Article

Linguistic Interval-Valued Spherical Fuzzy Soft Set and Its Application in Decision Making

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Abstract: Under uncertain environments, how to characterize individual preferences more naturally and aggregate parameters better have been hot research topics in multiple attribute decision making (MADM). Fuzzy set theory provides a better mathematical tool to deal with uncertain data, which promotes substantial extended studies. In this paper, we propose a hybrid fuzzy set model by combining a linguistic interval-valued spherical fuzzy set with a soft set for MADM. The emergence of a linguistic interval-valued spherical fuzzy soft set (LIVSFSS) not only handles qualitative information and provides more freedom to decision makers, but also solves the inherent problem of insufficient parameterization tools for fuzzy set theory. To tackle the application challenges, we introduce the basic concepts and define some operations of LIVSFSS, e.g., the "complement", the "AND", the "OR", the "necessity", the "possibility" and so on. Subsequently, we prove De Morgan's law, associative law, distribution law for operations on LIVSFSS. We further propose the linguistic weighted choice value and linguistic weighted overall choice value for MADM by taking parameter weights into account. Finally, the MADM algorithm and parameter reduction algorithm are provided based on LIVSFSS, together with examples and comparisons with some existing algorithms to illustrate the rationality and effectiveness of the proposed algorithms.

Keywords: fuzzy set; linguistic interval-valued spherical fuzzy soft set; multiple attribute decision-making; parameter reduction

1. Introduction

In real life, we often encounter uncertain and fuzzy data when dealing with decision-making problems in various fields such as economics, engineering, etc. How to deal with imprecise information is always a huge challenge. Many scholars are committed to optimizing the description of uncertainty by constructing different models or representations, such as fuzzy set theory [1], rough set theory [2], vague set theory [3] and so on. Fuzzy set describes the uncertainty in the data nicely by utilizing membership functions. It provides a better mathematical tool for decision-making problems with uncertainty. In order to optimize the problems of single membership degree of a fuzzy set, more scholars have extended their theories on the basis of the fuzzy set. Atanassov [4] proposed an intuitionistic fuzzy set (IFS), which described more uncertainty in terms of the membership degree and non-membership degree. Atanassov and Gargov [5] developed an interval-valued intuitionistic fuzzy set, which described the membership degree and non-membership degree by interval values. Yager [6] developed the Pythagorean fuzzy set as a generalization of the intuitionistic fuzzy set, which ensured that the square summation of the membership degree and non-membership degree was less than or equal to 1. This development has been beneficial for decision makers in solving attribute problems where the sum of

the membership degree and non-membership degree exceeds 1. Cuong and Kreinovich [7] initiated the picture fuzzy set, which dealt with the situation where the neutral membership degree was considered independently in real decision-making problems. Hesitation of the decision maker plays a significant role in practical decision-making problems. To enhance decision makers' flexibility when dealing with situations where the sum of the membership degree, hesitancy and non-membership degree exceeds 1, Gündoğdu and Kahraman [8] proposed the concept of the spherical fuzzy set. This concept is a generalization of the picture fuzzy set and ensures that the square summation of the membership degree, non-membership degree and hesitancy is less than or equal to 1. As a result, spherical fuzzy sets have been extensively explored by various scholars and applied in diverse fields [9–11].

In the real word, it is always more natural for people to use natural language to express their preferences than to use numerical values. In light of this, Zadeh proposed linguistic variables [12] to enable people to solve problems through qualitative evaluation. Furthermore, Herrera et al. [13] proposed the linguistic-term set (LTS) to describe all the discrete linguistic variables. Xu [14] studied aggregation operators based on probabilistic linguistic information. Thereafter, increasing numbers of studies have begun to explore hybrid models that combine LTS with extensions of fuzzy sets. Ünever et al. [15] proposed the linguistic single- and interval-valued hybrid intuitionistic fuzzy multi-set and linguistic single- and interval-valued hybrid intuitionistic fuzzy multi-value, which provided a qualitative and sensitive assessment tool for multi-criterion group decision making. Liu et al. [16] proposed the linguistic interval-valued spherical fuzzy set (LIVSFS), which was the extension of the interval-valued spherical fuzzy set and linguistic term set to address the difficulty of obtaining quantitative evaluation in decision making. Gurmani et al. [17] proposed the linguistic interval-valued T-spherical fuzzy set, which allowed decision makers to provide their evaluations in a wider space and to deal with vague information better. However, an inherent limitation of all these fuzzy set theories is the inadequacy of the parametrization tools associated with them, that is, the inability to capture information in the form of approximations by different parameters.

Molodtsov [18] pointed out the inherent difficulties of the above theories and introduced a new model called a soft set, which aimed to overcome the lack of parametrization tools in traditional uncertainty theories. The soft set is a parameterized family of subsets defined over a universe and can allow us to use any parameterization, which gives us more flexibility in practical application. Because of the advantages of the soft set as parametrization tools, it provides many excellent methods in different fields [19–21]. Since hybrid models combine the advantages of each model, researchers have conducted in-depth studies on the combined models of soft set theory and other mathematical models. Maji et al. [22] presented the fuzzy soft set by combining a fuzzy set with a soft set, which helped decision makers to deal with fuzzy data better. The main advantage of the fuzzy soft set is that it can solve MADM better when fuzzy data arise in the form of approximations by different parameters. Hence, more and more scholars studied and proposed new hybrid models, such as the interval-valued fuzzy soft set [23], spherical fuzzy soft set [24], intertemporal hesitant fuzzy soft set [25], possibility fermatean fuzzy soft set [26], generalized interval-valued intuitionistic fuzzy soft set [27] and so on [28–30]. These hybrid models effectively combine the description of imprecise and ambiguous data by different fuzzy sets with the advantage of the soft set as parametrization tools, and achieve promising results in some tasks, such as MADM [31–34], parameter reduction [35–37], approximate reasoning [38,39] and so on [40]. The geometric representations of the intuitionistic fuzzy set, Pythagorean fuzzy set, picture fuzzy set, spherical fuzzy, linguistic interval-valued spherical fuzzy set and LIVSFSS are given in Figure 1.
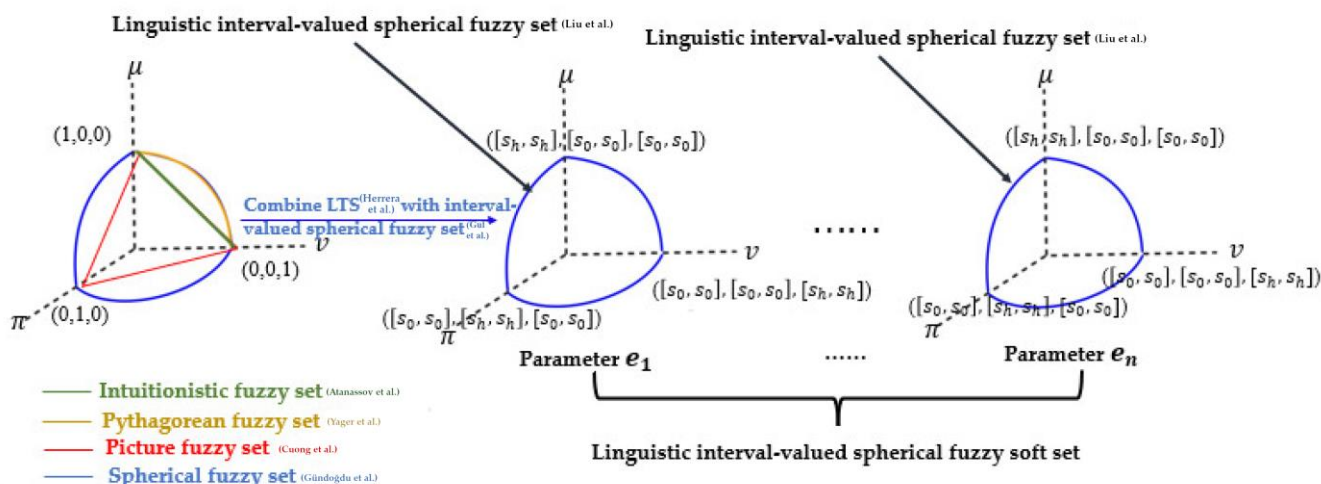
Figure 1. Geometric representations of intuitionistic fuzzy set, Pythagorean fuzzy set, picture fuzzy set, spherical fuzzy, linguistic interval-valued spherical fuzzy set and LIVSFSS [4,6–9,13,16].

The purpose of this paper is to propose a new hybrid model: the linguistic interval-valued spherical fuzzy soft set (LIVSFSS), which can not only achieve the preferences of decision makers by using linguistic terms and provide a greater degree of freedom for the decision makers, but also captures fuzzy information better in the form of approximations by different parameters under MADM. Hence, we first define the basic concepts of LIVSFSS. Then, we discuss various operational laws and the proofs of properties about LIVSFSS, such as the "AND" operation, the "OR" operation, the possibility operation, the necessity operation and so on. Subsequently, we redefine the linguistic weighted choice value and the linguistic soft weighted overall choice value after analyzing the decision-making algorithms based on different interval-valued fuzzy soft set models. Finally, we propose the MADM algorithm and the parameter reduction algorithm. Furthermore, we illustrate the algorithms' rationality and effectiveness by examples and comparative analysis. The main contributions of this paper are summarized as follows.

(1) LIVSFSS is proposed for the first time by combining the linguistic interval-valued spherical fuzzy set with the soft set, and its basic concepts, operations and properties are discussed.

(2) In order to solve MADM problems and consider the influence of the parameter weight, the linguistic weighted choice value and the linguistic soft weighted overall choice value are redefined by analyzing other models. Then the MADM algorithm and parameter reduction algorithm are proposed.

(3) We apply the MADM algorithm and parameter reduction algorithm to examples and compare them with some existing algorithms to illustrate their rationality and effectiveness.

The remainder of this paper is organized as follows. In Section 2, we recall the basic concepts required for this paper. In Section 3, we introduce the proposed concepts and operations of LIVSFSS. In Section 4, we introduce the multi-attribute decision-making algorithm and parameter reduction algorithm based on LIVSFSS. The conclusions and directions for future work are outlined in Section 5.

2. Preliminaries

In this section, we review some basic concepts briefly, including the linguistic interval-valued spherical fuzzy set and fuzzy soft set, which are very helpful in the remaining study of the paper.

2.1. Linguistic Interval-Valued Spherical Fuzzy Set

Definition 1. *[13] Let* $S = \{s_t | t = 0,1,2,\ldots,g\}$ *be a finite linguistic term set (LTS) with an odd number of linguistic terms, where* g *is a positive integer. The LTS* S *satisfies the following characteristics.*

(1) *order relation:* $s_i \geq s_k$, *if* $i \geq k$;
(2) *negation operator:* $Neg(s_i) = s_k$, *where* $k = g - i$;
(3) *maximization operator:* $max\{s_i, s_k\} = s_i$, *if* $i \geq k$;
(4) *minimization operator:* $max\{s_i, s_k\} = s_k$, *if* $i \geq k$.

For example, an LTS with five linguistic terms can be represented as $S = \{s_0 = none, s_1 = low, s_2 = medium, s_3 = high, s_4 = perfect\}$.

Definition 2. *[14] Let* $S = \{s_t | t = 0,1,2,\ldots,g\}$ *be an LTS. The discrete LTS* S *is extended to a continuous linguistic term set* $\bar{S} = \{s_\alpha | s_0 \leq s_\alpha \leq s_h, \alpha \in [0, h]\}$, *where* $h(h > g)$ *is a sufficiently large positive integer. Consider any linguistic terms* s_i, $s_k \in \bar{S}$, *and* $\lambda, \lambda_1, \lambda_2 \in [0,1]$, *some operation rules are as follows.*

$$s_i \oplus s_k = s_k \oplus s_i = s_{i+k} \tag{1}$$

$$s_i \otimes s_k = s_k \otimes s_i = s_{i*k} \tag{2}$$

$$\lambda s_i = s_{\lambda i} \tag{3}$$

$$(\lambda_1 + \lambda_2)s_i = \lambda_1 s_i \oplus \lambda_2 s_i \tag{4}$$

$$\lambda(s_i \oplus s_k) = \lambda s_i \oplus \lambda s_k \tag{5}$$

Definition 3. *[16] Let* $U \neq \emptyset$ *be an initial universe and* $\bar{S} = \{s_\alpha | s_0 \leq s_\alpha \leq s_h, \alpha \in [0, h]\}$ *be a continuous linguistic term set. An LIVSFS* A *in* U *is defined as*

$$A = \{\langle x, (\tilde{s}_{uA}(x), \tilde{s}_{\pi A}(x), \tilde{s}_{vA}(x))\rangle | x \in U\} \tag{6}$$

where $\tilde{s}_{uA}(x) = [s_{uA}^L(x), s_{uA}^U(x)] \subseteq [s_0, s_h], \tilde{s}_{\pi A}(x) = [s_{\pi A}^L(x), s_{\pi A}^U(x)] \subseteq [s_0, s_h]$, $\tilde{s}_{vA}(x) = [\tilde{s}_{vA}^L(x), \tilde{s}_{vA}^U(x)] \subseteq [s_0, s_h]$ *are the membership degree, the hesitancy degree and the non-membership degree of* x *to* A *such that, for all* $x \in U$, *respectively,*

$$0 \leq (\tilde{s}_{uA}^U(x))^2 + (\tilde{s}_{\pi A}^U(x))^2 + (\tilde{s}_{vA}^U(x))^2 \leq h^2 \tag{7}$$

then $\tilde{s}_{rA}(x) = [\tilde{s}_{rA}^L(x), \tilde{s}_{rA}^U(x)]$ *is the waiver degree of* x *to* U, *where* $\tilde{s}_{rA}^L(x) = s_{\sqrt{h^2 - (uA^L(x))^2 + (\pi A^L(x))^2 + (vA^L(x))^2}}$, $\tilde{s}_{rA}^U(x) = s_{\sqrt{h^2 - (uA^U(x))^2 + (\pi A^U(x))^2 + (vA^U(x))^2}}$. *The set of all LIVSFS on* U *is denoted by* $LIVSF(U)$.

For notational simplicity, the linguistic interval-valued spherical fuzzy number (LIVSFN) is denoted by $\alpha = ([s_a, s_b], [s_c, s_d], [s_e, s_o])$, *where* $[s_a, s_b] \subseteq [s_0, s_h], [s_b, s_c] \subseteq [s_0, s_h], [s_e, s_o] \subseteq [s_0, s_h]$, $0 \leq b^2 + d^2 + o^2 \leq h^2$ *and* $s_a, s_b, s_c, s_d, s_e, s_o \in \bar{S}$. *The basic operations of LIVSFS are defined as follows.*

Let $A = \{\langle x, (\tilde{s}_{uA}(x), \tilde{s}_{\pi A}(x), \tilde{s}_{vA}(x))\rangle | x \in U\}$, $B = \{\langle x, (\tilde{s}_{uB}(x), \tilde{s}_{\pi B}(x), \tilde{s}_{vB}(x))\rangle | x \in U\}$ $\in LIVSF(U)$, *then*

(1) $A \subseteq B \Leftrightarrow \tilde{s}_{uA}^L(x) \leq \tilde{s}_{uB}^L(x), \tilde{s}_{uA}^U(x) \leq \tilde{s}_{uB}^U(x), \tilde{s}_{\pi A}^L(x) \leq \tilde{s}_{\pi B}^L(x), \tilde{s}_{\pi A}^U(x) \leq \tilde{s}_{\pi B}^U(x), \tilde{s}_{vA}^L(x) \geq \tilde{s}_{vB}^L(x), \tilde{s}_{vA}^U(x) \geq \tilde{s}_{vB}^U(x)$;

(2) $A = B \Leftrightarrow A \subseteq B, B \subseteq A$;

(3) $A \cup B = \{< x, ([max\{\tilde{s}_{uA}^L(x), \tilde{s}_{uB}^L(x)\}, max\{\tilde{s}_{uA}^U(x), \tilde{s}_{uB}^U(x)\}], [min\{\tilde{s}_{\pi A}^L(x), \tilde{s}_{\pi B}^L(x)\}, min\{\tilde{s}_{\pi A}^U(x), \tilde{s}_{\pi B}^U(x)\}], [min\{\tilde{s}_{vA}^L(x), \tilde{s}_{vB}^L(x)\}, min\{\tilde{s}_{vA}^U(x), \tilde{s}_{vB}^U(x)\}]) > | x \in U\}$;

(4) $A \cap B = \{< x, ([min\{\tilde{s}_{uA}^L(x), \tilde{s}_{uB}^L(x)\}, min\{\tilde{s}_{uA}^U(x), \tilde{s}_{uB}^U(x)\}], [min\{\tilde{s}_{\pi A}^L(x), \tilde{s}_{\pi B}^L(x)\}, min\{\tilde{s}_{\pi A}^U(x), \tilde{s}_{\pi B}^U(x)\}], [max\{\tilde{s}_{vA}^L(x), \tilde{s}_{vB}^L(x)\}, max\{\tilde{s}_{vA}^U(x), \tilde{s}_{vB}^U(x)\}]) > | x \in U\}$;

(5) $A^C = \{< x, (\tilde{s}_{vA}(x), \tilde{s}_{\pi A}(x), \tilde{s}_{uA}(x)) > | x \in U\}$.

2.2. Fuzzy Soft Set

Definition 4. [18] *Let U be an initial universe set, E be a set of parameters, $A \subseteq E$ and $P(U)$ be the power set of U. A pair $< F, A >$ is called a soft set on U, where F is the mapping given by*

$$F: A \rightarrow P(U) \tag{8}$$

In other words, a soft set can be regarded as a parameterized family of subsets over the universe U.

Definition 5. [22] *Let U be an initial universe, E be a set of parameters, $F(U)$ be the set of all fuzzy sets of U. A pair $< \tilde{F}, E >$ is called a fuzzy soft set on U, where $\tilde{F}$ is the mapping given by*

$$\tilde{F}: E \rightarrow F(U) \tag{9}$$

Obviously, the combination of a fuzzy set and a soft set is a fuzzy soft set, i.e., the fuzzy soft set is a mapping from parameters to $F(U)$. It is a parameterized family of fuzzy subsets over the universe U.

3. Linguistic Interval-Valued Spherical Fuzzy Soft Set

By combining the linguistic interval-valued spherical fuzzy set with a soft set, it is natural to define the linguistic interval-valued spherical fuzzy soft set (LIVSFSS) model. In this section, we introduce the concepts of LIVSFSS.

3.1. Basic Concept of LIVSFSS

Definition 6. *Let U be an initial universe, E be a set of parameters, $LIVSF(U)$ be the set of all LIVSFS of U. A pair $< \tilde{F}, E >$ is called a linguistic interval-valued spherical fuzzy soft set on U, where $\tilde{F}$ is the mapping given by*

$$\tilde{F}: E \rightarrow LIVSF(U) \tag{10}$$

An LIVSFSS is a parameterized family of linguistic interval-valued spherical fuzzy subsets of U, thus, its universe is the set of all LIVSFS on U. In other words, because LIVSFSS is still a mapping from parameters to $LIVSF(U)$, it is regarded as a powerful extention of soft set.

For $\forall \varepsilon \in E$, $\tilde{F}(\varepsilon)$ is referred to as the LIVSFS with parameter ε, and it is actually an LIVSFS on U. For $x \in U$ and $\varepsilon \in E$, it can be written as follows.

$$\tilde{F}(\varepsilon) = \left\{ \left\langle x, \left(\tilde{s}_{u\tilde{F}(\varepsilon)}(x), \tilde{s}_{\pi\tilde{F}(\varepsilon)}(x), \tilde{s}_{v\tilde{F}(\varepsilon)}(x) \right) \right\rangle \Big| x \in U \right\} \tag{11}$$

where $\tilde{F}_{x_i}(\varepsilon) = ([s^L_{u\tilde{F}(\varepsilon)}(x_i), s^U_{u\tilde{F}(\varepsilon)}(x_i)], [s^L_{\pi\tilde{F}(\varepsilon)}(x_i), s^U_{\pi\tilde{F}(\varepsilon)}(x_i)], [s^L_{v\tilde{F}(\varepsilon)}(x_i), s^U_{v\tilde{F}(\varepsilon)}(x_i)])$ are the membership degree, the hesitancy degree and the non-membership degree of object x_i respectively.

Example 1. *Under uncertain environments, how do we deal with the situation where the sum of membership degree, hesitancy and non-membership degree exceeds 1 after the experts evaluate each parameter? And experts prefer to evaluate each parameter in natural language. LIVSFSS can facilitate the handing of the above situations. Let $U = \{x_1, x_2, x_3, x_4\}$ be a set of teachers and $E = \{e_1, e_2, e_3\} = \{teaching\ quality, blackboard\ writing, research\ ability\}$ be a set of parameters for evaluation indicators of teachers. Experts assign a value to each parameter according to a continuous LTS $\bar{S}$ defined as $\bar{S} = \{s_0 = "extremely\ poor", s_1 = "very\ poor", s_2 = "poor", s_3 = "slightly poor", s_4 = "medium", s_5 = "slightly\ good", s_6 = "good", s_7 = "very\ good", s_8 = "extremely\ good"\}$. The result described by the LIVSFSS $(\tilde{F}, E)$ is presented in Table 1 and $(\tilde{F}, E)$ is defined as follows.*

$\tilde{F}(e_1) = \{< x_1, ([s_3, s_5], [s_0, s_3], [s_2, s_3]) >, < x_2, ([s_3, s_5], [s_0, s_3], [s_2, s_3]) >, < x_3, ([s_5, s_6], [s_0, s_2], [s_1, s_2]) >, < x_4, ([s_4, s_5], [s_0, s_2], [s_2, s_3]) >\};$

$\tilde{F}(e_2) = \{< x_1, ([s_5, s_6], [s_0, s_2], [s_1, s_2]) >, < x_2, ([s_4, s_5], [s_0, s_2], [s_2, s_3]) >, < x_3, ([s_3, s_4], [s_1, s_3], [s_2, s_3]) >, < x_4, ([s_4, s_5], [s_0, s_2], [s_2, s_3]) >\};$

$\tilde{F}(e_3) = \{< x_1, ([s_5, s_6], [s_0, s_2], [s_1, s_2]) >, < x_2, ([s_2, s_4], [s_0, s_3], [s_3, s_4]) >, < x_3, ([s_3, s_4], [s_1, s_3], [s_2, s_3]) >, < x_4, ([s_4, s_5], [s_0, s_2], [s_1, s_3]) >\}.$

Table 1. LIVSFSS $(\tilde{F}, E)$ of Example 1.

U	e_1	e_2	e_3
x_1	$([s_3, s_5], [s_0, s_3], [s_2, s_3])$	$([s_5, s_6], [s_0, s_2], [s_1, s_2])$	$([s_5, s_6], [s_0, s_2], [s_1, s_2])$
x_2	$([s_3, s_5], [s_0, s_3], [s_2, s_3])$	$([s_4, s_5], [s_0, s_2], [s_2, s_3])$	$([s_2, s_4], [s_0, s_3], [s_3, s_4])$
x_3	$([s_5, s_6], [s_0, s_2], [s_1, s_2])$	$([s_3, s_4], [s_1, s_3], [s_2, s_3])$	$([s_3, s_4], [s_1, s_3], [s_2, s_3])$
x_4	$([s_4, s_5], [s_0, s_2], [s_2, s_3])$	$([s_4, s_5], [s_0, s_2], [s_2, s_3])$	$([s_4, s_5], [s_0, s_2], [s_1, s_3])$

Definition 7. *Let U be an initial universe set, E be a set of parameters and suppose that A, $B \subset E$, $(\tilde{F}, A)$ and $(\tilde{G}, B)$ are two linguistic interval-valued spherical fuzzy soft sets; $(\tilde{F}, A)$ is a linguistic interval-valued spherical fuzzy soft subset of $(\tilde{G}, B)$, which can be denoted by $(\tilde{F}, A) \tilde{\subset} (\tilde{G}, B)$, if and only if*

(1) $A \subset B$;
(2) $\forall \varepsilon \in A$, $\tilde{F}(\varepsilon)$ *is a linguistic interval-valued spherical fuzzy subset of $\tilde{G}(\varepsilon)$.*

Definition 8. *Let $(\tilde{F}, A)$ and $(\tilde{G}, B)$ be two linguistic interval-valued spherical fuzzy soft sets; $(\tilde{F}, A)$ and $(\tilde{G}, B)$ are said to be a linguistic interval-valued spherical fuzzy soft equal, which can be denoted by $(\tilde{F}, A) \cong (\tilde{G}, B)$, if and only if*

(1) $(\tilde{F}, A)$ *is a linguistic interval-valued spherical fuzzy soft subset of $(\tilde{G}, B)$;*
(2) $(\tilde{G}, B)$ *is a linguistic interval-valued spherical fuzzy soft subset of $(\tilde{F}, A)$.*

3.2. Operations on LIVSFSS

Definition 9. *The complement of a linguistic interval-valued spherical fuzzy soft set $(\tilde{F}, A)$ is denoted by $(\tilde{F}, A)^C$. It is defined by*

$$(\tilde{F}, A)^C = (\tilde{F}^C, \neg A) \tag{12}$$

where $\forall \alpha \in A$, $\neg \alpha = not\ \alpha$, is the not set of parameters α, which holds the opposite meanings to parameter α;

$$\tilde{F}^C : \neg A \rightarrow LIVSF(U) \tag{13}$$

is a mapping given by $\tilde{F}^C(\varepsilon) = (\tilde{F}(\varepsilon))^C$ for all $x \in U$ and $\varepsilon \in \neg A$.

Definition 10. *A linguistic interval-valued spherical fuzzy soft set $(\tilde{F}, A)$ over U is said to be a null linguistic interval-valued spherical fuzzy soft set if $\tilde{F}_{x_i}(\varepsilon) = ([s_0, s_0], [s_0, s_0], [s_h, s_h])$ for $\forall \varepsilon \in A, x \in U$.*

Definition 11. *A linguistic interval-valued spherical fuzzy soft set $(\tilde{F}, A)$ over U is said to be an absolute linguistic interval-valued spherical fuzzy soft set if $\tilde{F}_{x_i}(\varepsilon) = ([s_h, s_h], [s_0, s_0], [s_0, s_0])$ for $\forall \varepsilon \in A, x \in U$.*

Definition 12. *The union of two linguistic interval-valued spherical fuzzy soft sets $(\tilde{F}, A)$ and $(\tilde{G}, B)$ over U is a linguistic interval-valued spherical fuzzy soft set $(\tilde{H}, C)$, where $C = A \cup B$, $\forall \varepsilon \in C$,*

$$\tilde{H}(\varepsilon) = \begin{cases} \tilde{F}(\varepsilon), \varepsilon \in A - B; \\ \tilde{G}(\varepsilon), \varepsilon \in B - A; \\ \tilde{F}(\varepsilon) \cup \tilde{G}(\varepsilon), \varepsilon \in A \cap B; \end{cases} \tag{14}$$

where $\varepsilon \in A \cap B$, then, $\widetilde{H}(\varepsilon) = \tilde{F}(\varepsilon) \cup \tilde{G}(\varepsilon) = \{< x, ([max\{\tilde{s}^L_{u\tilde{F}(\varepsilon)}(x), \tilde{s}^L_{u\tilde{G}(\varepsilon)}(x)\}, max\{\tilde{s}^U_{u\tilde{F}(\varepsilon)}(x), \tilde{s}^U_{u\tilde{G}(\varepsilon)}(x)\}], [min\{\tilde{s}^L_{\pi\tilde{F}(\varepsilon)}(x), \tilde{s}^L_{\pi\tilde{F}(\varepsilon)}(x)\}, min\{\tilde{s}^U_{\pi\tilde{F}(\varepsilon)}(x), \tilde{s}^U_{\pi\tilde{G}(\varepsilon)}(x)\}], [min\{\tilde{s}^L_{v\tilde{F}(\varepsilon)}(x), \tilde{s}^L_{v\tilde{G}(\varepsilon)}(x)\}, min\{\tilde{s}^U_{v\tilde{F}(\varepsilon)}(x), \tilde{s}^U_{v\tilde{G}(\varepsilon)}(x)\}]) > |x \in U\}$. We denote it by $(\tilde{F}, A) \widetilde{\cup} (\tilde{G}, B) = (\widetilde{H}, C)$.

Definition 13. *The intersection of two linguistic interval-valued spherical fuzzy soft sets $(\tilde{F}, A)$ and $(\tilde{G}, B)$ over U is a linguistic interval-valued spherical fuzzy soft set $(\widetilde{H}, C)$, where $C = A \cap B$, $\forall \varepsilon \in C$, $\widetilde{H}(\varepsilon) = \tilde{F}(\varepsilon) \cap \tilde{G}(\varepsilon) = \{< x, ([min\{\tilde{s}^L_{u\tilde{F}(\varepsilon)}(x), \tilde{s}^L_{u\tilde{G}(\varepsilon)}(x)\}, min\{\tilde{s}^U_{u\tilde{F}(\varepsilon)}(x), \tilde{s}^U_{u\tilde{G}(\varepsilon)}(x)\}], [min\{\tilde{s}^L_{\pi\tilde{F}(\varepsilon)}(x), \tilde{s}^L_{\pi\tilde{F}(\varepsilon)}(x)\}, min\{\tilde{s}^U_{\pi\tilde{F}(\varepsilon)}(x), \tilde{s}^U_{\pi\tilde{G}(\varepsilon)}(x)\}], [max\{\tilde{s}^L_{v\tilde{F}(\varepsilon)}(x), \tilde{s}^L_{v\tilde{G}(\varepsilon)}(x)\}, max\{\tilde{s}^U_{v\tilde{F}(\varepsilon)}(x), \tilde{s}^U_{v\tilde{G}(\varepsilon)}(x)\}]) > |x \in U\}$. We denote it by $(\tilde{F}, A) \widetilde{\cap} (\tilde{G}, B) = (\widetilde{H}, C)$.*

Theorem 1. *Let $(\tilde{F}, A)$, $(\tilde{G}, B)$ and $(\widetilde{M}, C)$ be three linguistic interval-valued spherical fuzzy soft sets over U, then we have the following properties.*

(1) *Transitive law: If $(\tilde{F}, A) \widetilde{\subset} (\tilde{G}, B)$ and $(\tilde{G}, B) \widetilde{\subset} (\widetilde{M}, C)$, then $(\tilde{F}, A) \widetilde{\subset} (\widetilde{M}, C)$;*

(2) *Commutative law: $(\tilde{F}, A) \widetilde{\cup} (\tilde{G}, B) = (\tilde{G}, B) \widetilde{\cup} (\tilde{F}, A)$ and*
$(\tilde{F}, A) \widetilde{\cap} (\tilde{G}, B) = (\tilde{G}, B) \widetilde{\cap} (\tilde{F}, A)$;

(3) *Idempotent law: $(\tilde{F}, A) \widetilde{\cup} (\tilde{F}, A) = (\tilde{F}, A)$ and $(\tilde{F}, A) \widetilde{\cap} (\tilde{F}, A) = (\tilde{F}, A)$;*

(4) *Associative law: $(\tilde{F}, A) \widetilde{\cup} ((\tilde{G}, B) \widetilde{\cup} (\widetilde{M}, C)) = ((\tilde{F}, A) \widetilde{\cup} (\tilde{G}, B)) \widetilde{\cup} (\widetilde{M}, C)$ and*
$(\tilde{F}, A) \widetilde{\cap} ((\tilde{G}, B) \widetilde{\cap} (\widetilde{M}, C)) = ((\tilde{F}, A) \widetilde{\cap} (\tilde{G}, B)) \widetilde{\cap} (\widetilde{M}, C)$;

(5) *Distributive law: $(\tilde{F}, A) \widetilde{\cup} ((\tilde{G}, B) \widetilde{\cap} (\widetilde{M}, C)) = ((\tilde{F}, A) \widetilde{\cup} (\tilde{G}, B)) \widetilde{\cap} ((\tilde{F}, A) \widetilde{\cup} (\widetilde{M}, C))$ and*
$(\tilde{F}, A) \widetilde{\cap} ((\tilde{G}, B) \widetilde{\cup} (\widetilde{M}, C)) = ((\tilde{F}, A) \widetilde{\cap} (\tilde{G}, B)) \widetilde{\cup} ((\tilde{F}, A) \widetilde{\cap} (\widetilde{M}, C))$;

(6) *Absorption law: $((\tilde{F}, A) \widetilde{\cup} (\tilde{G}, B)) \widetilde{\cap} (\tilde{F}, A) = (\tilde{F}, A)$ and*
$((\tilde{F}, A) \widetilde{\cap} (\tilde{G}, B)) \widetilde{\cup} (\tilde{F}, A) = (\tilde{F}, A)$;

(7) *De Morgan's law: $((\tilde{F}, A) \widetilde{\cup} (\tilde{G}, B))^C = (\tilde{F}, A)^C \widetilde{\cap} (\tilde{G}, B)^C$ and*
$((\tilde{F}, A) \widetilde{\cap} (\tilde{G}, B))^C = (\tilde{F}, A)^C \widetilde{\cup} (\tilde{G}, B)^C$.

Proof. We can obtain (1)–(4) easily according to Definitions 7, 12 and 13. And we prove the other properties as follows.

Suppose that, for $(\tilde{F}, A)$, $\forall \alpha \in A$, $\tilde{F}(\alpha) = \{\langle x, (\tilde{s}_{u\tilde{F}(\alpha)}(x), \tilde{s}_{\pi\tilde{F}(\alpha)}(x), \tilde{s}_{v\tilde{F}(\alpha)}(x))\rangle|x \in U\}$, for $(\tilde{G}, B)$, $\forall \beta \in B$, $\tilde{G}(\beta) = \{\langle x, (\tilde{s}_{u\tilde{G}(\beta)}(x), \tilde{s}_{\pi\tilde{G}(\beta)}(x), \tilde{s}_{v\tilde{G}(\beta)}(x))\rangle|x \in U\}$ and, for $(\widetilde{M}, C)$, $\forall \gamma \in C$, $\widetilde{M}(\gamma) = \{\langle x, (\tilde{s}_{u\widetilde{M}(\gamma)}(x), \tilde{s}_{\pi\widetilde{M}(\gamma)}(x), \tilde{s}_{v\widetilde{M}(\gamma)}(x))\rangle|x \in U\}$. For convenience, we denote $\tilde{F}(\alpha) = ([s_{a1}, s_{b1}], [s_{c1}, s_{d1}], [s_{e1}, s_{o1}])$, $\tilde{G}(\beta) = ([s_{a2}, s_{b2}], [s_{c2}, s_{d2}], [s_{e2}, s_{o2}])$ and $\widetilde{M}(\gamma) = ([s_{a3}, s_{b3}], [s_{c3}, s_{d3}], [s_{e3}, s_{o3}])$, respectively.

(5). Suppose that $(\tilde{G}, B) \widetilde{\cap} (\widetilde{M}, C) = (\tilde{Q}, Y)$, where $Y = B \cap C$, $Y \neq \emptyset$ and $\forall \varphi \in Y$, $\tilde{Q}(\varphi) = \tilde{G}(\varphi) \cap \widetilde{M}(\varphi) = ([min\{s_{a2}, s_{a3}\}, min\{s_{b2}, s_{b3}\}], [min\{s_{c2}, s_{c3}\}, min\{s_{d2}, s_{d3}\}], [max\{s_{e2}, s_{e3}\}, max\{s_{o2}, s_{o3}\}])$. Then we have $(\tilde{F}, A) \widetilde{\cup} ((\tilde{G}, B) \widetilde{\cap} (\widetilde{M}, C)) = (\tilde{F}, A) \widetilde{\cup} (\tilde{Q}, Y)$. According to Definition 12, $(\tilde{F}, A) \widetilde{\cup} (\tilde{Q}, Y) = (\tilde{L}, Z)$, where $Z = A \cup Y$ and $\forall \omega \in Z$,

$$\tilde{L}(\omega) = \begin{cases} \tilde{F}(\omega), \omega \in A - Y = A - B \cap C; \\ \tilde{Q}(\omega), \omega \in Y - A = B \cap C - A; \\ \tilde{F}(\omega) \cup \tilde{Q}(\omega), \omega \in A \cap Y = A \cap B \cap C; \end{cases}$$

where $\tilde{F}(\omega) \cup \tilde{Q}(\omega) = ([max\{s_{a1}, min\{s_{a2}, s_{a3}\}\}, max\{s_{b1}, min\{s_{b2}, s_{b3}\}\}], [min\{s_{c1}, min\{s_{c2}, s_{c3}\}\}, min\{s_{d1}, min\{s_{d2}, s_{d3}\}\}], [min\{s_{e1}, max\{s_{e2}, s_{e3}\}\}, min\{s_{o1}, max\{s_{o2}, s_{o3}\}\}])$.

Assume that $(\tilde{F}, A) \widetilde{\cup} (\tilde{G}, B) = (\widetilde{N}, D)$ and $(\tilde{F}, A) \widetilde{\cup} (\widetilde{M}, C) = (\tilde{I}, E)$. According to Definition 12, for $(\widetilde{N}, D)$, where $D = A \cup B$ and $\forall \rho \in D$,

$$\widetilde{N}(\rho) = \begin{cases} \tilde{F}(\rho), \rho \in A - B; \\ \tilde{G}(\rho), \rho \in B - A; \\ \tilde{F}(\rho) \cup \widetilde{N}(\rho), \omega \in A \cap B; \end{cases}$$

where $\tilde{F}(\rho) \cup \tilde{G}(\rho) = ([\,max\{s_{a1}, s_{a2}\}, max\{s_{b1}, s_{b2}\}\,], [min\{s_{c1}, s_{c2}\}, min\{s_{d1}, s_{d2}\}], [min\{s_{e1}, s_{e2}\}, min\{s_{o1}, s_{o2}\}])$. For $(\tilde{I}, E)$, we have $E = A \cup C$ and $\forall \sigma \in E$,

$$\tilde{I}(\sigma) = \begin{cases} \tilde{F}(\sigma), \sigma \in A - C; \\ \tilde{M}(\sigma), \sigma \in C - A; \\ \tilde{F}(\sigma) \cup \tilde{M}(\sigma), \sigma \in A \cap C; \end{cases}$$

where $\tilde{F}(\sigma) \cup \tilde{M}(\sigma) = ([\,max\{s_{a1}, s_{a3}\}, max\{s_{b1}, s_{b3}\}\,], [min\{s_{c1}, s_{c3}\}, min\{s_{d1}, s_{d3}\}], [min\{s_{e1}, s_{e3}\}, min\{s_{o1}, s_{o3}\}])$. Hence $((\tilde{F}, A) \,\tilde{\cup}\, (\tilde{G}, B)) \,\tilde{\cap}\, ((\tilde{F}, A) \,\tilde{\cup}\, (\tilde{M}, C)) = (\tilde{N}, D) \,\tilde{\cap}\, (\tilde{I}, E)$, suppose $(\tilde{N}, D) \,\tilde{\cap}\, (\tilde{I}, E) = (\tilde{J}, F)$, where $F = D \cap E$ and $F \neq \emptyset$, then

$$\tilde{J}(z) = \begin{cases} \tilde{F}(z), \forall z \in D \cap E = \rho \cap \sigma = (A - C) \cap (A - B) = A - B \cap C; \\ \tilde{G}(z), \forall z \in B \cap C - A; \\ \tilde{N}(z) \cap \tilde{I}(z), \forall z \in D \cap E = \rho \cap \sigma = (A \cap B) \cap (A \cap C) = A \cap B \cap C; \end{cases}$$

where $\tilde{N}(z) \cap \tilde{I}(z) = ([min\{max\{s_{a1}, s_{a2}\}, max\{s_{a1}, s_{a3}\}\}, min\{max\{s_{b1}, s_{b2}\}, max\{s_{b1}, s_{b3}\}\}], [min\{min\{s_{c1}, s_{c2}\}, min\{s_{c1}, s_{c3}\}\}, min\{min\{s_{d1}, s_{d2}\}, min\{s_{d1}, s_{d3}\}\}], [max\{min\{s_{e1}, s_{e2}\}, min\{s_{e1}, s_{e3}\}\}, max\{min\{s_{o1}, s_{o2}\}, min\{s_{o1}, s_{o3}\}\}])$.

Consequently, we can obtain $(\tilde{L}, Z) = (\tilde{J}, F)$. Thus $(\tilde{F}, A) \,\tilde{\cup}\, ((\tilde{G}, B) \,\tilde{\cap}\, (\tilde{M}, C)) = ((\tilde{F}, A) \,\tilde{\cup}\, (\tilde{G}, B)) \,\tilde{\cap}\, ((\tilde{F}, A) \,\tilde{\cup}\, (\tilde{M}, C))$.

Similarly, $(\tilde{F}, A) \,\tilde{\cap}\, ((\tilde{G}, B) \,\tilde{\cup}\, (\tilde{M}, C)) = ((\tilde{F}, A) \,\tilde{\cap}\, (\tilde{G}, B)) \,\tilde{\cup}\, ((\tilde{F}, A) \,\tilde{\cap}\, (\tilde{M}, C))$.

(6). The proof is similar to that of (5).

(7). Suppose that $(\tilde{F}, A) \,\tilde{\cup}\, (\tilde{G}, B) = (\tilde{H}, C)$, where $C = A \cup B$ and $\forall \varepsilon \in C$,

$$\tilde{H}(\varepsilon) = \begin{cases} \tilde{F}(\varepsilon), \varepsilon \in A - B; \\ \tilde{G}(\varepsilon), \varepsilon \in B - A; \\ \tilde{F}(\varepsilon) \cup \tilde{G}(\varepsilon), \varepsilon \in A \cap B; \end{cases}$$

where $\tilde{F}(\varepsilon) \cup \tilde{G}(\varepsilon) = ([max\{s_{a1}, s_{a2}\}, max\{s_{b1}, s_{b2}\}], [min\{s_{c1}, s_{c2}\}, min\{s_{d1}, s_{d2}\}], [min\{s_{e1}, s_{e2}\}, min\{s_{o1}, s_{o2}\}])$. Then we have $((\tilde{F}, A) \,\tilde{\cup}\, (\tilde{G}, B))^C = (\tilde{H}, C)^C$, where $\forall \varepsilon \in C$,

$$(\tilde{H}(\varepsilon))^C = \begin{cases} (\tilde{F}(\varepsilon))^C = ([s_{e1}, s_{o1}], [s_{c1}, s_{d1}], [s_{a1}, s_{b1}]), \varepsilon \in A - B; \\ (\tilde{G}(\varepsilon))^C = ([s_{e2}, s_{o2}], [s_{c2}, s_{d2}], [s_{a2}, s_{b2}]), \varepsilon \in B - A; \\ (\tilde{F}(\varepsilon) \cup \tilde{G}(\varepsilon))^C, \varepsilon \in A \cap B; \end{cases}$$

where $\left(\tilde{F}(\varepsilon) \cup \tilde{G}(\varepsilon)\right)^C = ([min\{s_{e1}, s_{e2}\}, min\{s_{o1}, s_{o2}\}], [min\{s_{c1}, s_{c2}\}, min\{s_{d1}, s_{d2}\}], [max\{s_{a1}, s_{a2}\}, max\{s_{b1}, s_{b2}\}])$.

Assume that $(\tilde{F}, A)^C \,\tilde{\cap}\, (\tilde{G}, B)^C = (\tilde{I}, D)$, where $D = A \cap B$, $D \neq \emptyset$ and $\forall \varepsilon \in D$, then $\tilde{I}(\varepsilon) = (\tilde{F}, A)^C \,\tilde{\cap}\, (\tilde{G}, B)^C = ([min\{s_{e1}, s_{e2}\}, min\{s_{o1}, s_{o2}\}], [min\{s_{c1}, s_{c2}\}, min\{s_{d1}, s_{d2}\}], [max\{s_{a1}, s_{a2}\}, max\{s_{b1}, s_{b2}\}])$.

Consequently, $(\tilde{H}, C) = (\tilde{I}, D)$. Thus $((\tilde{F}, A) \,\tilde{\cup}\, (\tilde{G}, B))^C = (\tilde{F}, A)^C \,\tilde{\cap}\, (\tilde{G}, B)^C$.

Similarly, we can obtain $((\tilde{F}, A) \,\tilde{\cap}\, (\tilde{G}, B))^C = (\tilde{F}, A)^C \,\tilde{\cup}\, (\tilde{G}, B)^C$. □

Definition 14. *The "AND" operation on the two linguistic interval-valued spherical fuzzy soft sets $(\tilde{F}, A)$ and $(\tilde{G}, B)$ is defined by*

$$(\tilde{F}, A) \wedge (\tilde{G}, B) = (\tilde{H}, A \times B) \tag{15}$$

where $\tilde{H}(\alpha, \beta) = \tilde{F}(\alpha) \cap \tilde{G}(\beta)$, $\forall(\alpha, \beta) \in A \times B$.

Definition 15. *The "OR" operation on the two linguistic interval-valued spherical fuzzy soft sets $(\tilde{F}, A)$ and $(\tilde{G}, B)$ is defined by*

$$(\tilde{F}, A) \vee (\tilde{G}, B) = (\tilde{H}, A \times B) \tag{16}$$

where $\tilde{H}(\alpha, \beta) = \tilde{F}(\alpha) \cup \tilde{G}(\beta)$, $\forall(\alpha, \beta) \in A \times B$.

Theorem 2. *Let* $(\tilde{F}, A)$, $(\tilde{G}, B)$ *and* $(\tilde{M}, C)$ *be three linguistic interval-valued spherical fuzzy soft sets over* U, *then we have the following properties.*

(1) *Associative law:* $(\tilde{F}, A) \vee ((\tilde{G}, B) \vee (\tilde{M}, C)) = ((\tilde{F}, A) \vee (\tilde{G}, B)) \vee (\tilde{M}, C)$ *and*
   $(\tilde{F}, A) \wedge ((\tilde{G}, B) \wedge (\tilde{M}, C)) = ((\tilde{F}, A) \wedge (\tilde{G}, B)) \wedge (\tilde{M}, C);$

(2) *Distributive law:* $(\tilde{F}, A) \vee ((\tilde{G}, B) \wedge (\tilde{M}, C)) = ((\tilde{F}, A) \vee (\tilde{G}, B)) \wedge ((\tilde{F}, A) \vee (\tilde{M}, C))$ *and*
   $(\tilde{F}, A) \wedge ((\tilde{G}, B) \vee (\tilde{M}, C)) = ((\tilde{F}, A) \wedge (\tilde{G}, B)) \vee ((\tilde{F}, A, A) \wedge (\tilde{M}, C));$

(3) *De Morgan's law:* $((\tilde{F}, A) \vee (\tilde{G}, B))^C = (\tilde{F}, A)^C \wedge (\tilde{G}, B)^C$ *and* $((\tilde{F}, A) \wedge (\tilde{G}, B))^C = (\tilde{F}, A)^C \vee (\tilde{G}, B)^C.$

Proof. We can obtain (1) easily according to Definitions 14 and 15. And we prove the other properties as follows.

(2). Suppose $(\tilde{G}, B) \wedge (\tilde{M}, C) = (\tilde{H}, B \times C)$, where $\forall (\beta, \gamma) \in B \times C$, $\tilde{H}(\beta, \gamma) = \tilde{G}(\beta) \cap \tilde{M}(\gamma)$. Then we have $(\tilde{F}, A) \vee ((\tilde{G}, B) \wedge (\tilde{M}, C)) = (\tilde{F}, A) \vee (\tilde{H}, B \times C)$. And suppose $(\tilde{F}, A) \vee (\tilde{H}, B \times C) = (\tilde{I}, A \times (B \times C))$, $\forall (\alpha, (\beta, \gamma)) \in A \times (B \times C)$, $\tilde{I}(\alpha, (\beta, \gamma)) = \tilde{F}(\alpha) \cup (\tilde{G}(\beta) \cap \tilde{M}(\gamma))$.

Assume that $(\tilde{F}, A) \vee (\tilde{G}, B) = (\tilde{J}, A \times B)$ and $(\tilde{F}, A) \vee (\tilde{M}, C) = (\tilde{K}, A \times C)$, where $\forall (\alpha, \beta) \in A \times B$, $\tilde{J}(\alpha, \beta) = \tilde{F}(\alpha) \cup \tilde{G}(\beta)$ and $\forall (\alpha, \gamma) \in A \times C$, $\tilde{K}(\alpha, \gamma) = \tilde{F}(\alpha) \cup \tilde{M}(\gamma)$. Then we have $((\tilde{F}, A) \vee (\tilde{G}, B)) \wedge ((\tilde{F}, A) \vee (\tilde{M}, C)) = (\tilde{J}, A \times B) \wedge (\tilde{K}, A \times C)$.

Suppose $(\tilde{J}, A \times B) \wedge (\tilde{K}, A \times C) = (\tilde{L}, (A \times B) \times (A \times C))$, where $\forall (\alpha, \gamma) \in A \times C \times B$, then, $\tilde{L}((\alpha, \beta), (\alpha, \gamma)) = (\tilde{F}(\alpha) \cup \tilde{G}(\beta)) \cap (\tilde{F}(\alpha) \cup \tilde{M}(\gamma))$. According to Theorem 1(5), $(\tilde{F}, A) \tilde{\cup} ((\tilde{G}, B) \tilde{\cap} (\tilde{M}, C)) = ((\tilde{F}, A) \tilde{\cup} (\tilde{G}, B)) \tilde{\cap} ((\tilde{F}, A) \tilde{\cup} (\tilde{M}, C))$.

Consequently, $(\tilde{F}, A) \vee ((\tilde{G}, B) \wedge (\tilde{M}, C)) = ((\tilde{F}, A) \vee (\tilde{G}, B)) \wedge ((\tilde{F}, A) \vee (\tilde{M}, C))$.

Similarly, $(\tilde{F}, A) \wedge ((\tilde{G}, B) \vee (\tilde{M}, C)) = ((\tilde{F}, A) \wedge (\tilde{G}, B)) \vee ((\tilde{F}, A) \wedge (\tilde{M}, C))$.

(3). Suppose $(\tilde{F}, A) \vee (\tilde{G}, B) = (\tilde{H}, A \times B)$, where $\forall (\alpha, \beta) \in A \times B$, $\tilde{H}(\alpha, \beta) = \tilde{F}(\alpha) \cup \tilde{G}(\beta)$. Then we have $((\tilde{F}, A) \vee (\tilde{G}, B))^C = (\tilde{H}, A \times B)^C$. According to Definition 15, $\forall (\alpha, \beta) \in A \times B$, $(\tilde{H}(\alpha, \beta))^C = (\tilde{F}(\alpha) \cup \tilde{G}(\beta))^C$.

Assume that $(\tilde{F}, A) \wedge (\tilde{G}, B) = (\tilde{I}, A \times B)$, where $\forall (\alpha, \beta) \in A \times B$, $\tilde{I}(\alpha, \beta) = \tilde{F}(\alpha) \cap \tilde{G}(\beta)$. According to Definition 9, $(\tilde{F}, A)^C = (\tilde{F}^C, \neg A)$, $(\tilde{G}, B)^C = (\tilde{G}^C, \neg B)$. Then we have $(\tilde{F}, A)^C \wedge (\tilde{G}, B)^C = (\tilde{I}^C, \neg (A \times B))$, where $\forall (\alpha, \beta) \in \neg (A \times B)$, $\tilde{H}^C(\alpha, \beta) = (\tilde{F}(\alpha))^C \cap (\tilde{G}(\beta))^C$. According to Theorem 1(7), $(\tilde{F}(\alpha) \cup \tilde{G}(\beta))^C = (\tilde{F}(\alpha))^C \cap (\tilde{G}(\beta))^C$.

Consequently, $((\tilde{F}, A) \vee (\tilde{G}, B))^C = (\tilde{F}, A)^C \wedge (\tilde{G}, B)^C$.

Similarly, we can obtain $((\tilde{F}, A) \wedge (\tilde{G}, B))^C = (\tilde{F}, A)^C \vee (\tilde{G}, B)^C$. □

Definition 16. *The necessity operation on a linguistic interval-valued spherical fuzzy soft set* $(\tilde{F}, E)$ *is denoted as* $\Box(\tilde{F}, E)$, $\forall \varepsilon \in E$,

$$\Box \tilde{F}(\varepsilon) = \left\{ \left\langle x, \left(\tilde{s}_{\Box u \tilde{F}(\varepsilon)}(x), \tilde{s}_{\Box \pi \tilde{F}(\varepsilon)}(x), \tilde{s}_{\Box v \tilde{F}(\varepsilon)}(x) \right) \right\rangle \middle| x \in U \right\} \tag{17}$$

Here, $\tilde{s}_{\Box u \tilde{F}(\varepsilon)}(x) = [\tilde{s}_{u\tilde{F}(\varepsilon)}^L(x), \tilde{s}_{u\tilde{F}(\varepsilon)}^U(x)]$ is the necessary membership degree that object x holds on parameter ε, $\tilde{s}_{\Box \pi \tilde{F}(\varepsilon)}(x) = [s_0, s_0]$ is the empty hesitancy degree that object x holds on parameter ε, $\tilde{s}_{\Box v \tilde{F}(\varepsilon)}(x) = [\tilde{s}_{\sqrt{(h)^2 - ((u\tilde{F}(\varepsilon))^U)^2}}(x), \tilde{s}_{\sqrt{(h)^2 - ((u\tilde{F}(\varepsilon))^L)^2}}(x)]$ is the possible non-membership degree that object x does not hold on parameter ε.

Theorem 3. *Let* $(\tilde{F}, A)$ *and* $(\tilde{G}, B)$ *be two linguistic interval-valued spherical fuzzy soft sets over* U, *then we have following properties.*

(1) $\Box \Box (\tilde{F}, A) = \Box (\tilde{F}, A);$

(2) $\Box ((\tilde{F}, A) \tilde{\cup} (\tilde{G}, B)) = \Box (\tilde{F}, A) \tilde{\cup} \Box (\tilde{G}, B);$

(3) $\Box ((\tilde{F}, A) \tilde{\cap} (\tilde{G}, B)) = \Box (\tilde{F}, A) \tilde{\cap} \Box (\tilde{G}, B);$

(4) $\Box ((\tilde{F}, A) \wedge (\tilde{G}, B)) = \Box (\tilde{F}, A) \wedge \Box (\tilde{G}, B);$

(5) $\Box ((\tilde{F}, A) \vee (\tilde{G}, B)) = \Box (\tilde{F}, A) \vee \Box (\tilde{G}, B).$

Proof. We can obtain (1) easily according to Definition 16. And we prove the other properties as follows.

Suppose that, for $(\tilde{F}, A)$, $\forall \alpha \in A$, $\tilde{F}(\alpha) = ([s_{a1}, s_{b1}], [s_{c1}, s_{d1}], [s_{e1}, s_{o1}])$ and for $(\tilde{G}, B)$, $\forall \beta \in B$, $\tilde{G}(\beta) = ([s_{a2}, s_{b2}], [s_{c2}, s_{d2}], [s_{e2}, s_{o2}])$.

(2). Suppose that $(\tilde{F}, A) \, \widetilde{\cup} \, (\tilde{G}, B) = (\tilde{H}, C)$, where $C = A \cup B$ and $\forall \varepsilon \in C$,

$$\tilde{H}(\varepsilon) = \begin{cases} \tilde{F}(\varepsilon) = ([s_{a1}, s_{b1}], [s_{c1}, s_{d1}], [s_{e1}, s_{o1}]), \varepsilon \in A - B; \\ \tilde{G}(\varepsilon) = ([s_{a2}, s_{b2}], [s_{c2}, s_{d2}], [s_{e2}, s_{o2}]), \varepsilon \in B - A; \\ \tilde{F}(\varepsilon) \cup \tilde{G}(\varepsilon), \varepsilon \in A \cap B; \end{cases}$$

where $\tilde{F}(\varepsilon) \cup \tilde{G}(\varepsilon) = ([max\{s_{a1}, s_{a2}\}, max\{s_{b1}, s_{b2}\}], [min\{s_{c1}, s_{c2}\}, min\{s_{d1}, s_{d2}\}],$ $[min\{s_{e1}, s_{e2}\}, min\{s_{o1}, s_{o2}\}])$. Then we have $\square((\tilde{F}, A) \, \widetilde{\cup} \, (\tilde{G}, B)) = \square(\tilde{H}, C)$. According to Definition 16, we have

$$\square\tilde{H}(\varepsilon) = \begin{cases} \square\tilde{F}(\varepsilon) = ([s_{a1}, s_{b1}], [s_0, s_0], [s_{\sqrt{(h)^2-(b1)^2}}, s_{\sqrt{(h)^2-(a1)^2}}]), \varepsilon \in A - B; \\ \square\tilde{G}(\varepsilon) = ([s_{a2}, s_{b2}], [s_0, s_0], [s_{\sqrt{(h)^2-(b2)^2}}, s_{\sqrt{(h)^2-(a2)^2}}]), \varepsilon \in B - A; \\ \square(\tilde{F}(\varepsilon) \cup \tilde{G}(\varepsilon)), \varepsilon \in A \cap B; \end{cases}$$

where $\square(\tilde{F}(\varepsilon) \cup \tilde{G}(\varepsilon)) = ([max\{s_{a1}, s_{a2}\}, max\{s_{b1}, s_{b2}\}], [s_0, s_0], [min\{s_{\sqrt{(h)^2-(b1)^2}},$ $s_{\sqrt{(h)^2-(b2)^2}}\}, min\{s_{\sqrt{(h)^2-(a1)^2}}, s_{\sqrt{(h)^2-(a2)^2}}\}])$.

According to Definitions 12 and 16, $\square(\tilde{F}, A) \, \widetilde{\cup} \, \square(\tilde{G}, B) = (\tilde{O}, C)$, $C = A \cup B$ and $\forall \varepsilon \in C$,

$$\tilde{O}(\varepsilon) = \begin{cases} \square\tilde{F}(\varepsilon) = ([s_{a1}, s_{b1}], [s_0, s_0], [s_{\sqrt{(h)^2-(b1)^2}}, s_{\sqrt{(h)^2-(a1)^2}}]), \varepsilon \in A - B; \\ \square\tilde{G}(\varepsilon) = ([s_{a2}, s_{b2}], [s_0, s_0], [s_{\sqrt{(h)^2-(b2)^2}}, s_{\sqrt{(h)^2-(a2)^2}}]), \varepsilon \in B - A; \\ \square\tilde{F}(\varepsilon) \cup \square\tilde{G}(\varepsilon), \varepsilon \in A \cap B; \end{cases}$$

where $\square\tilde{F}(\varepsilon) \cup \square\tilde{G}(\varepsilon) = ([max\{s_{a1}, s_{a2}\}, max\{s_{b1}, s_{b2}\}], [s_0, s_0], [min\{s_{\sqrt{(h)^2-(b1)^2}},$ $s_{\sqrt{(h)^2-(b2)^2}}\}, min\{s_{\sqrt{(h)^2-(a1)^2}}, s_{\sqrt{(h)^2-(a2)^2}}\}])$.

Consequently, $\square(\tilde{H}, C) = (\tilde{O}, C)$. Thus $\square((\tilde{F}, A) \, \widetilde{\cup} \, (\tilde{G}, B)) = \square(\tilde{F}, A) \, \widetilde{\cup} \, \square(\tilde{G}, B)$.

(3). The proof is similar to that of (2).

(4). Suppose that $(\tilde{F}, A) \wedge (\tilde{G}, B) = (\tilde{H}, A \times B)$, where $\forall(\alpha, \beta) \in A \times B$, $\tilde{H}(\alpha, \beta) = \tilde{F}(\alpha) \cap \tilde{G}(\beta) = ([min\{s_{a1}, s_{a2}\}, min\{s_{b1}, s_{b2}\}], [min\{s_{c1}, s_{c2}\}, min\{s_{d1}, s_{d2}\}],$ $[max\{s_{e1}, s_{e2}\}, max\{s_{o1}, s_{o2}\}])$. Then we have $(\tilde{F}, A) \wedge (\tilde{G}, B) = (\tilde{H}, A \times B)$. According to Definition 16, $\forall \varepsilon \in A \times B$, $\square\tilde{H}(\varepsilon) = ([min\{s_{a1}, s_{a2}\}, min\{s_{b1}, s_{b2}\}], [s_0, s_0], [s_{\sqrt{(h)^2-(min\{s_{b1}, s_{b2}\})^2}}, s_{\sqrt{(h)^2-(min\{s_{a1}, s_{a2}\})^2}}])$.

According to Definition 14 and Definition 16, assume that $\square(\tilde{F}, A) \wedge \square(\tilde{G}, B) = (\tilde{I}, A \times B)$, where $\forall(\alpha, \beta) \in A \times B$, $\tilde{I}(\alpha, \beta) = \square\tilde{F}(\alpha) \cap \square\tilde{G}(\beta) = ([min\{s_{a1}, s_{a2}\}, min\{s_{b1}, s_{b2}\}], [s_0, s_0], [max\{s_{\sqrt{(h)^2-(b1)^2}},$ $s_{\sqrt{(h)^2-(b2)^2}}\}, max\{s_{\sqrt{(h)^2-(a1)^2}}, s_{\sqrt{(h)^2-(a2)^2}}\}]) = ([min\{s_{a1}, s_{a2}\}, min\{s_{b1}, s_{b2}\}], [s_0, s_0],$ $[s_{\sqrt{(h)^2-(min\{s_{b1}, s_{b2}\})^2}}, s_{\sqrt{(h)^2-(min\{s_{a1}, s_{a2}\})^2}}]) = \square((\tilde{F}, A) \wedge (\tilde{G}, B))$.

(5). The proof is similar to that of (4). $\square$

Definition 17. *The possibility operation on a linguistic interval-valued spherical fuzzy soft set* $(\tilde{F}, E)$ *is denoted as* $\circ (\tilde{F}, E)$, $\forall \varepsilon \in E$,

$$\circ \tilde{F}(\varepsilon) = \left\{ \left\langle x, \left(\tilde{s}_{\circ u \tilde{F}(\varepsilon)}(x), \tilde{s}_{\circ \pi \tilde{F}(\varepsilon)}(x), \tilde{s}_{\circ v \tilde{F}(\varepsilon)}(x) \right) \right\rangle \Big| x \in U \right\} \tag{18}$$

Here, $\tilde{s}_{\circ u \tilde{F}(\varepsilon)}(x) = [\tilde{s}_{\sqrt{(h)^2-((v\tilde{F}(\varepsilon))^U)^2}}(x), \tilde{s}_{\sqrt{(h)^2-((v\tilde{F}(\varepsilon))^L)^2}}(x)]$ is the possible membership degree that object x holds on parameter ε, $\tilde{s}_{\circ \pi \tilde{F}(\varepsilon)}(x) = [s_0, s_0]$ is the empty hesitancy degree that object x holds on parameter ε, $\tilde{s}_{\circ v \tilde{F}(\varepsilon)}(x) = [\tilde{s}^L_{v\tilde{F}(\varepsilon)}(x), \tilde{s}^U_{v\tilde{F}(\varepsilon)}(x)]$ is the necessary non-membership degree that object x does not hold on parameter ε.

Theorem 4. *Let* $(\tilde{F}, A)$ *and* $(\tilde{G}, B)$ *be two linguistic interval-valued spherical fuzzy soft sets over* U, *then we have the following properties.*

(1) $\circ\circ (\tilde{F}, E) = \circ (\tilde{F}, E)$;

(2) $\circ ((\tilde{F}, A) \,\tilde{\cup}\, (\tilde{G}, B)) = \circ (\tilde{F}, A) \,\tilde{\cup}\, \circ (\tilde{G}, B)$;

(3) $\circ ((\tilde{F}, A) \,\tilde{\cap}\, (\tilde{G}, B)) = \circ (\tilde{F}, A) \,\tilde{\cap}\, \circ (\tilde{G}, B)$;

(4) $\circ ((\tilde{F}, A) \wedge (\tilde{G}, B)) = \circ (\tilde{F}, A) \wedge \circ (\tilde{G}, B)$;

(5) $\circ ((\tilde{F}, A) \vee (\tilde{G}, B)) = \circ (\tilde{F}, A) \vee \circ (\tilde{G}, B)$.

Proof. We can obtain (1) easily according to Definition 17. And we prove the other properties as follows.

(2). Suppose that $(\tilde{F}, A) \,\tilde{\cup}\, (\tilde{G}, B) = (\tilde{H}, C)$, where $C = A \cup B$ and $\forall \varepsilon \in C$,

$$\tilde{H}(\varepsilon) = \begin{cases} \tilde{F}(\varepsilon) = ([s_{a1}, s_{b1}], [s_{c1}, s_{d1}], [s_{e1}, s_{o1}]), \varepsilon \in A - B; \\ \tilde{G}(\varepsilon) = ([s_{a2}, s_{b2}], [s_{c2}, s_{d2}], [s_{e2}, s_{o2}]), \varepsilon \in B - A; \\ \tilde{F}(\varepsilon) \cup \tilde{G}(\varepsilon), \varepsilon \in A \cap B; \end{cases}$$

where $\tilde{F}(\varepsilon) \cup \tilde{G}(\varepsilon) = ([max\{s_{a1}, s_{a2}\}, max\{s_{b1}, s_{b2}\}], [min\{s_{c1}, s_{c2}\}, min\{s_{d1}, s_{d2}\}], [min\{s_{e1}, s_{e2}\}, min\{s_{o1}, s_{o2}\}])$. Then we have $\circ ((\tilde{F}, A) \,\tilde{\cup}\, (\tilde{G}, B)) = \circ (\tilde{H}, C)$. According to Definition 17, we have

$$\circ \tilde{H}(\varepsilon) = \begin{cases} \circ \tilde{F}(\varepsilon) = ([s_{\sqrt{(h)^2 - (o1)^2}}, s_{\sqrt{(h)^2 - (e1)^2}}], [s_0, s_0], [s_{e1}, s_{o1}]), \varepsilon \in A - B; \\ \circ \tilde{G}(\varepsilon) = ([s_{\sqrt{(h)^2 - (o2)^2}}, s_{\sqrt{(h)^2 - (e2)^2}}], [s_0, s_0], [s_{e2}, s_{o2}]), \varepsilon \in B - A; \\ \circ (\tilde{F}(\varepsilon) \cup \tilde{G}(\varepsilon)), \varepsilon \in A \cap B; \end{cases}$$

where $\circ (\tilde{F}(\varepsilon) \cup \tilde{G}(\varepsilon)) = ([max\{s_{\sqrt{(h)^2 - (o1)^2}}, s_{\sqrt{(h)^2 - (o2)^2}}\}, max\{s_{\sqrt{(h)^2 - (e1)^2}}, s_{\sqrt{(h)^2 - (e2)^2}}\}], [s_0, s_0], [min\{s_{e1}, s_{e2}\}, min\{s_{o1}, s_{o2}\}])$.

According to Definitions 12 and 17, suppose $\circ (\tilde{F}, A) \,\tilde{\cup}\, \circ (\tilde{G}, B) = (\tilde{O}, C)$, where $C = A \cup B$ and $\forall \varepsilon \in C$,

$$\tilde{O}(\varepsilon) = \begin{cases} \circ \tilde{F}(\varepsilon) = ([s_{\sqrt{(h)^2 - (o1)^2}}, s_{\sqrt{(h)^2 - (e1)^2}}], [s_0, s_0], [s_{e1}, s_{o1}]), \varepsilon \in A - B; \\ \circ \tilde{G}(\varepsilon) = ([s_{\sqrt{(h)^2 - (o2)^2}}, s_{\sqrt{(h)^2 - (e2)^2}}], [s_0, s_0], [s_{e2}, s_{o2}]), \varepsilon \in B - A; \\ \circ \tilde{F}(\varepsilon) \cup \circ \tilde{G}(\varepsilon), \varepsilon \in A \cap B; \end{cases}$$

where $\circ \tilde{F}(\varepsilon) \cup \circ \tilde{G}(\varepsilon) = ([max\{s_{\sqrt{(h)^2 - (o1)^2}}, s_{\sqrt{(h)^2 - (o2)^2}}\}, max\{s_{\sqrt{(h)^2 - (e1)^2}}, s_{\sqrt{(h)^2 - (e2)^2}}\}], [s_0, s_0], [min\{s_{e1}, s_{e2}\}, min\{s_{o1}, s_{o2}\}])$.

Consequently, $((\tilde{F}, A) \,\tilde{\cup}\, (\tilde{G}, B)) = \circ (\tilde{F}, A) \,\tilde{\cup}\, \circ (\tilde{G}, B)$.

(3). The proof is similar to that of (2).

(4). Suppose that $(\tilde{F}, A) \wedge (\tilde{G}, B) = (\tilde{H}, A \times B)$, where $\forall (\alpha, \beta) \in A \times B$, $\tilde{H}(\alpha, \beta) = ([min\{s_{a1}, s_{a2}\}, min\{s_{b1}, s_{b2}\}], [min\{s_{c1}, s_{c2}\}, min\{s_{d1}, s_{d2}\}], [max\{s_{e1}, s_{e2}\}, max\{s_{o1}, s_{o2}\}])$. Then we have $\circ ((\tilde{F}, A) \wedge (\tilde{G}, B)) = \circ (\tilde{H}, A \times B)$. According to Definition 17, we have

$\circ \tilde{H}(\varepsilon) = ([s_{\sqrt{(h)^2 - (max\{o1, o2\})^2}}, s_{\sqrt{(h)^2 - (max\{e1, e2\})^2}}], [s_0, s_0], [max\{s_{e1}, s_{e2}\}, max\{s_{o1}, s_{o2}\}])$.

According to Definitions 14 and 17, assume that $\circ (\tilde{F}, A) \wedge \circ (\tilde{G}, B) = (\tilde{I}, A \times B)$, where $\forall (\alpha, \beta) \in A \times B$, $\tilde{I}(\alpha, \beta) = \circ \tilde{F}(\alpha) \cap \circ \tilde{G}(\beta) = ([min\{s_{\sqrt{(h)^2 - (o1)^2}}, s_{\sqrt{(h)^2 - (o2)^2}}\}, min\{s_{\sqrt{(h)^2 - (e1)^2}}, s_{\sqrt{(h)^2 - (e2)^2}}\}], [s_0, s_0], [max\{s_{e1}, s_{e2}\}, max\{s_{o1}, s_{o2}\}]) = ([s_{\sqrt{(h)^2 - (max\{o1, o2\})^2}}, s_{\sqrt{(h)^2 - (max\{e1, e2\})^2}}], [s_0, s_0], [max\{s_{e1}, s_{e2}\}, max\{s_{o1}, s_{o2}\}]) = \circ ((\tilde{F}, A) \wedge (\tilde{G}, B))$.

(5). The proof is similar to that of (4). $\square$

Theorem 5. *Let* $(\tilde{F}, A)$ *be a linguistic interval-valued spherical fuzzy soft set over* U, *then we have the following properties.*

(1) $\Box (\tilde{F}, A) \,\tilde{\subseteq}\, (\tilde{F}, A)$;

(2) $\circ \Box (\tilde{F}, A) = \Box (\tilde{F}, A)$;

(3)　　$\square \circ (\tilde{F}, A) = \circ (\tilde{F}, A)$.

Proof. (1). According to Definitions 16 and 17, we have $\square \tilde{F}(\alpha) = ([s_a, s_b],$ $[s_0, s_0], [s_{\sqrt{(h)^2-(b)^2}}, s_{\sqrt{(h)^2-(a)^2}}])$ and $\circ \tilde{F}(\alpha) = ([s_{\sqrt{(h)^2-(o)^2}}, s_{\sqrt{(h)^2-(e)^2}}], [s_0, s_0], [s_e, s_o])$.

　　Since $0 \le b^2 + d^2 + f^2 \le h^2$, then we have $s_{\sqrt{(h)^2-(b)^2}} \ge s_o$. Thus, $s_{\sqrt{(h)^2-(a)^2}} \ge s_{\sqrt{(h)^2-(b)^2}} \ge s_o \ge s_e$. Since $s_a \le s_a, s_b \le s_b, s_0 \le s_c, s_0 \le s_d$, hence $\square (\tilde{F}, A) \cong (\tilde{F}, A)$.

　　(2). Assume that $\forall \alpha \in A$, $\square \tilde{F}(\alpha) = \left([s_a, s_b], [s_0, s_0], [s_{\sqrt{(h)^2-(b)^2}}, s_{\sqrt{(h)^2-(a)^2}}]\right)$. Then

$$\circ \square \tilde{F}(\alpha) = \left(\left[s_{\sqrt{(h)^2-\left(\sqrt{(h)^2-(a)^2}\right)^2}}, s_{\sqrt{(h)^2-\left(\sqrt{(h)^2-(b)^2}\right)^2}} \right], [s_0, s_0], \left[s_{\sqrt{(h)^2-(b)^2}}, s_{\sqrt{(h)^2-(a)^2}} \right] \right)$$

$= \square \tilde{F}(\alpha)$.

　　(3). The proof is similar to that of (2). $\square$

4. Application of Linguistic Interval-Valued Spherical Fuzzy Soft Set

　　Inspired by the work of [23,32], we first define some of the necessary concepts we used. And then we introduce the decision-making and parameter reduction algorithms based on LIVSFSS. We apply the proposed algorithms to examples and we apply the MADM algorithm and parameter reduction algorithm to examples and compare them with some existing algorithms to illustrate their rationality and effectiveness.

4.1. Multi-Attribute Decision Making

Definition 18. *For a linguistic interval-valued spherical fuzzy soft set* $(\tilde{F}, E)$, $U = \{x_1, x_2, \ldots, x_n\}$, $E = \{e_1, e_2, \ldots, e_m\}$, $\bar{S} = \{s_\alpha | s_0 \le s_\alpha \le s_h, \alpha \in [0, h]\}$, $\omega = \{\omega_1, \omega_2, \ldots, \omega_m\}$ *contains the weighted values of every parameter* e_j. $\forall e_j \in E$, $\tilde{F}_{x_i}(e_j) = ([s_{aj}, s_{bj}], [s_{cj}, s_{dj}],$ $[s_{ej}, s_{oj}])$, $i \in [1, n], j \in [1, m]$, *which represents the membership degree of object* x_i *for parameter* e_j. *The linguistic weighted choice value* (ωc_i) *for each object* x_i *is defined as*

$$\omega c_i = [\omega c_i^L, \omega c_i^U]$$
$$= [\sum_{j=1}^m \omega_j \left(\sqrt{\frac{h^2 + (a_j)^2 - (e_j)^2}{3}} \right), \sum_{i=1}^m \omega_j \left(\sqrt{\frac{h^2 + (b_j)^2 - (f_j)^2}{3}} \right)] \tag{19}$$

where ωc_i^L *is the lower linguistic weighted choice value for* x_i *and* ωc_i^U *is the upper linguistic weighted choice value for* x_i.

Definition 19. *For a linguistic interval-valued spherical fuzzy soft set* $(\tilde{F}, E)$, $U = \{x_1, x_2, \ldots, x_n\}$, $E = \{e_1, e_2, \ldots, e_m\}$, $\bar{S} = \{s_\alpha | s_0 \le s_\alpha \le s_h, \alpha \in [0, h]\}$. $\omega c_i = [\omega c_i^L, \omega c_i^U]$ *is the linguistic weighted choice value for* x_i. *The linguistic weighted overall choice value* $\omega c_i^{overall}$ *for* x_i *is defined as*

$$\omega c_i^{overall} = \omega c_i^L + \omega c_i^U \tag{20}$$

　　That is to say, $\omega c_i^{overall}$, as the linguistic weighted overall choice value for x_i is the sum of the lower linguistic weighted choice value and the upper linguistic weighted choice value for x_i.

　　Based on the above given definitions, we describe our proposed MADM Algorithm 1 as follows.

Algorithm 1: MADM algorithm based on LIVSFSS

Step1: Input an LIVSFSS $(\tilde{F}, E)$ and $\omega = \{\omega_1, \omega_2, \ldots, \omega_m\}$, $\sum_{i=1}^m \omega_i = 1$.

Step2: $\forall x_i \in U$ and $\forall e_j \in A$, compute the linguistic weighted choice value (ωc_i) for h_i by Formula (19).

Step3: $\forall x_i \in U$, compute the linguistic weighted overall choice value $\omega c_i^{overall}$ for x_i by Formula (20).

Step4: Obtain k such that $x_k = max_{x_i \in U}\{\omega c_i^{overall}\}$.

Step5: Return $x_k \in U$ as the best choice candidate.
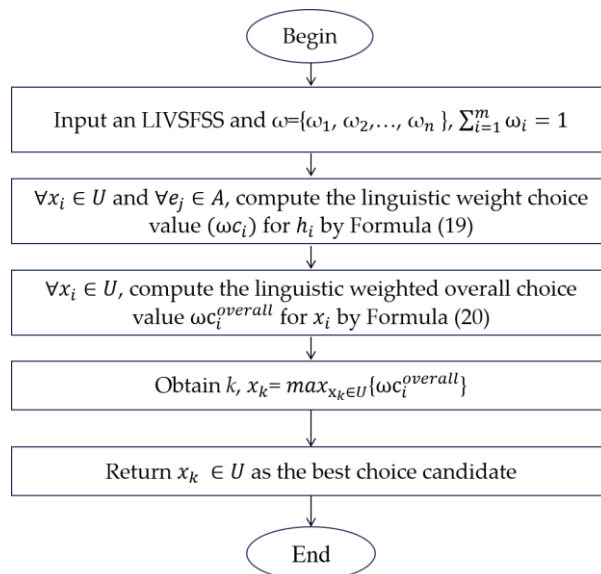
The flow chart of the MADM algorithm is shown in Figure 2.

Figure 2. Flow chart of Algorithm 1.

Example 2. *We use* $(\tilde{F}, E), U = \{h_1, h_2, h_3, h_4\}$, $E = \{e_1, e_2, e_3\}$ *shown in Table 1 and set* $\omega = \{\omega_1, \omega_2, \omega_3\} = \{1/3, 1/3, 1/3\}$ *to demonstrate how Algorithm 1 chooses the best teacher considering three parameters. For each* $x_i \in U$, *the algorithm determines the linguistic weighted choice value using Formula (19), the linguistic weighted overall choice value using Formula (20). It then returns* $x_k = \{x_k \in U | max_{x_k \in U}\{\omega c_i^{overall}\}, 1 \leq k \leq n\}$. *Table 2 shows the results of Algorithm 1 based on the above settings. The algorithm returns* x_1 *as the best teacher.*

Table 2. LIVSFSS $(\tilde{F}, E)$ of Example 2.

U	e_1	e_2	e_3	ωc_i	$\omega c_i^{overall}$
x_1	$([s_3, s_5], [s_0, s_3], [s_2, s_3])$	$[s_5, s_6], [s_0, s_2], [s_1, s_2])$	$([s_5, s_6], [s_0, s_2], [s_1, s_2])$	[4.214,4.562]	8.776
x_2	$([s_3, s_5], [s_0, s_3], [s_2, s_3]$	$([s_4, s_5], [s_0, s_2], [s_2, s_3])$	$([s_2, s_4], [s_0, s_3], [s_3, s_4])$	[3.637,3.931]	7.568
x_3	$([s_5, s_6], [s_0, s_2], [s_1, s_2])$	$([s_3, s_4], [s_1, s_3], [s_2, s_3])$	$([s_3, s_4], [s_1, s_3], [s_2, s_3])$	[3.955,4.111]	8.066
x_4	$([s_4, s_5], [s_0, s_2], [s_2, s_3])$	$([s_4, s_5], [s_0, s_2], [s_2, s_3])$	$([s_4, s_5], [s_0, s_2], [s_1, s_3])$	[4.040,4.164]	8.204

4.2. Parameter Reduction

In the process of decision making, some redundant parameters are not necessary. Thus, the parameter reduction becomes important. According to Algorithm 1, we can obtain a descending queue based on the linguistic weighted overall choice value. In real life, not all decision-making problems require only one choice, sometimes they require several choices. Therefore, it is necessary to consider multiple scenarios in the context of parameter reduction.

Definition 20. *Let* $(\tilde{F}, E)$ *be a linguistic interval-valued spherical fuzzy soft set over* U, $U = \{x_1, x_2, ..., x_n\}$, $E = \{e_1, e_2, ..., e_m\}$, $\bar{S} = \{s_\alpha | s_0 \leq s_\alpha \leq s_h, \alpha \in [0, h]\}$, $D_E = \{x_i, ...x_j | i, j \in [1, n]\}$ *be the set of all objects sorted in descending order by linguistic weighted overall choice values,* $\lambda \in [1, n]$ *be called decision-making number,* D_E^λ *be the set of* λ *objects sorted in descending order by linguistic weighted overall choice values. If a subset* $A=\{e_1', e_2', ..., e_n'\} \subset E$ *satisfies* $D_{E-A}^\lambda = D_E^\lambda$, A *is called the unnecessary parameter set in* E *with* λ *candidates, otherwise* A *is called the necessary parameter set in* E *with* λ *candidates.*

In other words, if A is the unnecessary parameter set with λ candidates, it means that A can be reduced, otherwise A cannot be reduced. Based on the above definition, the parameter reduction algorithm of keeping λ candidates (PRKAC) (Algorithm 2) is proposed as follows.

Algorithm 2: PRAKC
Step1: Input an LIVSFSS$(\tilde{F}, E)$, $\omega = \{\omega_1, \omega_2, \dots, \omega_m\}$, $\sum_{i=1}^{m} \omega_i = 1$, and $\lambda \in [1, \text{n}]$.
Step2: $\forall x_i \in U$ and $\forall e_j \in A$, compute the linguistic weight choice value (ωc_i) for h_i by Formula (19).
Step3: $\forall x_i \in U$, compute the linguistic weighted overall choice value $\omega c_i^{overall}$ for x_i by Formula (20).
Step4: Obtain a descending set D_E^{λ} according to the $\omega c_i^{overall}$.
Step5: Find A, where $A = \{e_1', e_2', \dots, e_n'\} \subset E \wedge D_{E-A}^{\lambda} = D_E^{\lambda}$.
Step6: Return $E - A$ as the parameter reduction with λ candidates.

The flow chart of the PRAKC algorithm is shown in Figure 3.
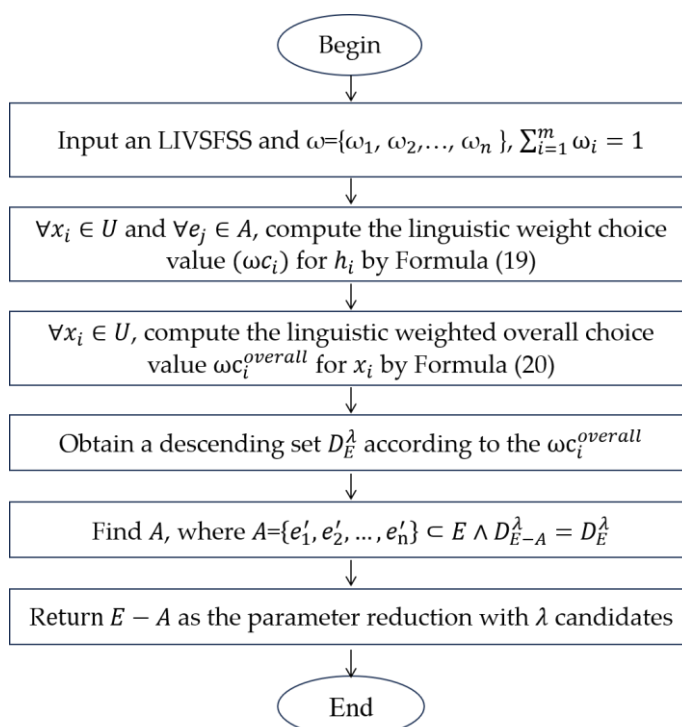
Figure 3. Flow chart of Algorithm 2.

Example 3. *We use* $(\tilde{F}, E)$, $U = \{h_1, h_2, h_3, h_4\}$, $E = \{e_1, e_2, e_3\}$, $\omega = \{\omega_1, \omega_2, \omega_3\} = \{1/3, 1/3, 1/3\}$ *shown in Table 2 and set* $\lambda = 4$ *to demonstrate how PRAKC reduces unnecessary parameters whilst maintaining the four candidates as invariable. We obtain that* x_1, x_4, x_3, x_2 *are the four choice candidates from Example 2. So,* $D_E^4 = \{x_1, x_4, x_3, x_2\}$. *We can find* $A = \{e_1\}$ *satisfies* $D_{E-A}^2 = D_E^2$. *Therefore,* $E - A = \{e_2, e_3\}$ *is the parameter reduction with two candidates, which is given in Table 3.*

Table 3. LIVSFSS $(\tilde{F}, E)$ of Example 3.

U	e_2	e_3	ωc_i	$\omega c_i^{overall}$
x_1	$([s_5, s_6], [s_0, s_2], [s_1, s_2])$	$([s_5, s_6], [s_0, s_2], [s_1, s_2])$	$[2.981, 3.174]$	6.155
x_2	$([s_4, s_5], [s_0, s_2], [s_2, s_3])$	$([s_2, s_4], [s_0, s_3], [s_3, s_4])$	$[2.405, 2.543]$	4.948
x_3	$([s_3, s_4], [s_1, s_3], [s_2, s_3])$	$([s_3, s_4], [s_1, s_3], [s_2, s_3])$	$[2.465, 2.524]$	4.989
x_4	$([s_4, s_5], [s_0, s_2], [s_2, s_3])$	$([s_4, s_5], [s_0, s_2], [s_1, s_3])$	$[2.707, 2.776]$	5.483

4.3. Comparative Analysis

In this subsection, we compare the MADM algorithms presented by Yang et al. [23] and Ma et al. [32]. for an interval-valued fuzzy soft set with our proposed algorithm. In [23], Yang selected the best choice by calculating the score. The idea of our MADM algorithm is similar to [32]. We use Example 1 to demonstrate how these three MADM algorithms choose the best choice. Since each parameter is weighted equally, it does not affect the results. Orderings of the alternatives for each method are given in Table 4.

Table 4. Orderings of the alternatives according to four MADM algorithms.

MADM Algorithms	Orderings of the Alternatives
Yang et al. [23]	$x_1 > x_4 > x_3 > x_2$
Ma et al. [32]	$x_1 > x_4 > x_3 > x_2$
Our proposed algorithm	$x_1 > x_4 > x_3 > x_2$

We see that the proposed algorithm is comparable to other algorithms in terms of the selection of optimal elements, which illustrates the rationality and effectiveness of our proposed algorithm. The main difference between the proposed algorithm and previous algorithms is the use of linguistic interval-valued spherical fuzzy numbers to evaluate parameters. And we also consider the influence of parameter weight. Hence, in real life, we can deal with MADM problems described by linguistic interval-valued spherical fuzzy numbers according to our proposed algorithm.

Since the goal of parameter reduction algorithms based on different fuzzy soft set models is to maintain the decision results unchanged, we compare with the parameter reduction algorithms presented by Ma et al. [31]. Ma proposed the Keeping optimal choice parameter reduction algorithm (KOCPR), the Keeping top three choice parameter reduction algorithm (KTTCPR) and the Standard parameter reduction algorithm (SPR) to reduce parameters. We also compare our proposed PRAKC with the above three algorithms in terms of the number of objects that keep the decision choice result unchanged (KDCRU) and the case of retaining parameters after reducing (RPR) using Example 1. The comparison results are shown in Table 5.

Table 5. Comparison of four parameter reduction algorithms.

Algorithms	KDCRU	RPR
KOCPR	keep the decision result of the top choices	$\{e_2\}, \{e_3\}, \{e_1, e_2\}, \{e_1, e_3\}, \{e_2, e_3\};$
KTTCPR	keep the decision result of the top three choices	$\{e_3\}, \{e_2, e_3\};$
SPR	keep the decision result of all objects choices	$\{e_2, e_3\};$
PRAKC	keep the decision result of the top λ choices, $\lambda \in [1,4]$	$\lambda = 1$, we can obtain $\{e_2\}, \{e_3\}, \{e_1, e_2\}, \{e_1, e_3\}, \{e_2, e_3\};$ $\lambda = 2$, we can obtain $\{e_3\}, \{e_2, e_3\};$ $\lambda = 3$, we can obtain $\{e_3\}, \{e_2, e_3\};$ $\lambda = 4$, we can obtain $\{e_2, e_3\}.$

In real life, not all decision-making problems require a certain number of choices, but sometimes different numbers are required according to different problems. We find that PRAKC can deal with parameter reduction more flexibly than other algorithms by setting a different λ.

5. Conclusions

The soft set theory is a general mathematical tool to deal with uncertainty. In this paper, firstly, we have proposed the concept of the linguistic interval-valued spherical fuzzy soft set. It is a combination of a linguistic interval-valued spherical fuzzy set and a soft set. It optimizes the problem where the linguistic interval-valued spherical fuzzy set is tedious due to the lack of parameterization tools in the decision process. Then, we have defined the basic concepts and discussed various operational laws, related properties and their proofs. In addition, in order to deal with multi-attribute decision-making problems, we have proposed a multi-attribute decision-making algorithm and a parameter reduction algorithm. Finally, the effectiveness and rationality of the algorithms have been verified and illustrated by examples and comparisons with some existing algorithms.

The method proposed in this paper favors theoretical research. In the future, we intend to further explore not only the application of the linguistic interval-valued spherical fuzzy soft set in group decision making, but also its practical applications in combination with machine learning and deep learning such as forecasting and data analysis.

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