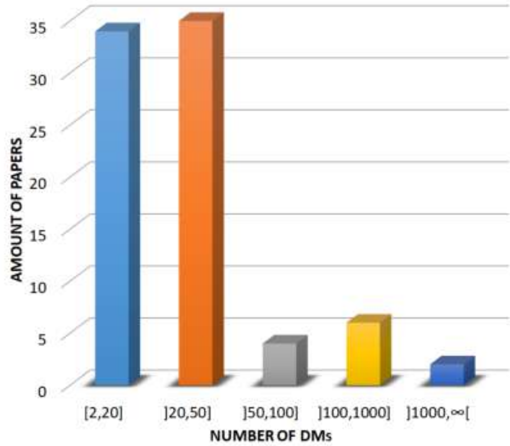

Fig. A.5: Papers reviewed in Section 4.1 according to the number of decision-makers.

However, considering excessively complex preference structures significantly increases the number of variables in the LSGDM problem, leading to higher resource consumption [27]. Therefore, in LSGDM, improvements in preference modeling should not imply an increase in the number of variables but focus on better modeling of human psychology. In this regard, it should be remarked that existing proposals for preference representation assume that decision-makers provide their preferences in a linear way. However, recent studies suggest that by using nonlinear scales to reallocate decision makers' preferences, more psychologically realistic collective solutions are obtained [6, 20]. Therefore, further studies on the impact of nonlinear scales in LSGDM are needed.

A.6 DISCUSSION OF RESULTS

This section presents a summary of the proposals that make up this research report, as well as the results and conclusions derived from them.

A.6.1 *Critical analysis of the matter*

The aim of the first paper in this report, included in Section 4.1, is to serve as an updated state-of-the-art for researchers. With a better understanding of the concept of LSGDM, researchers may present new proposals aimed at addressing the challenges in the area related to technological developments (such as Big Data or social networks), and pay more attention to the validity of their models in these contexts.

In this work, a systematic review of the existing literature on LSGDM has been carried out, following the indications proposed by Kitchenham et al. [11] for the development of bibliographic analysis in Software Engineering. Using this methodology, the existing proposals have been reviewed from four different perspectives: Preference Structure, Group Decision Rules, Quality Assessment, and Applications. These points of view contain the most relevant keywords in the LSGDM literature and represent the different steps to consider when proposing models. Since the analysis performed revealed several important limitations in the current research in the area, this paper also provides a further detailed critical analysis of these bad practices found in the literature, as well as some indications on how to redirect future research toward a more realistic LSGDM, related to proposing methodologies to cope with decision situations involving a really large number of decision-makers.

Generally speaking, it is important to note that the definition of theoretical models and the testing of their performance on very simple examples (toy examples), in which from twenty to fifty decision makers are considered, could hardly be applied in real practical situations if they do not explicitly specify the number of decision-makers they are able to manage and demonstrate their good performance in these contexts. In such an applied field as LSGDM, researchers should focus future studies on addressing real-world problems involving a large group of decision-makers (e.g., Netflix manages 209 million paid subscriptions) rather than proposing "large-scale" models that work with twenty decision-makers and do not clarify how they would perform if that number were increased.

In order to bring more transparency to LSGDM processes, this paper also proposes the definition of $m-$LSGDM models as those proposals that can efficiently handle at least m decision makers.

This not only provides a fairer view of the performance of each proposal but also allows differentiating models oriented to handle a few hundred decision-makers from models designed to handle millions.

A.6.2 *Non-linear preferences*

Even though there are several proposals of CRPs in the literature, they usually assume linear scales for experts' preferences [10]. However, recent studies suggest that the use of nonlinear scales can improve the results of GDM processes [6, 20].

The work presented in Section 4.2 explores the use of nonlinear scales to define more realistic preference models from the original preferences of experts, even in large-scale situations. In that work, we have made a comprehensive study of the analytical properties of such nonlinear scales and obtained the main mathematical characteristics of the functions that may be suitable for adapting expert preferences according to this psychological factor. We have called these functions Extreme Values Amplifications (EVAs) and they allow reallocating fuzzy preference relations given on a linear scale into a nonlinear scale by increasing the distance between extreme values and decreasing the distance between intermediate values. In addition, the dual definition of Extreme Values Reductions (EVR), which reduce the distance between extreme values and amplify the distance between intermediate values, has also been provided.

A general method for constructing EVAs and EVRs has been introduced and several families of EVAs have been proposed. The use of the nonlinear scales provided by EVAs has been found to improve the performance of the consensus models used in the study. Specifically, in addition to obtaining more psychologically realistic results, simulations show that the EVA approach reduces the average number of rounds required to reach consensus in both models and increases the level of consensus.

A.6.3 *Aggregation of extreme values in CRPs*

In GDM, it is necessary to combine the preferences of the experts in an aggregation phase to obtain a collective opinion before moving on to the exploitation phase. In the literature on aggregation operators, Ordered Weighted Average (OWA) operators stand out because they allow merging information according to the magnitude of the values to be aggregated [38]. To calculate the corresponding weights, several alternatives have been proposed, including the method given by Yager, which is based on the use of a parametric family of linear fuzzy linguistic quantifiers [39].

The work developed in Section 4.3 proves that, although the method proposed by Yager [39] is simple and effective, it has significant drawbacks in terms of the choice of parameters. For example, aggregations could produce biased results (orness measure [2] not equal to 0.5) or even not aggregate enough information (entropy measure [2] low). Furthermore, the OWA operator constructed from these quantifiers completely ignores the most extreme values in the aggregation process, which could result in unrealistic aggregations.

These biased aggregations are a major drawback in real-world applications such as CRPs [10, 23], since an OWA operator whose orness is greater than 0.5 would tend to prioritize extreme values close to 1 over those close to 0, which is unreasonable since these values should be equally important. Moreover, a theoretical consensus that completely ignores the most extreme values would be unrealistic. Since it has also been shown that less extreme information has a cohesive effect and facilitates agreement among experts [30, 31], the paper in Section 4.3 explores new ways to generate OWA weights that prioritize intermediate information without neglecting extreme data by taking more information into account in the aggregation process and avoiding biased aggregations in the results.

To overcome these limitations, the EVR-OWA operator, which uses an EVR as a fuzzy linguistic quantifier, has been proposed. This OWA operator takes into account the most extreme values but gives more importance to the intermediate ones. Moreover, aggregations performed by EVR-OWA operators are better for certain real-world applications such as consensus models for GDM [10], because these operators aggregate preferences in an unbiased way and allow taking into account more information in the aggregation process.

The proposed EVR-OWA operator not only provides a simple and general method to obtain OWA weights but also provides a characterization that relates those families of symmetric, positive OWA weights that prioritize intermediate values to the EVRs.

A.6.4 *Optimization of LSGDM processes*

The Minimum Cost Consensus (MCC) models, which are based on convex optimization problems, are automatic CRPs that do not need a recommendation mechanism, and, consequently, they are particularly interesting for LSGDM problems [3, 27]. These models minimize the cost of modifying experts' preferences to reach a consensus and guarantee that the distance between the modified individual preferences and the collective opinion is bounded by a threshold $\varepsilon > 0$. Recently, Comprehensive Minimum Cost Consensus (CMCC) models have been proposed, which add an additional constraint related to the consensus threshold $\gamma \in [0, 1]$ associated with a consensus measure.

The work presented in Section 4.4 analyzes the relationship between the aforementioned constraints in CMCC models from two different perspectives. The first one is based on inequalities and allows determining simple bounds to relate the parameters ε and γ. The second perspective is based on Convex Polytope Theory and provides algorithms that compute more precise and complex constraints to relate these parameters.

Since in CMCC models the parameter values are fixed a priori, the proposed method allows identifying parameter configurations that can simplify the optimization model, eliminating those constraints that are redundant and, consequently, significantly improving the efficiency of these models in LSGDM.

A.6.5 *Generalized MCC models for LSGDM*

MCC models have been widely used to reach agreed solutions to GDM problems. However, the relationship between previous extensions of these models has not yet been studied, which limits their practical application. In this paper, a reformulation of MCC models using Fuzzy Set Theory is presented. The proposed approach, called FZZ-MCC, provides a clearer understanding of MCC models and their extensions, as well as a rigorous and flexible methodology to address various types of GDM problems. In addition, the applicability of the FZZ-MCC approach is demonstrated through three practical examples related to e-democracy, personnel selection, and green supplier selection.

The FZZ-MCC approach introduces three main advantages:

- A rigorous and unified notation based on Fuzzy Sets that allows generalizing previous studies on MCC,
- Generalization of classical notions related to GDM such as preference structure, consensus measure, or cost function,
- Abstract nature that implies flexibility to adapt the FZZ-MCC scheme to address various decision situations.

In this proposal, the flexibility of the FZZ-MCC approach is exploited to propose several models based on MCC:

- An FZZ-MCC model is defined to cope with an e-democracy scenario involving urban planning by managing thousands of preferences through MCC models and multiplicative preference relations.
- An FZZ-MCC model is used to efficiently persuade a hiring committee to select a particular manager. This is done by analyzing the associated cost of driving the decision-makers toward an agreement on the predefined target solution.
- A hybrid FZZ-MCC model is proposed that combines valuations in a database with managers' pairwise comparisons, integrating expert knowledge and data in a green supplier selection problem.

Moreover, all these models have been proposed in terms of linear and absolute-based objective functions and constraints, which facilitates their linearization to improve both their accuracy and computational efficiency aspects, which are essential for dealing with LSGDM problems.

A.6.6 *Metric for large-scale linguistic consensus modeling*

Even though CRPs based on linguistic information have been the subject of extensive research and numerous solutions have been proposed in the specialized literature, there is not an objective metric to compare these models and decide which one is the best for each decision problem.

In the work developed in Section 4.6, we introduce a metric to evaluate the performance of linguistic CRPs that takes into account both the resulting degree of consensus and the cost of modifying the initial opinions of the participants.

This metric is based on a linguistic CCM model that uses ELICIT (Extended Comparative Linguistic Expressions with Symbolic Translation) [14] information to model participant indecision and ensure accurate word computation processes. Furthermore, this metric is defined based on a linear optimization model to speed up the computational model and improve its accuracy, thus being able to be applied in LSGDM processes with several thousands of experts in a few seconds.

The proposed metric for evaluating linguistic CRPs compares the optimal cost required to achieve the desired level of consensus with the changes made by the CRP. If the degree of consensus achieved by the CRP is below the desired threshold, the metric will rate the CRP as ineffective. In the event that the CRP exceeds the consensus threshold, the metric will score the CRP higher or lower based on the extent of the unnecessary modifications made.

This metric has also been used to evaluate the performance of two linguistic consensus models defined in the literature [13, 26] to demonstrate their applicability in practice.

A.7 CONCLUSIONS AND FUTURE WORKS

To conclude this report, we present the conclusions we have drawn during this research, as well as possible future works that could be undertaken based on the results obtained.

A.7.1 *Conclusions*

LSGDM arises to solve real-world decision problems in contexts of uncertainty that require considering the opinions of a large number of decision-makers. By taking into account the opinions of many people, more objective results can be achieved than when using small groups, but at the same time, the complexity increases.

The first objective of this thesis was to propose new approaches to model the nonlinear nature of preferences given by human beings. To accomplish this, we have defined the EVAs and EVRs functions, which allow transforming decision-makers' preferences in a nonlinear way. We have found that using EVAs in CRP, besides being more realistic from a psychological point of view, generally improves the results of the CRP. On the other hand, it has been shown that EVRs can be applied to merge decision-makers' opinions by giving higher priority to intermediate values, which are the most relevant for reaching a consensus.

The following objective was related to improving LSGDM models to address problems that require a large number of decision-makers. To this end, we focused on MCC models, which are particularly useful for handling problems in LSGDM as they do not require several rounds of negotiation to reach a consensual solution. In this aspect, we have analyzed in detail the mathematical structure of CMCC models, concluding that it is possible to considerably improve their efficiency when constraints that may be redundant are removed. In addition, we have proposed the FZZ-MCC models, which generalize existing MCC models in the literature in a flexible way using Fuzzy Set Theory. From a practical point of view, FZZ-MCC models have been used to define several consensus models that allow addressing LSGDM problems such as e-democracy and the integration of data and expert knowledge in decision-making.

The last objective consisted of defining metrics that enable objectively evaluating the performance of LSCRPs. To achieve this, we have developed a metric that evaluates CRPs by simultaneously taking into account the level of agreement reached in the CRP and the modifications made to the original preferences.

In conclusion, it is important to highlight that this thesis has achieved all the objectives established at the beginning of this research, which has allowed us to develop tools, models, and results that go beyond the previous state of the art and open up new research possibilities, as described below.

A.7.2 *Future works*

The results of this research open the possibility of exploring new areas of study that can be addressed in future works. Some of the possible lines of research that can continue the work done in this doctoral thesis are:

- Apply the theory of EVAs and EVRs to deal with LSGDM challenges such as the polarization of opinions or the management of minority opinions.
- Study the use of EVR-OWA Operators in LSGDM problems.
- Develop new scalable LSGDM models to solve real-world decision-making problems involving millions of people.
- Extend FZZ-MCC models to address new decision situations, e.g., integrating social network analysis.
- Explore new types of decision problems from the LSGDM perspective, such as sorting problems that seek to classify alternatives into different categories.

A.7.3 *Additional publications*

In the course of this research, other publications have been presented that have not been included in this report. They are listed below:

- Publications in international journals:
  - H. Song, D. García-Zamora, A. Labella Romero, X. Jia, Y. Wang, and L. Martínez. "Handling multi-granular hesitant information: A group decision-making method based on cross-efficiency with regret theory". In: *Expert Systems with Applications*, 2023, 120332. Impact factor 8.665, Q1.
  - Y. Wang, S. He, D. García Zamora, X. Pan, and L.Martínez, "A Large Scale Group Three-Way Decision-based consensus model for site selection of New Energy Vehicle charging stations" In: *Expert Systems with Applications*, 214, 119107 (2023). Impact factor 8.665, Q1.
  - S. Feng, Y. Xin, S. Xiong, Z. Cheng, M. Devecy, D. García-Zamora, and W. Pedrycz. "Safety Perception Evaluation of Civil Aviation Based on Weibo Posts in China: An Enhanced Large-Scale Group Decision-Making Framework". *Int. J. Fuzzy Syst.* (2023). Impact factor 4.085, Q2.
  - M. Zhou, Z. Chen, J. Jiang, G. Qian, D. García-Zamora, B. Dutta, Q. Zhan, and L. Jin. "Auto-generated Relative

Importance for Multi-agent Inducing Variable in Uncertain and Preference Involved Evaluation". In: *International Journal of Computational Intelligence Systems* 15, 108 (2022). Impact factor 2.259, Q3.

- Y. Wang, X. Pan, S. He, B. Dutta, D. García-Zamora, and L. Martínez, "A New Decision-Making Framework for Site Selection of Electric Vehicle Charging Station with Heterogeneous Information and Multi-Granular Linguistic Terms". In: *IEEE Transactions on Fuzzy Systems* (2022). Impact factor 12.253, Q1.

- Book chapters:
  - D. García-Zamora, A. Labella, R. M. Rodríguez and L. Martínez. "Nonlinear Scaled Preferences in Linguistic Multi-Criteria Group Decision Making". In: *Real Life Applications of Multiple Criteria Decision-Making Techniques in Fuzzy Domain*. Springer Nature Singapore Pte Ltd; 2022.

- Internacional conferences:
  - Á. Labella, D. García-Zamora, W. He, R.M. Rodríguez, and L. Martínez (2022). "Grouping representative points in AHP-FuzzySort with agglomerative hierarchical clustering". In: The International Symposium on the Analytic Hierarchy Process (ISAHP2022), Online, December 15-18, 2022.
  - D. García-Zamora, A. Labella, R. M. Rodríguez and L. Martínez. "Comprehensive Minimum Cost Consensus for Analyzing the Cost of Different Agreed Solutions". In: *15th International FLINS Conference on Machine Learning, Multi agent and Cyber physical systems and the 17th International ISKE Conference* (FLINS/ISKE 2022), 26-28 August Tianjin (China) 2022.
  - D. García-Zamora, A. Labella, R. M. Rodríguez and L. Martínez. "Comprehensive Minimum Cost Consensus Models for ELICIT Information". In: *2022 Word Congress on Computational Intelligence* (IEEE WCCI2022), Padua, Italy 18-23 July 2022.
  - J. Baz, D. García-Zamora, I. Díaz, S. Montes, L. Martínez. "Flexible-Dimensional EVR-OWA as Mean Estimator for

Symmetric Distributions". In: *The 19th International Conference On Information Processing And Management Of Uncertainty In Knowledge-Based Systems* (IPMU 2022), Milan (Italy), July 11-15, 2022.

- Á. Labella, D. García-Zamora, R.M. Rodríguez, and L. Martínez (2022). "Fuzzy TODIM for ELICIT Information". In: *International Conference on Intelligent and Fuzzy Systems* (INFUS 2022), Istanbul (Turkey), July 19-21, 2022 .
- A. Labella, D. García-Zamora, R. M. Rodríguez and L. Martínez. "A Consensus-based Best-Worst Method for Multi-criteria Group Decision-Making". In: *The Third International Workshop on Best-Worst Method* (BWM-2022), Delft, The Netherlands, 09-10 June 2022.
- D. García-Zamora, A. Labella, R. M. Rodríguez and L. Martínez. "Modelling non linear preferences in Consensus Reaching Processes. A sustainability application". In: *16th International Conference on Intelligent Systems and Knowledge Engineering* (ISKE 2021), 26-28 November Chengdu (China) 2021.
- D. García-Zamora, A. Labella, R. M. Rodríguez and L. Martínez. "An Ordered Weighted Averaging Operator based on Extreme Values Reductions´´. In: *The 19th World Congress of the International Fuzzy Systems Association. The 12th Conference of the European Society for Fuzzy Logic and Technology jointly with the AGOP, IJCRS, and FQAS conferences*. Bratislava (Slovakia), September 19-24, 2021.
- D. García-Zamora, A. Labella, R. M. Rodríguez and L. Martínez. "Non Linear Scales in GDM: Extreme Values Amplifications". In: *International Virtual Workshop on Business Analytics Eureka*, 2-4 Junio, Ciudad Juarez (Mexico), 2021.
- A. Labella, D. García-Zamora, R. M. Rodríguez and L. Martínez. "A Novel Linguistic Cohesion Measure based on Restricted Equivalence Functions for Weighting Experts' subgroups in Large-scale Group Decision Making Problems´´. In: *International Virtual Workshop on Business Analytics Eureka*, 2-4 Junio, Ciudad Juarez (Mexico), 2021.

- National conferences:
  - D. García-Zamora, A. Labella, R. M. Rodríguez and L. Martínez. "Nonlinear Preferences in Consensus Reaching Processes. Extreme Values Amplifications". In: *XXI Congreso español sobre Tecnologías y Lógica Fuzzy* (ESTYLF 2022), celebrado en Toledo el 4 –7 de Septiembre 2022.
  - D. García-Zamora, A. Labella, P. Nuñez-Cacho, R. M. Rodríguez and L. Martínez. "Modelos de consenso basados en Mínimo Coste para expresiones ELICIT". In: *XXI Congreso español sobre Tecnologías y Lógica Fuzzy* (ESTYLF 2022), celebrado en Toledo el 4 –7 de Septiembre 2022.
  - D. García-Zamora, A. Labella, R. M. Rodríguez and L. Martínez. "Preferencias no lineales en Toma de Decisión en Grupo. Amplificación de Valores Extremos". In: *CEDI 20/21: VI Congreso Español de Informática*, celebrado el 22 – 24. Septiembre 2021.

A.7.4 *Awards*

It is important to mention that some of the works developed during this doctoral thesis have been recognized by the scientific community:

1. The contribution entitled "Modelling Non Linear Preferences in Consensus Reaching Processes: A Sustainability Application", presented at the 16th International Conference on Intelligent Systems and Knowledge Engineering (ISKE 2021), held in Chengdu, China, in November 2021, received the best student paper award.

2. The poster entitled " Non-linear preferences in Group Decision Making: Extreme Values Amplifications" received a special mention at the Doctoral Conference for young researchers of the University of Jaen held in November 2021.

3. The contribution "Comprehensive Minimum Cost Consensus for Analyzing the Cost of Different Agreed Solutions", presented at the 15th International FLINS Conference on Machine Learning, Multi-agent and Cyber-physical systems and the 17th International ISKE Conference (FLINS/ISKE 2022), held in Tian-

jin, China, during August 2022, also received the best student paper award.

All the diplomas associated with the aforementioned awards are included at the end of Chapter 5.

A.7.5 *Research stays*

During the doctoral thesis, two research stays were carried out in order to improve the research training of the doctoral student through the knowledge and experience of experts in the field. Thanks to the mobility aid for beneficiaries of the FPU contract EST22/00031, granted by the Ministry of Universities, a three-month stay (from 15/04/2022 to 14/07/2022) was made at the School of Computing, Ulster University, in the United Kingdom. In addition, a one-month stay (from 23/01/2023 to 24/02/2023) was spent at the Artificial Intelligence and Approximate Reasoning Research Group of the Public University of Navarra.

REFERENCIAS

[1] U. Awan, S. Shamim, Z. Khan, N. Zia, S. Shariq, and M. Khan. "Big data analytics capability and decision-making: The role of data-driven insight on circular economy performance." In: *Technological forecasting and social change* 168 (2021).

[2] G. Beliakov, H. Bustince, and T. Calvo. *A practical guide to averaging functions*. Springer, 2016.

[3] D. Ben-Arieh and T. Easton. "Multi-criteria group consensus under linear cost opinion elasticity." In: *Decision Support Systems* 43.3 (2007). Integrated Decision Support, pp. 713–721.

[4] C. Butler and A. Rothstein. *On Conflict and Consensus: a Handbook on Formal Consensus Decision Making*. Portland: Food Not Bombs Publishing, 1987.

[5] G. Carvalho, A. S. Vivacqua, J. M. Souza, and S. P. J. Medeiros. "LaSca: a Large Scale Group Decision Support System." In: *Journal of Universal Computer Science* 17.2 (2011), pp. 261–275.

[6] F. Delic A.and Ricci and J. Neidhardt. "Preference Networks and Non-Linear Preferences in Group Recommendations." In: *2019 IEEE/WIC/ACM International Conference on Web Intelligence (WI)*. 2019, pp. 403–407.

[7] R. X. Ding, I. Palomares, X. Q. Wang, G. R. Yang, B. S. Liu, Y. C. Dong, E. Herrera-Viedma, and F. Herrera. "Large-Scale decision-making: Characterization, taxonomy, challenges and future directions from an Artificial Intelligence and applications perspective." In: *Information Fusion* 59 (2020), pp. 84–102.

[8] Y. C. Dong, S. H. Zhao, H. J. Zhang, F. Chiclana, and E. Herrera-Viedma. "A Self-Management Mechanism for Noncooperative Behaviors in Large-Scale Group Consensus Reaching Processes." In: *IEEETransactions on Fuzzy Systems* 26.6 (2018), pp. 3276–3288.

[9] Y. Du and X. Shen. "Group hierarchical DEMATEL method for reaching consensus." In: *Computers & Industrial Engineering* 175 (2023), p. 108842.

[10] E. Herrera-Viedma, F. Herrera, and F. Chiclana. "A consensus model for multiperson decision making with different preference structures." In: *Systems, Man and Cybernetics, Part A: Systems and Humans, IEEE Transactions on* 32 (2002), pp. 394 –402.

[11] B. Kitchenham and S. Charters. *Guidelines for performing Systematic Literature Reviews in Software Engineering*. Tech. rep. EBSE 2007-001. Keele University, 2007.

[12] Á. Labella, Y. Liu, R. Rodríguez, and L. Martínez. "Analyzing the performance of classical consensus models in large scale group decision making: A comparative study." In: *Applied Soft Computing* 67 (2018), pp. 677–690.

[13] Á. Labella, R. M. Rodríguez, A. A. Alzahrani, and L. Martínez. "A Consensus Model for Extended Comparative Linguistic Expressions with Symbolic Translation." In: *Mathematics* 8.12 (2020), p. 2198.

[14] Á. Labella, R. M. Rodríguez, and L. Martínez. "Computing with comparative linguistic expressions and symbolic translation for decision making: ELICIT information." In: *IEEE Transactions on Fuzzy Systems* 28.10 (2019), pp. 2510–2522.

[15] Álvaro Labella, Hongbin Liu, Rosa M. Rodríguez, and Luis Martínez. "A Cost Consensus Metric for Consensus Reaching Processes based on a comprehensive minimum cost model." In: *European Journal of Operational Research* 281.2 (2020), pp. 316–331.

[16] B. Liu, Q. Zhou, R. Ding, I. Palomares, and F. Herrera. "Large-scale group decision making model based on social network analysis: Trust relationship-based conflict detection and elimination." In: *European Journal of Operational Research* 275.2 (2019), pp. 737–754.

[17] L. Liu, W. Cao, B. Shi, and M. Tang. "Large-Scale Green Supplier Selection Approach under a Q-Rung Interval-Valued Orthopair Fuzzy Environment." In: *Processes* 7.9 (2019).

[18] J. Lu, G. Zhang, D. Ruan, and F. Wu. *Multi-Objective Group Decision Making: Methods, Software and Applications with Fuzzy Set Techniques*. Imperial College Press, 2007.

[19] Y. Lu, Y. Xu, E. Herrera-Viedma, and Y. Han. “Consensus of large-scale group decision making in social network: the minimum cost model based on robust optimization.” In: *Information Sciences* 547 (2021), pp. 910–930.

[20] J. Masthoff. “Group Modeling: Selecting a Sequence of Television Items to Suit a Group of Viewers.” In: *User Model. User-Adapt. Interact.* 14 (2004), pp. 37–85.

[21] S. Modgil, S. Gupta, U. Sivarajah, and B. Bhushan. “Big data-enabled large-scale group decision making for circular economy: An emerging market context.” In: *Technological Forecasting and Social Change* 166 (2021), p. 120607.

[22] S. Orlovsky. “Decision-making with a fuzzy preference relation.” In: *Fuzzy sets and systems* 1.3 (1978), pp. 155–167.

[23] I. Palomares. “Consensus model for large-scale group decision support in IT services management.” In: *Intelligent Decision Technologies-Netherlands* 8.2 (2014), pp. 81–94.

[24] I. Palomares, F. Estrella, L. Martínez, and F. Herrera. “Consensus under a fuzzy context: Taxonomy, analysis framework AFRYCA and experimental case of study.” In: *Information Fusion* 20 (2014), pp. 252–271.

[25] I. Palomares, L. Martinez, and F. Herrera. “A Consensus Model to Detect and Manage Noncooperative Behaviors in Large-Scale Group Decision Making.” In: *IEEE Transactions on Fuzzy Systems* 22.3 (2014), pp. 516–530.

[26] R. M. Rodríguez, Á. Labella, M. Sesma-Sara, H. Bustince, and L. Martínez. “A cohesion-driven consensus reaching process for large scale group decision making under a hesitant fuzzy linguistic term sets environment.” In: *Computers & Industrial Engineering* 155 (2021), p. 107158.

[27] R. Rodríguez, Á. Labella, B. Dutta, and L. Martínez. “Comprehensive minimum cost models for large scale group decision making with consistent fuzzy preference relations.” In: *Knowledge-Based Systems* 215 (2021), p. 106780.

[28] M. Roubens. “Fuzzy sets and decision analysis.” In: *Fuzzy Sets and Systems* 90.2 (1997). Fuzzy Sets: Where Do We Stand? Where Do We Go?

[29] Z. Shi, X. Wang, I. Palomares, S. Guo, and R. Ding. "A novel consensus model for multi-attribute large-scale group decision making based on comprehensive behavior classification and adaptive weight updating." In: *Knowledge-Based Systems* 158 (2018), pp. 196–208.

[30] A. Sîrbu, V. Loreto, and V.D.P. Servedio. "Opinion Dynamics with Disagreement and Modulated Information." In: *Journal of Statistical Physics* 151 (2013), pp. 218–237.

[31] A. Sîrbu, V. Loreto, V.D.P. Servedio, and F. Tria. "Cohesion, Consensus and extreme information in opinion dynamics." In: *Advances in Complex Systems* 16.06 (2013), p. 1350035.

[32] H. Song, Z. Gong, J. Forrest, W. Guo, and E. Herrera-Viedma. "Social network utility consensus model with empathic and fuzzy interactions." In: *Computers & Industrial Engineering* 175 (2023), p. 108904.

[33] M. Tang and H. Liao. "From conventional group decision making to large-scale group decision making: What are the challenges and how to meet them in big data era? A state-of-the-art survey." In: *Omega* 100 (2021), p. 102141.

[34] Q. Wan, X. Xu, X. Chen, and J. Zhuang. "A Two-Stage Optimization Model for Large-Scale Group Decision-Making in Disaster Management: Minimizing Group Conflict and Maximizing Individual Satisfaction." In: *Group Decision and Negotiation* 29.5 (2020), pp. 901–921.

[35] T. Wu, X. Liu, and J. Qin. "A linguistic solution for double large-scale group decision-making in E-commerce." In: *Computers & Industrial Engineering* 116 (2018), pp. 97–112.

[36] J. Xiao, X. Wang, and H. Zhang. "Coping with diversity ratings in prioritizing design requirements in quality function deployment: A consensus-based approach with minimum-maximum adjustments." In: *Computers & Industrial Engineering* 163 (2022), p. 107799.

[37] Y. Xu, X. Wen, and W. Zhang. "A two-stage consensus method for large-scale multi-attribute group decision making with an application to earthquake shelter selection." In: *Computers & Industrial Engineering* 116 (2018), pp. 113–129.

[38] R. Yager. “On ordered weighted averaging aggregation operators in multicriteria decisionmaking.” In: *IEEE Transactions on Systems, Man, and Cybernetics* 18.1 (1988), pp. 183–190.

[39] R. Yager. “Quantifier guided aggregation using OWA operators.” In: *International Journal of Intelligent Systems* 11.1 (1996), pp. 49–73.

[40] B. Zhang, Y. Dong, and Y. Xu. “Maximum expert consensus models with linear cost function and aggregation operators.” In: *Computers & Industrial Engineering* 66.1 (2013), pp. 147–157.

[41] H. Zhang, F. Wang, and X. Song. “From collective opinion dependence to personalization: The impacts of social trust relationship in consensus formation.” In: *Computers & Industrial Engineering* 172.A (2022), p. 108541.

[42] Z. Zhang, C. Guo, and L. Martínez. “Managing Multigranular Linguistic Distribution Assessments in Large-Scale Multiattribute Group Decision Making.” In: *IEEE Transactions on Systems, Man, and Cybernetics: Systems* 47.11 (2017), pp. 3063–3076.