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# Aggregating Interrelated Attributes in Multi-Attribute Decision-Making With ELICIT Information Based on Bonferroni Mean and Its Variants

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**ABSTRACT**

In recent times, to improve the interpretability and accuracy of computing with words processes, a rich linguistic representation model has been developed and referred to as Extended Comparative Linguistic Expressions with Symbolic Translation (ELICIT). This model extends the definition of the comparative linguistic expressions into a continuous domain due to the use of the symbolic translation concept related to the 2-tuple linguistic model. The aggregation of ELICIT information via a suitable rule that reflects the underlying interrelation among the aggregated information in output is the key tool to design decision-making algorithm for solving multi-attribute decision-making problems under linguistic information. In this study, we introduce three aggregation operators for aggregating ELICIT information in aim of capturing three different types of interrelationship patterns among inputs, which we refer to as ELICIT Bonferroni mean, ELICIT extended Bonferroni mean and ELICIT partitioned Bonferroni mean. Further, the key aggregation properties of these proposed operators are investigated with the proposal of weighted forms. Based on the proposed aggregation operators, an approach for solving multi-attribute decision-making problems, in which attributes are interrelated is developed. Finally, a didactic example is presented to illustrate the working of the proposal and demonstrate its feasibility.



## 1. INTRODUCTION

With the growing complexity of the socio-economic environment, it is quite common to prevail the uncertainty and vagueness in the decision-making process, in particular, the situations, where human judgments/assessments/perceptions are inevitable to reach a final decision over a set of alternatives [1]. The emergence of such scenarios involving human cognition leads us to use linguistic information based on the fuzzy linguistic approach [2] to effectively manage uncertainty in such decision-making processes. The *fuzzy linguistic approach* uses fuzzy set theory [3] to manage uncertainty and model linguistic information by using linguistic variables described by Zadeh [2] as "*A variable whose values are not numbers but words or sentences in a natural or artificial language.*" A linguistic variable is characterized by a syntactic value or *label* and a semantic value. Whereas the label is a word that belongs to a set of linguistic terms, semantics is provided by a fuzzy set in a discourse universe. Over the years, the fuzzy linguistic approach has been applied successfully in solving many practical multi-attribute decision-making (MADM) problems from the different domains [1,4] and many linguistic computational models have been put forwarded to improve and enhance the information modeling and computation process capability of the Zadeh's approach [2]. They can be broadly classified into two distinct categories: symbolic computational models [5–7] and semantic-based computational models [8]. In terms of simplicity and interpretability, symbolic models stand out semantic models. The symbolic models have evolved enormously over the years. The first proposals [4,9,10] made use of single linguistic variables, for instance, *good, horrible, very bad, perfect*, to provide the decision makers' preferences and carried out the linguistic computations. Among these symbolic models, 2-tuple linguistic computational model [4,5], which enhanced the interpretability of the fuzzy linguistic approach by introducing the concept of symbolic translation, has got wide speared acceptance among the community and successfully applied in solving the MADM problems [11,12]. However, in spite of many of these approaches have been applied successfully in decision-making problems, the modeling of linguistic information is limited when experts provide their preferences by using just single terms. To overcome this drawback, several proposals that obtain richer linguistic expressions than single linguistic terms have been proposed [13]. One of the most outstanding proposals is the so-called *Hesitant Fuzzy Linguistic Term Sets* (HFLTSs) [14], which were introduced to model the hesitancy of the experts when they doubt among several linguistic terms at the same time. HFLTSs are also based on the fuzzy linguistic approach that will serve as bases to increase the flexibility of the elicitation of linguistic information. An example of HFLTS might be $\{good, very\ good, excellent\}$. Furthermore,

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several decision-making proposals have been put forwarded in the literature [15]. Although HFLTS can be directly used by the experts to elicit several linguistic values for a linguistic variable, they are not close to the way of expressing opinions used by human beings. For this reason, Rodríguez *et al.* [14,16] proposed a formalization process to generate linguistic expressions close to the common language used by human beings in decision-making problems. Such expressions, so-called *comparative linguistic expressions* (CLEs), are based on HFLTSs and model the decision maker's hesitancy by means of the use of context-free grammars. An example of CLEs might be *between good and very good, at most bad, at least medium*, etc. Several decision-making models in CLEs environment have been proposed by adopting different computational approaches [17–20]. Up to this point, the CLEs are the closest to the way of thinking of the decision makers but the interpretability of the results in the existing computational approaches and information loss in the linguistic computations are the two key concerns that restrict their use as a decision tool under uncertainty. For this reason, in recent times, Labella *et al.* [21] proposed a new fuzzy linguistic representation for CLEs, which they referred as ***E****xtended Comparative* ***LI****nguisti****C*** *Expressions with Symbol****I****c* ***T****ranslation* (ELICIT) information. This representation takes advantage of the main characteristic of the CLEs, their interpretability, and improves the precision of the results by extending the representation of CLEs generated by a context-free grammar into a continuous domain to perform computing with words (CW) processes without any kind of approximation. In this way, the proposed ELICIT computational model overcomes the drawbacks of the earlier proposals. Some examples of ELICIT information might be *between* $\left(good, 0.23\right)^{0.12}$ *and* $\left(very\,good, 0.1\right)^{0.3}$, *at most* $(bad, 0)^{0}$, *at least* $(medium, -0.1)^{0.11}$, etc.

In the same way that representing information in the decision process is key, the aggregation of such information, which comes from different sources via a suitable rule (aggregation operator), plays also a pivotal role in decision-making process by combining several pieces of information into a single information, which represents overall overview [22]. In the context of MADM, aggregation operators are generally used to find overall performance of the alternatives from their performances against the predefined set of criteria. The need of modeling specific interaction among the attributes and computational formalization with different types of linguistic information to conduct decision-making process under specific linguistic environment were the cornerstone behind the development of several classes of aggregation operators in MADM context.

In this vein, to aggregate interrelated linguistic information represented by 2-tuple linguistic information, several 2-tuple linguistic aggregation operators have been proposed in the literature [4,23–27]. On the other hand, to fuse linguistic information, expressed by HFLTSs, many aggregation operators have been developed considering the nature of the interaction (independent/interrelated) among the aggregated HFLTSs [28–33]. Despite many successful uses of the hesitant fuzzy linguistic computational model in decision-making, it has limitations in modeling complex linguistic expressions by HFLTS [34] and can be overcome with the capability of ELICIT expression. The use of ELICIT information in the decision-making makes it necessary to consider the issue of aggregation of ELICIT information. In this view, Labella *et al.* [21] defined an aggregation operator, which we can refer to as ELICIT arithmetic mean, to aggregate ELICIT expressions in the decision-making process. However, the proposed aggregation operator does not consider the interrelationship among the aggregated ELICIT expressions that are connected with the underlying interrelationship structure of associated concepts/objects, like the attributes' interrelationship and the corresponding ratings. Further, considering the importance/weights of the inputs in the aggregation process is vital to take into account in many decision-making processes and that have not been considered by Labella *et al.* [21]. Therefore, in spite of ELICIT information advantages, there is an evident lack of proposals about ELICIT aggregation operators that consider the interrelation among the ELICIT expressions and their importance in the aggregation process. For this reason, this study aims:

- Develop several aggregation operator to aggregate ELICIT information by capturing different interrelationship patterns (homogeneous, heterogeneous and partitioned structure) among the aggregated arguments.
- Capture the homogeneous relationship among ELICIT expressions by developing the ELICIT Bonferroni mean (ELICITBM) operator.
- Reflect the heterogeneous interaction among the aggregated ELICIT expressions by developing the ELICIT extended Bonferroni mean (ELICITEBM) operator
- Capture the partitioned structured interrelationship among aggregated ELICIT expressions by developing the ELICIT partitioned Bonferroni mean (ELICITPBM) operator.
- Study the proposed aggregation operators properties and weighted form to take into account weight information in the aggregation process.
- Based on the proposed aggregation operators, present an approach for solving MADM problems in which attributes follow the different interrelationship patterns.

To this end, the paper is organized as follows. In Section 2, we provide a brief primer of classical aggregation operator that captures interrelationship of among the aggregated arguments along with fuzzy set theory. A brief overview of the ELICIT representation and computational model is also included in Section 2. In Section 3, we develop three aggregation operators to fuse the ELICIT information according to their underlying interrelationship structures, namely, ELICITBM, ELICITEBM and ELICITPBM. The key properties of these operators are also studied along with the weighted forms: ELICITWBM, weighted ELICITEBM (ELICITWEBM) and WELICITPBM. In Section 4, an aggregation operator-based approach to solving the MADM problems, in which attributes are interrelated with different patterns is proposed. A didactic example is presented in Section 5 to illustrate the working of our approach and feasibility. Finally concluding remarks are made in Section 6.

## 2. PRELIMINARIES

In this section, we overlay the key concepts related to Bonferroni mean (BM), arithmetic operational laws of fuzzy numbers and ELICIT information for easy understanding of our subsequent proposals on aggregation of interrelated ELICIT information and linguistic decision-making process.

## 2.1. Aggregation Operators for Interrelated Information

In this section, we briefly introduce the BM and its variants, which are capable of capturing different kinds of interrelationship patterns among the aggregated information. We start by recalling the definition of the BM operator.

**Definition 1.** [35] Let $p$ and $q \geq 0$, $p+q>0$. For an input vector $\mathbf{a}=(a_1,a_2,...,a_n) \in [0,1]^n$, the BM can be defined as a mapping $BM:[0,1]^n \rightarrow [0,1]$ and given by

$$BM_{p,q}(a_1,a_2,...,a_n)=\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n} a_i^p a_j^q\right)^{\frac{1}{p+q}} \tag{1}$$

Although, BM was introduced by Bonferroni [35] in 1950, it is analyzed and interpreted in decision-making context by Yager [36]. Specifically, BM captures a homogeneous interrelationship pattern among the inputs that every input $a_i \in \mathbf{a}$ is related to the rest of the inputs of $\mathbf{a}$. But in many real-life contexts, such homogeneous connections among the inputs may not exist rather the inputs are related to each other in a heterogeneously related fashion. To capture such heterogeneous connections among the inputs, Dutta *et al.* [25] developed a new aggregation operator, which is referred to as extended Bonferroni mean (EBM). Based on heterogeneous connection among the inputs, they classified inputs $\mathbf{a}$ into two categories $U$ and $V$, where every input of $U$ is related to a subset of the rest of the inputs, i.e., $E_i \subset \mathbf{a} \setminus \{a_i\}$ and the inputs of $V$ are not related to each other. Having this interpretation of the heterogeneous interrelationship pattern, the rule for the EBM aggregation operator is given by

**Definition 2.** [25] For any $p>0$ and $q \geq 0$, the EBM operator of dimension $n$ is a mapping $EBM:[0,1]^n \rightarrow [0,1]$ such that

$$EBM_{p,q}(a_1,a_2,\cdots,a_3) \tag{2}$$
$$=\left(\frac{n-|I'|)}{n}\left(\frac{1}{n-|I'|)}\sum_{i\in I'} a_i^p\left(\frac{1}{|I_i|}\sum_{j\in I_i} a_j^p\right)\right)^{\frac{p}{p+q}} +\frac{|I'|}{n}\left(\frac{1}{|I'|}\sum_{i\in I'} a_i^p\right)\right)^{\frac{1}{p}}$$

where $I_i$ is the set of indices of the elements of $E_i$, $I'$ is the collection indices of the inputs of $V$, $|I'|$ denotes the cardinality of the set $I'$ and empty sum is zero by convention with $\frac{0}{0}=0$.

Partitioned Bonferroni mean (PBM) is another variant of BM, which is capable of capturing partition structure interrelationship pattern among the input set in the aggregation process and reflects it in the aggregated value [24]. In the following, we provide a brief description of the specific partition structure interrelationship pattern and PBM operator.

Let $\mathbf{a}=(a_1,a_2,...,a_n)$ be the collection of inputs, with $a_i$'s being non-negative real numbers. Suppose, on the basis of the interrelationship pattern, the input set $\mathbf{a}$ is partitioned into $d$ distinct classes $P_1,P_2,...,P_d$ such that $P_i \cap P_j = \phi$ for all $i \neq j$, $i,j \in \{1,2,...,d\}$, $\cup_{r=1}^{d} P_r = \mathbf{a}$ and$|P_i| \geq 2$ for all $i=1,2,...,d$. We further assume that the inputs of each $P_i$ are interrelated and there is no interrelationship among the inputs of any two partitions $P_i$ and $P_j$ whenever $i,j \in \{1,2,...,d\}$ and $i \neq j$. With these assumptions and notations, the PBM operator of the collection of inputs $(a_1,a_2,...,a_n)$ is defined as follows:

**Definition 3.** [24] For $p,q \geq 0$ with $p+q>0$, the PBM operator is a mapping $PBM:[0,1]^n \rightarrow [0,1]$ such that

$$PBM(a_1,a_2,...,a_n) = \frac{1}{d}\left(\sum_{r=1}^{d}\left(\frac{1}{|P_r|}\sum_{i\in P_r} a_i^p\left(\frac{1}{|P_r|-1}\sum_{\substack{j\neq i\\ j\in P_r}} a_j^q\right)\right)^{\frac{1}{p+q}}\right) \tag{3}$$

where $|P_r|$ denotes cardinality of $P_r$.

It is evident from the Definitions 1 and 3 that BM is a special case PBM when all the inputs belong to same class [24]. To establish more concrete link between BM and PBM, we can write Eq. (3) as follows:

$$PBM_{p,q}(a_1,a_2,...,a_n) = \frac{1}{d}\left(\sum_{r=1}^{d}\left(\frac{1}{|P_r|(|P_r|-1)}\sum_{\substack{i,j\in P_r\\ i\neq j}} a_i^p a_j^q\right)^{\frac{1}{p+q}}\right) = \frac{1}{d}\sum_{r=1}^{d} BM_r(a_i \in P_r) \tag{4}$$

where,

$$BM_r(a_i \in P_r) = \left(\left(\frac{1}{|P_r|(|P_r|-1)}\sum_{\substack{i,j\in P_r\\ i\neq j}} a_i^p a_j^q\right)\right)^{\frac{1}{p+q}}$$

and $(a_i \in P_r)$ denotes the set of inputs belongs to the partition $P_r$. With the help of Eq. (4), we can interpret PBM as arithmetic average of BM over different partition of the given input set. Therefore, one can compute the aggregated value of an input set by PBM via computing BM over different partitions.

## 2.2. Arithmetic Operations of Fuzzy Numbers

In this section, key concepts associated with the fuzzy numbers and their operational laws are briefly described. We start by recalling the definition of a fuzzy set, which is well known to model the concept that does not possess the sharp boundaries. Throughout this article, we will restrict ourselves to the class of fuzzy sets over the universe of discourse $X$ which is a subset of the set of real numbers $\mathbb{R}$.

**Definition 4.** [3] A fuzzy set $\tilde{A}$ over the universe of discourse $X$ is characterized by a membership function, which associates every element of $x \in X$ to a real number from the interval $[0, 1]$ and denoted as

$$\mu_{\tilde{A}} : X \to [0, 1] \tag{5}$$

A fuzzy set $\tilde{A}$ can also be defined with help of ordered pairs of generic element $x \in X$ and the corresponding membership degree $(\mu_{\tilde{A}}(x))$ and represented as

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \,|x \in X\} \tag{6}$$

**Definition 5.** [3] The support of the fuzzy set $\tilde{A}$ over the universe of discourse $X$ is the set of all elements $x \in X$, such that, the membership degree is greater than 0, i.e.,

$$Supp\left(\tilde{A}\right) = \{x \in X | \mu_{\tilde{A}}(x) > 0\}. \tag{7}$$

**Definition 6.** [37] A fuzzy set $\tilde{A}$ is said to be normal if there exists a $x_0 \in X$ such that $\mu_{\tilde{A}}(x_0) = 1$.

**Definition 7.** [37] A fuzzy set $A$ over a convex universe of discourse $X$ is said to be convex if

$$\mu_A\left(\lambda x + (1 - \lambda) y\right) \geq \min\{\mu_A(x), \mu_A\left(y\right)\},$$

for all $x, y \in supp(A)$ and $\lambda \in [0, 1]$.

**Definition 8.** [37] A fuzzy number $\tilde{A}$ over the universe of discourse $X \subset \mathbb{R}$ is a special fuzzy set, which is convex and normal.

As a fuzzy set is completely characterized by its membership function, we can say the membership functions are synonyms of the fuzzy sets. Although any function $f : X \to [0, 1]$ can serve as a membership function, in practice trapezoidal and triangular membership functions are widely used to quantify the fuzzy meaning of the linguistic terms used by the decision maker to express their opinions in natural language.

**Definition 9.** A trapezoidal fuzzy number (TrFN) $\tilde{A} = (a, b, c, d)$ with four parameters $a, b, c, d\,(a \leq b \leq c \leq d)$ is a fuzzy subset of the real line $\mathbb{R}$ and described by its membership function $\mu_{\tilde{A}}$ as follows:

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{x-a}{b-a} & \text{if} \quad a \leq x < b \\ 1 & \text{if} \quad b < x \leq c \\ \dfrac{d-x}{d-c} & \text{if} \quad c < x \leq d \\ 0, & \text{otherwise} \end{cases} \tag{8}$$

**Definition 10.** A triangular fuzzy number (TFN) $\tilde{A} = (a, b, c)$ with three parameters $a, b, c\,(a \leq b \leq c)$ is a fuzzy subset of the real line $\mathbb{R}$ and described by its membership function $\mu_{\tilde{A}}$ as follows:

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{x-a}{b-a} & \text{if} \quad a \leq x < b \\ 1 & \text{if} \quad x = b \\ \dfrac{c-x}{c-b} & \text{if} \quad b < x \leq c \\ 0, & \text{otherwise} \end{cases} \tag{9}$$

The obvious motivations behind the use of trapezoidal and TFNs come from the simplicity of the membership functions and their characterization requires reasonably limited information about the linguistic term [38,39]. For example, when a triangular $\tilde{A} = (a, b, c)$ is used to quantify a linguistic term, the triplet $(a, b, c)$ represents the lower, most likely and upper values of that linguistic term with varied membership degree, described via membership function $\mu_{\tilde{A}}(x)$.

The fuzzy arithmetic operational laws allow us to facilitate the computation over linguistic information. There are several ways to derive the arithmetic operational laws of the fuzzy numbers based on the Zadeh's *extension principle* [37]. As in the ELICIT computational model [21] the meaning of the primary linguistic term sets are represented by using TFNs or TrFNs, we restrict ourselves on fuzzy arithmetic operational laws, which preserve the shape of the original fuzzy numbers. In this view, we adopt Chen's *function principle* based arithmetic operational laws, which is given as follows [40]:

**Definition 11.** Let $\tilde{A} = (a_1, b_1, c_1, d_1)$ and $\tilde{B} = (a_2, b_2, c_2, d_3)$ be the two positive TrFNs. Following Chen's function Then arithmetic operations between $\tilde{A}$ and $\tilde{B}$ can be defined as follows:

- Addition: $\tilde{A} \oplus \tilde{B} = (a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2)$
- Multiplication: $\tilde{A} \otimes \tilde{B} = (a_1 a_2, b_1 b_2, c_1 c_2, d_1 d_2)$
- Scalar multiplication: $r\tilde{A} = (ra_1, rb_1, rc_1)$, $r > 0$
- Exponent: $\tilde{A}^r = \left(a_1^r, b_1^r, c_1^r\right)$, $r > 0$.

Note that the *function principle* based arithmetic laws differ from extension principle-based arithmetic laws in multiplication operation as the former approximate resultant fuzzy number shape. Further, one may observe that with the increment of the number of aggregated fuzzy numbers in the aggregation process, the difference between *function principle* based aggregation and *extension principle* based aggregation results diminishes.

## 2.3. ELICIT Information

Despite the evolution of the symbolic approaches over the time [4,14,16], there exists several drawbacks in terms of interpretability and/or accuracy. ELICIT information allows us to keep the interpretability and precision of the results in MADM problems under linguistic environments thanks to the extension of CLEs into a continuous domain. To carry out such extension, the ELICIT expressions are generated by means of a context-free grammar by using the symbolic translation concept used by the 2-tuple linguistic model.

**Definition 12.** [21] Let $G_H$ be a context-free grammar and $S = \{s_0, \ldots, s_g\}$ a linguistic term set. The elements of $G_H = (V_N, V_T, I, P)$ are defined as follows.

$$V_N = \{(\textit{continuous primary term}), (\textit{composite term}), (\textit{unary relation}), (\textit{binary relation}), (\textit{conjunction})\}$$
$$V_T = \{\textit{at least}, \textit{at most}, \textit{between}, \textit{and}, (s_0, \alpha)^\gamma, \ldots, (s_g, \alpha)^\gamma\}$$
$$I \in V_N$$

The production rules defined in an extended Backus–Naur Form are:

$$\begin{aligned}
&P = \{I ::= (\textit{continuous primary term}) \mid (\textit{composite term})\\
&(\textit{composite term}) ::= (\textit{unary relation})\\
&(\textit{continuous primary term}) \mid (\textit{binary relation})\\
&(\textit{continuous primary term})\ (\textit{conjunction})\\
&(\textit{continuous primary term})\\
&(\textit{continuous primary term}) ::= (s_0, \alpha)^\gamma \mid (s_1, \alpha)^\gamma \mid \ldots \mid (s_g, \alpha)^\gamma\\
&(\textit{unary relation}) ::= \textit{at least} \mid \textit{at most}\\
&(\textit{binary relation}) ::= \textit{between}\\
&(\textit{conjunction}) ::= \textit{and}\}
\end{aligned}$$

Therefore, the possible ELICIT expressions generated according to the previous context-free grammar are: "*at least* $(s_i, \alpha)^\gamma$", "*at most* $(s_i, \alpha)^\gamma$" and "*between* $(s_i, \alpha_1)^{\gamma_1}$ *and* $(s_j, \alpha_2)^{\gamma_2}$" (see Figure 1).

To obtain linguistic results represented by ELICIT information in decision-making processes, a novel approach was introduced in [21]. This approach starts from linguistic preferences provided by the experts modeled by CLEs and/or ELICIT information. Afterward, CLEs and ELICIT information are transformed into TrFNs. Whereas the CLEs are transformed into TrFNs through the computation of their fuzzy envelope [18], the transformation of the ELICIT information into TrFNs is carried by means an *inverse function*.

**Definition 13.** [21] Let $EL_1$ be an ELICIT expression and $T(a, b, c, d)$ a TrFN. The function $\zeta^{-1}$ is defined as:

$$\zeta^{-1}: EL_1 \rightarrow T(a, b, c, d) \tag{10}$$

Such that, from an ELICIT expression, it returns its equivalent TrFN.

In this point, the *adjustment*, $\gamma$, of the ELICIT expression plays a key role. The *adjustment* is an additional parameter included in the ELICIT expression, which will be used to obtain the respective fuzzy number from an ELICIT expression by using its inverse function, $\zeta^{-1}$, preserving as much information as possible in the fuzzy representation and facilitating accurate computations. Depending on the ELICIT expression, the $\zeta^{-1}$ function is defined in different ways.

A. *At least expression*: The function $\zeta^{-1}$ for an ELICIT expression whose relation is *at least* is defined as follows:

**Definition 14.** [21] Let *at least* $(s_i, \alpha)^\gamma$ be an ELICIT expression and $T_{ELICIT}(a', b', 1, 1)$ the fuzzy envelope of such ELICIT expression. There is a function $\zeta^{-1}$:

$$\zeta^{-1}\left(\textit{at least } (s_i, \alpha)^\gamma\right) = T(a, b, 1, 1)$$
$$a = a' + \gamma$$
$$b = b'$$

B. *At most expression*: The function $\zeta^{-1}$ for an ELICIT expression whose relation is *at most* is defined as follows:

**Definition 15.** [21] Let *at most* $(s_i, \alpha)^\gamma$ be an ELICIT expression and $T_{ELICIT}(0, 0, c', d')$ the fuzzy envelope of such ELICIT expression. There is a function $\zeta^{-1}$:

$$\zeta^{-1}\left(\textit{at most } (s_i, \alpha)^\gamma\right) = T(0, 0, c, d)$$
$$c = c'$$
$$d = d' + \gamma$$

C. *Between expression*: The function $\zeta^{-1}$ for an ELICIT expression whose relation is *between* is defined as follows:

**Definition 16.** [21] Let *between* $(s_i, \alpha_1)^{\gamma_1}$ and $(s_j, \alpha_2)^{\gamma_2}$ be an ELICIT expression and $T_{ELICIT}(a', b', c', d')$ the fuzzy envelope of such ELICIT expression. There is a function $\zeta^{-1}$:

$$\zeta^{-1}\left(\textit{between } (s_i, \alpha_1) \text{ and } (s_j, \alpha_2)\right) = T(a, b, c, d)$$
$$a = a' + \gamma_1$$
$$b = b'$$
$$c = c'$$
$$d = d' + \gamma_2$$

## Remark 1.

Appendix A.1 has been included in order to show the performance of $\zeta^{-1}$ through a practical example.

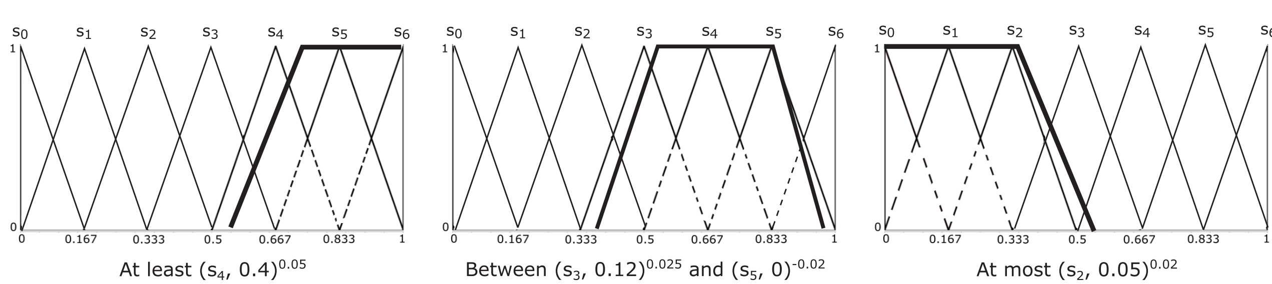


**Figure 1** ELICIT information examples.

Once the TrFNs are obtained, they are manipulated and aggregated by means of fuzzy operations that keep the fuzzy parametric representation of such TrFNs [41]. Finally, the resulting TrFNs, noted as $\beta$, are retranslated into ELICIT information. This process consists of several steps, which are briefly described below:

1. *Identify relation*: The relation of the ELICIT expression is determined by the fuzzy number $\tilde{\beta}$ and the $\zeta$ function, defined in [21] as follows:

   **Definition 17.** Let $S = \{s_0, \dots, s_g\}$ be a set of linguistic terms and $\tilde{\beta}$ a fuzzy number. The function $\zeta$ is given by Eq. (11) as shown in the beginning of the next page.

   For sake of space, it is assumed that the ELICIT expression is composed by a "*between*" relation (see [21] for further detail about the construction of other ELICIT expressions).

$$\zeta\left(\tilde{\beta}\right) = EL,\ where \begin{cases} EL = at\ least\ (s_i, \alpha)^{\gamma}\ if \tilde{\beta} = T(a, b, 1, 1) \\ EL = at\ most\ (s_i, \alpha)^{\gamma}\ if \tilde{\beta} = T(0, 0, c, d) \\ EL = between\ (s_i, \alpha_1)^{\gamma_1}\ and\ \left(s_j, \alpha_2\right)^{\gamma_2} \\ if \tilde{\beta} = T(a, b, c, d) \end{cases} \tag{11}$$

2. *2-tuple linguistic terms computation*: The ELICIT expression with the relation "*between*" is composed by two continuous primary terms $(s_i, \alpha_1)^{\gamma_1}$ and $\left(s_j, \alpha_2\right)^{\gamma_2}$. The process of obtaining such terms is divided into different steps:
   (a) *Compute linguistic terms*: To select the linguistic terms $s_i$ and $s_j \in S, i, j \in \{0, \dots g\}$, whose distance between the coordinates $x$ of their respective centroids [42], $\overline{x}_i$ and $\overline{x}_j$, and the points $b$ and $c$ belonging to $\tilde{\beta}$ is minimal.

$$i = \arg\min_h \left|b - \overline{x}_h\right|,\ \ h \in \{0, \dots, g\} \tag{12}$$
$$j = \arg\min_h \left|c - \overline{x}_h\right|,\ \ h \in \{0, \dots, g\}$$

   The ELICIT expression so far is "*between* $(s_i, ?)^?$ *and* $\left(s_j, ?\right)^?$".

   (b) *Compute symbolic translations*: According to [4,43], $1/2g$ represents the distance equivalent to a symbolic translation equal to 0.5 in $S$, where $g + 1$ is the cardinality of $S$:

$$\begin{aligned} \alpha_1 &= g \cdot \left(b - \overline{x}_i\right) \quad \alpha_1 \in [-0.5, 0.5) \\ \alpha_2 &= g \cdot \left(c - \overline{x}_j\right) \quad \alpha_2 \in [-0.5, 0.5) \end{aligned} \tag{13}$$

   The ELICIT expression so far is "*between* $(s_i, \alpha_1)^?$ *and* $\left(s_j, \alpha_2\right)^?$".

3. *Compute adjustments*: The steps to compute the adjustments for the ELICIT expression are:
   (a) *Compute HFLTS*: The HFLTS of an ELICIT expression whose relation is *between* would be composed by:

$$E_{ELICIT}\left(between\ (s_i, \alpha)\ \text{and}\ \left(s_j, \alpha\right)\right) = \left\{s_k \mid (s_i, \alpha)\ \text{and}\ \left(s_j, \alpha\right)\ \text{and}\ s_i < s_k < s_j\ \text{where}\ s_k \in S\right\}$$

   (b) *Compute fuzzy envelope*: The fuzzy envelope [18] of the computed HFLTS is computed and noted as $T_{ELICIT} = T(a', b', c', d')$.

   (c) *Compute adjustments $\gamma_1$ and $\gamma_2$*: The adjustments $\gamma_1$ and $\gamma_2$ are determined by the subtraction between the points $a$ and $d$ of $\tilde{\beta} = T(a, b, c, d)$ and the points $a'$ and $d'$ of $T_{ELICIT}(a', b', c', d')$, so that:

$$\begin{aligned} \gamma_1 &= a - a' \quad \gamma_1 \in [0, 1] \\ \gamma_2 &= d - d' \quad \gamma_2 \in [0, 1] \end{aligned} \tag{14}$$

   Finally, the ELICIT expression is completed "*between* $(s_i, \alpha_1)^{\gamma_1}$ *and* $\left(s_j, \alpha_2\right)^{\gamma_2}$".

## Remark 2.

Appendix B.1 has been included in order to show the retranslation process through a practical example.

## 3. AGGREGATION OF INTERRELATED ELICIT EXPRESSIONS

The fusion of linguistic information that is represented by CLEs and/or ELICIT expressions according to underlying interrelationship structure of the information is essential to design a variety of linguistic decision-making processes. In this section, we extend the classical interrelated aggregation operators described in the previous section to aggregate the ELICIT expressions with certain underlying interrelationship pattern. From now onward, we are going to use $\mathcal{F}$ to denote the set of all possible ELICIT expressions over a linguistic term set $S$.

### 3.1. ELICIT Bonferroni Operators

Based on the Definition 1, the homogeneously interrelated ELICIT expressions can be aggregated as follows:

**Definition 18.** Let $\mathbf{EL} = (EL_1, EL_2, \dots, EL_n)$ be the collection of $n$ ELICIT expressions from $\mathcal{F}$. For any $p, q \geq 0$ with $p + q > 0$, the ELICITBM operator is a mapping $ELICITBM: \mathcal{F}^n \rightarrow \mathcal{F}$ and defined as follows:

$$ELICITBM_{p,q}(EL_1, EL_2, \dots, EL_n) = \zeta\left(\frac{1}{n(n-1)} \bigoplus_{\substack{i,j=1 \\ i \neq j}} \left(\zeta^{-1}(EL_i)\right)^p \otimes \left(\zeta^{-1}\left(EL_j\right)\right)^q\right)^{\frac{1}{p+q}} \tag{15}$$

where $\oplus$ represents the addition of fuzzy numbers and $\otimes$ denotes the multiplication of fuzzy numbers.

Based on the arithmetic operational laws of fuzzy numbers, we illustrate the computational formula of ELICITBM in the following theorem:

**Theorem 1.** *Let* $\mathbf{EL} = (EL_1, EL_2, \dots, EL_n)$ *be the collection of* $n$ *ELICIT expressions from* $\mathcal{F}$*. For any* $p, q \geq 0$ *with* $p + q > 0$*, the* *aggregated value of ELICIT expressions by ELICITBM is a ELICIT expression and given by*

$$ELICITBM_{p,q}(EL_1, EL_2, ..., EL_n) = \zeta\left(\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ j\neq i}}^{n} a_i^p a_j^q\right)^{\frac{1}{p+q}}, \left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ j\neq i}}^{n} b_i^p b_j^q\right)^{\frac{1}{p+q}}, \left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ j\neq i}}^{n} c_i^p c_j^q\right)^{\frac{1}{p+q}}, \left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ j\neq i}}^{n} d_i^p d_j^q\right)^{\frac{1}{p+q}}\right) \tag{16}$$

*where* $\zeta^{-1}(EL_i) = (a_i, b_i, c_i, d_i)$ *is the equivalent fuzzy number of the ELICIT expression* $EL_i$ *for all* $i = 1, 2, ..., n$.

**Proof.** Please see Appendix C.1

## Remark 3.

With the notation of the BM operator, the computational formula for ELICITBM (Eq. 16) can be rewritten as follows:

$$\begin{aligned} &ELICITBM_{p,q}(EL_1, EL_2, ..., EL_n) \\ &= \zeta\left(BM_{p,q}(a_1, a_2, ..., a_n), BM_{p,q}(b_1, b_2, ..., b_n),\right. \\ &\quad \left. BM_{p,q}(c_1, c_2, ..., c_n), BM_{p,q}(d_1, d_2, ..., d_n)\right) \end{aligned} \tag{17}$$

## Example 1.

Let us consider the aggregation of homogeneously interrelated ELICIT information: $EL_1 = at\,least\,(s_4, 0)^0$, $EL_2 = at\,least\,(s_5, 0)^0$, $EL_3 = at\,most\,(s_3, 0)^0$ $EL_4 = between\,(s_3, 0)^0\ and\ (s_4, 0)^0$. To capture the homogeneous interrelation pattern in the aggregation process, we are going to employ ELICITBM operator with parameters $p = q = 1$. As per Theorem 1, we first obtain the fuzzy numbers corresponding to the given ELICIT by utilizing Definitions 14–16 with the semantics of linguistic terms defined in Figure 1 as follows: $\zeta^{-1}(EL_1) = (0.5, 0.86, 1, 1)$, $\zeta^{-1}(EL_2) = (0.67, 0.98, 1, 1)$, $\zeta^{-1}(EL_3) = (0, 0, 0.36, 0.67)$, $\zeta^{-1}(EL_4) = (0.34, 0.5, 0.67, 0.84)$). With the help of Eq. (17), we obtain

$$\begin{aligned} &ELICITBM_{1,1}(EL_1, EL_2, EL_3, EL_4) \\ &= \zeta\left(BM_{1,1}(0.5, 0.67, 1, 0.34), BM_{1,1}(0.86, 0.98, 1, 0.5),\right. \\ &\quad \left. BM_{1,1}(1, 1, 0.36, 0.67), BM_{1,1}(1, 1, 0.67, 0.84)\right) \end{aligned}$$

From Eq. (1), we have

$$\begin{aligned} &ELICITBM_{1,1}(EL_1, EL_2, EL_3, EL_4) \\ &= \zeta(0.35, 0.54, 0.74, 0.87) \end{aligned}$$

By utilizing Eq. (11) with the retranslation steps of ELICIT information, we obtain

$$\begin{aligned} &ELICITBM_{1,1}(EL_1, EL_2, EL_3, EL_4) \\ &= \text{between } (s_3, -0.28)^{0.02} \text{ and } (s_4, 0.42)^{0.04}. \end{aligned}$$

**Theorem 2.** *The ELICIT expressions aggregation operator ELICITBM satisfies the following properties:*

- *ELICITBM* $: \mathcal{F}^n \to \mathcal{F}$ is commutative, i.e.,

  $$\begin{aligned} &ELICITBM_{p,q}(EL_1, EL_2, ..., EL_n) \\ &= ELICITBM_{p,q}\left(EL_{\sigma(1)}, EL_{\sigma(2)}, ..., EL_{\sigma(n)}\right) \end{aligned}$$

  where $EL_{\sigma(1)}, EL_{\sigma(2)}, ..., EL_{\sigma(n)}$ is a permutation of the ELICIT expressions $EL_1, EL_2, ..., EL_n$.
- *ELICITBM* $: \mathcal{F}^n \to \mathcal{F}$ is idempotent, i.e.,

  $$ELICITBM_{p,q}(EL, EL, ..., EL) = EL$$
- *ELICITBM* $: \mathcal{F}^n \to \mathcal{F}$ is ratio-scale invariant, i.e. for any real number $r > 0$

  $$\begin{aligned} &ELICITBM_{p,q}(rEL_1, rEL_2, ..., rEL_n) \\ &= rELICITBM_{p,q}(EL_1, EL_2, ..., EL_n). \end{aligned}$$

**Proof.** Please see Appendix C.2.

**Theorem 3.** *Let* $\mathbf{EL} = (EL_1, EL_2, ..., EL_n)$ *be the collection of ELICIT expressions and* $\zeta^{-1}(EL_i) = (a_i, b_i, c_i, d_i)$ $(i = 1, 2, ..., n)$ *be the equivalent fuzzy numbers of the ELICIT expression* $EL_i$ $(i = 1, 2, ..., n)$. *Then the operator* $ELICITBM: \mathcal{F}^n \to \mathcal{F}$ *is bounded, i.e.*

$$\begin{aligned} &\zeta\left(\min_i a_i, \min_i b_i, \min_i c_i, \min_i d_i\right) \\ &\leq ELICITBM_{p,q}(EL_1, EL_2, ..., EL_n) \\ &\leq \zeta\left(\max_i a_i, \max_i b_i, \max_i c_i, \max_i d_i\right). \end{aligned}$$

**Proof.** Please see Appendix C.3.

In the above, we have not considered the weight of the aggregated ELICIT expressions. But, in many practical applications, we need to consider the weight of input arguments in the aggregation process. In this view, we define the weighted form of ELICITBM as follows:

**Definition 19.** Let $\mathbf{EL} = (EL_1, EL_2, ..., EL_n)$ be the collection of $n$ ELICIT expressions from $\mathcal{F}$. For any $p, q \geq 0$ with $p + q > 0$, the ELICITBM operator is a mapping $ELICITBM: \mathcal{F}^n \to \mathcal{F}$ and defined as follows:

$$ELICITWBM_{p,q}(EL_1, EL_2, ..., EL_n) = \zeta\left(\frac{1}{n(n-1)} \bigoplus_{\substack{i,j=1\\ i\neq j}} \left(w_i\left(\zeta^{-1}(EL_i)\right)^p\right) \otimes \left(\frac{w_j}{1-w_i}\left(\zeta^{-1}(EL_j)\right)^q\right)\right)^{\frac{1}{p+q}} \tag{18}$$

where $(w_1, w_2, ..., w_n)$ be the weights of the input ELICIT expressions and $w_i > 0$ $(i = 1, 2, ..., n)$ with $\sum_{i=1}^{m} w_i = 1$.

With the operational laws of the fuzzy numbers, we derive the computational formula of the *ELICITWBM* as follows:

**Theorem 4.** *Let* $\mathbf{EL} = (EL_1, EL_2, ..., EL_n)$ *be the collection of n ELICIT expressions from* $\mathcal{F}$. *For any* $p, q \geq 0$ *with* $p + q > 0$, *the aggregated value of ELICIT expressions by ELICITBM is a ELICIT expression and given by*

$$\begin{aligned} &ELICITBM_{p,q}(EL_1, EL_2, ..., EL_n) \\ &= \zeta\left(\left(\sum_{\substack{i,j=1\\ j\neq i}}^{n} \frac{w_i w_j}{1-w_i} a_i^p a_j^q\right)^{\frac{1}{p+q}}, \left(\sum_{\substack{i,j=1\\ j\neq i}}^{n} \frac{w_i w_j}{1-w_i} b_i^p b_j^q\right)^{\frac{1}{p+q}},\right. \\ &\left.\left(\sum_{\substack{i,j=1\\ j\neq i}}^{n} \frac{w_i w_j}{1-w_i} c_i^p c_j^q\right)^{\frac{1}{p+q}}, \left(\sum_{\substack{i,j=1\\ j\neq i}}^{n} \frac{w_i w_j}{1-w_i} d_i^p d_j^q\right)^{\frac{1}{p+q}}\right) \end{aligned} \tag{19}$$

*where,* $\zeta^{-1}(EL_i) = (a_i, b_i, c_i, d_i)$ *is the equivalent fuzzy number of the ELICIT expression* $EL_i$ *for all* $i = 1, 2, ..., n$ *and* $(w_1, w_2, ..., w_n)$ *is the weight vector of the inputs and* $w_i > 0$ $(i = 1, 2, ..., n)$ *with* $\sum_{i=1}^{m} w_i = 1$.

**Proof.** It follows in the lines of Theorem 1.

## 3.2. ELICIT Extended Bonferroni Mean

This section focuses on aggregating ELICIT expressions that are heterogeneously interrelated in the fashion described in Section 2 and define ELICITEBM operator as follows:

**Definition 20.** Let $\mathbf{EL} = (EL_1, EL_2, ..., EL_n)$ be the collection of $n$ ELICIT expressions from $\mathcal{F}$ such that the input set $\mathbf{EL}$ is heterogeneously interrelated (as described in Section 2). For any $p, q \geq 0$ with $p + q > 0$, the ELICITEBM operator is a mapping *ELICITPBM* $: \mathcal{F}^n \rightarrow \mathcal{F}$ and defined as follows:

$$\begin{aligned} &ELICITEBM_{p,q}(EL_1, EL_2, ..., EL_n) \\ &= \zeta\left(\frac{n-|I'|}{n}\left(\frac{1}{n-|I'|} \bigoplus_{i\notin I'} \left(\zeta^{-1}(EL_i)\right)^p \otimes \left(\frac{1}{|I_i|} \bigoplus_{j\in I_i}\right.\right.\right. \\ &\left.\left.\left.\left(\zeta^{-1}(EL_j)\right)^q\right)\right)^{\frac{p}{p+q}} \oplus \frac{|I'|}{n}\left(\frac{1}{|I'|} \bigoplus_{i\in I'} \left(\zeta^{-1}(EL_i)\right)^p\right)\right)^{\frac{1}{p}} \end{aligned} \tag{20}$$

where empty sum of fuzzy numbers ($\oplus$) is set as fuzzy zero (with TrFN representation $(0, 0, 0, 0)$) in the lines of convention of classic crisp system with $(0, 0, 0, 0)/0 = (0, 0, 0, 0)$.

For the computational purpose, we derive the explicit mathematical formulae based on the arithmetic operational laws of TrFNs and ELICIT computational model as follows:

**Theorem 5.** *Let* $\mathbf{EL} = (EL_1, EL_2, ..., EL_n)$ *be the collection of n ELICIT expressions from* $\mathcal{F}$, *which are heterogeneously interrelated. For any* $p, q \geq 0$ *with* $p + q > 0$, *the aggregated value of ELICIT expressions is a ELICIT expression and given by*

$$\begin{aligned} &ELICITEBM_{p,q}(EL_1, EL_2, ..., EL_n) \\ &= \zeta\left(EBM(a_1, a_2, ..., a_n), EBM(b_1, b_2, ..., b_n),\right. \\ &\quad\left. EBM(c_1, c_2, ..., c_n), EBM(d_1, d_2, ..., d_n)\right) \end{aligned} \tag{21}$$

*wher,* $\zeta^{-1}(EL_i) = (a_i, b_i, c_i, d_i)$ *is the equivalent fuzzy number of the ELICIT expression* $EL_i$ *for all* $i = 1, 2, ..., n$ *and the heterogeneous interrelationship structure of* $EL_i$*'s is inherited into* $\zeta^{-1}(EL_i)$*'s in component-wise fashion.*

It is not difficult to show that ELICITEBM satisfies commutative, idempotency and ratio-scale invariant properties of the aggregation operator as those properties holds for classic EBM. Further, it is bounded by $\zeta\left(\min_i a_i, \min_i b_i, \min_i c_i, \min_i d_i\right)$ and $\zeta\left(\max_i a_i, \max_i b_i, \max_i c_i, \max_i d_i\right)$. To take into account the relative importance of the aggregated arguments in the aggregation process, we define the weighted form of the ELICITEBM as follows:

**Definition 21.** Let $\mathbf{EL} = (EL_1, EL_2, ..., EL_n)$ be the collection of $n$ ELICIT expressions from $\mathcal{F}$, which are heterogeneously interrelated in the fashion described Section 2. For any $p, q \geq 0$ with $p + q > 0$ and weight vector $w = (w_1, w_2, ..., w_n)$, such that $w_i > 0$ with $\sum_{i=1}^{n} w_i = 1$, the ELICITWEBM operator is a mapping *ELICITWEBM* $: \mathcal{F}^n \rightarrow \mathcal{F}$ and defined as follows:

$$\begin{aligned} &ELICITWEBM_{p,q}(EL_1, EL_2, ..., EL_n) \\ &= \zeta\left(\left(\left(1 - \sum_{i\in I'} w_i\right)\left(\bigoplus_{i\notin I'} \frac{w_i}{1-\sum_{i\in I'} w_i} \left(\zeta^{-1}(EL_i)\right)^p \otimes\right.\right.\right. \\ &\left.\left.\left(\frac{1}{|I_i|} \bigoplus_{j\in I_i} \frac{w_j}{\sum_{j\in I} w_j} \left(\zeta^{-1}(EL_j)\right)^q\right)\right)^{\frac{p}{p+q}} \\ &\left.\left.\oplus\left(\sum_{i\in I'} w_i \bigoplus_{i\in I'} \frac{w_i}{\sum_{i\in I'} w_i} \left(\zeta^{-1}(EL_i)\right)^p\right)\right)^{\frac{1}{p}}\right) \end{aligned} \tag{22}$$

The explicit computational formula of ELICITWEBM could be obtained by using the arithmetic laws of fuzzy numbers with ELICIT computational model and summarized in the following:

**Theorem 6.** *Let* $\mathbf{EL} = (EL_1, EL_2, ..., EL_n)$ *be the collection of n ELICIT expressions from* $\mathcal{F}$, *which are heterogeneously related. For any* $p, q \geq 0$ *with* $p + q > 0$ *and weight vector* $w = (w_1, w_2, ..., w_n)$, *such that* $w_i > 0$ *and* $\sum_{i=1}^{n} w_i = 1$, *the aggregated value of ELICIT expressions by ELICITWEBM is a ELICIT expression and given by*

$$\begin{aligned} &ELICITWEBM_{p,q}(EL_1, EL_2, ..., EL_n) \\ &= \zeta\left(WEBM(a_1, a_2, ..., a_n), WEBM(b_1, b_2, ..., b_n),\right. \\ &\quad\left. WEBM(c_1, c_2, ..., c_n), WEBM(d_1, d_2, ..., d_n)\right) \end{aligned} \tag{23}$$

*where,* $\zeta^{-1}(EL_i) = (a_i, b_i, c_i, d_i)$ *is the equivalent fuzzy number of the ELICIT expression* $EL_i$ *for all* $i = 1, 2, ..., n$ *and the heterogeneous interrelationship structure of* $EL_i$*'s is inherited into* $\zeta^{-1}(EL_i)$*'s* *in component-wise fashion. The WEBM: $[0,1]^n \to [0,1]$ is the weighted form of EBM aggregation operator, which is given by*

$$WEBM_{p,q}(a_1, a_2, ...a_n) = \left(\left(1 - \sum_{i\in I'} w_i\right)\left(\sum_{i\notin I'} \frac{w_i}{1-\sum_{i\in I'} w_i} a_i^p \left(\frac{1}{|I_i|}\sum_{j\in I_i} \frac{w_j}{\sum_{j\in I} w_j} a_j^q\right)\right)^{\frac{p}{p+q}} \oplus \sum_{i\in I'} w_i \left(\sum_{i\in I'} \frac{w_i}{\sum_{i\in I'} w_i} a_i^p\right)^{\frac{1}{p}}\right. \tag{24}$$

## 3.3. ELICIT Partitioned Bonferroni Mean

In this section, we consider the aggregation of ELICIT expressions, which follows a partitioned structure interrelationship pattern described in Section 2. Based on the fact in Eq. (4) and Definition 18, we define ELICITPBM operator in the following:

**Definition 22.** Let $\mathbf{EL} = (EL_1, EL_2, ..., EL_n)$ be the collection of $n$ ELICIT expressions from $\mathcal{F}$ such that the input set $\mathbf{EL}$ is partitioned into $d$ distinct classes $P_1, P_2,..., P_d$ (as described in Section 2). For any $p, q \geq$ with $p + q > 0$, the ELICITPBM operator is a mapping $ELICITPBM: \mathcal{F}^n \to \mathcal{F}$ and defined as follows:

$$ELICITPBM_{p,q}(EL_1, EL_2, ..., EL_n) = \zeta\left(\frac{1}{d}\bigoplus_{r=1}^{d} \zeta^{-1}(ELCITBM(EL_i : i \in P_r))\right) \tag{25}$$

where $(EL_i : i \in P_r)$ denotes the set of ELICIT expressions $EL_i$s that belong to the partition $P_r$.

From the Definition 22, we note that by repeated application of ELICITBM over the partitions of the input set we can obtain the aggregated value of ELICITPBM. The more explicit computational formula to find the aggregated value of the ELICICTPBM in terms of BM is given below:

**Theorem 7.** *Let $\mathbf{EL} = (EL_1, EL_2, ..., EL_n)$ be the collection of $n$ ELICIT expressions from $\mathcal{F}$,which are partitioned into $d$ classes $P_1$, $P_2$,..., $P_d$. For any $p, q \geq 0$ with $p + q > 0$, the aggregated value of ELICIT expressions is a ELICIT expression and given by*

$$ELICITPBM_{p,q}(EL_1, EL_2, ..., EL_n) = \zeta\left(\frac{1}{d}\left(\sum_{r=1}^{d} BM_{p,q}(a_i : i \in P_r), \sum_{r=1}^{d} BM_{p,q}(b_i : i \in P_r), \sum_{r=1}^{d} BM_{p,q}(c_i : i \in P_r), \sum_{r=1}^{d} BM(d_i : i \in P_r)\right)\right) \tag{26}$$

*where, $\zeta^{-1}(EL_i) = (a_i, b_i, c_i, d_i)$ is the equivalent fuzzy number of the ELICIT expression $EL_i$ for all $i = 1, 2, ..., n$ and the partitioned structure interrelationship of $EL_i$'s is inherited into $\zeta^{-1}(EL_i)$'s in component-wise fashion.*

As the ELICITPBM operator is composed of a set of ELICITBM operators with different dimensions, we can easily exhibit that the ELICITPBM operator satisfies commutative, idempotent and ratio-scale invariant properties with help of Theorem 2. Further, the ELCITPBM operator is bounded as follows:

$$\zeta\left(\min_i a_i, \min_i b_i, \min_i c_i, \min_i d_i\right) \leq ELICITPBM_{p,q}(EL_1, EL_2, ..., EL_n) \leq \zeta\left(\max_i a_i, \max_i b_i, \max_i c_i, \max_i d_i\right).$$

When the inputs ELICIT expressions have different relative importance, we need to take account it in the aggregation process and to reflect on the aggregated value. In this view, the weighted form of the ELICITPBM can be defined as follows:

**Definition 23.** Let $\mathbf{EL} = (EL_1, EL_2, ..., EL_n)$ be the collection of $n$ ELICIT expressions from $\mathcal{F}$ such that the input set $\mathbf{EL}$ is partitioned into $d$ distinct classes $P_1$, $P_2$, ..., $P_d$ (as described in Section 2). For any $p, q \geq 0$ with $p + q > 0$ and weight vector $w = (w_1, w_2, ..., w_n)$, such that $w_i > 0$ and $\sum_{i=1}^n w_i = 1$, the ELICITWPBM operator is a mapping $ELICITWPBM: \mathcal{F}^n \to \mathcal{F}$ and defined as follows:

$$ELICITWPBM_{p,q}(EL_1, EL_2, ..., EL_n) = \zeta\left(\frac{1}{d}\bigoplus_{r=1}^{d} \zeta^{-1}(ELCITWBM(EL_i \in P_r))\right) \tag{27}$$

where $(EL_i : i \in P_r)$ denotes the set of ELICIT expressions $EL_i$s that belong to the partition $P_r$.

**Theorem 8.** *Let $\mathbf{EL} = (EL_1, EL_2, ..., EL_n)$ be the collection of $n$ ELICIT expressions from $\mathcal{F}$. For any $p, q \geq 0$ with $p + q > 0$ and weight vector $w = (w_1, w_2, ..., w_n)$, such that $w_i > 0$ and $\sum_{i=1}^n w_i = 1$, the aggregated value of ELICIT expressions by ELICITWBM is a ELICIT expression and given by*

$$ELICITWBM_{p,q}(EL_1, EL_2, ..., EL_n) = \zeta\left(\frac{1}{d}\left(\sum_{r=1}^{d} WBM_{p,q}(a_i : i \in P_r), \sum_{r=1}^{d} WBM_{p,q}(b_i : i \in P_r), \sum_{r=1}^{d} WBM_{p,q}(c_i : i \in P_r), \sum_{r=1}^{d} WBM_{p,q}(d_i : i \in P_r)\right)\right) \tag{28}$$

*where $\zeta^{-1}(EL_i) = (a_i, b_i, c_i, d_i)$ is the equivalent fuzzy number of the ELICIT expression $EL_i$ for all $i = 1, 2, ..., n$ and*

$$WBM_{p,q}(a_i : i \in P_r) = \left(\sum_{\substack{i,j \in P_r \\ j \neq i}} \frac{w_i w_j}{\left(\sum_{i\in P_r} w_i\right)\left(\sum_{\substack{j \in P_r \\ j \neq i}} w_j\right)} a_i^p a_j^q\right)^{\frac{1}{p+q}}$$

## 4. APPROACHES TO MADM WITH ELICIT ASSESSMENTS

In this section, we develop an approach based on ELICIT expressions aggregation operators to solve MADM problem in which attributes follow a typical interrelationship pattern, and the decision maker provides his/her assessments by using CLEs and/or ELICIT expressions.

We consider a typical MADM problem, where a finite set of alternatives are evaluated against a predefined set of performance measuring attributes in the aim of ranking the alternatives from best to worst on their suitability. In such a decision-making problem two pieces of information are required to find the ranking of the alternatives. One is assessment information of the alternatives against the criteria, which we often refer to as decision information. Another one is related to the relative importance of the criteria that is referred to as weight information. Mathematically, we can describe the MADM problem with all the relevant information as follows:

- A finite set of $m\,(\geq 2)$ alternatives: $X = \{X_i | i \in I\}$, where $I = \{1, 2, ..., m\}$
- A fixed set of criteria: $A = \left\{A_j | j \in J\right\}$ where $J = \{1, 2, ..., n\}$
- The weight vector of the criteria: $w = (w_1, w_2, ..., w_n)$ such that $w_j \geq 0$ and $\sum_{j=1}^{n} w_j = 1$.
- The alternatives are assessed over criteria and evaluations are summarized in the following decision matrix:

$$D = \begin{array}{c} \\ X_1 \\ X_2 \\ \vdots \\ X_m \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & \cdots & A_n \end{array} \\ \begin{pmatrix} EL_{11} & EL_{12} & \cdots & EL_{1n} \\ EL_{21} & EL_{22} & \cdots & EL_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ EL_{m1} & EL_{m2} & \cdots & EL_{mn} \end{pmatrix} \end{array}$$

where $EL_{ij}$ is the ELICIT expression that has been obtained from the decision maker's linguistic opinions to provide his/her assessment for the alternative $X_i$ against the criteria $A_j$. Specifically, decision maker uses CLEs to express his/her assessments against the alternatives under different attributes.

Apart from these binding pieces of information, the decision maker needs to provide the typical pattern of the interrelationship among the attributes. As interrelationship is vital in the selection of an appropriate aggregation operator, this information is crucial to make a reliable decision.

$$\begin{aligned} &\mathrm{WEBM}_{\mathrm{p,q}}(a_1, a_2, ...a_n) \\ &= \left( \left(1 - \sum_{s\in I'} w_s\right) \left( \sum_{s\notin I'} \frac{w_s}{1 - \sum_{s\in I'} w_s} a_{is}^p \left( \frac{1}{|I_s|} \sum_{t\in I_s} \frac{w_t}{\sum_{t\in I_s} w_j} a_{it}^q \right)\right)^{\frac{p}{p+q}} \right. \\ &\left. + \sum_{s\in I'} w_s \left( \sum_{s\in I'} \frac{w_s}{\sum_{s\in I'} w_s} a_{is}^p \right) \right)^{\frac{1}{p}} \end{aligned} \tag{29}$$

With this available information in hand, we intend to design an algorithm based on the aggregation operators, developed in the previous section, to find the most desirable alternative(s) from the alternatives' pool $\{X_1, X_2, ..., X_m\}$. Our proposed algorithm takes following steps to find ranking order of the alternatives:

## Step 1

Give the decision maker's preference summarized in the decision matrix $D = \left(EL_{ij}\right)_{m\times n}$ and weight information $w = (w_1, w_2, ..., w_n)$.

## Step 2

Provide the interrelationship patter among the attributes, i.e., whether, the attributes follows homogeneous interrelationship pattern, heterogeneously interrelation patter or partitioned structured interrelationship pattern. In the cases of heterogeneous and partitioned interrelationship, specific structure of interrelationship data need to be provided.

## Step 3

Based on the interrelationship pattern, the suitable aggregation operator is selected to obtain the overall performance of the alternative $X_i$ from the alternative's individual performances under different attributes $E_{ij}\ \left(j = 1, 2, ..., n\right)$. Specifically, three scenarios arise here:

- *attributes are homogeneously related* in this case, we utilize ELICITBM operator to find the alternatives $X_i$ overall performance $r_i\ (i = 1, 2, ..., m)$ as follows:

$$\begin{aligned} r_i &= ELICITBM(EL_{i1}, EL_{i2}, ..., EL_{in}) \\ &= \zeta \left( \left( \sum_{\substack{s,t=1 \\ t\neq s}}^{n} \frac{w_s w_t}{1 - w_s} a_{is}^p a_{it}^q \right)^{\frac{1}{p+q}}, \left( \sum_{\substack{s,t=1 \\ t\neq s}}^{n} \frac{w_s w_t}{1 - w_s} b_{is}^p b_{it}^q \right)^{\frac{1}{p+q}}, \right. \\ &\quad \left. \left( \sum_{\substack{s,t=1 \\ t\neq s}}^{n} \frac{w_s w_t}{1 - w_s} c_{is}^p c_{it}^q \right)^{\frac{1}{p+q}}, \left( \sum_{\substack{s,t=1 \\ t\neq s}}^{n} \frac{w_s w_t}{1 - w_s} d_{is}^p d_{it}^q \right)^{\frac{1}{p+q}} \right) \end{aligned} \tag{30}$$

where $\zeta^{-1}\left(EL_ij\right) = \left(a_{ij}, b_{ij}, c_{ij}, d_{ij}\right)$ is the equivalent fuzzy number of the ELICIT expression $EL_ij$ for all $i = 1, 2, ..., m$.

- *attributes are heterogeneously interrelated*, in this case, we employ ELCITWEBM operator to obtain overall performance $r_i$ of the alternative $X_i$ as follows:

$$\begin{aligned} &ELICITWEBM_{p,q}(EL_{i1}, EL_{i2}, ..., EL_{in}) \\ &= \zeta\,(WEBM(a_{i1}, a_{i2}, ..., a_{in}), \\ &WEBM(b_{i1}, b_{i2}, ..., b_{in}), \\ &WEBM(c_{i1}, c_{i2}, ..., c_{in}), \\ &WEBM(d_{i1}, d_{i2}, ..., d_{in})) \end{aligned} \tag{31}$$

where, $\zeta^{-1}\left(EL_{ij}\right) = \left(a_{ij}, b_{ij}, c_{ij}, d_{ij}\right)$ is the equivalent fuzzy number of the ELICIT expression $EL_{ij}$ for all $j = 1, 2, ..., n$ and $WEBM(a_{i1}, a_{i2}, ..., a_{in})$ is given by Eq. (29).

- *attributes are partitioned structured*, in this case, WELCITPBM operator is utilized to obtain overall performance $r_i$ of the alternative $X_i$ as follows:

$$\begin{aligned} r_i &= WELICITPBM\left(EL_{i1}, EL_{i2}, ..., EL_{in}\right) \\ &= \zeta\left(\frac{1}{d}\left(\sum_{r=1}^{d} WBM\left(a_{ij} \in P_r\right), \sum_{r=1}^{d} WBM\left(b_{ij} : j \in P_r\right), \right.\right. \\ &\left.\left. \sum_{r=1}^{d} WBM\left(c_{ij} : j \in P_r\right), \sum_{r=1}^{d} WBM\left(d_{ij} : j \in P_r\right)\right)\right) \end{aligned} \tag{32}$$

where, $\zeta^{-1}\left(EL_i\right) = \left(a_{ij}, b_{ij}, c_{ij}, d_{ij}\right)$ is the equivalent fuzzy number of the ELICIT expression $EL_{ij}$ for all $j = 1, 2, ..., n$ and

$$WBM\left(a_{ij} : j \in P_r\right) = \left(\sum_{\substack{k,j \in P_r \\ j \neq k}} \frac{w_k w_j}{\left(\sum_{k \in P_r} w_k\right)\left(\sum_{\substack{j \in P_r \\ j \neq k}} w_j\right)} a_k^p a_j^q\right)^{\frac{1}{p+q}}$$

## Step 4

The overall performance of the alternatives $r_i\,(i = 1, 2, ..., m)$ are ELICIT expressions. To facilitate the comparisons, we first transformed them into fuzzy numbers $T_{r_i} = \zeta^{-1}\left(r_i\right) = \left(t_{i1}, t_{i2}, t_{i3}, t_{i4}\right)$ for $i = 1, 2, ..., m$ and then defuzzified them into real number $Mag\left(T_{r_i}\right)(i = 1, 2, ..., m)$ by using the approach proposed by Abbasbandy and Hajri [44].

## Step 5

Based on the $Mag\left(T_{r_i}\right)(i = 1, 2, ..., m)$, we rank the alternatives $X_i\,(i = 1, 2, ..., m)$ in the sense that better the magnitude, better the rank.

# 5. PRACTICAL EXAMPLE

In this section, we provide a practical example to demonstrate the working and feasibility of the proposed decision-making algorithm.

In the face of a trade war, a major company is considering to shift its manufacturing plant from the current location. After, initial screening the company has identified five possible locations around the world to step up the new manufacturing plant. We name this potential locations as $\{X_1, X_2, X_3, X_4, X_5\}$. To prioritize further these locations, the company has identified seven assessment attributes: market $(A_1)$, business climate $(A_2)$, labour characteristic $(A_3)$, infrastructure $(A_4)$, availability of raw materials $(A_5)$, investment cost $(A_6)$ and possibility for the further extensions $(A_7)$. These performance measuring attributes have some intrinsic connections/interrelations and that could be described as follows: $A_1$ is interrelated with $A_4$; $A_2$ with $\{A_6, A_7\}$; $A_3$ with $A_7$; $A_4$ with $\{A_1, A_6\}$; $A_5$ with $A_7$; $A_6$ with $\{A_2, A_4\}$ and $A_7$ with $\{A_3, A_5\}$. The information regarding the attributes for all possible options are collected and presented to the key managerial responsible for taking a decision.

Due to the presence of vagueness and uncertainty, the decision maker uses linguistic information to assess the locations against the attributes. According to the expertise of the decision maker, a linguistic term set with 7 labels is provided, $S = \{s_0$*: unfeasible (UF),* $s_1$*: very unsuitable (VUS),* $s_2$*: unsuitable (US),* $s_3$*: fair (F),* $s_4$*: suitable (S),* $s_5$*: very suitable (VS),* $s_6$*: excellent (E)\}*.

Decision maker uses a single linguistic term or complex linguistic expression, modeled by CLEs to rate the alternatives against the attributes. The decision maker's preferences are represented by CLEs (Table 1 Rating in CLEs) that are transformed into ELICIT information and modeled by the decision matrix $D$ and presented at the beginning of the next page.

$$D = \begin{array}{c} \\ X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \end{array} \left( \begin{array}{lllllll} A_1 & A_2 & A_3 & A_4 & A_5 & A_6 & A_7 \\ at\,least\ (s_4,0)^0 & at\,least\ (s_5,0)^0 & (s_4,0)^0 & (s_4,0)^0 & at\,least\ (s_3,0)^0 & at\,least\ (s_4,0)^0 & bt\ (s_3,0)^0\ and\ (s_4,0)^0 \\ at\,most\ (s_1,0)^0 & (s_3,0)^0 & bt\ (s_3,0)^0\ and\ (s_4,0)^0 & bt\ (s_0,0)^0\ and\ (s_1,0)^0 & at\,least\ (s_3,0)^0 & (s_3,0)^0 & (s_3,0)^0 \\ (s_5,0)^0 & at\,least\ (s_5,0)^0 & (s_4,0)^0 & (s_5,0)^0 & bt\ (s_2,0)^0\ and\ (s_3,0)^0 & at\,most\ (s_3,0)^0 & (s_3,0)^0 \\ (s_0,0)^0 & (s_0,0)^0 & (s_1,0)^0 & bt\ (s_3,0)^0\ and\ (s_4,0)^0 & at\,most\ (s_2,0)^0 & (s_3,0)^0 & (s_2,0)^0 \\ (s_6,0)^0 & (s_3,0)^0 & (s_6,0)^0 & at\,least\ (s_4,0)^0 & (s_2,0)^0 & bt\ (s_4,0)^0\ and\ (s_5,0)^0 & (s_5,0)^0 \end{array} \right)$$

$$D = \begin{array}{c} \\ X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \end{array} \left( \begin{array}{lllllll} A_1 & A_2 & A_3 & A_4 & A_5 & A_6 & A_7 \\ T(0.5, 0.86, 1, 1) & T(0.67, 0.98, 1, 1) & T(0.5, 0.67, 0.84) & T(0.5, 0.67, 0.84) & T(0.34, 0.64, 1, 1) & T(0.5, 0.86, 1, 1) & T(0.34, 0.5, 0.67, 0.84) \\ T(0, 0, 0.03, 0.34) & T(0.34, 0.5, 0.67) & T(0.34, 0.5, 0.67, 0.84) & T(0, 0, 0.17, 0.34) & T(0.34, 0.65, 1, 1) & T(0.34, 0.5, 0.67) & T(0.34, 0.5, 0.67) \\ T(0.67, 0.84, 1) & T(0.67, 0.98, 1, 1) & T(0.5, 0.67, 0.84) & T(0.67, 0.84, 1) & T(0.17, 0.34, 0.5, 0.67) & T(0, 0, 0.36, 0.67) & T(0.34, 0.5, 0.67) \\ T(0, 0, 0.17) & T(0, 0, 0.17) & T(0.67, 0.84, 1) & T(0.4, 0.5, 0.67, 0.84) & T(0, 0, 0.15, 0.5) & T(0.34, 0.5, 067) & T(0.17, 0.34, 0.5) \\ T(0.84, 1, 1) & T(0.34, 0.5, 0.67) & T(0.84, 1, 1) & T(0.5, 0.84, 1, 1) & T(0.17, 0.34, 0.5) & T(0.5, 0.67, 0.84, 1) & T(0.67, 0.84, 1) \end{array} \right)$$

Further all performance measuring attributes are not equally important. To take into account the variation in relative importance of the attributes, weight information is set as $w = (0.2, 0.1, 0.15, 0.15, 0.2, 0.1, 0.1)$.

With this available information about the locations' choices problem, we employ the proposed decision-making algorithm to prioritize the locations and to find the most suitable one.

## Step 1

To carry out the linguistic computations, all the ELICIT expressions are required to transform into machine manipulative format, i.e., TrFNs. Decision maker's opinions in terms of ELICIT expressions given in $D$ are converted into TrFNs and summarized in the matrix $\overline{D}$ and given at the beginning of the next page.

Here, the first entry of $\overline{D}$, $T(0.5, 0.86, 1, 1)$ is the equivalent TrFN corresponding to the ELICIT expression at least $(s_4, 0)^0$, i.e., $\zeta^{-1}\left(\text{at least } (s_4, 0)^0\right) = T(0.5, 0.86, 1, 1)$.

## Step 2

From the description of the attributes interrelationship pattern, it is quite evident that the attributes are heterogeneously related with no independent arguments. In the aim of capturing this heterogeneous interaction among the attributes and its reflection in the aggregated value, we choose ELICITWEBM (Eq. 29), to compute the overall performance of the alternatives. We set the associated parameter $p$ and $q$ to 1 in ELICITWEBM and compute the overall performance with the translated information $\overline{D}$ and weight information $w$. The results are summarized in the following Table 2. From Table 2 decision maker obtains the overall performance of alternatives expressed in terms of linguistic ELICIT expressions, which is quite intuitive to interpret. It is also clear to the decision maker from the Table 2 that $X_3$ is better than $\{X_2, X_4\}$ and $X_2$ is better than $X_4$. Undoubtedly, $X_1$ and $X_5$ are better than rest of the alternatives but it is not very clear about the order of the $X_1$ and $X_5$ from the linguistic overall performances. We are going to the next step for finding the exact ranking order of the alternatives.

## Step 3

From the overall performances $r_i\,(i = 1, 2, 3, 4, 5)$, we compute the magnitude of the corresponding TrFNs, $T_{r_i}\,(i = 1, 2, 3, 4, 5)$ of the $r_i$ as follows: $Mag\left(T_{r_1}\right) = 0.7416$, $Mag\left(T_{r_2}\right) = 0.4486$, $Mag\left(T_{r_3}\right) = 0.6242$, $Mag\left(T_{r_4}\right) = 00.3144$ and $Mag\left(T_{r_5}\right) = 0.7928$. Based on the $Mag\left(T_{r_i}\right)\,(i = 1, 2, 3, 4, 5)$, the ranking of the alternatives are as follows: $X_5 \succ X_1 \succ X_3 \succ X_2 \succ X_4$. Hence the location $X_5$ is the most suitable to set up the manufacturing plant followed by location $X_1$.

In the above analysis, we have set the parameters associated with ELICITWEBM as $(p, q) = (1, 1)$. But this choice of the parameters $p$ and $q$ associated with ELICITWEBM may have an impact on the final ranking of the locations. Thus, it is necessary to check the robustness of the ranking result concerning the parameters. For this purpose, we adopt the simulation-based approach, specifically, the framework of stochastic multi-criteria acceptability analysis [45]. As there is no preference over the parameters' values, we assume that the parameters are uniformly distributed in the space $[0.1, 100]^2$. By randomly drawing the parameters from the space $[0.1, 100]^2$, we solve the decision-making problem and find the ranking of the locations. Further, repeating this process for the sufficient numbers of times (10, 000) within Monte Carlo framework, we collect the evidence in terms of probability of occupying a ranking position by an alternative. We report the result of the Monte Carlo in the Table 3, where $\mathbf{b}^r$ corresponding the alternative $X_i$ denotes the probability of occupying $r$-th ranking position by $X_i$. It is quite evident that for the almost all configuration of the parameters from the space $[0.1, 100]^2$, the $X_5$ occupied the first ranking positions followed by $X_1$. Unanimously, $X_3$ is always occupied the third-ranking positions followed by $X_2$ and $X_4$. But there is a possibility of switching the ranking position between $X_2$ and $X_4$ for some configurations of the parameters. In nutshell, we can conclude that present ranking results are robust and not much sensitive to the parameters. Note that the exact estimation of the appropriate parameters associated with ELICITEBM could also be stem from the decision maker's perceived view towards aggregation process [22,46].

**Table 1** Alternatives rating under different criteria.

| | $A_1$ | $A_2$ | $A_3$ | $A_4$ | $A_5$ | $A_6$ | $A_7$ |
|---|---|---|---|---|---|---|---|
| $X_1$ | *at least S* | *at least VS* | *S* | *S* | *at least F* | *at least S* | *bt F and S* |
| $X_2$ | *at most VUS* | *F* | *bt F and S* | *bt UF and VUS* | *at least F* | *F* | *F* |
| $X_3$ | *VS* | *at least VS* | *S* | *VS* | *bt US and F* | *at most F* | *F* |
| $X_4$ | *UF* | *UF* | *VUS* | *bt F and S* | *at most US* | *F* | *US* |
| $X_5$ | *E* | *F* | *E* | *at least S* | *US* | *bt S and VS* | *VS* |

**Table 2** Alternatives overall performance.

| **Alternative** | $T_{r_i}$ **(TrFN)** | $r_i = \zeta^{-1}\left(T_{r_i}\right)$ |
|---|---|---|
| $X_1$ | $T(0.4545, 0.6963, 0.8106, 0.9102)$ | *between* $(s_4, 0.1758)^{-0.0455}$ *and* $(s_5, -0.1362)^{-0.0898}$ |
| $X_2$ | $T(0.2612, 0.4084, 0.4889, 0.6351)$ | *between* $(s_2, 0.4524)^{0.0942}$ *and* $(s_3, -0.0666)^{0.1351}$ |
| $X_3$ | $T(0.4401, 0.5869, 0.6567, 0.8315)$ | *between* $(s_4, -0.4806)^{-0.0599}$ *and* $(s_4, -0.0618)^{-0.0015}$ |
| $X_4$ | $T(0.1862, 0.2874, 0.3241, 0.5291)$ | *between* $(s_2, -0.2736)^{0.0192}$ *and* $(s_2, -0.0540)^{0.0291}$ |
| $X_5$ | $T(0.5639, 0.7797, 0.8286, 0.9084)$ | *between* $(s_5, -0.3198)^{-0.1031}$ *and* $(s_5, -0.0264)^{-0.0916}$ |

**Table 3** Percentage of occupying different ranking positions by alternatives.

| Alternative | $b^1$ | $b^2$ | $b^3$ | $b^4$ | $b^5$ |
|---|---|---|---|---|---|
| $X_1$ | 0.0040 | 99.9960 | 0 | 0 | 0 |
| $X_2$ | 0 | 0 | 0 | 88.7300 | 11.2700 |
| $X_3$ | 0 | 0 | 100 | 0 | 0 |
| $X_4$ | 0 | 0 | 0 | 11.2700 | 88.7300 |
| $X_5$ | 99.9960 | 0.0040 | 0 | 0 | 0 |

As we have emphasized on the fact that capturing the underlying interrelationship pattern in the aggregated ELICIT information is vital to make a reliable decision, it is worthy here to investigate the consequence if we do not consider the interrelationship in the information fusion process. For this purpose, we use the weighted ELICIT arithmetic mean operators, which assume that the input arguments are independent, in place of ELICITWBM in the proposed decision-making algorithm to compute the overall performances of the alternatives. Rest of the steps in our proposed MADM algorithm to find the ranking of the alternatives is kept unaltered. With this new configuration of the algorithm, we re-execute the step of the MADM algorithms and found the following ranking order of the alternatives $X_1 \succ X_5 \succ X_3 \succ X_2 \succ X_4$. It is evident that the ranking positions for $X_1$ and $X_5$ are reversed, which due to not capturing the underlying interrelationship structure among the attributes.

## 6. CONCLUSION

In this study, we have investigated the aggregation of linguistic information that is represented by ELICIT expressions and followed some specific interrelationship patterns. Specifically, we have considered three types of interrelationship patterns, namely, heterogeneous, homogeneous and partition structure among the aggregated arguments and such relationships are captured via direct conjunctions among the aggregated arguments with the core of three classical aggregation operators: BM, EBM, and PBM. In this view, we have extended these classical operators in ELICIT information environment and developed three new aggregation operators for aggregation ELICIT expressions, which we have referred to as ELICITBM, ELICITEBM, and ELICITPBM. Furthermore, we have investigated the properties of these aggregation operators and proposed the weighted form of these aggregation operators to deal with the situations where inputs have different relative importance. Using these aggregation operators as an information fusion tool, an algorithm for solving the MADM problems, in which attributes follow some specific interrelationship patterns. Finally, we have presented numerical examples to illustrate the feasibility and applicability of our proposed approach.

In the future, it would be interesting to investigate the more complex interaction among the ELICIT expressions via Choquet integral [47]. Further, one may consider extending the aggregation of ELICIT expressions for other class of averaging aggregation operators, such as ordered weighted average operators [48] , power averaging operator [49], prioritize aggregation operator [50] and their different variants.

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## REFERENCES


[1] R. Parreiras, W. Pedrycz, P. Ekel, Fuzzy Multicriteria Decision Making: Models, Methods and Applications, Wiley, 2011.

[2] L.A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning–I, Inf. Sci. 8 (1975), 199–249.

[3] L.A. Zadeh, Fuzzy sets, Inf. Control. 8 (1965), 338–353.

[4] L. Martínez, R.M. Rodríguez, F. Herrera, The 2-Tuple Linguistic Model, Springer International Publishing, 2015.

[5] F. Herrera, L. Martínez, A 2-tuple fuzzy linguistic representation model for computing with words, IEEE Trans. Fuzzy Syst. 8 (2000), 746–752.

[6] J.H. Wang, J. Hao, A new version of 2-tuple fuzzy linguistic representation model for computing with words, IEEE Trans. Fuzzy Syst. 14 (2006), 435–445.

[7] Z. Xu, A method based on linguistic aggregation operators for group decision making with linguistic preference relations, Inf. Sci. 166 (2004), 19–30.

[8] R. Degani, G. Bortolan, The problem of linguistic approximation in clinical decision making, Int. J. Approx. Reason. 2 (1988), 143–162.

[9] M. Delgado, J.L. Verdegay, M.A. Vila, On aggregation operations of linguistic labels, Int. J. Intell. Syst. 8 (1993), 20.

[10] R.R. Yager, A new methodology for ordinal multiobjective decisions based on fuzzy sets, in: D. Dubois, H. Prade, R.R.Yager (Eds.), Readings in Fuzzy Sets for Intelligent Systems, Elsevier, 1993, pp. 751–756.

[11] Y. Liu, J. Liu, Z. Hong, A multiple attribute decision making approach based on new similarity measures of interval-valued hesitant fuzzy sets, Int. J. Comput. Intell. Syst. 11 (2018), 15–32.

[12] L. Martínez, F. Herrera, An overview on the 2-tuple linguistic model for computing with words in decision making: extensions, applications and challenges, Inf. Sci. 207 (2012), 1–18.

[13] R.M. Rodríguez, Á. Labella, L. Martínez, An overview on fuzzy modelling of complex linguistic preferences in decision making, Int. J. Comput. Intell. Syst. 9 (2016), 81–94.

[14] R.M. Rodriguez, L. Martínez, F. Herrera, Hesitant fuzzy linguistic term sets for decision making, IEEE Trans. Fuzzy Syst. 20 (2011), 109–119.

[15] H. Wang, Z. Xu, X.J. Zeng, Hesitant fuzzy linguistic term sets for linguistic decision making: current developments, issues and challenges, Inf. Fusion. 43 (2018), 1–12.

[16] R.M. Rodríguez, L. Martínez, F. Herrera, A group decision making model dealing with comparative linguistic expressions based on hesitant fuzzy linguistic term sets, Inf. Sci. 241 (2013), 28–42.

[17] Z.S. Chen, K.S. Chin, Y.L. Li, Y. Yang, Proportional hesitant fuzzy linguistic term set for multiple criteria group decision making, Inf. Sci. 357(2016), 61–87.

[18] H. Liu, R.M. Rodríguez, A fuzzy envelope for hesitant fuzzy linguistic term set and its application to multicriteria decision making, Inf. Sci. 258 (2014), 220–238.

[19] Q. Pang, H. Wang, Z. Xu, Probabilistic linguistic term sets in multi-attribute group decision making, Inf. Sci. 369 (2016), 128–143.

[20] J.V. Riera, S. Massanet, E. Herrera-Viedma, J. Torrens, Some interesting properties of the fuzzy linguistic model based on discrete fuzzy numbers to manage hesitant fuzzy linguistic information, Appl. Soft Comput. 36 (2015), 383–391.

[21] Á. Labella, R.M. Rodríguez, L. Martínez, Computing with comparative linguistic expressions and symbolic translation for decision making: ELICIT information, IEEE Trans. Fuzzy Syst. (2019).

[22] B. Dutta, F.T.S. Chan, D. Guha, B. Niu, J.H. Ruan, Aggregation of heterogeneously related information with extended geometric bonferroni mean and its application in group decision making, Int. J. Intell. Syst. 33 (2018), 487–513.

[23] Z.-S. Chen, K.-S. Chin, K.-L. Tsui, Constructing the geometric bonferroni mean from the generalized bonferroni mean with several extensions to linguistic 2-tuples for decision-making, Appl. Soft Comput. 78 (2019), 595–613.

[24] B. Dutta, D. Guha, Partitioned bonferroni mean based on linguistic 2-tuple for dealing with multi-attribute group decision making, Appl. Soft Comput. 37 (2015), 166–179.

[25] B. Dutta, D. Guha, R. Mesiar, A model based on linguistic 2-tuples for dealing with heterogeneous relationship among attributes in multi-expert decision making, IEEE Trans. Fuzzy Syst. 23 (2015), 1817–1831.

[26] M. Feng, P. Liu, Y. Geng, A method of multiple attribute group decision making based on 2-tuple linguistic dependent maclaurin symmetric mean operators, Symmetry. 11 (2019), 31.

[27] Y. Liu, J. Liu, Y. Qin, Y. Xu, A novel method based on extended uncertain 2-tuple linguistic muirhead mean operators to magdm under uncertain 2-tuple linguistic environment, Int. J. Comput. Intell. Syst. 12 (2019), 498–512.

[28] X. Gou, Z. Xu, H. Liao, Multiple criteria decision making based on bonferroni means with hesitant fuzzy linguistic information, Soft Comput. 21 (2017), 6515–6529.

[29] D. Ju, Y. Ju, A. Wang, Multi-attribute group decision making based on power generalized heronian mean operator under hesitant fuzzy linguistic environment, Soft Comput. 23 (2019), 3823–3842.

[30] C. Liu, G. Tang, P. Liu, C. Liu, Hesitant fuzzy linguistic archimedean aggregation operators in decision making with the dempster–shafer belief structure, Int. J. Fuzzy Syst. 21 (2019), 1330–1348.

[31] L. Wang, Y. Wang, X. Liu, Prioritized aggregation operators and correlated aggregation operators for hesitant 2-tuple linguistic variables, Symmetry. 10 (2018), 39.

[32] L. Wang, Y. Wang, W. Witold, Hesitant 2-tuple linguistic bonferroni operators and their utilization in group decision making, App. Soft Comput. 77 (2019), 653–664.

[33] N. Zhang, J. Jian, J. Su, Dual hesitant fuzzy linguistic power-average operators based on archimedean t-conorms and t-norms, IEEE Access. 7 (2019), 40602–40624.

[34] R.M. Rodríguez, L. Martínez, An analysis of symbolic linguistic computing models in decision making, Int. J. Gen. Syst. 42 (2013), 121–136.

[35] C. Bonferroni, Sulle medie multiple di potenze, Boll. Mat. Ital. 5 (1950), 267–270.

[36] R.R. Yager, On generalized bonferroni mean operators for multi-criteria aggregation, Int. J. Approx. Reason. 50 (2009), 1279–1286.

[37] H.J. Zimmermann, Fuzzy Set Theory-and its Applications, Springer, Netherlands, 2011.

[38] M. Delgado, M.A. Vila, W. Voxman, On a canonical representation of fuzzy numbers, Fuzzy Sets Syst. 93 (1998), 125–135.

[39] W. Pedrycz, Why triangular membership functions?, Fuzzy Sets Syst. 64 (1994), 21–30.

[40] S.H. Chen, Operations of fuzzy numbers with step form membership function using function principle, Inf. Sci. 108 (1998), 149–155.

[41] S. Rezvani, M. Molani, Representation of trapezoidal fuzzy numbers with shape function, Ann. Fuzzy Math. Informat. 8 (2014), 89–112.

[42] C. Cheng, A new approach for ranking fuzzy numbers by distance method, Fuzzy Sets Syst. 95 (1998), 307–317.

[43] Z. Xu. Deviation measures of linguistic preference relations in group decision making, Omega. 33 (2005), 249–254.

[44] S. Abbasbandy, T. Hajjari, A new approach for ranking of trapezoidal fuzzy numbers, Comput. Math. Appl. 57 (2009), 413–419.

[45] R. Lahdelma, J. Hokkanen, P. Salminen, Smaa-stochastic multiobjective acceptability analysis, Eur. J. Oper. Res. 106 (1998), 137–143.

[46] B. Dutta, J.R. Figueira, S. Das, On the orness of bonferroni mean and its variants, Int. J. Intell. Syst. 34 (2019), 1889–1919.

[47] G Choquet, Theory of capacities, Ann. de l'Institut Fourier. 5 (1954), 131–295.

[48] R.R. Yager, On ordered weighted averaging aggregation operators in multicriteria decisionmaking, IEEE Trans. Syst. Man Cybern. 18 (1988), 183–190.

[49] R.R. Yager, The power average operator, IEEE Trans. Syst. Man Cybern. Part A Syst. Hum. 31 (2001), 724–731.

[50] R.R. Yager, Prioritized aggregation operators, Int. J. Approx. Reason. 48 (2008), 263–274.

# APPENDIX A

## A.1 ELICIT Inverse Function Example

In order to facilitate the understanding of the inverse function, $\zeta^{-1}$, for ELICIT information, let us suppose a linguistic term set with seven labels, $S = \{s_0 : horrible, s_1 : very\ bad, s_2 : bad, s_3 : medium, s_4 : good, s_5 : very\ good, s_6 : perfect\}$ and an ELICIT expression *between* $(s_3, 0.432)^{0.024}$ *and* $(s_4, 0.144)^{-0.023}$ (see Figure A.1).

First, it is necessary to compute the fuzzy envelope [18] of the ELICIT expression. To do that, the HFLTS of the expression is obtained through the transformation function defined in [21]:

$$E_{ELICIT}\left(between\ (s_i, \alpha) \text{ and } (s_j, \alpha)\right) = \\ \left\{s_k \middle| (s_i, \alpha) \text{ and } (s_j, \alpha) \text{ and } s_i < s_k < s_j \text{ where } s_k \in S\right\}$$

For our example:

$$E_{ELICIT}(between\ (s_3, 0.432) \text{ and } (s_4, 0.144)) = \\ \{s_k | (s_3, 0.43) \text{ and } (s_4, 0.14) \text{ and } s_3 < s_k < s_4 \\ \text{where } s_k \in S\} = \{(s_3, 0.432), (s_4, 0.144)\}$$

Once the HFLTS is computed, the different fuzzy memberships functions of the linguistic terms that belong to the HFLTS are aggregated with the OWA operator [48]. The OWA operator assigns different importance to the linguistic terms that compose the HFLTS through the *orness measure* thus, the way of computing the OWA weights affect directly to the resulting fuzzy envelopes. This process is carried out in [21] by means of a parameter, noted as $\epsilon \in [0, 1]$, which allows modifying the way to compute the OWA weights. The variation of $\epsilon$ modifies the importance of the linguistic terms of the HFLTS, in order to reduce the interval whose height is 1 in the fuzzy envelope. In [21], several fixed orness values provided by $\epsilon$ are used in order to compute fuzzy envelopes that preserve as much information as possible. The fixed values of $\epsilon$ are: $\epsilon = 0$ for *at least* relations, $\epsilon = 1$ for *at most* relations and $\epsilon_1 = 0$ and $\epsilon_2 = 1$ for *between* relations. Following this process, the resulting fuzzy envelope for the ELICIT expression is $T(0.405, 0.572, 0.691, 0.857)$.

Finally, the corresponding TrFN of the respective ELICIT expression is obtained by applying Prop. 16:

$$\begin{aligned} &\zeta^{-1}\left(between\ (s_3, 0.432)^{0.024} \text{ and } (s_4, 0.144)^{-0.023}\right) \\ &= T(0.429, 0.572, 0.691, 0.834) \\ &a = 0.405 + 0.024 = 0.429 \\ &b = 0.572 \\ &c = 0.691 \\ &d = 0.857 + (-0.023) = 0.834 \end{aligned} \tag{A.1}$$

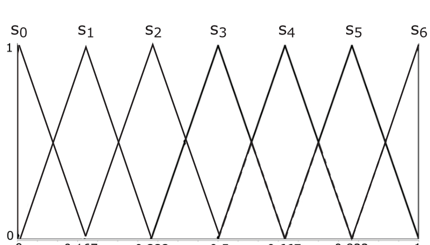


**Figure A.1** | Extended Comparative LInguistiC Expressions with SymbolIc Translation (ELICIT) information examples.

# APPENDIX B

## B.1 ELICIT Retranslation Process Example

In order to facilitate the understanding of the retranslation process to obtain an ELICIT expression from a TrFN, let us suppose the TrFN computed in A.1, $\tilde{\beta} = T(0.429, 0.572, 0.691, 0.834)$. The process to obtain an ELICIT expression is composed by several steps:

1. *Identify relation*: The relation of the ELICIT expression is determined by the fuzzy number $\tilde{\beta} = T(0.429, 0.572, 0.691, 0.834)$ and the $\zeta$ function (see Eq. 11).

$$\zeta(T(0.429, 0.572, 0.691, 0.834)) = EL, \tag{B.1}$$
$$where \begin{cases} EL = at\ least\ (s_i, \alpha)^{\gamma}\ if\ \tilde{\beta} = T(a, b, 1, 1) \\ EL = at\ most\ (s_i, \alpha)^{\gamma}\ if\ \tilde{\beta} = T(0, 0, c, d) \\ EL = between\ (s_i, \alpha_1)^{\gamma_1}\ and\ (s_j, \alpha_2)^{\gamma_2} \\ \qquad if\ \tilde{\beta} = T(a, b, c, d) \end{cases}$$

   According to the fuzzy number $\tilde{\beta}$, the relation of the ELICIT expression is "between".

2. *2-tuple linguistic terms computation* (see Figure B.1): The ELICIT expression with the relation "*between*" is composed by two continuous terms, $(s_i, \alpha_1)^{\gamma_1}$ and $(s_j, \alpha_2)^{\gamma_2}$.
   (a) *Compute linguistic terms*: First, we select the linguistic terms $s_i$ and $s_j \in S, i, j \in \{0, 1, 2, 3, 4, 5, 6\}$, whose distance between the coordinates $x$ of their respective centroids [42], $\overline{x}_i$ and $\overline{x}_j$, and the points $b = 0.572$ and $c = 0.691$ belonging to $\tilde{\beta}$ is minimal. In this case, such centroids are $\overline{x}_3$ and $\overline{x}_4$:

$$\begin{aligned} i &= \underset{h \in \{0,1,2,3,4,5,6\}}{\arg\min} \left|0.572 - \overline{x}_h\right| = 3 \\ j &= \underset{h \in \{0,1,2,3,4,5,6\}}{\arg\min} \left|0.691 - \overline{x}_h\right| = 4 \end{aligned} \tag{B.2}$$

   The ELICIT expression so far is "*between* $(s_3, ?)^{?}$ *and* $(s_4, ?)^{?}$."

   (b) *Compute symbolic translations*: Once the linguistic terms have been selected, the symbolic translations of the continuous terms are computed as follows:

$$\begin{aligned} \alpha_1 &= 6 \cdot (0.57 - 0.5) = 0.432 \\ \alpha_2 &= 6 \cdot (0.691 - 0.667) = 0.144 \\ \alpha_1, \alpha_2 &\in [-0.5, 0.5), \end{aligned} \tag{B.3}$$

   The ELICIT expression so far is "*between* $(s_3, 0.432)^{?}$ *and* $(s_4, 0.144)^{?}$."

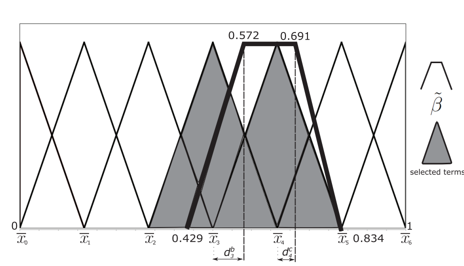


**Figure B.1** | Select linguistic terms.

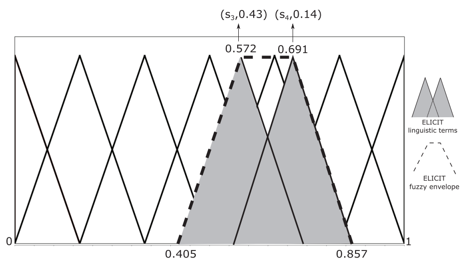


**Figure B.2** Extended Comparative LInguistiC Expressions with SymbolIc Translation (ELICIT) fuzzy envelope.

3. *Compute adjustments*: Finally, to complete the ELICIT expression, we compute the adjustments for the ELICIT expression following the steps below:
   (a) *Compute HFLTS*:

$$\begin{aligned} &E_{ELICIT}\left(between\ (s_3, 0.432)\ \text{and}\ (s_4, 0.144)\right) \\ &= \{s_k | (s_3, 0.432)\ \text{and}\ (s_4, 0.144)\ \text{and} \\ &\qquad s_3 < s_k < s_4\ \text{where}\ s_k \in S\} \\ &= \{(s_3, 0.432), (s_4, 0.144)\} \end{aligned}$$

   (b) *Compute fuzzy envelope* (see Figure B.2): The fuzzy envelope [18] of the HFLTS $\{(s_3, 0.432), (s_4, 0.144)\}$ is:

$$T_{ELICIT} = T(0.405, 0.572, 0.691, 0.857)$$

   (c) *Compute adjustments $\gamma_1$ and $\gamma_2$*:

$$\begin{aligned} \gamma_1 &= 0.429 - 0.405 = 0.024 \\ \gamma_2 &= 0.834 - 0.857 = -0.023 \\ gamma_1, \gamma_2 &\in [0, 1] \end{aligned} \tag{B.4}$$

   Finally, the ELICIT expression is completed "*between* $(s_3, 0.432)^{0.024}$ *and* $(s_4, 0.144)^{-0.023}$."

# APPENDIX C

## C.1 Proof of Theorem 1

By using operational laws of fuzzy numbers, we have

$$\begin{aligned} &\left(\zeta^{-1}(EL_i)\right)^p \otimes \left(\zeta^{-1}(EL_j)\right)^q \\ &= \left(a_i^p a_j^q, b_i^p b_j^q, c_i^p c_j^q, d_i^p d_j^q\right) \end{aligned} \tag{C.1}$$

Clearly, the right-hand side of Eq. (C.1) is a TrFN due to the assumption $0 \le a_i \le b_i \le c_i \le d_i\, (i = 1, 2, ..., n)$ on the parameters of the envelope of ELICIT expression $\zeta^{-1}(EL_i)$. Further the Eq. (C.1) is true for any pair of ELICIT expressions $\left(EL_i, EL_j\right)$ $\left(i, j \in \{1, 2, ..., n\}\right)$. As the addition of TrFNs is associative, we can extend easily to the addition of $n(n-1)$ TrFNs of the form $\left(\zeta^{-1}(EL_i)\right)^p \otimes \left(\zeta^{-1}(EL_j)\right)^q$ $\left(i, j \in \{1, 2, ..., n\}, i \ne j\right)$ and obtain

$$\begin{aligned} &\bigoplus_{\substack{i,j=1 \\ i \ne j}} \left(\zeta^{-1}(EL_i)\right)^p \otimes \left(\zeta^{-1}(EL_j)\right)^q \\ &= \left( \sum_{\substack{i,j=1 \\ j \ne i}}^{n} a_i^p a_j^q, \sum_{\substack{i,j=1 \\ j \ne i}}^{n} b_i^p b_j^q, \sum_{\substack{i,j=1 \\ j \ne i}}^{n} c_i^p c_j^q, \sum_{\substack{i,j=1 \\ j \ne i}}^{n} d_i^p d_j^q \right) \end{aligned} \tag{C.2}$$

With the help of scalar multiplication laws of TrFNs, we get

$$\begin{aligned} &\frac{1}{n(n-1)} \bigoplus_{\substack{i,j=1 \\ i \ne j}} \left(\zeta^{-1}(EL_i)\right)^p \otimes \left(\zeta^{-1}(EL_j)\right)^q \\ &= \left( \frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ j \ne i}}^{n} a_i^p a_j^q, \frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ j \ne i}}^{n} b_i^p b_j^q, \right. \\ &\qquad \left. \frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ j \ne i}}^{n} c_i^p c_j^q, \frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ j \ne i}}^{n} d_i^p d_j^q \right) \end{aligned} \tag{C.3}$$

Finally by using exponential operational laws of TrFN from Eq. (C.3), we obtain

$$\begin{aligned} &\left( \frac{1}{n(n-1)} \bigoplus_{\substack{i,j=1 \\ i \ne j}} \left(\zeta^{-1}(EL_i)\right)^p \otimes \left(\zeta^{-1}(EL_j)\right)^q \right)^{\frac{1}{p+q}} \\ &= \left( \left( \frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ j \ne i}}^{n} a_i^p a_j^q \right)^{\frac{1}{p+q}}, \left( \frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ j \ne i}}^{n} b_i^p b_j^q \right)^{\frac{1}{p+q}}, \right. \\ &\qquad \left. \left( \frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ j \ne i}}^{n} c_i^p c_j^q \right)^{\frac{1}{p+q}}, \left( \frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ j \ne i}}^{n} d_i^p d_j^q \right)^{\frac{1}{p+q}} \right) \end{aligned} \tag{C.4}$$

Since $a_i \le b_i \le c_i \le d_i$ for all $i = 1, 2, ..., n$, the monotonocity property of the $BM_{p,q} : [0,1]^n \to [0,1]$ implies Eq. (C.5).

$$\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ j\neq i}}^{n} a_i^p a_j^q\right)^{\frac{1}{p+q}} \leq \left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ j\neq i}}^{n} b_i^p b_j^q\right)^{\frac{1}{p+q}} \leq \left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ j\neq i}}^{n} c_i^p c_j^q\right)^{\frac{1}{p+q}} \leq \left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ j\neq i}}^{n} d_i^p d_j^q\right)^{\frac{1}{p+q}} \tag{C.5}$$

$$\begin{aligned} & ELICITBM_{p,q}\left(EL_{\sigma(1)}, EL_{\sigma(2)}, ..., EL_{\sigma(n)}\right) \\ & = \zeta\left(BM_{p,q}\left(a_{\sigma(1)}, a_{\sigma(2)}, ..., a_{\sigma(n)}\right),\right. \\ & BM_{p,q}\left(b_{\sigma(1)}, b_{\sigma(2)}, ..., b_{\sigma(n)}\right), \\ & BM_{p,q}\left(c_{\sigma(1)}, c_{\sigma(2)}, ..., c_{\sigma(n)}\right), \\ & \left. BM_{p,q}\left(d_{\sigma(1)}, d_{\sigma(2)}, ..., d_{\sigma(n)}\right)\right) \end{aligned} \tag{C.6}$$

It infers that $\left(\frac{1}{n(n-1)} \bigoplus_{\substack{i,j=1\\ i\neq j}} \left(\zeta^{-1}(EL_i)\right)^p \otimes \left(\zeta^{-1}\left(EL_j\right)\right)^q\right)^{\frac{1}{p+q}}$ is TrFN and therefore $ELICITBM_{p,q}(EL_1, EL_2, ..., EL_n)$ is an ELICIT expression. Hence the results.

## C.2 Proof of Theorem 2

(i) First we will show that *ELICITBM* is commutative. Let $EL_{\sigma(1)}, EL_{\sigma(2)}, ..., EL_{\sigma(n)}$ is a permutation of the ELICIT expressions $EL_1, EL_2, ..., EL_n$. With the help of computational formula Eq. (17), we can express $ELICITBM\left(EL_{\sigma(1)}, EL_{\sigma(2)}, ..., EL_{\sigma(n)}\right)$ in the form of Eq. (C.6)

The values of the parameters $p$ and $q$, and the underlying interrelationship structure among the aggregated ELICIT expressions remain intact in the permutation $\left(EL_{\sigma(1)}, EL_{\sigma(2)}, ..., EL_{\sigma(n)}\right)$. Further such interrelationship is also inherited in the parameters of the TrFNs $\left(\left(a_{\sigma(i)}, b_{\sigma(i)}, c_{\sigma(i)}, d_{\sigma(i)}\right)(i = 1, 2, ..., n)\right.$, which are the envelope of the ELICIT expression $EL_{\sigma(i)}$, $(i = 1, 2, ..., n)$. Thus, the components of the envelopes of $EL_{\sigma(i)}$, $(i = 1, 2, ..., n)$ become connected. Under this circumstance, BM exhibits the commutative property, i.e.,

$$BM_{p,q}\left(a_{\sigma(1)}, a_{\sigma(2)}, ..., a_{\sigma(n)}\right) = BM_{p,q}(a_1, a_2, ..., a_n)$$

It follows that

$$\begin{aligned} & ELICITBM_{p,q}\left(EL_{\sigma(1)}, EL_{\sigma(2)}, ..., EL_{\sigma(n)}\right) \\ & = \zeta\left(BM_{p,q}(a_1, a_2, ..., a_n), BM_{p,q}(b_1, b_2, ..., b_n),\right. \\ & \left. BM_{p,q}(c_1, c_2, ..., c_n), BM_{p,q}(d_1, d_2, ..., d_n)\right) \\ & = ELICITBM_{p,q}(EL_1, EL_2, ..., EL_n) \end{aligned} \tag{C.7}$$

(ii) Now we will show that *ELICITBM* operator is idempotent. Let $\zeta^{-1}(EL) = (a, b, c, d)$ be the envelope of the ELICIT expression $EL$. From the Eq. (17), we have

$$\begin{aligned} & ELICITBM_{p,q}(EL, EL, ..., EL) \\ & = \zeta\left(BM_{p,q}(a, a, ..., a), BM_{p,q}(b, b, ..., b),\right. \\ & \left. BM_{p,q}(c, c, ..., c), BM_{p,q}(d, d, ..., d)\right) \end{aligned} \tag{C.8}$$

Since the BM operator is idempotent, i.e., $BM_{p,q}(e, e, ..., e) = e$, we obtain from Eq. (C.8)

$$\begin{aligned} & ELICITBM_{p,q}(EL, EL, ..., EL) \\ & = \zeta(a, b, c, d) = \zeta\left(\zeta^{-1}(EL)\right) = EL \end{aligned}$$

(iii) Now we will prove that *ELICITBM* is ratio-scale invariant. Let $r > 0$ be a scalar. From the scalar multiplication law of TrFN, we have $r\zeta^{-1}(EL_i) = (ra_i, rb_i, rc_i, rd_i)$. From the definition of ELICITBM, we obtain

$$\begin{aligned} & ELICITBM_{p,q}(rEL_1, rEL_2, ..., rEL_n) \\ & = \left(\frac{1}{n(n-1)} \bigoplus_{\substack{i,j=1\\ i\neq j}} \left(\zeta^{-1}(rEL_i)\right)^p \otimes \left(\zeta^{-1}\left(rEL_j\right)\right)^q\right)^{\frac{1}{p+q}} \\ & = \zeta\left(BM_{p,q}(ra_1, ra_2, ..., ra_n), BM_{p,q}(rb_1, rb_2, ..., rb_n),\right. \\ & \quad \left. BM_{p,q}(rc_1, rc_2, ..., rc_n), BM_{p,q}(rd_1, rd_2, ..., rd_n)\right) \end{aligned} \tag{C.9}$$

As the BM operator is ratio-scale invariant i.e. $BM_{p,q}(re_1, re_2, ..., re_n) = rBM_{p,q}(e_1, e_2, ..., e_n)$, from Eq. (C.9) we have

$$\begin{aligned} & ELICITBM_{p,q}(rEL_1, rEL_2, ..., rEL_n) \\ & = \zeta\left(rBM_{p,q}(a_1, a_2, ..., a_n), rBM_{p,q}(b_1, b_2, ..., b_n),\right. \\ & \left. rBM_{p,q}(c_1, c_2, ..., c_n), rBM_{p,q}(d_1, d_2, ..., d_n)\right) \\ & = r\zeta\left(BM_{p,q}(a_1, a_2, ..., a_n), BM_{p,q}(b_1, b_2, ..., b_n),\right. \\ & \left. BM_{p,q}(c_1, c_2, ..., c_n), BM_{p,q}(d_1, d_2, ..., d_n)\right) \\ & = rELICITBM_{p,q}(EL_1, EL_2, ..., EL_n) \end{aligned}$$

## C.3 Proof of the Theorem 3

We will show that ELICITBM is bounded. Since $a_i \geq \min_i a_i$ for all $i$, the monotonocity and idempotency of properties of the BM operator implies that

$$\begin{aligned} & BM_{p,q}(a_1, a_2, ..., a_n) \leq BM\left(\min_i a_i, \min_i a_i, ..., \min_i a_i\right) \\ & = \min_i a_i \end{aligned}$$

Similarly, we can obtain

$$\begin{aligned} BM_{p,q}(b_1, b_2, ..., b_n) &\geq \min_i b_i \\ BM_{p,q}(c_1, c_2, ..., c_n) &\geq \min_i c_i \\ BM_{p,q}(d_1, d_2, ..., d_n) &\geq \min_i d_i. \end{aligned}$$

From these inequalities, we have

$$\begin{aligned} & \left(BM_{p,q}(a_1, a_2, ..., a_n), BM_{p,q}(b_1, b_2, ..., b_n),\right. \\ & \left. BM_{p,q}(c_1, c_2, ..., c_n), BM_{p,q}(d_1, d_2, ..., d_n)\right) \\ & \geq \left(\min_i a_i, \min_i b_i, \min_i c_i, \min_i d_i\right) \end{aligned} \tag{C.10}$$

Note that the inequality Eq. (C.10) is in the sense of lexicographic ordering of TrFNs, i.e., $(a_1, b_1, c_1, d_1) \geq (a_2, b_2, c_2, d_2)$ iff $a_1 \geq a_2$, $b_1 \geq b_2$, $c_1 \geq c_2$ and $d_1 \geq d_2$. From Eq. (C.10), we have

$$\begin{aligned} & ELICITBM_{p,q}\left(EL_1, EL_2, ..., EL_n\right) \\ & = \zeta\left(BM_{p,q}\left(a_1, a_2, ..., a_n\right), BM_{p,q}\left(b_1, b_2, ..., b_n\right), \right. \\ & \quad \left. BM_{p,q}\left(c_1, c_2, ..., c_n\right), BM_{p,q}\left(d_1, d_2, ..., d_n\right)\right) \\ & \geq \zeta\left(\min_i a_i, \min_i b_i, \min_i c_i, \min_i d_i\right) \end{aligned}$$

Similarly, we can show that

$$\begin{aligned} & ELICITBM_{p,q}\left(EL_1, EL_2, ..., EL_n\right) \\ & \leq \zeta\left(\max_i a_i, \max_i b_i, \max_i c_i, \max_i d_i\right) \end{aligned}$$

Hence the result.