IEEE Access
Multidisciplinary | Rapid Review | Open Access Journal

Received November 4, 2019, accepted November 25, 2019, date of publication November 27, 2019,
date of current version December 12, 2019.

Digital Object Identifier 10.1109/ACCESS.2019.2956393

Paradigm Shift Toward Aggregation Strategies in Proportional Hesitant Fuzzy Multi-Criteria Group Decision Making Models of Advanced Practice for Selecting Electric Vehicle Battery Supplier

JIAN-PENG CHANG[1,3], ZHEN-SONG CHEN[2], (Member, IEEE), XIAO-LU LIU[2], WEN-TAO KONG[2], SHENG-HUA XIONG[4], AND LUIS MARTÍNEZ[5], (Senior Member, IEEE)

[1]Collaborative Innovation Center for Chongqing's Modern Trade Logistics & Supply Chain, Chongqing Technology and Business University, Chongqing 400067 China
[2]School of Civil Engineering, Wuhan University, Wuhan 430072, China
[3]School of Business Planning, Chongqing Technology and Business University, Chongqing 400067, China
[4]Civil Aviation Safety Engineering Institute, Civil Aviation Flight University of China, Guanghan 618307, China
[5]Department of Computer Science, University of Jaén, 23071 Jaén, Spain

Corresponding authors: Zhen-Song Chen (zschen@whu.edu.cn) and Wen-Tao Kong (kongwt73@whu.edu.cn)

This work was supported in part by the National Natural Science Foundation of China under Grant 71801175, Grant 71871171, Grant 71971182, and Grant 71373222, in part by the Theme-based Research Projects of the Research Grants Council under Grant T32-101/15-R, in part by the Fundamental Research Funds for the Central Universities under Grant 2042018kf0006, in part by the Spanish Government Project under Grant TIN2015-66524-P and Grant PGC2018-099402-B-I00, in part by the Scientific Research Start-up Foundation of Chongqing Technology and Business University under Grant 1855016, in part by the Chongqing Doctoral Program of Social and Scientific Planning under Grant 2018BS79, in part by the National Social Science Fund of China under Grant 18BGL007, in part by the National Key Research and Development Program of China under Grant 2018YFC0810600, and in part by the Civil Aviation Flight University of China Scientific Research Foundation under Grant J2018-12.

ABSTRACT Since its initiation, hesitant fuzzy sets (HFSs) have gained prominence thanks to their capability to describe the hesitation of experts to assign membership degrees to objects belonging to a concept. Proportional hesitant fuzzy sets (PHFSs) are an important extension of HFSs and are characterized by the combination of possible membership degrees and their associated proportional information. PHFSs have a huge application potential for hesitant fuzzy GDM problems, because the proportional information in PHFSs can be determined objectively and the introduction of this new information dimension can effectively reduce the uncertainty. Nevertheless, PHFSs have not yet attracted sufficient attention from researchers and practitioners, which motivates us to expand the theory of PHFSs and explore its application potential. The main work comprises the following three aspects: First, we define some basis operations on PHFSs, develop aggregation operators for PHFSs, and demonstrate their properties and interrelationships to lay the theoretical foundations for the application of PHFSs. Next, we construct two multicriteria group decision making (MCGDM) models based on the proposed PHFS-based aggregation operators to bridge between theory and practice for PHFSs. In this step, we propose a method for transforming HFSs or fuzzy sets (FSs) into PHFSs, and two methods based on the maximum entropy principle are proposed for specifying criterion weights. Finally, we investigate a practical case study of the problem of selecting an electric vehicle battery (EVB) supplier to validate the outstanding advantages of PHFSs, explore the compensation characteristics and the applicability of the PHFS-based aggregation operators, and demonstrate the effectiveness and feasibility of the proposed MCGDM models. This paper provides a useful reference for MCGDM in a hesitant fuzzy context.

INDEX TERMS Hesitant fuzzy set (HFS), proportional hesitant fuzzy set (PHFS), aggregation operators, multicriteria group decision making (MCGDM), electric vehicle battery (EVB) supplier selection.

I. INTRODUCTION

A fuzzy set is a class of objects with a continuum of membership degrees and is represented mathematically by a membership function that assigns a membership degree in the

The associate editor coordinating the review of this manuscript and approving it for publication was Jerry Chun-Wei Lin.

interval [0,1] to each object [1]. Starting with the original work of Zadeh [1], fuzzy sets have gained significant attention because of their outstanding ability to model uncertainty. At present,several generalizations exist to relax the requirement that only a single value between 0 and 1 be assigned to the membership of an element. Examples include type-2 fuzzy sets [2], intuitionistic fuzzy sets [3], Pythagorean fuzzy sets [4],and hesitant fuzzy sets [5]. Type-2 fuzzy sets incorporate uncertainty about the membership function into fuzzy sets to address the drawback of the original fuzzy sets whereby the membership function has no uncertainty associated with it [6], [7]. Type-n fuzzy sets generalize type-2 fuzzy sets by permitting the membership to be a type-$(n-1)$ fuzzy set. Intuitionistic fuzzy sets are characterized by membership degree and nonmembership degree which satisfy the condition that their sum is equal to or less than one. As a generalization of IFSs, PFSs accommodate more relaxed condition that the square sum of the membership degree and nonmembership degree is equal to or less than one [8]–[10]. Hesitant fuzzy sets (HFSs) are introduced to model the scenario in which a set of values are possible for assigning membership degree to an object belonging to a concept. Scenarios involving hesitation can be classified into the following two types:

- **Scenario 1**: When an expert is required to assign a membership degree to an object belonging to a concept, the expert hesitates among a set of possible values. To characterize the hesitation of the expert, all possible values are retained in a set instead of selecting a single value.
- **Scenario 2**: When each member of an expert team is asked to assign a membership degree to an object belonging to a concept, the experts have different opinions and fail to reach an agreement. Instead of using an aggregation operator to fuse these values into a single value, all values are retained in a set to characterize the hesitation of this expert team.

Both types of hesitant scenarios are common in real decision-making problems. Thus, since their introduction, HFSs have attracted significant research attention [11]–[17]. Existing theoretical research on HFSs mainly focuses on diverse extensions, information measures, and aggregation operators.

Currently, various extensions of HFSs have been proposed in a bid to model uncertainty from different perspectives [12]; for example, dual hesitant fuzzy sets [18], interval-valued hesitant fuzzy sets [19],Pythagorean hesitant fuzzy sets [20]–[22], hesitant fuzzy linguistic terms sets [23], and proportional hesitant fuzzy sets (PHFSs) [24]. In the field of information measurement, researchers mainly concentrate on the construction of distance measures, correlation coefficients, entropy, and cross entropy [19], [25]. In addition, some scholars have turned their attention to aggregation operators for HFSs and facilitate the decision-making process; such operators include hesitant fuzzy averaging operators [26], hesitant fuzzy geometric operators [26], hesitant fuzzy power aggregation operators [27], and hesitant fuzzy geometric Bonferroni means [28].

PHFSs are an extension of HFSs that were introduced by [24] and are characterized by a predefined set of possible membership degrees for elements and the proportional information of each membership degree. These are mainly used to address group-decision-making (GDM) problems in a fuzzy hesitant context. Initially, PHFSs were introduced to model ***Scenario 2*** with the proportion of each membership degree measurable. In this scenario, each member of an expert team is required to output individual assessment information in the form of a fuzzy set (FS), and PHFSs can be used to characterize the collective assessment information in which membership degrees comprise all values given by all the experts, and the proportion of each membership degree equals the proportion of experts who output it. In practical decision-making scenarios, it is hard for experts to provide assessment information in the form of PHFSs, but outputting assessment information in the form of HFSs is feasible. ***Scenario 1*** describes a GDM context in which an expert may prefer to articulate her preferences in the form of a HFS when she hesitates between possible values. Thus, the initial condition should be relaxed to enable PHFSs to characterize the collective assessment information synthesized from the individual assessment information in the form of FSs or HFSs. In addition, the linguistic counterpart of PHFSs, which is called a proportional hesitant fuzzy linguistic term set [29], is an effective alternative when linguistic uncertainty needs to be modeled in real-life scenarios.

PHFSs are easily mistaken by the concepts of probability hesitant fuzzy sets proposed by Zhu and Xu [30] and weighted hesitant fuzzy sets proposed by Zhang and Wu [31] owing to the fact that the three concepts share the same mathematical structure, but there are obvious differences between them. Subsequently, we highlight the merits of PHFSs by differentiating these similar concepts. For probability hesitant fuzzy sets, the authors clearly state that the probability information is assigned to each possible membership degree with the sum of all the probability information equaling 1. The probability information is derived from the subjective judgement of expert to measure the likelihood of each membership degree in HFS. For weighted hesitant fuzzy sets, different weights are assigned to all the possible membership degrees and the sum of these weights is equal to 1. The weight information is also specified by experts to depict the relative importance ratings of each membership degree. Thus, both the probability information in probability hesitant fuzzy sets and the weight information in weighted hesitant fuzzy sets are subjectively given by experts, and probability hesitant fuzzy sets and weighted hesitant fuzzy sets can be used to model assessment information in hesitant fuzzy GDM settings both individually and collectively. Nevertheless, it is difficult for decision makers (DMs) to further output the corresponding probability information or weighting information besides hesitant fuzzy evaluations in practical decision-making context. For PHFSs, proportional information is introduced to depict the collective preferences on all the possible membership degrees. Thus, PHFSs are only used to model collective assessment information. More importantly, the proportional information can be objectively calculated from the assessment information given by all the DMs, thus PHFSs are more preferable than probability hesitant fuzzy sets and weighted hesitant fuzzy sets in hesitant fuzzy GDM settings. Furthermore, the introduction of a new dimension of information about proportional information can reduce the uncertainty of characterizing group evaluations while preserving the original assessment information, effectively improving the reliability of decision-making results. Thus, PHFSs has a significant application potential for hesitant fuzzy GDM problems.

Xiong *et al.* [24] defined a normalized Hamming distance measure and a comparison law for PHFSs and developed a multiple criterion GDM method in the context of PHFSs. However, few studies have since followed up on the theory and the application of PHFSs, which motivates us to expand the theory of PHFSs and explore its application potential.

Information aggregation functions, which are used to combine various inputs coming from different sources into a single representative value [32], [33], have been widely applied in the aspects of decision-making, expert systems, risk analysis, and image processing [33]–[36]. This paper concentrates on the proposal of proportional hesitant fuzzy weighted averaging (PHFWA) operator, proportional hesitant fuzzy weighted geometric (PHFWG) operator, proportional hesitant fuzzy ordered weighted averaging (PHFOWA) operator, proportional hesitant fuzzy ordered weighted geometric (PHFOWG) operator, and further puts forward their generalized types, including the generalized PHFWA operator (GPHFWA), generalized PHFWG operator (GPHFWG), generalized PHFOWA operator (GPHFOWA), and generalized PHFOWG operator (GPHFOWG), to extend the weighted averaging and geometric aggregation operators, the ordered weighted averaging and geometric aggregations to accommodate the PHFS environments and to lay the theoretical foundations for the application of PHFSs. In addition, the PHFS-based aggregation operators are preceded by the operation laws for PHFSs to serve as the basis of developing these aggregation operators.

As an important extension of multi-criteria decision making (MCDM) [37]–[41] and one of the main application fields of aggregation functions [42]–[44], multi-criteria GDM (MCGDM) has garnered considerable attention from scholars and practitioners and been widely applied to various socioeconomic fields including management science, social science, economics, public administration, military research, etc. [45]–[52] In this paper, we further develop two MCGDM models that use the PHFS-based aggregation operators to connect theory and practice for PHFS. One model is constructed based on the GPHFWA or GPHFWG operator, and the other model is based on the GPHFOWA or GPHFOWG operator. The main reason for developing two different MCGDM models is that clear differences exist between the prerequisites of applications of the GPHFWA or GPHFWG operator and those of the GPHFOWA or GPHFOWG operator. Additionally, because PHFSs exist only as collective assessment information during hesitant fuzzy GDM processes,we propose a method to transform individual assessment information outputted by DMs in the form of FSs or HFSs to collective assessment information in the form of PHFSs during the construction of MCGDM models. Furthermore, for the proposed MCGDM model based on the GPHFWA or GPHFWG operator, we provide a method based on maximum entropy to specify the weights of the criteria. For the proposed MCGDM model based on the GPHFOWA or GPHFOWG operator, we provide a similar method based on maximum entropy and the attitudinal character given by DMs. Finally, we conduct a thorough practical case study of the selection of a strategic supplier of electric-vehicle batteries (EVBs), which involves multiple qualitative and quantitative criteria and necessitates a multifunctional expert team. Therefore, this problem can be considered as an MCGDM problem in a hesitant fuzzy context. Through this case study, we not only test the properties of the PHFS-based aggregation operators and demonstrate the effectiveness and feasibility of the proposed MCGDM models but also explore the compensation characteristics and the applicability of these aggregation operators and validate the advantages of PHFSs.

This paper is organized as follows: Section 2 briefly reviews several basic concepts of HFSs and PHFSs and presents some basic laws for operations on PHFSs. Section 3 presents a series of PHFS-based averaging operators and their properties and interrelationships. Two MCGDM models based on PHFS-based averaging operators are developed in Sec. 4, in which we propose a method for transforming FSs or HFSs into PHFSs and methods to specify criterion weights. Section 5 undertakes a practical case study to validate the effectiveness and practicality of the proposed techniques. Finally, Sec. 6 concludes this paper.

II. PRELIMINARIES

This section reviews the basic concepts of HFSs and PHFSs and gives some basic laws for operations on PHFSs.

A. HESITANT FUZZY SETS

A hesitant fuzzy set is defined by a function that returns a set of possible membership degrees for each element in the domain [5], [53].

Definition 1 [5], [53]: Let X be a reference set: a HFS on X is a function that, when applied to X, returns a subset of $[0, 1]$.

The HFS can be mathematically expressed as [25], [26]

$$E = \{\langle x, h_E(x)\rangle \,|x \in X\},$$

where $h_E(x)$ is a set of values in $[0, 1]$ that denotes the possible membership degrees of the element $x \in X$ to the set E. For convenience, [26] called $h = h_E(x)$ a hesitant fuzzy element (HFE).

Torr [5], Torra and Narukawa [53], and Xia and Xu [26] defined the following operations on HFEs:

Definition 2 [5], [26], [53]: Let h, h_1, and h_2 be three HFEs on a fixed set X; then

(1) $h^c = \cup_{\gamma\in h}\{1-\gamma\}$;
(2) $h_1 \cup h_2 = \cup_{\gamma_1\in h,\gamma_2\in h} \max\{\gamma_1, \gamma_2\}$;
(3) $h_1 \cap h_2 = \cup_{\gamma_1\in h,\gamma_2\in h} \min\{\gamma_1, \gamma_2\}$;
(4) $h^\lambda = \cup_{\gamma\in h}\{\gamma^\lambda\},\ \lambda > 0$;
(5) $\lambda h = \cup_{\gamma\in h}\{1-(1-\gamma)^\lambda\},\ \lambda > 0$;
(6) $h_1 \oplus h_2 = \cup_{\gamma_1\in h,\gamma_2\in h}\{\gamma_1+\gamma_2-\gamma_1\gamma_2\}$;
(7) $h_1 \otimes h_2 = \cup_{\gamma_1\in h,\gamma_2\in h}\{\gamma_1\gamma_2\}$.

B. PROPORTIONAL HESITANT FUZZY SETS

A HFS and its related operations can be used to deal with GDM problems in a fuzzy, hesitant context. However, if the proportion of each membership degree is measurable, the information characterized by the HFS is incomplete. To bridge this gap, Xiong et al. [24] proposed PHFSs based on HFSs.

Definition 3 [24]: Let X be a reference set; the PHFS E on X is then represented as

$$E = \{\langle x, \mathfrak{P}_E(x)\rangle \,|x\in X\} = \{\langle x, (h_E(x), p_E(x))\rangle \,|x\in X\},$$

where

(a) $h_E(x) = \{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ is a set of values in [0, 1] that represents n possible degrees of membership of the element x to the set X; and
(b) $p_E(x) = \{\tau_1, \tau_2, \ldots, \tau_n\}$ is a set of values in [0, 1], where $\tau_i\ (i = 1, 2, \ldots, n)$ is the proportion of membership degree $\gamma_i\ (i = 1, 2, \ldots, n)$ and $\sum_{i=1}^n \tau_i = 1$.

For convenience, Xiong et al. [24] called $\mathfrak{P} = \mathfrak{P}_E(x)$ a PHFE. This paper similarly uses "PHS" to refer to the set of all PHFEs.

Definition 4 [24]: Let X be a reference set; for any $x \in X$,

(1) $\mathfrak{P}_E(x) = \{(0, 1)\}$ is the empty PHFS, denoted by $\emptyset$;
(2) $\mathfrak{P}_E(x) = \{(1, 1)\}$ is the full PHSF, denoted by Ω.

Definition 5 [24]: Given a PHFS represented by its PHFE $\mathfrak{P}$, the complement of $\mathfrak{P}$ is

$$\mathfrak{P}^c = \cup_{(\gamma,\tau)\in\mathfrak{P}}\{(1-\gamma, \tau)\}.$$

If $l(\mathfrak{P})$ represents the number of elements in PHFE $\mathfrak{P}$, it is difficult to calculate the distance measure between PHFSs A and B because $l(\mathfrak{P}_A(x))$ is usually not equal to $l(\mathfrak{P}_B(x))$ for any $x \in X$. Supposing $l_x = \max\{l(\mathfrak{P}_A(x)),\ l(\mathfrak{P}_B(x))\}$, this problem can be handled by the following two steps:

(1) Ordering: Arrange the elements in $l(\mathfrak{P}_A(x))$ and $l(\mathfrak{P}_B(x))$ in decreasing order according to the product values of the membership degrees and their associated proportions.
(2) Adding: Add several times the PHFE with smaller $l(*)$ with element (0, 0) until both have the same number of elements, i.e., l_x.

The distance measure of PHFS is then given as follows:

Definition 6 [24]: Let A and B be two PHFSs on the reference set $X = \{x_1, x_2, \ldots, x_n\}$, then the proportional hesitant normalized Hamming distance is

$$d(A,B) = \frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{2l_{x_i}}\sum_{j=1}^{l_{x_i}}\left(\left|\gamma_A^{\sigma(j)}(x_i)\cdot\tau_A^{\sigma(j)}(x_i)-\gamma_B^{\sigma(j)}(x_i)\cdot\tau_B^{\sigma(j)}(x_i)\right| + \left|\tau_A^{\sigma(j)}(x_i)-\tau_B^{\sigma(j)}(x_i)\right|\right)\right]$$

where $l_{x_i} = \max\{l(\mathfrak{P}_A(x_i)),\ l(\mathfrak{P}_B(x_i))\}$, and $\gamma_A^{\sigma(j)}(x_i)$, $\tau_A^{\sigma(j)}(x_i)$ and $\gamma_B^{\sigma(j)}(x_i)$, $\tau_B^{\sigma(j)}(x_i)$ are the jth largest product value in PHFEs $\mathfrak{P}_A(x_i)$ and $\mathfrak{P}_B(x_i)$, respectively.

The following two definitions are used to compare PHFSs:

Definition 7 [24]: Let $\mathfrak{P}$ be a PHFE on the reference set X; the score function of $\mathfrak{P}$ is then defined as

$$s(\mathfrak{P}) = \sum_{(\gamma,\tau)\in\mathfrak{P}} \gamma\cdot\tau,$$

and the deviation function of $\mathfrak{P}$ is defined as

$$t(\mathfrak{P}) = \sum_{(\gamma,\tau)\in\mathfrak{P}} \tau[\gamma - s(\mathfrak{P})]^2.$$

Definition 8 [24]: Let $\mathfrak{P}_1$ and $\mathfrak{P}_2$ be two PHFEs on the reference set X:

(1) if $s(\mathfrak{P}_1) > s(\mathfrak{P}_2)$, then $\mathfrak{P}_1 > \mathfrak{P}_2$;
(2) if $s(\mathfrak{P}_1) = s(\mathfrak{P}_2)$ and $t(\mathfrak{P}_1) < t(\mathfrak{P}_2)$, then $\mathfrak{P}_1 > \mathfrak{P}_2$;
(3) if $s(\mathfrak{P}_1) = s(\mathfrak{P}_2)$, $t(\mathfrak{P}_1) = t(\mathfrak{P}_2)$,
  (a) and $d(\{\mathfrak{P}_1\}, \Omega) = d(\{\mathfrak{P}_2\}, \Omega)$, then $\mathfrak{P}_1 = \mathfrak{P}_2$;
  (b) and $d(\{\mathfrak{P}_1\}, \Omega) < d(\{\mathfrak{P}_2\}, \Omega)$, then $\mathfrak{P}_1 > \mathfrak{P}_2$;

where Ω is the full proportional hesitant fuzzy set and $d(A, B)$ is the distance measure for PHFSs, as defined by Xiong et al. [24].

C. OPERATIONAL LAWS FOR PROPORTIONAL HESITANT FUZZY ELEMENTS

In this section, we present some basic operations on PHFEs to lay the foundations for the development of PHFS-based aggregation operators.

Let $\mathfrak{P}_1$ and $\mathfrak{P}_2$ be two PHFEs in the reference set X and suppose the membership degree of the $x \in X$ for the set "1" and that for the set "2" are mutually independent. Based on the operation laws for HFSs [26], the following operations are defined from the angle of probability:

Definition 9: Let $\mathfrak{P}$, $\mathfrak{P}_1$, and $\mathfrak{P}_2$ be three PHFEs in the fixed set X, then

(1) $\mathfrak{P}^\lambda = \cup_{(\gamma,\tau)\in\mathfrak{P}}\{(\gamma^\lambda, \tau)\},\ \lambda > 0$;
(2) $\lambda\mathfrak{P} = \cup_{(\gamma,\tau)\in\mathfrak{P}}\{(1-(1-\gamma)^\lambda, \tau)\},\ \lambda > 0$;
(3) $\mathfrak{P}_1\oplus\mathfrak{P}_2 = \cup_{(\gamma_1,\tau_1)\in\mathfrak{P}_1,(\gamma_2,\tau_2)\in\mathfrak{P}_2}\{(\gamma_1+\gamma_2-\gamma_1\gamma_2, \tau_1\tau_2)\}$;
(4) $\mathfrak{P}_1\otimes\mathfrak{P}_2 = \cup_{(\gamma_1,\tau_1)\in\mathfrak{P}_1,(\gamma_2,\tau_2)\in\mathfrak{P}_2}\{(\gamma_1\gamma_2, \tau_1\tau_2)\}$.

Theorem 1: For three PHFEs β, β_1, and β_2 in the fixed set X, we have

(1) $(\mathfrak{P}^C)^\lambda = (\lambda\mathfrak{P})^C$, $\lambda > 0$;
(2) $\lambda\mathfrak{P}^C = (\mathfrak{P}^\lambda)^C$, $\lambda > 0$;
(3) $\mathfrak{P}_1^C \oplus \mathfrak{P}_2^C = (\mathfrak{P}_1 \otimes \mathfrak{P}_2)^C$;
(4) $\mathfrak{P}_1^C \otimes \mathfrak{P}_2^C = (\mathfrak{P}_1 \oplus \mathfrak{P}_2)^C$.

Proof: (1) For any $\lambda > 0$,

$$\begin{aligned}\left(\mathfrak{P}^C\right)^\lambda &= \left[\cup_{(\gamma,\tau)\in\mathfrak{P}} \{(1-\gamma,\tau)\}\right]^\lambda \\ &= \cup_{(\gamma,\tau)\in\mathfrak{P}} \left\{\left((1-\gamma)^\lambda, \tau\right)\right\} = (\lambda\mathfrak{P})^C.\end{aligned}$$

(2) For any $\lambda > 0$,

$$\begin{aligned}\lambda\mathfrak{P}^C &= \cup_{(\gamma,\tau)\in\mathfrak{P}} \left\{\left(1-(1-(1-\gamma))^\lambda, \tau\right)\right\} \\ &= \cup_{(\gamma,\tau)\in\mathfrak{P}} \left\{\left(1-\gamma^\lambda, \tau\right)\right\} \\ &= \left(\mathfrak{P}^\lambda\right)^C.\end{aligned}$$

(3)

$$\begin{aligned}\mathfrak{P}_1^C \oplus \mathfrak{P}_2^C &= \cup_{(\gamma_1,\tau_1)\in\mathfrak{P}_1,(\gamma_2,\tau_2)\in\mathfrak{P}_2} \{((1-\gamma_1)+(1-\gamma_2) \\ &\quad -(1-\gamma_1)(1-\gamma_2), \tau_1\tau_2)\} \\ &= \cup_{(\gamma_1,\tau_1)\in\mathfrak{P}_1,(\gamma_2,\tau_2)\in\mathfrak{P}_2} \{(1-\gamma_1\gamma_2, \tau_1\tau_2)\} \\ &= (\mathfrak{P}_1\otimes\mathfrak{P}_2)^C.\end{aligned}$$

(4)

$$\begin{aligned}&\mathfrak{P}_1^C \otimes \mathfrak{P}_2^C \\ &= \cup_{(\gamma_1,\tau_1)\in\mathfrak{P}_1,(\gamma_2,\tau_2)\in\mathfrak{P}_2} \{((1-\gamma_1)(1-\gamma_2), \tau_1\tau_2)\} \\ &= \cup_{(\gamma_1,\tau_1)\in\mathfrak{P}_1,(\gamma_2,\tau_2)\in\mathfrak{P}_2} \{(1-\gamma_1-\gamma_2+\gamma_1\gamma_2, \tau_1\tau_2)\} \\ &= (\mathfrak{P}_1\oplus\mathfrak{P}_2)^C.\end{aligned}$$

Theorem 2: Let $\mathfrak{P}_j\,(j = 1, 2, \ldots, n)$ be a collection of PHFEs and let $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the weight vector of $\mathfrak{P}_j\,(j = 1, 2, \ldots, n)$ with $w_j \in [0, 1]$ and $\sum_{j=1}^n w_j = 1$, $\lambda > 0$. Then

(1) $\oplus_{j=1}^n(\omega_j\mathfrak{P}_j^C) = (\otimes_{j=1}^n \mathfrak{P}_j^{\omega_j})^C$;
(2) $\otimes_{j=1}^n(\mathfrak{P}_j^C)^{\omega_j} = (\oplus_{j=1}^n \omega_j\mathfrak{P}_j)^C$;
(3) $\{\oplus_{j=1}^n[\omega_j(\mathfrak{P}_j^C)^\lambda]\}^{1/\lambda} = \{\frac{1}{\lambda}[\otimes_{j=1}^n(\lambda\mathfrak{P}_j)^{\omega_j}]\}^C$;
(4) $\frac{1}{\lambda}[\otimes_{j=1}^n(\lambda\mathfrak{P}_j^C)^{\omega_j}] = \{[\oplus_{j=1}^n(\omega_j\mathfrak{P}_j^\lambda)]^{1/\lambda}\}^C$.

Proof: For any $(\gamma_1, \tau_1) \in \mathfrak{P}_1$, $(\gamma_2, \tau_2) \in \mathfrak{P}_2, \ldots, (\gamma_n, \tau_n) \in \mathfrak{P}_n$, we have

(1)

$$\begin{aligned}&\oplus_{j=1}^n\left(\omega_j\mathfrak{P}_j^C\right) \\ &= \cup_{(\gamma_1,\tau_1)\in\mathfrak{P}_1,\ldots,(\gamma_n,\tau_n)\in\mathfrak{P}_n}\left\{\left(1-\prod_{j=1}^n\gamma_j^{\omega_j}, \prod_{j=1}^n\tau_j\right)\right\} \\ &= \left[\cup_{(\gamma_1,\tau_1)\in\mathfrak{P}_1,\ldots,(\gamma_n,\tau_n)\in\mathfrak{P}_n}\left\{\left(\prod_{j=1}^n\gamma_j^{\omega_j}, \prod_{j=1}^n\tau_j\right)\right\}\right]^C \\ &= \left(\otimes_{j=1}^n\mathfrak{P}_j^{\omega_j}\right)^C,\end{aligned}$$

(2)

$$\begin{aligned}&\otimes_{j=1}^n\left(\mathfrak{P}_j^C\right)^{\omega_j} \\ &= \cup_{(\gamma_1,\tau_1)\in\mathfrak{P}_1,\ldots,(\gamma_n,\tau_n)\in\mathfrak{P}_n}\left\{\left(\prod_{j=1}^n(1-\gamma_j)^{\omega_j}, \prod_{j=1}^n\tau_j\right)\right\} \\ &= \left[\cup_{(\gamma_1,\tau_1)\in\mathfrak{P}_1,\ldots,(\gamma_n,\tau_n)\in\mathfrak{P}_n}\left\{\left(1-\prod_{j=1}^n(1-\gamma_j)^{\omega_j}, \prod_{j=1}^n\tau_j\right)\right\}\right]^C \\ &= \left(\oplus_{j=1}^n\omega_j\mathfrak{P}_j\right)^C,\end{aligned}$$

(3)

$$\begin{aligned}&\left\{\oplus_{j=1}^n\left[\omega_j\left(\mathfrak{P}_j^C\right)^\lambda\right]\right\}^{1/\lambda} \\ &= \cup_{(\gamma_1,\tau_1)\in\mathfrak{P}_1,\ldots,(\gamma_n,\tau_n)\in\mathfrak{P}_n}\left\{\left(1-\prod_{j=1}^n\left[1-(1-\gamma_j)^\lambda\right]^{\omega_j}\right)^{1/\lambda}, \prod_{J=1}^n\tau_j\right\} \\ &= \cup_{(\gamma_1,\tau_1)\in\mathfrak{P}_1,\ldots,(\gamma_n,\tau_n)\in\mathfrak{P}_n}\left\{\left[1-\left(1-\prod_{j=1}^n\left[1-(1-\gamma_j)^\lambda\right]^{\omega_j}\right)^{1/\lambda}, \prod_{J=1}^n\tau_j\right]\right\}^C \\ &= \left\{\frac{1}{\lambda}\left[\otimes_{j=1}^n(\lambda\mathfrak{P}_j)^{\omega_j}\right]\right\},\end{aligned}$$

(4)

$$\begin{aligned}&\frac{1}{\lambda}\left[\otimes_{j=1}^n\left(\lambda\mathfrak{P}_j^C\right)^{\omega_j}\right] \\ &= \cup_{(\gamma_1,\tau_1)\in\mathfrak{P}_1,\ldots,(\gamma_n,\tau_n)\in\mathfrak{P}_n}\left\{\left\{1-\left[1-\prod_{j=1}^n\left(1-\gamma_j^\lambda\right)^{\omega_j}\right]^{1/\lambda}, \prod_{J=1}^n\tau_j\right\}\right\} \\ &= \left(\cup_{(\gamma_1,\tau_1)\in\mathfrak{P}_1,\ldots,(\gamma_n,\tau_n)\in\mathfrak{P}_n}\left\{\left\{\left[1-\prod_{j=1}^n\left(1-\gamma_j^\lambda\right)^{\omega_j}\right]^{1/\lambda}, \prod_{J=1}^n\tau_j\right\}\right\}\right)^C \\ &= \left\{\left[\oplus_{j=1}^n\left(\omega_j\mathfrak{P}_j^\lambda\right)\right]^{1/\lambda}\right\}^C.\end{aligned}$$

III. AGGREGATION OPERATORS FOR PROPORTIONAL HESITANT FUZZY SETS

This section presents a series of aggregation operators for PHFSs to lay the foundation for the construction of MCGDM models, including the PHFWA, PHFWG, PHFOWA, and PHFOWG operators and their generalized forms. In addition, we validate the properties and interrelationships of these aggregation operators.

A. PROPORTIONAL HESITANT FUZZY WEIGHTED AVERAGING OPERATOR

This section defines the PHFWA operator based on PHFEs and the traditional weighted averaging operator and gives the properties of the PHFWA operator.

Definition 10: Let $E = \{\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_n\}$ be n PHFEs defined for a fixed set X, and let Θ be a function of E, $\Theta : [0,1]^n \to [0,1]$. In this case,

$$\Theta_E = \cup_{(\gamma,\tau)\in\{\mathfrak{P}_1\times\mathfrak{P}_2\times\cdots\times\mathfrak{P}_n\}} \{\Theta(\gamma,\tau)\}.$$

Following **Definition 10**, we begin our discussion.

Definition 11: Let $\mathfrak{P}_j\ (j = 1, 2, \ldots, n)$ be a collection of PHFEs. A PHFWA operator is a mapping $PH^n \to PH$ such that

$$PHFWA(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_n) = \mathop{\oplus}_{j=1}^{n} (\omega_j \mathfrak{P}_j),$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weighting vector of $\mathfrak{P}_j\ (j = 1, 2, \ldots, n)$ with $\omega_j \in [0,1]$ and $\sum_{j=1}^{n} \omega_j = 1$.

In particular, if $\omega = (1/n, 1/n, \ldots, 1/n)^T$, then the PHFWA operator reduces to the PHFA operator:

$$PHFA(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_n) = \mathop{\oplus}_{j=1}^{n} \left(\frac{1}{n}\mathfrak{P}_j\right).$$

Theorem 3: Let $\mathfrak{P}_j\ (j = 1, 2, \ldots, n)$ be a collection of PHFEs and let $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the weighting vector of $\mathfrak{P}_j\ (j = 1, 2, \ldots, n)$ with $\omega_j \in [0,1]$ and $\sum_{j=1}^{n} \omega_j = 1$. The aggregated value using the PHFWA is also a PHFE, and

$$PHFWA(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_n) = \cup_{(\gamma_1,\tau_1)\in\mathfrak{P}_1,\ldots,(\gamma_n,\tau_n)\in\mathfrak{P}_n} \left\{\left(1-\prod_{j=1}^{n}(1-\gamma_j)^{\omega_j}, \prod_{j=1}^{n}\tau_j\right)\right\}.$$

Proof: The proof is by mathematical induction on n. First, we show that

$$\oplus_{j=1}^{n}(\omega_j\mathfrak{P}_j) = \cup_{\substack{(\gamma_1,\tau_1)\in\mathfrak{P}_1,\\ \ldots,(\gamma_n,\tau_n)\in\mathfrak{P}_n}} \left\{\left(1-\prod_{j=1}^{n}(1-\gamma_j)^{\omega_j}, \prod_{j=1}^{n}\tau_j\right)\right\}$$

holds for $n = 2$.

Following the operations of PHFEs, we have

$$\begin{aligned}\mathop{\oplus}_{j=1}^{2}(\omega_j\mathfrak{P}_j) &= \cup_{(\gamma_1,\tau_1)\in\mathfrak{P}_1,(\gamma_2,\tau_2)\in\mathfrak{P}_2} \{1-(1-\gamma_1)^{\omega_1}+1-(1-\gamma_2)^{\omega_2} \\ &\quad -\left(1-(1-\gamma_1)^{\omega_1}\right)\left(1-(1-\gamma_2)^{\omega_2}\right), \tau_1\tau_2\} \\ &= \cup_{(\gamma_1,\tau_1)\in\mathfrak{P}_1,(\gamma_2,\tau_2)\in\mathfrak{P}_2} \{1-(1-\gamma_1)^{\omega_1}(1-\gamma_2)^{\omega_2}, \tau_1\tau_2\}.\end{aligned}$$

If

$$\oplus_{j=1}^{n}(\omega_j\mathfrak{P}_j) = \cup_{\substack{(\gamma_1,\tau_1)\in\mathfrak{P}_1,\\ \ldots,(\gamma_n,\tau_n)\in\mathfrak{P}_n}} \left\{\left(1-\prod_{j=1}^{n}(1-\gamma_j)^{\omega_j}, \prod_{j=1}^{n}\tau_j\right)\right\}$$

holds for $n = k$, then

$$\mathop{\oplus}_{j=1}^{k}(\omega_j\mathfrak{P}_j) = \cup_{(\gamma_1,\tau_1)\in\mathfrak{P}_1,\ldots,(\gamma_k,\tau_k)\in\mathfrak{P}_k} \left\{\left(1-\prod_{j=1}^{k}(1-\gamma_j)^{\omega_j}, \prod_{j=1}^{k}\tau_j\right)\right\}.$$

When $n = k+1$, PHFE operations yield

$$\begin{aligned}\mathop{\oplus}_{j=1}^{k+1}(\omega_j\mathfrak{P}_j) &= \mathop{\oplus}_{j=1}^{k}(\omega_j\mathfrak{P}_j) \oplus \omega_{k+1}\mathfrak{P}_{k+1} \\ &= \cup_{(\gamma_1,\tau_1)\in\mathfrak{P}_1,\cdots,(\gamma_{k+1},\tau_{k+1})\in\mathfrak{P}_{k+1}} \left\{\left(1-\prod_{j=1}^{k}(1-\gamma_j)^{\omega_j}\right.\right. \\ &\quad +1-(1-\gamma_{k+1})^{\omega_{k+1}} - \left(1-\prod_{j=1}^{k}(1-\gamma_j)^{\omega_j}\right) \\ &\quad \left.\left.\times\left(1-(1-\gamma_{k+1})^{\omega_{k+1}}\right), \left(\prod_{j=1}^{k}\tau_j\right)\tau_{k+1}\right)\right\} \\ &= \cup_{(\gamma_1,\tau_1)\in\mathfrak{P}_1,\cdots,(\gamma_{k+1},\tau_{k+1})\in\mathfrak{P}_{k+1}} \left\{\left(1-\prod_{j=1}^{k+1}(1-\gamma_j)^{\omega_j},\right.\right. \\ &\quad \left.\left.\prod_{j=1}^{k+1}\tau_j\right)\right\}\end{aligned}$$

Then,

$$\oplus_{j=1}^{n}(\omega_j\mathfrak{P}_j) = \cup_{\substack{(\gamma_1,\tau_1)\in\mathfrak{P}_1,\\ \ldots,(\gamma_n,\tau_n)\in\mathfrak{P}_n}} \left\{\left(1-\prod_{j=1}^{n}(1-\gamma_j)^{\omega_j}, \prod_{j=1}^{n}\tau_j\right)\right\}$$

holds for $n = k+1$.

Thus,

$$\begin{aligned}\oplus_{j=1}^{n}(\omega_j\mathfrak{P}_j) &= \cup_{(\gamma_1,\tau_1)\in\mathfrak{P}_1,\ldots,(\gamma_n,\tau_n)\in\mathfrak{P}_n} \left\{\left(1-\prod_{j=1}^{n}(1-\gamma_j)^{\omega_j},\right.\right. \\ &\quad \left.\left.\prod_{j=1}^{n}\tau_j\right)\right\}\end{aligned}$$

holds for all n.

Similarly, we have the following theorem:

Theorem 4: Let $\mathfrak{P}_j\ (j = 1, 2, \ldots, n)$ be a collection of PHFEs and let $\omega = (1/n, 1/n, \ldots, 1/n)^T$ be the weight vector of $\mathfrak{P}_j$ $(j = 1, 2, \ldots, n)$. The aggregated value using the PHFA is also a PHFE, and

$$PHFA(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_n) = \cup_{(\gamma_1,\tau_1)\in\mathfrak{P}_1,\ldots,(\gamma_n,\tau_n)\in\mathfrak{P}_n} \left\{\left(1-\prod_{j=1}^{n}(1-\gamma_j)^{1/n}, \prod_{j=1}^{n}\tau_j\right)\right\}.$$

Proof: The proof of **Theorem 4** is similar to that of **Theorem 3**.

Theorem 5 (Commutativity): Let $\mathfrak{P}_j$ $(j = 1, 2, \ldots, n)$ be a collection of PHFEs. If $\mathfrak{P}'_j$ $(j = 1, 2, \ldots, n)$ is any permutation of $\mathfrak{P}_j$ $(j = 1, 2, \ldots, n)$, then we have

$$PHFA(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_n) = PHFA(\mathfrak{P}'_1, \mathfrak{P}'_2, \ldots, \mathfrak{P}'_n).$$

Proof: Since $\mathfrak{P}'_j$ $(j = 1, 2, \ldots, n)$ is any permutation of $\mathfrak{P}_j$ $(j = 1, 2, \ldots, n)$, then there exists for any $\mathfrak{P}_j$ one and only one $\mathfrak{P}'_i$ such that $\mathfrak{P}_j = \mathfrak{P}'_i$, and vice versa. We then have

$$PHFA(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_n) = \mathop{\oplus}_{j=1}^{n}\left(\frac{1}{n}\mathfrak{P}_j\right) = \mathop{\oplus}_{i=1}^{n}\left(\frac{1}{n}\mathfrak{P}_i\right) = PHFA(\mathfrak{P}'_1, \mathfrak{P}'_2, \ldots, \mathfrak{P}'_n).$$

The following example shows that the PHFA operator is neither idempotent, bounded, nor monotonic.

Example 1: Let $\mathfrak{P}_1 = \{(0.8, 0.6), (0.3, 0.4)\}$, $\mathfrak{P}_2 = \{(0.9, 0.4), (0.4, 0.6)\}$, $\mathfrak{P}_3 = \{(0.5, 0.1), (0.2, 0.9)\}$, and $\mathfrak{P}_4 = \{(0.6, 0.8), (0.4, 0.2)\}$ be four PHFEs. From **Theorem 4**, we obtain

$$\begin{aligned}
&PHFA(\mathfrak{P}_1, \mathfrak{P}_1, \mathfrak{P}_1)\\
&= \{(0.8, 0.2160), (0.6963, 0.432), (0.539, 0.288), (0.3, 0.064)\},\\
&PHFA(\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_3)\\
&= \{(0.7846, 0.024), (0.7480, 0.216), (0.6085, 0.036), (0.5421, 0.324), (0.6729, 0.016), (0.6174, 0.144), (0.4056, 0.024), (0.3048, 0.216)\},\\
&PHFA(\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_4)\\
&= \{(0.8, 0.1920), (0.7711, 0.048), (0.6366, 0.288), (0.584, 0.072), (0.6963, 0.128), (0.6524, 0.032), (0.4482, 0.192), (0.3684, 0.048)\}.
\end{aligned}$$

Following **Definition 7**, we have

$$\begin{aligned}
s(\mathfrak{P}_1) = s(\mathfrak{P}_2) &= 0.6,\\
t(\mathfrak{P}_1) = t(\mathfrak{P}_2) &= 0.06,\\
s(\mathfrak{P}_3) &= 0.23,\\
s(\mathfrak{P}_4) &= 0.56,\\
s(PHFA(\mathfrak{P}_1, \mathfrak{P}_1, \mathfrak{P}_1)) &= 0.6480,\\
s(PHFA(\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_3)) &= 0.5532,\\
s(PHFA(\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_4)) &= 0.6297,
\end{aligned}$$

then

(1) the PHFA operator is not idempotent because

$$s(PHFA(\mathfrak{P}_1, \mathfrak{P}_1, \mathfrak{P}_1)) \neq s(\mathfrak{P}_1);$$

(2) the PHFA operator is not bounded because

$$s(PHFA(\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_4)) > \max_{i=1,2,4}\{s(\mathfrak{P}_i)\};$$

(3) the PHFA operator is not monotonic because

$$\begin{aligned}
s(PHFA(\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_3)) &< s(PHFA(\mathfrak{P}_1, \mathfrak{P}_1, \mathfrak{P}_1))\\
&< s(PHFA(\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_4)).
\end{aligned}$$

B. PROPORTIONAL HESITANT FUZZY WEIGHTED GEOMETRIC OPERATOR

This section defines the PHFWG operator based on PHFEs and the traditional weighted geometric operator and also presents the properties of the PHFWG operator.

Definition 12: Let $\mathfrak{P}_j$ $(j = 1, 2, \ldots, n)$ be a collection of PHFEs. A PHFWG operator is a mapping $PH^n \to PH$ such that

$$PHFWG(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_n) = \mathop{\otimes}_{j=1}^{n}\left(\mathfrak{P}_j^{\omega_j}\right),$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weighting vector of $\mathfrak{P}_j$ $(j = 1, 2, \ldots, n)$ with $\omega_j \in [0, 1]$ and $\sum_{j=1}^{n}\omega_j = 1$.

In particular, if $\omega = (1/n, 1/n, \ldots, 1/n)^T$, the PHFWG operator reduces to the PHFG operator:

$$PHFG(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_n) = \mathop{\otimes}_{j=1}^{n}\left(\mathfrak{P}_j^{1/n}\right).$$

Theorem 6: Let $\mathfrak{P}_j$ $(j = 1, 2, \ldots, n)$ be a collection of PHFEs and let $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the weighting vector of $\mathfrak{P}_j$ $(j = 1, 2, \ldots, n)$ with $\omega_j \in [0, 1]$ and $\sum_{j=1}^{n}\omega_j = 1$. The aggregated value using the PHFWG operator is also a PHFE, and

$$PHFWG(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_n) = \cup_{(\gamma_1,\tau_1)\in\mathfrak{P}_1,\ldots,(\gamma_n,\tau_n)\in\mathfrak{P}_n} \left\{\left(\prod_{j=1}^{n}\gamma_j^{\omega_j}, \prod_{j=1}^{n}\tau_j\right)\right\}.$$

Proof: The proof of **Theorem 6** is similar to that of **Theorem 3**.

Similar to the PHFA operator, the PHFG operator is neither idempotent, bounded, nor monotonic, but it is commutative.

Lemma 1: Let $x_j > 0$, $\lambda_j > 0$, $j = 1, 2, \ldots, n$ and $\sum_{j=1}^{n}\lambda_j = 1$. Then

$$\prod_{j=1}^{n}x_j^{\lambda_j} \leq \sum_{j=1}^{n}\lambda_j x_j,$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$.

Theorem 7: Let $\mathfrak{P}_j$ $(j = 1, 2, \ldots, n)$ be a collection of PHFEs and $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the weight vector of $\mathfrak{P}_j$ $(j = 1, 2, \ldots, n)$ with $\omega_j \in [0, 1]$ and $\sum_{j=1}^{n}\omega_j = 1$. Then

$$PHFWG(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_n) \leq PHFWA(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_n).$$

Proof: For any $(\gamma_1, \tau_1) \in \mathfrak{P}_1, \ldots, (\gamma_n, \tau_n) \in \mathfrak{P}_n$, based on **Lemma 1**, we obtain

$$\prod_{j=1}^{n} \gamma_j^{\omega_j} \leq \sum_{j=1}^{n} \omega_j \gamma_j = 1 - \sum_{j=1}^{n} \omega_j \left(1 - \gamma_j\right) \leq 1 - \prod_{j=1}^{n} \left(1 - \gamma_j\right)^{\omega_j},$$

then

$$\left(\prod_{j=1}^{n} \gamma_j^{\omega_j}\right) \prod_{j=1}^{n} \tau_j \leq \left(1 - \prod_{j=1}^{n} \left(1 - \gamma_j\right)^{\omega_j}\right) \prod_{j=1}^{n} \tau_j.$$

Following **Definitions 7** and **8**, we have

$$PHFWG\left(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_n\right) \leq PHFWA\left(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_n\right).$$

Theorem 8 shows that the aggregated value obtained by the PHFWG operator is no more than that obtained by the PHFWA operator.

C. GENERALIZED PROPORTIONAL HESITANT FUZZY WEIGHTED AVERAGING OPERATOR

This section generalizes the PHFWA operator to define the GPHFWA operator and its properties.

Definition 13: Let $\mathfrak{P}_j$ $(j = 1, 2, \ldots, n)$ be a collection of PHFEs. A GPHFWA operator is a mapping $PH^n \to PH$ such that

$$GPHFWA_\lambda\left(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_n\right) = \left(\overset{n}{\underset{j=1}{\oplus}} \left(\omega_j \mathfrak{P}_j^\lambda\right)\right)^{1/\lambda}, \quad \lambda > 0,$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weighting vector of $\mathfrak{P}_j$ $(j = 1, 2, \ldots, n)$ with $\omega_j \in [0, 1]$ and $\sum_{j=1}^{n} \omega_j = 1$.

In particular, if $\lambda = 1$, then the GPHFWA operator reduces to the PHFWA operator.

Theorem 8: Let $\mathfrak{P}_j$ $(j = 1, 2, \ldots, n)$ be a collection of PHFEs and let $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the weighting vector of $\mathfrak{P}_j$ $(j = 1, 2, \ldots, n)$ with $\omega_j \in [0, 1]$ and $\sum_{j=1}^{n} \omega_j = 1$. For any $\lambda > 0$, the aggregated value using the GPHFWA operator is also a PHFE, and

$$GPHFWA_\lambda\left(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_n\right) = \cup_{\substack{(\gamma_1, \tau_1) \in \mathfrak{P}_1, \ldots \\ , (\gamma_n, \tau_n) \in \mathfrak{P}_n}} \left\{\left(\left[1 - \prod_{j=1}^{n} \left(1 - \gamma_j^\lambda\right)^{\omega_j}\right]^{1/\lambda}, \prod_{j=1}^{n} \tau_j\right)\right\}, \quad \lambda > 0.$$

Proof: The proof of **Theorem 8** is similar to that of **Theorem 3**.

Theorem 9: Let $\mathfrak{P}_j$ $(j = 1, 2, \ldots, n)$ be a collection of PHFEs and let $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the weighting vector of $\mathfrak{P}_j$ $(j = 1, 2, \ldots, n)$ with $\omega_j \in [0, 1]$ and $\sum_{j=1}^{n} \omega_j = 1$, $\lambda > 0$. Then,

$$PHFWG(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_n) \leq GPHFWA_\lambda(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_n).$$

Proof: For any $(\gamma_1, \tau_1) \in \mathfrak{P}_1$, $(\gamma_2, \tau_2) \in \mathfrak{P}_2, \ldots, (\gamma_n, \tau_n) \in \mathfrak{P}_n$, on the basis of **Lemma 1**, we obtain

$$\prod_{j=1}^{n} \gamma_j^{\omega_j} \leq \left(\prod_{j=1}^{n} \left(\gamma_j^\lambda\right)^{\omega_j}\right)^{1/\lambda} \leq \left(\sum_{j=1}^{n} \omega_j \gamma_j^\lambda\right)^{1/\lambda} = \left(1 - \sum_{j=1}^{n} \omega_j \left(1 - \gamma_j^\lambda\right)\right)^{1/\lambda} \leq \left(1 - \prod_{j=1}^{n} \left(1 - \gamma_j^\lambda\right)^{\omega_j}\right)^{1/\lambda},$$

then

$$\left(\prod_{j=1}^{n} \gamma_j^{\omega_j}\right) \prod_{j=1}^{n} \tau_j \leq \left(1 - \prod_{j=1}^{n} \left(1 - \gamma_j^\lambda\right)^{\omega_j}\right)^{1/\lambda} \prod_{j=1}^{n} \tau_j.$$

From **Definitions 7** and **8** we obtain the result

$$PHFWG(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_n) \leq GPHFWA_\lambda(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_n).$$

Theorem 9 shows that the aggregated value obtained by the PHFWG operator is not greater than that obtained by the GPHFWA operator.

Theorem 10: Let $\mathfrak{P}_j$ $(j = 1, 2, \ldots, n)$ be a collection of PHFEs. The aggregated value obtained by the GPHFWA operator increases monotonically with the parameter λ, where $\lambda > 0$.

Before proving this theorem, we first introduce the following lemma which was proven by Zhang [27]:

Lemma 2: For any $x \in [0, 1]$,

$$g(x) = \frac{(1 - x)\ln(1 - x)}{x}$$

is strictly convex.

We now prove **Theorem 10**.

Proof: For any $(\gamma_1, \tau_1) \in \mathfrak{P}_1$, $(\gamma_2, \tau_2) \in \mathfrak{P}_2, \ldots, (\gamma_n, \tau_n) \in \mathfrak{P}_n$, let $f(\lambda) = [1 - \prod_{j=1}^{n} (1 - \gamma_j^\lambda)^{\omega_j}]^{1/\lambda}$. Because the proportion of the aggregated value is $\prod_{j=1}^{n} \tau_j$ for any $\lambda > 0$, then we only need to show that the function $f(\lambda)$ increases monotonically with λ $(\lambda > 0)$. Since

$$f'(\lambda) = \left[e^{\ln\left[1 - \prod_{j=1}^{n} \left(1 - \gamma_j^\lambda\right)^{\omega_j}\right]/\lambda}\right]' = \frac{f(\lambda) \prod_{j=1}^{n} \left(1 - \gamma_j^\lambda\right)^{\omega_j}}{\lambda^2 \left[1 - \prod_{j=1}^{n} \left(1 - \gamma_j^\lambda\right)^{\omega_j}\right]} \left(\sum_{j=1}^{n} \omega_j g\left(x_j\right) - g\left(x_0\right)\right),$$

where $x_j = 1 - \gamma_j^\lambda$ $(j = 1, 2, \ldots, n)$, $x_0 = \prod_{j=1}^{n} \left(1 - \gamma_j^\lambda\right)^{\omega_j}$ and $g(x) = \frac{(1-x)\ln(1-x)}{x}$.

Following **Lemma 2**, because $g(x)$ is strictly convex for any $x \in [0, 1]$, $g\left(x_j\right) > g\left(x_0\right) + \left(x_j - x_0\right) g'\left(x_0\right)$ holds for all $x_0 > 0$, $x_j \geq 0$ $(j = 1, 2, \ldots, n)$, $x_0 \neq x_j$. We then have

$$\begin{aligned}
&\sum_{j=1}^{n}\omega_j g\left(x_j\right)\\
&\quad > \sum_{j=1}^{n}\omega_j\left[g\left(x_0\right)+\left(x_j-x_0\right)g'\left(x_0\right)\right]\\
&\quad = \sum_{j=1}^{n}\omega_j g\left(x_0\right)+\sum_{j=1}^{n}\omega_j\left(x_j-x_0\right)g'\left(x_0\right)\\
&\quad = g\left(x_0\right)\sum_{j=1}^{n}\omega_j+g'\left(x_0\right)\left(\sum_{j=1}^{n}\omega_j x_j-x_0\sum_{j=1}^{n}\omega_j\right)\\
&\quad = g\left(x_0\right)+g'\left(x_0\right)\left[\sum_{j=1}^{n}\omega_j\left(1-h_j^{\lambda}\right)-\prod_{j=1}^{n}\left(1-h_j^{\lambda}\right)^{\omega_j}\right].
\end{aligned}$$

By **Lemma 1**, we have

$$\begin{aligned}
&\sum_{j=1}^{n}\omega_j g\left(x_j\right)\\
&\quad > g\left(x_0\right)+g'\left(x_0\right)\left[\sum_{j=1}^{n}\omega_j\left(1-h_j^{\lambda}\right)-\prod_{j=1}^{n}\left(1-h_j^{\lambda}\right)^{\omega_j}\right]\\
&\quad > g\left(x_0\right)+g'\left(x_0\right)\left[\prod_{j=1}^{n}\left(1-h_j^{\lambda}\right)^{\omega_j}-\prod_{j=1}^{n}\left(1-h_j^{\lambda}\right)^{\omega_j}\right]\\
&\quad = g\left(x_0\right).
\end{aligned}$$

Thus, $f'(\lambda) > 0$, and $f(\lambda)$ increases monotonically with λ $(\lambda > 0)$.

Theorem 11: Let $\mathfrak{P}_j$ $(j = 1, 2, \ldots, n)$ be a collection of PHFEs. As $\lambda \to 0$ $(\lambda > 0)$, the GPHFWA operator approaches the following limit:

$$\begin{aligned}
&\lim_{\lambda\to 0} GPHFWA_{\lambda}\left(\mathfrak{P}_1,\mathfrak{P}_2,\ldots,\mathfrak{P}_n\right)\\
&\quad = \cup_{(\gamma_1,\tau_1)\in\mathfrak{P}_1,\ldots,(\gamma_n,\tau_n)\in\mathfrak{P}_n}\left\{\left(e^{\prod_{j=1}^{n}(\ln\gamma_j)^{\omega_j}},\ \prod_{j=1}^{n}\tau_j\right)\right\}.
\end{aligned}$$

Proof: Following **Theorem 8**, we have

$$\begin{aligned}
&\lim_{\lambda\to 0} GPHFWA_{\lambda}\left(\mathfrak{P}_1,\mathfrak{P}_2,\ldots,\mathfrak{P}_n\right)\\
&= \bigcup_{(\gamma_1,\tau_1)\in\mathfrak{P}_1,\ldots,(\gamma_n,\tau_n)\in\mathfrak{P}_n}\left\{\left(\lim_{\lambda\to 0}\left[1-\prod_{j=1}^{n}\left(1-\gamma_j^{\lambda}\right)^{\omega_j}\right]^{1/\lambda},\right.\right.\\
&\qquad \left.\left.\prod_{j=1}^{n}\tau_j\right)\right\},
\end{aligned}$$

so we only need to consider the limit of the function $f(\lambda) = [1-\prod_{j=1}^{n}(1-h_j^{\lambda})^{\omega_j}]^{1/\lambda}$; namely, $\lim_{\lambda\to 0} f(\lambda)$.

With l'Hôpital's rule, we have

$$\begin{aligned}
\lim_{\lambda\to 0} f(\lambda) &= \lim_{\lambda\to 0} e^{\ln\left(1-\prod_{j=1}^{n}\left[1-h_j^{\lambda}\right]^{\omega_j}\right)/\lambda}\\
&= e^{\lim_{\lambda\to 0}\ln\left(1-\prod_{j=1}^{n}\left[1-h_j^{\lambda}\right]^{\omega_j}\right)/\lambda}\\
&= e^{\lim_{\lambda\to 0}\frac{\sum_{j=1}^{n}\left[\frac{\omega_j h_j^{\lambda}\ln h_j}{1-h_j^{\lambda}}\prod_{k=1}^{n}\left(1-h_k^{\lambda}\right)^{\omega_k}\right]}{1-\prod_{j=1}^{n}\left(1-h_j^{\lambda}\right)^{\omega_j}}}\\
&= e^{\sum_{j=1}^{n}\left[\omega_j\ln h_j\lim_{\lambda\to 0}\frac{\prod_{k=1}^{n}\left(1-h_k^{\lambda}\right)^{\omega_k}}{1-h_j^{\lambda}}\lim_{\lambda\to 0}\frac{h_j^{\lambda}}{1-\prod_{j=1}^{n}\left(1-h_j^{\lambda}\right)^{\omega_j}}\right]}\\
&= e^{\sum_{j=1}^{n}\left[\omega_j\ln h_j\lim_{\lambda\to 0}\frac{\prod_{k=1}^{n}\left(1-h_k^{\lambda}\right)^{\omega_k}}{1-h_j^{\lambda}}\right]}\\
&= e^{\sum_{j=1}^{n}\left[\omega_j\ln h_j\lim_{\lambda\to 0}\frac{\prod_{k=1}^{n}\left(1-h_k^{\lambda}\right)^{\omega_k}}{\prod_{k=1}^{n}\left(1-h_j^{\lambda}\right)^{\omega_k}}\right]}\\
&= e^{\sum_{j=1}^{n}\left[\omega_j\ln h_j\prod_{k=1}^{n}\left(\lim_{\lambda\to 0}\frac{1-h_k^{\lambda}}{1-h_j^{\lambda}}\right)^{\omega_k}\right]}\\
&= e^{\sum_{j=1}^{n}\left[\omega_j\ln h_j\prod_{k=1}^{n}\left(\lim_{\lambda\to 0}\frac{h_k^{\lambda}\ln h_k}{h_j^{\lambda}\ln h_j}\right)^{\omega_k}\right]}\\
&= e^{\sum_{j=1}^{n}\left[\omega_j\ln h_j\prod_{k=1}^{n}\left(\frac{\ln h_k}{\ln h_j}\right)^{\omega_k}\right]}\\
&= e^{\sum_{j=1}^{n}\left[\omega_j\ln h_j\frac{\prod_{k=1}^{n}(\ln h_k)^{\omega_k}}{\ln h_j}\right]}\\
&= e^{\sum_{j=1}^{n}\left[\omega_j\prod_{k=1}^{n}(\ln h_k)^{\omega_k}\right]} = e^{\prod_{k=1}^{n}(\ln h_k)^{\omega_k}}\\
&= e^{\prod_{j=1}^{n}(\ln h_j)^{\omega_j}}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\lim_{\lambda\to 0} GPHFWA_{\lambda}\left(\mathfrak{P}_1,\mathfrak{P}_2,\ldots,\mathfrak{P}_n\right)\\
&\quad = \cup_{(\gamma_1,\tau_1)\in\mathfrak{P}_1,\ldots,(\gamma_n,\tau_n)\in\mathfrak{P}_n}\left\{\left(e^{\prod_{j=1}^{n}(\ln\gamma_j)^{\omega_j}},\ \prod_{j=1}^{n}\tau_j\right)\right\}.
\end{aligned}$$

D. GENERALIZED PROPORTIONAL HESITANT FUZZY WEIGHTED GEOMETRIC OPERATOR

This section generalizes the PHFWG operator and defines the GPHFWG operator and its properties.

Definition 14: Let $\mathfrak{P}_j$ $(j = 1, 2, \ldots, n)$ be a collection of PHFEs. A GPHFWG operator is a mapping $PH^n \to PH$ such that

$$GPHFWG_{\lambda}\left(\mathfrak{P}_1,\mathfrak{P}_2,\ldots,\mathfrak{P}_n\right) = \frac{1}{\lambda}\bigotimes_{j=1}^{n}\left(\lambda\mathfrak{P}_j\right)^{\omega_j},\quad \lambda > 0,$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weighting vector of $\mathfrak{P}_j$ $(j = 1, 2, \ldots, n)$ with $\omega_j \in [0, 1]$ and $\sum_{j=1}^{n}\omega_j = 1$.

In particular, if $\lambda = 1$, then the GPHFWG operator reduces to the PHFWG operator.

Theorem 12: Let $\mathfrak{P}_j$ $(j = 1, 2, \ldots, n)$ be a collection of PHFEs and let $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the weighting vector of $\mathfrak{P}_j$ $(j = 1, 2, \ldots, n)$ with $\omega_j \in [0, 1]$, and $\sum_{j=1}^{n}\omega_j = 1$. For any $\lambda > 0$, the aggregated value using the GPHFWG operator is also a PHFE, and

$$\begin{aligned}
&GPHFWG_{\lambda}\left(\mathfrak{P}_1,\mathfrak{P}_2,\ldots,\mathfrak{P}_n\right)\\
&= \cup_{\substack{(\gamma_1,\tau_1)\in\mathfrak{P}_1,\\ \ldots,(\gamma_n,\tau_n)\in\mathfrak{P}_n}}\left\{\left(1-\left(1-\prod_{j=1}^{n}\left[1-\left(1-\gamma_j\right)^{\lambda}\right]^{\omega_j}\right)^{1/\lambda},\right.\right.\\
&\qquad \left.\left.\prod_{j=1}^{n}\tau_j\right)\right\},\quad \lambda > 0.
\end{aligned}$$

Proof: The proof of **Theorem 12** is similar to that of **Theorem 3**.

Theorem 13: emphLet $\mathfrak{P}_j$ $(j = 1, 2, \ldots, n)$ be a collection of PHFEs. The aggregated value obtained by the GPHFWG operator decreases monotonically with the parameter λ $(\lambda > 0)$.

Proof: Following **Theorem 10**, $\{1 - \prod_{j=1}^{n} [1 - (1 - \gamma_j)^{\lambda}]^{\omega_j}\}^{1/\lambda}$ increases with the parameter λ $(\lambda > 0)$, so $1 - \{1 - \prod_{j=1}^{n} [1 - (1 - \gamma_j)^{\lambda}]^{\omega_j}\}^{1/\lambda}$ decreases with the parameter λ $(\lambda > 0)$. The use of **Definitions 7** and **8** proves this theorem.

Theorem 14: Let $\mathfrak{P}_j$ $(j = 1, 2, \ldots, n)$ be a collection of PHFEs. As $\lambda \to 0$ $(\lambda > 0)$, the GPHFWG operator approaches the following limit:

$$\lim_{\lambda \to 0} GPHFWG_{\lambda}(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_n) = \cup_{(\gamma_1,\tau_1)\in\mathfrak{P}_1,\ldots,(\gamma_n,\tau_n)\in\mathfrak{P}_n} \left\{ \left(1 - e^{\prod_{j=1}^{n} [\ln(1-\gamma_j)]^{\omega_j}}, \prod_{j=1}^{n} \tau_j \right) \right\}.$$

Proof: The proof of **Theorem 14** is similar to that of **Theorem 11**.

Theorem 15: Let $\mathfrak{P}_j$ $(j = 1, 2, \ldots, n)$ be a collection of PHFEs and let $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the weighting vector of $\mathfrak{P}_j$ $(j = 1, 2, \ldots, n)$ with $\omega_j \in [0, 1]$ and $\sum_{j=1}^{n} \omega_j = 1$. For any $\lambda > 0$, we have

$$GPHFWG_{\lambda}(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_n) \leq PHFWA(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_n).$$

For any $(\gamma_1, \tau_1) \in \mathfrak{P}_1, (\gamma_2, \tau_2) \in \mathfrak{P}_2, \ldots, (\gamma_n, \tau_n) \in \mathfrak{P}_n$, we obtain from **Lemma 1**,

$$\begin{aligned} 1 - &\left\{ 1 - \prod_{j=1}^{n} \left[1 - \left(1 - h_j\right)^{\lambda} \right]^{\omega_j} \right\}^{1/\lambda} \\ &\leq 1 - \left\{ 1 - \sum_{j=1}^{n} \omega_j \left[1 - \left(1 - h_j\right)^{\lambda} \right] \right\}^{1/\lambda} \\ &= 1 - \left[\sum_{j=1}^{n} \omega_j \left(1 - h_j\right)^{\lambda} \right]^{1/\lambda} \\ &\leq 1 - \left[\prod_{j=1}^{n} \left(1 - h_j\right)^{\lambda\omega_j} \right]^{1/\lambda} \\ &= 1 - \prod_{j=1}^{n} \left(1 - h_j\right)^{\omega_j}, \end{aligned}$$

then

$$\left(1 - \left\{ 1 - \prod_{j=1}^{n} \left[1 - \left(1 - h_j\right)^{\lambda} \right]^{\omega_j} \right\}^{1/\lambda} \right) \prod_{j=1}^{n} \tau_j \leq \left[1 - \prod_{j=1}^{n} \left(1 - h_j\right)^{\omega_j} \right] \prod_{j=1}^{n} \tau_j.$$

From **Definitions 7** and **8**, we have

$$GPHFWG_{\lambda}(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_n) \leq PHFWA(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_n).$$

Theorem 15 shows that the aggregated value obtained by the PHFWA operator is no less than that obtained by the GPHFWG operator.

Combining **Theorems 7**, **9**, and **15**, we obtain the following theorem:

Theorem 16: Let $\mathfrak{P}_j$ $(j = 1, 2, \ldots, n)$ be a collection of PHFEs and let $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the weighting vector of $\mathfrak{P}_j$ $(j = 1, 2, \ldots, n)$ with $\omega_j \in [0, 1]$ and $\sum_{j=1}^{n} \omega_j = 1$. For any $\lambda > 0$, we have

$$GPHFWG_{\lambda}(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_n) \leq GPHFWA_{\lambda}(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_n).$$

We visually illustrate these relationships of the aggregated values obtained by the four operators by the following example:

Example 4: Let $\mathfrak{P}_1 = \{(0.8, 0.6), (0.3, 0.4)\}$, $\mathfrak{P}_2 = \{(0.9, 0.3), (0.3, 0.7)\}$, and $\mathfrak{P}_3 = \{(0.7, 0.2), (0.4, 0.8)\}$ be three PHFEs, and suppose that the weighting vector $\omega = (0.3, 0.5, 0.2)^T$. Following **Definitions 11–14**, we have

$$\begin{aligned} &GPHFWA_1(\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_3) \\ &= PHFWA(\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_3) \\ &= \bigcup_{\substack{(\gamma_1,\tau_1)\in\mathfrak{P}_1,(\gamma_2,\tau_2)\in\mathfrak{P}_2,\\(\gamma_3,\tau_3)\in\mathfrak{P}_3}} \left\{ \left(1 - \prod_{j=1}^{3} (1 - \gamma_j)^{\omega_j}, \prod_{j=1}^{3} \tau_j \right) \right\} \\ &= \{(0.8466, 0.036), (0.8238, 0.144), (0.7767, 0.024), \\ &\quad (0.7435, 0.096), (0.5942, 0.084), (0.5339, 0.336), \\ &\quad (0.4091, 0.056), (0.3213, 0.224)\}; \end{aligned}$$

$$\begin{aligned} &GPHFWA_6(P_1, P_2, P_3) \\ &= \bigcup_{\substack{(\gamma_1,\tau_1)\in\mathfrak{P}_1,(\gamma_2,\tau_2)\in\mathfrak{P}_2,\\(\gamma_3,\tau_3)\in\mathfrak{P}_3}} \left\{ \left(\left(1 - \prod_{j=1}^{3} \left(1 - h_j^6\right)^{\omega_j} \right)^{1/6}, \prod_{j=1}^{3} \tau_j \right) \right\} \\ &= \{(0.8550, 0.036), (0.8494, 0.144), (0.8324, 0.024), \\ &\quad (0.8254, 0.096), (0.6923, 0.084), (0.6672, 0.336), \\ &\quad (0.5418, 0.056), (0.3346, 0.224)\}. \end{aligned}$$

$$\begin{aligned} &GPHFWG_1(\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_3) \\ &= PHFWG(\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_3) \\ &= \bigcup_{\substack{(\gamma_1,\tau_1)\in\mathfrak{P}_1,(\gamma_2,\tau_2)\in\mathfrak{P}_2\\,(\gamma_3,\tau_3)\in\tau_3}} \left\{ \left(\prod_{j=1}^{3} h_j^{\omega_j}, \prod_{j=1}^{3} \tau_j \right) \right\} \\ &= \{(0.8262, 0.036), (0.7387, 0.144), (0.6156, 0.024), \\ &\quad (0.5504, 0.096), (0.4770, 0.084), (0.4265, 0.336), \\ &\quad (0.3554, 0.056), (0.3178, 0.224)\}. \end{aligned}$$

$$GPHFWG_6(\mathfrak{P}_1,\mathfrak{P}_2,\mathfrak{P}_3) = \bigcup_{\substack{(\gamma_1,\tau_1)\in\mathfrak{P}_1,\cdots,\\(\gamma_3,\tau_3)\in\mathfrak{P}_3}} \left\{\left(1-\left(1-\prod_{j=1}^{3}\left(1-(1-\gamma_j)^6\right)^{\omega_j}\right)^{1/6}, \prod_{j=1}^{3}\tau_j\right)\right\}$$
$$= \{(0.7657, 0.036), (0.5396, 0.144), (0.4228, 0.024), (0.4014, 0.096), (0.3729, 0.084), (0.3586, 0.336), (0.3240, 0.056), (0.3144, 0.224)\}.$$

By **Definition 7**, we have

$$s(PHFWA(\mathfrak{P}_1,\mathfrak{P}_2,\mathfrak{P}_3)) = 0.5633,$$
$$s(GPHFWA_6(\mathfrak{P}_1,\mathfrak{P}_2,\mathfrak{P}_3)) = 0.6399,$$
$$s(PHFWG(\mathfrak{P}_1,\mathfrak{P}_2,\mathfrak{P}_3)) = 0.4782,$$
$$s(GPHFWG_6(\mathfrak{P}_1,\mathfrak{P}_2,\mathfrak{P}_3)) = 0.3943.$$

Then,

$$GPHFWG_6(\mathfrak{P}_1,\mathfrak{P}_2,\mathfrak{P}_3) < PHFWG(\mathfrak{P}_1,\mathfrak{P}_2,\mathfrak{P}_3) < PHFWA(\mathfrak{P}_1,\mathfrak{P}_2,\mathfrak{P}_3) < GPHFWA_6(\mathfrak{P}_1,\mathfrak{P}_2,\mathfrak{P}_3).$$

E. EXTENDING PREVIOUS OPERATORS TO OPERATORS BASED ON OWA: PHFOWA, PHFOWG, GPHFOWA, AND GPHFOWG OPERATORS

This section extends the PHFWA, PHFWG, GPHFWA, and GPHFWG operators based on the OWA operator to define the PHFOWA, PHFOWG, GPHFOWA, and GPHFOWG operators, and also gives their properties and interrelationships.

Definition 15: Let $\mathfrak{P}_j$ $(j = 1, 2, \ldots, n)$ be a collection of PHFEs, let $\mathfrak{P}_{\delta(j)}$ be the jth largest PHFE, and let $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the aggregation-associated vector with $\omega_j \in [0, 1]$ and $\sum_{j=1}^{n}\omega_j = 1$. A PHFOWA operator is then a mapping $PH^n \to PH$ such that

$$PHFOWA(\mathfrak{P}_{\delta(1)},\mathfrak{P}_{\delta(2)},\ldots,\mathfrak{P}_{\delta(n)}) = \overset{n}{\underset{j=1}{\oplus}}(\omega_j\mathfrak{P}_{\delta(j)}).$$

In particular, if $\omega = (1/n, 1/n, \ldots, 1/n)^T$, then the PHFOWA operator reduces to the PHFA operator.

Theorem 17: Let $\mathfrak{P}_j$ $(j = 1, 2, \ldots, n)$ be a collection of PHFEs, let $\mathfrak{P}_{\delta(j)}$ be the largest PHFE, and let $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the aggregation-associated vector with $\omega_j \in [0, 1]$ and $\sum_{j=1}^{n}\omega_j = 1$. The aggregated value using a PHFOWA operator is also a PHFE, and

$$PHFOWA(\mathfrak{P}_{\delta(1)},\mathfrak{P}_{\delta(2)},\ldots,\mathfrak{P}_{\delta(n)}) = \cup_{(\gamma_{\delta(1)},\tau_{\delta(1)})\in\mathfrak{P}_{\delta(1)},\ldots,(\gamma_{\delta(n)},\tau_{\delta(n)})\in\mathfrak{P}_{\delta(n)}} \times\left\{\left(1-\prod_{j=1}^{n}(1-\gamma_{\delta(j)})^{\omega_j}, \prod_{j=1}^{n}\tau_{\delta(j)}\right)\right\}.$$

Proof: The proof of **Theorem 17** is similar to that of **Theorem 4**.

Definition 16: Let $\mathfrak{P}_j$ $(j = 1, 2, \ldots, n)$ be a collection of PHFEs, let $\mathfrak{P}_{\delta(j)}$ be the jth largest PHFE, and let $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the aggregation-associated vector with $\omega_j \in [0, 1]$ and $\sum_{j=1}^{n}\omega_j = 1$. A PHFOWG operator is then a mapping $PH^n \to PH$ such that

$$PHFOWG(\mathfrak{P}_{\delta(1)},\mathfrak{P}_{\delta(2)},\ldots,\mathfrak{P}_{\delta(n)}) = \overset{n}{\underset{j=1}{\otimes}}\left(\mathfrak{P}_{\delta(j)}^{\omega_j}\right).$$

In particular, if $\omega = (1/n, 1/n, \ldots, 1/n)^T$, then the PHFOWG operator reduces to the PHFG operator.

Theorem 18: Let $\mathfrak{P}_j$ $(j = 1, 2, \ldots, n)$ be a collection of PHFEs, let $\mathfrak{P}_{\delta(j)}$ be the jth largest PHFE, and let $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the aggregation-associated vector with $\omega_j \in [0, 1]$ and $\sum_{j=1}^{n}\omega_j = 1$. The aggregated value using a PHFOWG operator is also a PHFE, and

$$PHFOWG(\mathfrak{P}_{\delta(1)},\mathfrak{P}_{\delta(2)},\ldots,\mathfrak{P}_{\delta(n)}) = \cup_{(\gamma_{\delta(1)},\tau_{\delta(1)})\in\mathfrak{P}_{\delta(1)},\ldots,(\gamma_{\delta(n)},\tau_{\delta(n)})\in\mathfrak{P}_{\delta(n)}} \times\left\{\left(\prod_{j=1}^{n}\gamma_{\delta(j)}^{\omega_j}, \prod_{j=1}^{n}\tau_{\delta(j)}\right)\right\}.$$

Proof: The proof of **Theorem 18** is similar to that of **Theorem 4**.

Definition 19: Let $\mathfrak{P}_j$ $(j = 1, 2, \ldots, n)$ be a collection of PHFEs, let $\mathfrak{P}_{\delta(j)}$ be the jth largest PHFE, and let $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the aggregation-associated vector with $\omega_j \in [0, 1]$ and $\sum_{j=1}^{n}\omega_j = 1$. A GPHFOWA operator is a mapping $PH^n \to PH$ such that

$$GPHFOWA(\mathfrak{P}_{\delta(1)},\mathfrak{P}_{\delta(2)},\ldots,\mathfrak{P}_{\delta(n)}) = \overset{n}{\underset{j=1}{\oplus}}\left(\omega_j\mathfrak{P}_{\delta(j)}^{\lambda}\right)^{1/\lambda}, \quad \lambda > 0.$$

In particular, if $\lambda = 1$, the GPHFOWA operator reduces to the PHFOWA operator, and if $\omega = (1/n, 1/n, \ldots, 1/n)^T$ and $\lambda = 1$, the GPHFOWA operator reduces to the PHFA operator.

Theorem 19: Let $\mathfrak{P}_j$ $(j = 1, 2, \ldots, n)$ be a collection of PHFEs, let $\mathfrak{P}_{\delta(j)}$ be the jth largest PHFE, and let $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the aggregation-associated vector with $\omega_j \in [0, 1]$ and $\sum_{j=1}^{n}\omega_j = 1$. For any $\lambda > 0$, the aggregated value using a GPHFOWA operator is also a PHFE, and

$$GPHFOWA(\mathfrak{P}_{\delta(1)},\mathfrak{P}_{\delta(2)},\ldots,\mathfrak{P}_{\delta(n)}) = \cup_{\substack{(\gamma_{\delta(1)},\tau_{\delta(1)})\in\mathfrak{P}_{\delta(1)},\ldots,\\(\gamma_{\delta(n)},\tau_{\delta(n)})\in\mathfrak{P}_{\delta(n)}}} \left\{\left(\left[1-\prod_{j=1}^{n}\left(1-\gamma_{\delta(j)}^{\lambda}\right)^{\omega_j}\right]^{1/\lambda}, \prod_{j=1}^{n}\tau_{\delta(j)}\right)\right\}.$$

Proof: The proof of **Theorem 19** is similar to that of **Theorem 4**.

Definition 18: Let $\mathfrak{P}_j$ $(j = 1, 2, \ldots, n)$ be a collection of PHFEs, let $\mathfrak{P}_{\delta(j)}$ be the jth largest PHFE, and let $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the aggregation-associated vector with $\omega_j \in [0, 1]$ and $\sum_{j=1}^{n} \omega_j = 1$. A GPHFOWG operator is then a mapping $PH^n \to PH$ such that

$$GPHFOWG\left(\mathfrak{P}_{\delta(1)}, \mathfrak{P}_{\delta(2)}, \ldots, \mathfrak{P}_{\delta(n)}\right) = \frac{1}{\lambda}\left[\bigotimes_{j=1}^{n}\left(\lambda \mathfrak{P}_{\delta(j)}\right)^{\omega_j}\right].$$

In particular, if $\lambda = 1$, the GPHFOWG operator reduces to the PHFOWG operator, and if $\omega = \left(1/n, 1/n, \ldots, 1/n\right)^T$ and $\lambda = 1$, the GPHFOWG operator reduces to the PHFG operator.

Theorem 20: Let $\mathfrak{P}_j$ $(j = 1, 2, \ldots, n)$ be a collection of PHFEs, let $\mathfrak{P}_{\delta(j)}$ be the jth largest PHFE, and let $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the aggregation-associated vector with $\omega_j \in [0, 1]$ and $\sum_{j=1}^{n} \omega_j = 1$. For any $\lambda > 0$, the aggregated value using a GPHFOWG operator is also a PHFE, and

$$GPHFOWG\left(\mathfrak{P}_{\delta(1)}, \mathfrak{P}_{\delta(2)}, \ldots, \mathfrak{P}_{\delta(n)}\right) = \cup_{\substack{(\gamma_{\delta(1)}, \tau_{\delta(1)}) \in \mathfrak{P}_{\delta(1)}, \\ \ldots, (\gamma_{\delta(n)}, \tau_{\delta(n)}) \in \mathfrak{P}_{\delta(n)}}} \left\{\left(1 - \prod_{j=1}^{n}\left[1 - \left(1 - \gamma_{\delta(j)}\right)^{\lambda}\right]^{\omega_j}, \prod_{j=1}^{n} \tau_{\delta(j)}\right)\right\}.$$

Proof: The proof of **Theorem 20** is similar to that of **Theorem 4**.

The four ordered-weighted operators are expended based on the PHFWA, PHFWG, GPHFWA, and PHFWG operators. We then have the following theorems:

Theorem 21: Let $\mathfrak{P}_j$ $(j = 1, 2, \ldots, n)$ be a collection of PHFEs, let $\mathfrak{P}_{\delta(j)}$ be the jth largest PHFE, and let $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the aggregation-associated vector with $\omega_j \in [0, 1]$ and $\sum_{j=1}^{n} \omega_j = 1$. For any $\lambda > 0$,

$$GPHFOWG_{\lambda}\left(\mathfrak{P}_{\delta(1)}, \mathfrak{P}_{\delta(2)}, \ldots, \mathfrak{P}_{\delta(n)}\right) \leq GPHFOWA_{\lambda}\left(\mathfrak{P}_{\delta(1)}, \mathfrak{P}_{\delta(2)}, \ldots, \mathfrak{P}_{\delta(n)}\right).$$

Theorem 22: The GPHFOWA operator increases monotonically with λ $(\lambda > 0)$, and the GPHFOWA operator decreases monotonically with λ $(\lambda > 0)$.

IV. MULTIPLE-CRITERIA DECISION MAKING MODELS BASED ON PHFS-BASED AGGREGATION OPERATORS

In this section, we develop two models based on GPHFWA (or GPHFWG) and GPHFOWA (or GPHFOWG) operators to solve the MCGDM problem in an uncertain context. First, we use the following notations to denote the important indices, sets, and variables in the MCGDM problem in the proportional hesitant fuzzy context.

- m: total number of alternatives;
- n: total number of criteria;
- t: total number of DMs involved in the decision process;
- $i \in M = \{1, 2, \ldots, m\}$: index of alternative;
- $j \in N = \{1, 2, \ldots, n\}$: index of criterion;
- $k \in T = \{1, 2, \ldots, t\}$: index of DM involved in the decision process;
- A_i: ith alternative;
- $A = \{A_1, A_2, \ldots, A_m\}$: set of m alternatives;
- C_j: jth criterion;
- $C = \{C_1, C_2, \ldots, C_n\}$: set of n criteria, which are considered to be independent;
- D_k: kth DM;
- $D = \{D_1, D_2, \ldots, D_t\}$: set of t DMs;
- δ_k: Weight of kth DM;
- $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_t)^T$: vector of weights of DMs, where $\sum_{k=1}^{t} \varphi_k = 1$, $0 \leq \varphi_k \leq 1$, and $k = 1, 2, \ldots, t$;
- $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$: weighting vector of criteria with respect to DM D_k, where $\sum_{j=1}^{n} \omega_j = 1$, $0 \leq \omega_j \leq 1$, and $j = 1, 2, \ldots, n$;
- N_b: collection of benefit criteria;
- N_c: collection of cost criteria such that $N_b \cup N_c = N$;
- $s_{ij}^k = \{\gamma_{ij}^{k1}, \gamma_{ij}^{k2}, \ldots, \gamma_{ij}^{k\#s_{ij}^k}\}$: assessment information on the performance of alternative A_i with respect to criterion C_j that is given by decision maker D_k and takes the form of a FS or a HFS, in which each element of the set represents the possible membership degree to which the alternative should satisfy the criterion and $\#s_{ij}^k$ represents the number of elements in the set;
- $S = \left(S^1, S^2, \ldots, S^t\right)^T$: vector of proportional hesitant fuzzy decision matrices with respect to all DMs, where $S^k = \left(s_{ij}^k\right)_{m\times n}$, $k = 1, 2, \ldots, t$.

In group decision making under uncertainty individuals feel easier to elicit FSs or HFSs than PHFSs. But from FSs or HFSs we propose to fuse individual assessments into collective assessments represented by PHFSs.

In addition, the accurate specification of expert weights and criterion weights is a usual core prerequisite for a MCGDM problem. The existing methods for deriving expert weights and criterion weights can be classified into three categories: subjective methods, objective methods, and methods integrating subjective methods and objective methods. In our proposals, the expert weights are predetermined by a supervisor who is familiar with all of the DMs based on the richness of his relevant knowledge and experience, and the weights of criteria can be specified based on the principle of maximum entropy principle.

Firstly, we give the basic framework of the proposed MCGDM models, which is presented in Figure1. Subsequently,we build the MCGDM model based on the GPHFWA or GPHFWG operator.

Model 1:

Step 1-1: Normalize the evaluation information. In a MAGDM problem, benefit criteria frequently coexist with cost criteria. If $N_c = \emptyset$, the normalization for assessment information is unnecessary. If $N_c \neq \emptyset$, we transform the assessment values of cost type into values of benefit type. Assessment information $\{S^k = (s_{ij}^k)_{m\times n} | k \in T\}$ is then transformed to the normalized assessment information $\{R^k = (r_{ij}^k)_{m\times n} | k \in T\}$, where r_{ij}^k is also a HFE and can be

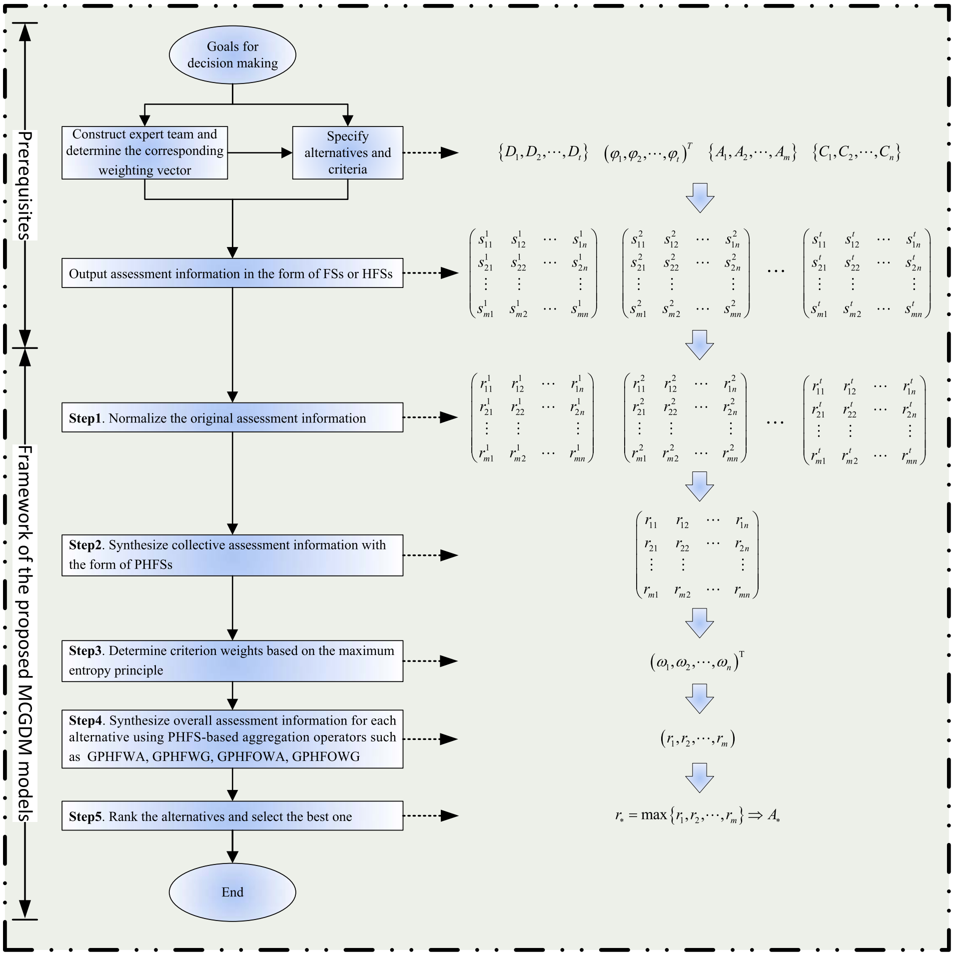


FIGURE 1. Framework of the proposed MCGDM models.

determined by

$$r_{ij}^{k}=\begin{cases} s_{ij}^{k}, & \text{for benefit criterion} C_j \\ \left(s_{ij}^{k}\right)^{C}, & \text{for cost criterion} C_j \end{cases}.$$

where $i \in M$, $j \in N$, and $k \in T$.

Without loss of generality, the normalized values r_{ij}^{k} are also represented as the set $r_{ij}^{k}=\{\gamma_{ij}^{k1},\gamma_{ij}^{k2},\ldots,\gamma_{ij}^{k\#r_{ij}^{k}}\}$, where $\#r_{ij}^{k}$ is the number of elements in the set r_{ij}^{k}.

Step 1-2: Aggregate individual arguments into collective assessment information by transforming FSs or HFSs into PHFSs. In this paper, we propose a method to achieve this transformation. This method simultaneously accommodates the membership degrees and the corresponding proportional information, thereby more accurately characterizing the hesitancy of DMs. Given the fact that r_{ij}^{k} $(k=1,2,\ldots,t)$ is made up of a set of values, we can determine the collective assessment information

$$r_{ij}=\left\{\left(\gamma_{ij}^{1},\tau_{ij}^{1}\right),\left(\gamma_{ij}^{2},\tau_{ij}^{2}\right),\ldots,\left(\gamma_{ij}^{\#r_{ij}},\tau_{ij}^{\#r_{ij}}\right)\right\},$$

in which $\{\gamma_{r_{ij}}^{1},\gamma_{r_{ij}}^{2},\ldots,\gamma_{r_{ij}}^{l_{ij}}\}$ is the union of the sets $r_{ij}^{1},r_{ij}^{2},\ldots,r_{ij}^{t}$, and τ_{ij}^{l} $(l=1,2,\ldots,l_{ij})$ denotes the proportion of the membership degree γ_{ij}^{l} $(l=1,2,\ldots,l_{ij})$ that can be determined by

$$\tau_{ij}^{l}=\sum\nolimits_{D_k\in D'}\left(\delta_k\Big/\#r_{ij}^{k}\right),$$

where D' is the set of DMs who provide the value γ_{ij}^{l}.

Step 1-3: Determine the weights of the criteria. An entropy measure of weight, which is called a "weight-dispersion" measure, is introduced to quantify the degree to which the corresponding weighted aggregation function takes into account all the inputs. For a given weighting vector $\omega=(\omega_1,\omega_2,\ldots,\omega_n)^T$, the entropy measure is determined by

$$\mathrm{Disp}\left(\omega\right)=-\sum_{i=1}^{n}\omega_i\log_2\omega_i,$$

with the convention $0\cdot\log_2 0=0$.

Remark: According to the principle of maximum entropy, we assume that the weighting vector with the largest entropy is the best because the corresponding weighted aggregation function can make full use of the information on all the criteria. For example, given the two weighting vectors $\omega_1=(0.5,0.5)$ and $\omega_2=(0.1,0.9)$, the former is assumed to be preferable because the weighted aggregation function based on the former can use information from two sources with equal degree and is less sensitive to inaccurate input.

During the decision process, the decision team can easily express the preference information on criteria, but the team has difficulty accurately specifying the weights of all the criteria. With respect to a MAGDM problem, let $\omega_j \in \Gamma$ $(j \in N)$, where Γ is a set of known information of criterion weights provided by the DM D_k $(k \in T)$ and can be constructed by the following ranking forms [54]: For $j_1 \neq j_2 \neq j_3$ and $k \in T$,

Form 1: a weak ranking: $\Gamma_1 = \left\{ \omega_{j_1} \geq \omega_{j_2} \middle| j_1, j_2 \in N \right\}$;

Form 2: a strict ranking: $\Gamma_2 = \left\{ \omega_{j_1} - \omega_{j_2} \geq \sigma_{j_1 j_2} \middle| j_1, j_2 \in N \right\}$;

Form 3: a ranking of differences:

$$\Gamma_3 = \left\{ \omega_{j_1} - \omega_{j_2} \geq \omega_{j_2} - \omega_{j_3} \middle| j_1, j_2, j_3 \in N \right\};$$

Form 4: a ranking with multiples:

$$\Gamma_4 = \left\{ \sigma_{j_1} \leq \omega_{j_1} \leq \sigma_{j_1} + \varepsilon_{j_1} \middle| j_1 \in N \right\};$$

Form 5: an interval form: $\Gamma_5 = \left\{ \omega_{j_1} \geq \sigma_{j_1} \omega_{j_2} \middle| j_1, j_2 \in N \right\}$, $0 \leq \sigma_{j_1} \leq 1$; where $\Gamma = \Gamma_1^k \cup \Gamma_2^k \cup \Gamma_3^k \cup \Gamma_4^k \cup \Gamma_5^k$.

Based on the principle of maximum entropy and the incomplete information given by DMs of criterion weights, we can determine the criterion weights for each DM by constructing the following programming model:

$$[\mathrm{M1}] \qquad \max\ \mathrm{Disp}(\omega) = -\sum_{j=1}^{n} \omega_j \log_2 \omega_j$$

$$s.t. \begin{cases} (\omega_1, \omega_2, \ldots, \omega_n)^T \in \Gamma \\ \sum_{j=1}^{n} \omega_j = 1 \\ \omega_j \geq 0. \end{cases}$$

By solving the method [M1], we can determine the optimal weighting vector $(\omega_1, \omega_2, \ldots, \omega_n)^T$ to maximize the information contained in the original decision information.

Step 1-4: For each alternative, aggregate assessment information on all criteria into overall assessment information. This model uses the GPHFWA operator,

$$\begin{aligned} r_i &= GPHFWA_\lambda (r_{i1}, r_{i2}, \ldots, r_{in}) \\ &= \cup_{\substack{(\gamma_{i1}, \tau_{i1}) \in r_{i1}, \\ \ldots, (\gamma_{in}, \tau_{in}) \in r_{in}}} \left\{ \left(\left[1 - \prod_{j=1}^{n} \left[1 - (\gamma_{ij})^\lambda \right]^{\omega_j} \right]^{1/\lambda}, \prod_{j=1}^{n} \tau_{ij} \right) \right\}, \end{aligned}$$

or the GPHFWG operator,

$$\begin{aligned} r_i &= GPHFWG_\lambda (r_{i1}, r_{i2}, \cdots, r_{in}) \\ &= \cup_{\substack{(\gamma_{i1}, \tau_{i1}) \in r_{i1}, \\ \cdots, (\gamma_{in}, \tau_{in}) \in r_{in}}} \left\{ \left(1 - \left(1 - \prod_{j=1}^{n} \left(1 - (1 - \gamma_{ij})^\lambda \right)^{\omega_j} \right)^{1/\lambda}, \prod_{k=1}^{t} \tau_{ij} \right) \right\} \end{aligned}$$

to determine the overall performance of each alternative $\{r_i | i \in M\}$, where ω_j is the weight of criterion k, $\sum_{j=1}^{n} \omega_j^k = 1$, $0 \leq \omega_j \leq 1$, $\lambda > 0$, $i \in M$, and $j \in N$.

Step 1-5: Rank the r_i $(i \in M)$ in descending order by **Definitions 7** and **8** and select the best alternative according to the ranking.

Finally, we formulate another MAGDM model based on the GPHFOWA or GPHFOWG operator.

Model 2:

Step 2-1: The information processing process is similar to **Step 1** of **Model 1**.

Step 2-2: The information processing process is the same as **Step 1** of **Model 1**.

Step 2-3: Determine the criterion weights. In this model, we also specify the criterion weights on the principle of maximum entropy. In addition, we introduce as constraint the desired ORness measure [42], [43] instead of the partial preference information. The ORness measure, also known as the attitudinal character (AC), is used to measure how far a given averaging function is from the max function and reflects changes in the DM's attitude. For an OWA function with a weighting vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$, its ORness measure is

$$\mathrm{ORness}(OWA_\omega) = \sum_{j=1}^{n} \frac{n-j}{n-1} \omega_j.$$

If $\omega^* = (1, 0, \ldots, 0)^T$, the ORness measure is unity and the OWA operator is reduced to the max operator, corresponding to the fully optimistic decision. If $\omega_* = (0, 0, \ldots, 1)^T$, the ORness measure is zero and the OWA function is reduced to the min function, corresponding to a fully pessimistic decision. Finally, if $\omega_A = \left(1/n, 1/n, \ldots, 1/n\right)^T$, the ORness measure is 0.5 and the OWA operator is reduced to the averaging operator, corresponding to a Laplace decision [42], [43].

During the decision process, each DM may have different attitudes towards each alternative. We can fuse individual attitudes into a collective attitude towards each alternative. If the decision team has an optimistic collective attitude on the overall performance of the alternative AC_i $(i \in M)$, the team can determine the AC value $AC_i \in (0.5, 1]$, where a more positive team reaction corresponds to a bigger AC. Similarly, if the team expresses a pessimistic attitude regarding the overall performance of the alternative AC_i $(i \in M)$, they can give an AC value $AC_i \in [0, 0.5)$, where a smaller AC represents a more pessimistic reaction of the DM. The AC is used to guide the OWA aggregation process by specifying the corresponding weighting vector for each alternative.

Based on the principle of maximum entropy and the perceived AC value AC_i $(i \in M)$ for each alternative, we determine the weighting vector for each alternative by calculating the following programming model:

$$[\text{M2}] \qquad \max \ \mathrm{Disp}\left(\omega_i\right) = -\sum_{j=1}^{n} \omega_{ij} \log \omega_{ij}$$

$$s.t. \begin{cases} \sum_{j=1}^{n} \frac{n-j}{n-1} \omega_{ij} = AC_i \\ \sum_{j=1}^{n} \omega_{ij} = 1 \\ \omega_{ij} \geq 0. \end{cases}$$

By solving the method [M2], we can determine the optimal weighting vector $\{(\omega_{i1}, \omega_{i2}, \ldots, \omega_{in})^T \,| i \in M\}$ for each alternative to maximize the information contained in the original decision information and to characterize the AC of each DM simultaneously.

Step 2-4: For each alternative, use the GPHFOWA operator,

$$\begin{aligned} r_i &= GPHFOWA_\lambda\left(r_{i1}, r_{i2}, \cdots, r_{in}\right) \\ &= \cup_{\left(\gamma_{i\delta(1)}, \tau_{i\delta(1)}\right) \in r_{i\delta(1)}, \cdots, \left(\gamma_{i\delta(n)}, \tau_{i\delta(n)}\right) \in r_{i\delta(n)}} \left\{\left(\left(1 - \prod_{j=1}^{n}\left(1 - \left(\gamma_{i\delta(j)}\right)^\lambda\right)^{\omega_j}\right)^{1/\lambda}, \prod_{j=1}^{n} \tau_{i\delta(j)}\right)\right\} \end{aligned}$$

or the GPHFOWG operator.

$$\begin{aligned} r_i &= GPHFOWG_\lambda\left(r_{i1}, r_{i2}, \cdots, r_{in}\right) \\ &= \cup_{\left(\gamma_{i\delta(1)}, \tau_{i\delta(1)}\right) \in r_{i\delta(1)}, \cdots, \left(\gamma_{i\delta(n)}, \tau_{i\delta(n)}\right) \in r_{i\delta(n)}} \\ &\quad \left\{\left(1 - \left(1 - \prod_{j=1}^{n}\left(1 - \left(1 - \gamma_{i\delta(j)}\right)^\lambda\right)^{\omega_j}\right)^{1/\lambda}, \prod_{j=1}^{n} \tau_{i\delta(j)}\right)\right\} \end{aligned}$$

to derive the overall assessment information $\{r_i \,| i \in M\}$, where $\left(r_{i\delta(1)}, r_{i\delta(2)}, \ldots, r_{i\delta(n)}\right)$ is the rearrangement of $(r_{i1}, r_{i2}, \ldots, r_{in})$ satisfying $r_{i\delta(1)} \geq r_{i\delta(2)} \geq \cdots \geq r_{i\delta(n)}$, $r_{i\delta(j)}$ is the jth largest in $(r_{i\delta(1)}, r_{i\delta(2)}, \ldots, r_{i\delta(n)})$, and ω_j denotes the weight of the kth position in $(r_{i\delta(1)}, r_{i\delta(2)}, \ldots, r_{i\delta(n)})$ with the following conditions satisfied: $\sum_{j=1}^{n} \omega_j^k = 1$, $0 \leq \omega_j \leq 1$, $\lambda > 0$, $i \in M$, and $j \in N$.

Step 2-5: Rank the r_i $(i \in M)$ in descending order by **Definitions 7** and **8**, and select the best alternative in accordance with the ranking.

V. CASE STUDY

Over the last decade, due to environmental issues and the fear of peak oil prices, battery electric vehicles (BEVs), which use for vehicle propulsion the chemical energy stored in rechargeable battery packs combined with electric motors and motor controllers, have garnered considerable attention from governments around the word, including China, the European Union, and the United States. These governments have issued a series of regulations to compel original equipment manufacturers (OEMs) to produce more EVs and encourage consumers to buy EVs. To date, the price and driving range are the biggest obstacles preventing consumers from purchasing EVs, and both obstacles are directly linked to the EVB. At present, most OEMs are inclined to outsource EVB manufacturing to a battery manufacturer. Thus, selecting a suitable EVB supplier is extremely important for OEMs.

In China, an OEM has made a strategic decision to invest in the research and development of EVs. During the development of a new BEV, the OEM tries to select the best EVB supplier on the market and hope to establish a long-term, stable, and mutually beneficial strategic partnership with the selected EVB supplier. The OEM builds a decision-making team that is responsible for selecting the EVB supplier. The team consists of five experts $\{D_1, D_2, D_3, D_4, D_5\}$ who are weighted by the weighting vector $\varphi = (0.15, 0.2, 0.3, 0.1, 0.25)^T$ on the basis of their relevant knowledge and experience. The first task assigned to the decision team is to specify the assessment criteria for selecting the EVB supplier. For the strategic-supplier-selection problem, the OEM should not only focus on short-term criteria, such as cost and quality, but also focus on long-term criteria, such as technical capability, company profile, and level of risk. Based on the objectives for the EVB supplier, the decision team specifies seven criteria: cost (C_1), quality (C_2), delivery and lead time (C_3), service level (C_4), technical capability (C_5), company profile (C_6), and risk level (C_7). Evidently, C_1, C_3, and C_7 are cost criteria whereas the others are benefit criteria. Table 1 gives detailed information on the seven criteria.

Once the criteria are specified, the alternative potential suppliers are identified from upstream of the EV industry chain, which is not limited to the domestic market. The decision team identifies a list of suppliers through a variety of channels and slims down the list to five potential suppliers thorough a preliminary review. These suppliers are represented as $\{A_1, A_2, A_3, A_4, A_5\}$. Next, each expert D_k $(k = 1, 2, \ldots, 5)$ assesses the performance of each potential supplier A_i $(i = 1, 2, \ldots, 5)$ with respect to each criterion C_j $(j = 1, 2, \ldots, 7)$. The following rules are used to facilitate the articulation of assessment information for DMs:

- The assessment value given by each DM should fall in the range of 0 to 1 and is used to embody the possible membership degree that alternatives should satisfy or to measure the performance of an alternative with respect to a specific criterion.
- The better an alternative performs with respect to a specific criterion, the larger assessment value given by DMs.
- DMs can assign only a single value to the performance of an alternative with respect to a specific criterion when

TABLE 1. Criteria for selection of electric vehicle battery (EVB) supplier.

Criteria	Content	References
Cost	Battery cost makes up a substantial part of BEV cost, but the increase in production scale and advances in battery technology have reduced the price of lithium-ion batteries. According to a report published by BloombergNEF in 2018, the average price of a battery pack has fallen 85% from 2010, reaching \$176/kW. However, the price of EVs is still not competitive with that of conventional vehicles powered by internal combustio nengine(ICE). In addition to purchase price, the cost also includes transportation and operational costs.	[55–59]
Quality	Refers mainly to the technical performance of EVBs, including specific energy, specific power, life span, energy density, safety, and charging time.	[55, 56, 58, 59]
Delivery and lead time	At present, unstable market requirements for EVs urges OEMs and traditional OEMs and EV start-ups to adopt agile manufacturing to rapidly respond to customer requirements, necessitating delivery reliability and short lead time for various components, especially for core components including battery, motor, converter, and electronic control system.	[57, 60]
Service level	The real requirement of OEMs is EVB solutions rather than EVB products. Thus, besides tangible products, EVB suppliers are also required to provide life-cycle services including training, development of battery management system, quality improvement, and battery recycling.	[56, 61–63]
Technical capability	Ensures future improvements in technical performances of EVB. At present, clear gaps remain between BEVs and conventional vehicles powered by ICE with respect to cost, energy density, safety., which necessitates long-term investment in research and development by EVB suppliers.	[58, 61–63]
Company profile	OEM attempts to develop a long-term and strategic partnership with EVB supplier and integrates the supplier into the new-product development process. Thus, it makes enormous sense to pay more attention to the supplier profile, which refers to the history and evolution of the company, mainly containing financial status, the staffing pattern, response of customers, the performance history, anticipated performance in the future, etc.	[56, 58, 61–63]
Risk level	Selecting a global supplier is much riskier than selecting a domestic supplier. Thus, if the scope of selection is worldwide, the OEM should pay more attention to risk factors such as geographical location and political stability.	[55, 61–63]

she affirms her judgment. She can also give a set of possible values when she hesitates about her judgment.

Thus, the assessment should be either a degree of membership of a FS or several degrees of membership of a HFS. The initial assessment information

$$\left\{S^k = \left(s_{ij}^k\right)_{5\times 7} \middle| k = 1, 2, \ldots, 5\right\}$$

is presented in Table 2.

Next, **Models 1** and **2** are used to address the problem of selecting the EVB supplier.

A. APPLICATION OF MODEL 1

Step 1-1: Transform the initial assessment information into normalized assessment information

$$\left\{R^k = \left(r_{ij}^k\right)_{5\times 7} \middle| k \in 1, 2, \ldots, 5\right\},$$

which is presented in Table 3.

Step 1-2: Aggregate individual assessment information into collective assessment information $R = \left(r_{ij}\right)_{5\times 7}$ by transforming the FS or HFS into a PHFS, the result of which is given in Table 4.

Step 1-3: Determine the weights of criteria based on the principle of maximum entropy. After thorough discussion, the decision team gives incomplete preference information on the importance of criteria, which is presented as follows:

$$\omega_2 - \omega_1 \geq 0.025, \quad \omega_2 = \omega_4, \quad \omega_4 - \omega_3 \geq 0.05,$$
$$\omega_3 \geq \omega_1, \quad \omega_5 \geq 1.25 \cdot \omega_2, \quad \omega_6 \geq \omega_5,$$
$$\omega_6 \geq 1.25 \cdot \omega_7.$$

The input of information into Method [M1], which was developed based on the maximum entropy principle, gives the optimal weighting vector for criteria,

$$\omega = (0.097, 0.147, 0.097, 0.147, 0.184, 0.184, 0.141)^T.$$

Step 1-4: Use the GPHFWA operator with $\lambda = 1$ to fuse collective assessment information on each criterion and obtain overall assessment information on each alternative $\{r_i \,|i = 1, 2, \ldots, 5\}$. Due to space limitations, we only present the scores for the overall assessment information on each alternative:

$$s(r_1) = 0.68, \quad s(r_2) = 0.675,$$
$$s(r_3) = 0.682, \quad s(r_4) = 0.753, \quad s(r_5) = 0.593.$$

Applying the GPHFWG operator, we obtain

$$s(r_1) = 0.524, \quad s(r_2) = 0.565,$$
$$s(r_3) = 0.653, \quad s(r_4) = 0.581, \quad s(r_5) = 0.55.$$

Step 1-5: Using **Definitions 7** and **8**, we find that the application of the GPHFWA operator leads to $A_4 \succ A_3 \succ A_1 \succ A_2 \succ A_5$, with A_4 being the best alternative, and the use of the GPHFWG operator leads to $A_3 \succ A_4 \succ A_2 \succ A_5 \succ A_1$, with A_3 being the best EVB supplier.

After negotiation with the OEM management, the decision team chooses A_3 as the best EVB supplier for the following reasons:

- From Table 4, we find that the alternative A_4, which is a well-known overseas EVB supplier, performs well with respect to criteria including quality (C_2), service level (C_4), technical capability (C_5), and company profile (C_6). But the cost (C_1) and the geopolitical risk (C_7)

TABLE 2. Initial assessment information provided by DMs.

		C_1	C_2	C_3	C_4	C_5	C_6	C_7
D_1	A_1	{0.1}	{0.8,0.7}	{0.4}	{0.3,0.2}	{0.8}	{0.3,0.2}	{0.1}
	A_2	{0.2,0.1}	{0.4,0.3}	{0.3}	{0.8,0.7}	{0.3,0.2}	{0.7,0.6}	{0.1}
	A_3	{0.5,0.4}	{0.8,0.7}	{0.4}	{0.7}	{0.8}	{0.9}	{0.5,0.4}
	A_4	{0.8,0.7}	{0.9}	{0.5}	{0.8}	{0.9}	{0.9,0.8}	{0.9,0.8}
	A_5	{0.2,0.1}	{0.6,0.5}	{0.4}	{0.6}	{0.5}	{0.6}	{0.2,0.1}
D_2	A_1	{0.2,0.1}	{0.7}	{0.5,0.4}	{0.3}	{0.8,0.7}	{0.2}	{0.1}
	A_2	{0.2,0.1}	{0.4,0.3}	{0.4,0.3}	{0.7,0.6}	{0.3}	{0.7}	{0.1}
	A_3	{0.5}	{0.7}	{0.5,0.4}	{0.7,0.6}	{0.8}	{0.9,0.8}	{0.5,0.4}
	A_4	{0.6}	{0.9,0.8}	{0.6,0.5}	{0.8}	{0.9,0.8}	{0.8,0.7}	{0.8,0.7}
	A_5	{0.1}	{0.6}	{0.3}	{0.7,0.6}	{0.7,0.6}	{0.6,0.5}	{0.1}
D_3	A_1	{0.3,0.2,0.1}	{0.8}	{0.5}	{0.3,0.2}	{0.8,0.7,0.6}	{0.3,0.2}	{0.1}
	A_2	{0.1}	{0.4}	{0.3}	{0.8}	{0.4,0.3}	{0.6}	{0.2,0.1}
	A_3	{0.4}	{0.7}	{0.4}	{0.7,0.6}	{0.7,0.6}	{0.7}	{0.5}
	A_4	{0.8,0.7}	{0.9,0.8}	{0.5}	{0.8,0.7}	{0.9}	{0.8}	{0.8}
	A_5	{0.1}	{0.5}	{0.4,0.3}	{0.6}	{0.7,0.6}	{0.6}	{0.2,0.1}
D_4	A_1	{0.1}	{0.8,0.7,0.6}	{0.5,0.4}	{0.2}	{0.7}	{0.3}	{0.1}
	A_2	{0.1}	{0.4,0.3}	{0.2}	{0.7}	{0.3}	{0.6,0.5}	{0.1}
	A_3	{0.5}	{0.7}	{0.5,0.4}	{0.7}	{0.7,0.6}	{0.8,0.7}	{0.5}
	A_4	{0.8,0.7}	{0.9,0.8}	{0.6}	{0.9,0.8}	{0.9}	{0.8,0.7}	{0.8}
	A_5	{0.1}	{0.5}	{0.4}	{0.7,0.6}	{0.5}	{0.6}	{0.2,0.1}
D_5	A_1	{0.2,0.1}	{0.8,0.7}	{0.4}	{0.4,0.3,0.2}	{0.7,0.6}	{0.3}	{0.1}
	A_2	{0.2,0.1}	{0.5}	{0.4}	{0.7,0.6}	{0.4,0.3,0.2}	{0.6,0.5}	{0.2,0.1}
	A_3	{0.4}	{0.8,0.7}	{0.5,0.4,0.3}	{0.7,0.6}	{0.7}	{0.8,0.7}	{0.6,0.5}
	A_4	{0.7}	{0.9}	{0.6}	{0.9,0.8}	{0.9,0.8}	{0.9}	{0.8}
	A_5	{0.1}	{0.6,0.5}	{0.4,0.3}	{0.7}	{0.6}	{0.6,0.5}	{0.2,0.1}

TABLE 3. Normalized assessment information.

		C_1	C_2	C_3	C_4	C_5	C_6	C_7
D_1	A_1	{0.9}	{0.8,0.7}	{0.6}	{0.3,0.2}	{0.8}	{0.3,0.2}	{0.9}
	A_2	{0.9,0.8}	{0.4,0.3}	{0.7}	{0.8,0.7}	{0.3,0.2}	{0.7,0.6}	{0.9}
	A_3	{0.6,0.5}	{0.8,0.7}	{0.6}	{0.7}	{0.8}	{0.9}	{0.6,0.5}
	A_4	{0.3,0.2}	{0.9}	{0.5}	{0.8}	{0.9}	{0.9,0.8}	{0.2,0.1}
	A_5	{0.9,0.8}	{0.6,0.5}	{0.6}	{0.6}	{0.5}	{0.6}	{0.9,0.8}
D_2	A_1	{0.9,0.8}	{0.7}	{0.6,0.5}	{0.3}	{0.8,0.7}	{0.2}	{0.9}
	A_2	{0.9,0.8}	{0.4,0.3}	{0.7,0.6}	{0.7,0.6}	{0.3}	{0.7}	{0.9}
	A_3	{0.5}	{0.7}	{0.6,0.5}	{0.7,0.6}	{0.8}	{0.9,0.8}	{0.6,0.5}
	A_4	{0.4}	{0.9,0.8}	{0.5,0.4}	{0.8}	{0.9,0.8}	{0.8,0.7}	{0.3,0.2}
	A_5	{0.9}	{0.6}	{0.7}	{0.7,0.6}	{0.7,0.6}	{0.6,0.5}	{0.9}
D_3	A_1	{0.9,0.8,0.7}	{0.8}	{0.5}	{0.3,0.2}	{0.8,0.7,0.6}	{0.3,0.2}	{0.9}
	A_2	{0.9}	{0.4}	{0.7}	{0.8}	{0.4,0.3}	{0.6}	{0.9,0.8}
	A_3	{0.6}	{0.7}	{0.6}	{0.7,0.6}	{0.7,0.6}	{0.7}	{0.5}
	A_4	{0.3,0.2}	{0.9,0.8}	{0.5}	{0.8,0.7}	{0.9}	{0.8}	{0.2}
	A_5	{0.9}	{0.5}	{0.7,0.6}	{0.6}	{0.7,0.6}	{0.6}	{0.9,0.8}
D_4	A_1	{0.9}	{0.8,0.7,0.6}	{0.6,0.5}	{0.2}	{0.7}	{0.3}	{0.9}
	A_2	{0.9}	{0.4,0.3}	{0.8}	{0.7}	{0.3}	{0.6,0.5}	{0.9}
	A_3	{0.5}	{0.7}	{0.6,0.5}	{0.7}	{0.7,0.6}	{0.8,0.7}	{0.5}
	A_4	{0.3,0.2}	{0.9,0.8}	{0.4}	{0.9,0.8}	{0.9}	{0.8,0.7}	{0.2}
	A_5	{0.9}	{0.5}	{0.6}	{0.7,0.6}	{0.5}	{0.6}	{0.9,0.8}
D_5	A_1	{0.9,0.8}	{0.8,0.7}	{0.6}	{0.4,0.3,0.2}	{0.7,0.6}	{0.3}	{0.9}
	A_2	{0.9,0.8}	{0.5}	{0.6}	{0.7,0.6}	{0.4,0.3,0.2}	{0.6,0.5}	{0.9,0.8}
	A_3	{0.6}	{0.8,0.7}	{0.7,0.6,0.5}	{0.7,0.6}	{0.7}	{0.8,0.7}	{0.5,0.4}
	A_4	{0.3}	{0.9}	{0.4}	{0.9,0.8}	{0.9,0.8}	{0.9}	{0.2}
	A_5	{0.9}	{0.6,0.5}	{0.7,0.6}	{0.7}	{0.6}	{0.6,0.5}	{0.9,0.8}

obtain a lower score, placing them beyond the scope of tolerance.

- The alternative A_3, which is also an overseas EVB supplier, obtains more balanced scores for each criterion. The cost (C_1) and the geopolitical risk (C_7) have moderate scores that fall within the scope of tolerance. The main reason that the decision team chooses A_3 is that the supplier has set up a factory for EVB manufacturing with a domestic manufacturer.
- The domestic EVB supplier A_1 performs well for cost (C_1), quality (C_2), and risk (C_7) but performs poorly on company profile (C_6), which plays a significant role in the establishment of a strategic partnership. The domestic alternative A_2 obtains a lower score for technical capability (C_5), which plays a significant role in a long-term partnership because EVB technology is changing very fast. The alternative A_5, which is also a domestic EVB supplier, obtains balanced scores on

TABLE 4. Normalized assessment information.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
A_1	{(0.9,0.8,0.7), (0.575,0.325,0.1)}	{(0.8,0.7,0.6), (0.533,0.433,0.033)}	{(0.6,0.5), (0.55,0.45)}	{(0.4,0.3,0.2), (0.083,0.508,0.408)}	{(0.8,0.7,0.6), (0.35,0.425,0.225)}	{(0.3,0.2), (0.575,0.425)}	{(0.9),(1)}
A_2	{(0.9,0.8), (0.7,0.3)}	{(0.5,0.4,0.3), (0.25,0.525,0.225)}	{(0.8,0.7,0.6), (0.1,0.55,0.35)}	{(0.8,0.7,0.6), (0.375,0.4,0.225)}	{(0.4,0.3,0.2), (0.233,0.608,0.158)}	{(0.7,0.6,0.5), (0.275,0.55,0.175)}	{(0.9,0.8), (0.725,0.275)}
A_3	{(0.6,0.5), (0.625,0.375)}	{(0.8,0.7), (0.2,0.8)}	{(0.7,0.6,0.5), (0.083,0.683,0.233)}	{(0.7,0.6), (0.625,0.375)}	{(0.8,0.7,0.6), (0.35,0.45,0.2)}	{(0.9,0.8,0.7), (0.25,0.275,0.475)}	{(0.6,0.5,0.4), (0.175,0.7,0.125)}
A_4	{(0.4,0.3,0.2), (0.2,0.525,0.275)}	{(0.9,0.8), (0.7,0.3)}	{(0.5,0.4), (0.55,0.45)}	{(0.9,0.8,0.7), (0.175,0.675,0.15)}	{(0.9,0.8), (0.775,0.225)}	{(0.9,0.8,0.7), (0.325,0.525,0.15)}	{(0.3,0.2,0.1), (0.1,0.825,0.075)}
A_5	{(0.9,0.8), (0.925,0.075)}	{(0.6,0.5), (0.4,0.6)}	{(0.7,0.6), (0.475,0.525)}	{(0.7,0.6), (0.4,0.6)}	{(0.7,0.6,0.5), (0.1,0.5,0.25)}	{(0.6,0.5), (0.775,0.225)}	{(0.9,0.8), (0.6,0.4)}

TABLE 5. Results obtained by the GPHFWA and GPHFWG operators.

Scenes	λ	Scores of overall assessment information for each alternative					Rankings of alternatives				
		A_1	A_2	A_3	A_4	A_5	A_1	A_2	A_3	A_4	A_5
Application of GPHFWA	0.1	0.654	0.657	0.677	0.732	0.587	4	3	2	1	5
	0.2	0.657	0.659	0.678	0.734	0.587	4	3	2	1	5
	0.5	0.666	0.665	0.679	0.741	0.589	3	4	2	1	5
	1	0.680	0.675	0.682	0.753	0.593	3	4	2	1	5
	2	0.705	0.694	0.687	0.771	0.601	2	3	4	1	5
	5	0.755	0.739	0.702	0.805	0.626	2	3	4	1	5
	10	0.796	0.783	0.724	0.830	0.662	2	3	4	1	5
Application of GPHFWG	0.1	0.572	0.600	0.663	0.644	0.565	4	3	1	2	5
	0.2	0.566	0.596	0.662	0.638	0.563	4	3	1	2	5
	0.5	0.551	0.584	0.659	0.617	0.558	5	3	1	2	4
	1	0.524	0.565	0.653	0.581	0.550	5	3	1	2	4
	2	0.477	0.529	0.641	0.514	0.536	5	3	1	4	2
	5	0.389	0.454	0.606	0.397	0.509	5	3	1	4	2
	10	0.329	0.396	0.568	0.322	0.488	4	3	1	5	2

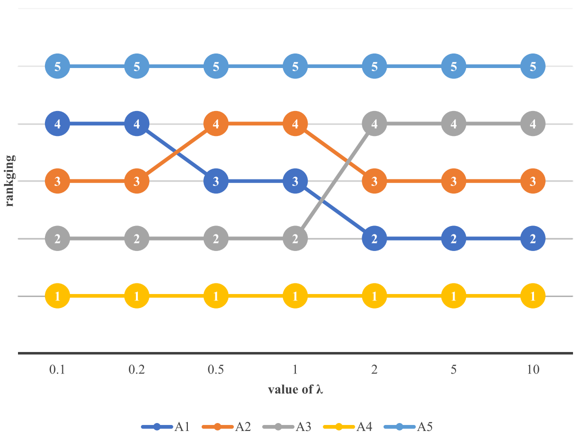


FIGURE 2. Results obtained by the GPHFWA operator with different values of λ.

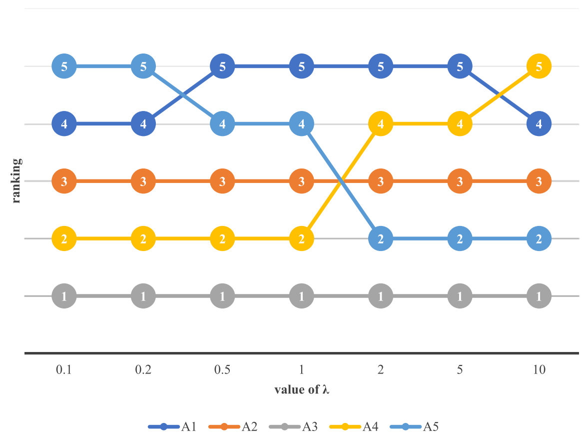


FIGURE 3. Results obtained by the GPHFWG operator with different values of λ.

each criterion. Compared with A_3, A_5 performs worse on quality (C_2), service level (C_4), technical capability (C_5), and company profile (C_6), which means that A_3 is the best supplier.

When the parameter λ varies, we attain different results. In this section, we assign 1/10, 1/5, 1/2, 1, 2, 5, and 10 to λ and use the GPHFWA operator to calculate the results, which are presented in Table 5 and Figure 2. Similarly, we use the GPHFWG operator with these seven values of λ to determine the results given in Table 5 and Figure 3.

(1) When the input information of the model remains the same, the score determined by the GPHFWA operator becomes larger with increasing λ, and the score value obtained by the GPHFWG operator becomes smaller with increasing λ. In addition, the scores obtained by the GPHFWA operator are always greater than those obtained by the GPHFWG operator.

(2) Applying the GPHFWA operator gives alternative A_4 as the best supplier, whereas applying the GPHFWG operator gives alternative A_3 as the best supplier. From Table 4, we see that the differences between the

TABLE 6. Weights of criteria in the case of different ACs for alternatives.

AC	Alternatives	C_1	C_2	C_3	C_4	C_5	C_6	C_7
0.3	A_1	0.044	0.061	0.084	0.117	0.161	0.224	0.31
0.4	A_2	0.086	0.1	0.117	0.136	0.159	0.185	0.216
0.6	A_3	0.216	0.185	0.159	0.136	0.117	0.1	0.086
0.4	A_4	0.086	0.1	0.117	0.136	0.159	0.185	0.216
0.5	A_5	0.143	0.143	0.143	0.143	0.143	0.143	0.143

performances of A_4 with respect to each criterion are greater than those of its counterpart A_3. In other words, the cohesiveness of the assessment information of A_3 exceeds that of A_4. We further conclude that, upon applying the GPHFWA operator, the criteria with lower scores obtain more compensation from the criteria with higher scores, thus leading to the result that A_4 is the best supplier. However, applying the GPHFWG operator yields the result that the criteria with lower scores obtain limited compensation from the criteria with higher scores, thus leading to the result that A_3 is the best supplier. Thus, we conclude that, for PHFSs with less difference between elements, the GPHFWG operator is more friendly than the GPHFWA operator. In this case, the OEM prefers A_3 to A_4, which indicates that, in practical applications, the GPHFWG operator is preferable when the compensation is limited between the performances with respect to criteria.

(3) For the GPHFWA operator, as the parameter λ increases, the ranking of A_1 rises and the ranking of A_3 drops. For the GPHFWG operator, increasing the parameter λ decreases the ranking of A_4 and increases the ranking of A_5. Table 4 reveals that A_1 and A_4 have a greater difference between performances for all criteria, and A_3 and A_5 have less difference. We thus infer that, upon increasing λ, the compensation of the GPHFWA operator increases, and the compensation of the GPHFWG operator decreases.

B. APPLICATION OF MODEL 2

This section describes the application of **Model 2** to the case described above.

Step 2-1: The results of the information processing procedure of this step are listed in Table 3.

Step 2-2: The results of this step are given in Table 4.

Step 2-3: Determine the weights of criteria based on the maximum entropy principle and the AC of DMs. In this case, different DMs have different ACs for each alternative. We simplify the information processing procedure and, after much deliberation, let the decision team output a collective AC for each alternative:

$$AC_1 = 0.3, \quad AC_2 = 0.4, \quad AC_3 = 0.6, \quad AC_4 = 0.4,$$
$$AC_5 = 0.5.$$

Based on Method [M2], we calculate the final weights of criteria with respect to each DM. The results are given in Table 6.

Step 2-4: Use the GPHFOWA operator with $\lambda = 1$ to synthesize the overall assessment information of each potential supplier, which gives the following scores:

$$s(r_1) = 0.543, \quad s(r_2) = 0.642,$$
$$s(r_3) = 0.692, \quad s(r_4) = 0.646, \quad s(r_5) = 0.611.$$

Instead of using the GPHFOWA operator, we use the GPHFOWG operator with $\lambda = 1$ to determine the overall assessment information for each alternative, which gives the following results:

$$s(r_1) = 0.407, \quad s(r_2) = 0.534,$$
$$s(r_3) = 0.665, \quad s(r_4) = 0.456, \quad s(r_5) = 0.564.$$

Step 2-5: Applying the GPHFOWA operator gives $A_3 \succ A_4 \succ A_2 \succ A_5 \succ A_1$, with A_3 being the best EVB supplier, and the introduction of the GPHFOWG operator gives $A_3 \succ A_5 \succ A_2 \succ A_4 \succ A_1$, with A_3 being the best alternative.

Evidently, the results obtained by the GPHFOWA and GPHFOWG operators are consistent with the desired result of the OEM. In a similar way, we assign the values 1/10, 1/5, 1/2, 1, 2, 5, 10 to λ and use the GPHFOWA operator and the GPHFOWG operator to determine the final ranking of alternatives. The results are given in Table 7 and Figs.4 and 5.

(1) Evidently, the scores obtained by the GPHFOWA operator increase as λ increases, whereas the scores obtained by the GPHFOWG operator decrease as λ increases. In addition, the scores obtained by the GPHFOWA operator are always greater than those obtained by the GPHFOWG operator.

(2) When the GPHFOWA operator is applied, if $\lambda = 0.1$, 0.2, 0.5, or 1, the alternative A_3 is the best supplier, followed by A_2 and A_4; if $\lambda = 5$ or 10, A_4 becomes the optimum supplier, followed by A_2. When the GPHFOWG operator is applied, the alternative A_3 remains the best supplier, followed by A_5. Table 4 shows that the differences for A_2 and A_4 between the performances for each alternative are greater than for the counterparts of A_3 and A_5. We can then reason that, compared with the GPHFWG operator, the GPHFWA operator has a higher level of compensation for the performance with respect to the given criteria, but the results obtained by the GPHFOWG operator are better, which is consistent with the preferences of the OEM.

(3) For the GPHFOWA operator, along with the parameter λ increasing, the rankings of A_3 and A_5 decline, but the rankings of A_1 and A_4 increase. Given that the differences for A_1 and A_4 are greater than those for A_3 and A_5, we infer that the compensation for the GPHFOWA operator increases along with λ.

(4) For the GPHFOWG operator, the data in Table 7 cannot reveal the relationship between the compensation level of the GPHFOWG operator and the value of λ, mainly because different ACs are assigned to different alternatives, and the AC critically impacts the compensation level of the GPHFOWG operator, as illustrated below.

TABLE 7. Results obtained by GPHFOWA and GPHFOWG operators.

Scenes	λ	Scores for overall assessment information for each alternative					Rankings of alternatives				
		A_1	A_2	A_3	A_4	A_5	A_1	A_2	A_3	A_4	A_5
Application of GPHFOWA	0.1	0.507	0.621	0.688	0.609	0.604	5	2	1	3	4
	0.2	0.511	0.623	0.689	0.613	0.605	5	2	1	3	4
	0.5	0.523	0.630	0.690	0.626	0.607	5	2	1	3	4
	1	0.543	0.642	0.692	0.646	0.611	5	3	1	2	4
	2	0.581	0.663	0.697	0.682	0.619	5	3	1	2	4
	5	0.666	0.715	0.710	0.747	0.642	4	2	3	1	5
	10	0.736	0.766	0.730	0.794	0.675	3	2	4	1	5
Application of GPHFOWG	0.1	0.435	0.565	0.675	0.505	0.581	5	3	1	4	2
	0.2	0.432	0.561	0.674	0.499	0.579	5	3	1	4	2
	0.5	0.422	0.551	0.671	0.482	0.574	5	3	1	4	2
	1	0.407	0.534	0.665	0.456	0.564	5	3	1	4	2
	2	0.381	0.502	0.653	0.412	0.547	5	3	1	4	2
	5	0.333	0.437	0.618	0.339	0.516	5	3	1	4	2
	10	0.297	0.386	0.579	0.290	0.492	4	3	1	5	2

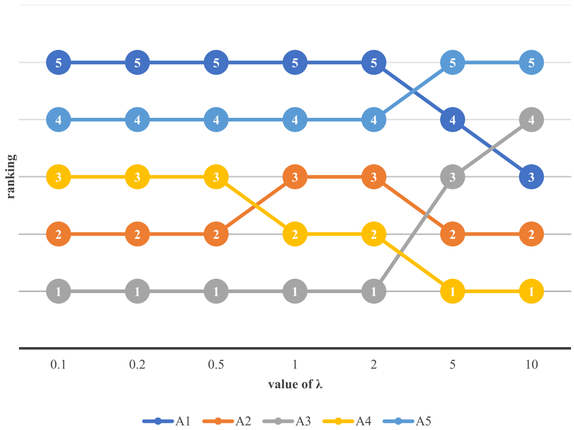


FIGURE 4. Results obtained by the GPHFOWA operator with different values of λ.

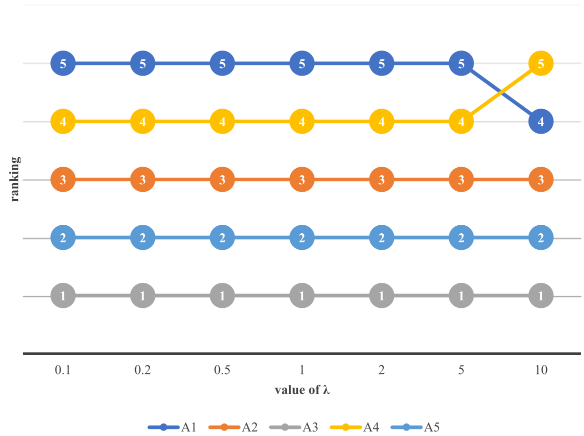


FIGURE 5. Results obtained by the GPHFOWG operator with different values of λ.

To eliminate the impact of AC, we assign 0.6 to the AC for each alternative and apply the GPHFOWG operator with different values of λ. The results are presented in Table 8 and Figure 6. Obviously, as λ increases, the rankings of A_3 and A_5 with lower compensation levels rise, but the rankings slip for A_2 and A_4, which have a higher level of compensation. Thus, we infer that the compensation level of the GPHFOWG operator decreases as λ increases.

To further explore how AC affects the final ranking of alternatives, we maintain $\lambda = 1$ and change the AC. In this case, the decision team may have different ACs for each alternative. For convenience, we assign the same AC to all alternatives. The results obtained by the GPHFOWA and GPHFOWG operators are given in Table 9 and Figs. 7 and 8.

(1) Table 9 shows clearly that the AC has a remarkable impact on the final ranking of alternatives. DMs should accurately output the AC according to their real feelings about alternatives.

(2) Upon applying the GPHFOWA operator, the increase in AC increases the rankings of A_1 and A_4 and decreases the rankings of A_3 and A_5. The results obtained by the GPHFOWG operator also lead to a similar conclusion. We therefore conclude that the compensation level of the GPHFOWA operator and of the GPHFOWG operator increases along with the AC.

C. COMPARISON BETWEEN PROPORTIONAL HESITANT FUZZY SETS AND HESITANT FUZZY SETS

In this section, we compare PHFSs with HFSs. [25] proposed a series of aggregation operators for HFS, including the GHFWA, GHFWG, GHFOWA, and GHFOWG operators. To delve into the differences between PHFSs and HFSs, we make the following comparisons: GPHFWA vs GHFWA, GPHFWG vs GHFWG, GPHFOWA vs GHFOWA, and GPHFOWG vs GHFOWG.

1) GPHFWA VS GHFWA AND GPHFWG VS GHFWG

In accordance with **Model 1**, we develop a new MCGDM model, called **Model 1′**, by replacing the GPHFWA operator or the GPHFWG operator with the GHFWA operator or the GHFWG operator, respectively. **Model 1′** is based

TABLE 8. Results obtained by GPHFOWG operator with $AC = 0.6$.

Scenes	λ	Scores for overall assessment information for each alternative					Rankings of alternatives				
		A_1	A_2	A_3	A_4	A_5	A_1	A_2	A_3	A_4	A_5
Application of GPHFOWG with $AC = 0.6$	0.1	0.676	0.695	0.675	0.682	0.615	3	1	4	2	5
	0.2	0.671	0.692	0.674	0.676	0.612	4	1	3	2	5
	0.5	0.655	0.680	0.671	0.657	0.606	4	1	2	3	5
	1	0.625	0.658	0.665	0.622	0.595	3	2	1	4	5
	2	0.567	0.614	0.653	0.554	0.573	4	2	1	5	3
	5	0.451	0.515	0.618	0.429	0.533	4	3	1	5	2
	10	0.367	0.435	0.579	0.345	0.503	4	3	1	5	2

TABLE 9. Results obtained by GPHFOWA and GPHFOWG operator with $\lambda = 1$.

Scenes	AC	Scores for overall assessment information for each alternative					Rankings of alternatives				
		A_1	A_2	A_3	A_4	A_5	A_1	A_2	A_3	A_4	A_5
Application of GPHFOWA with $\lambda = 1$	0	0.257	0.307	0.505	0.203	0.495	4	3	1	5	2
	0.1	0.334	0.397	0.540	0.310	0.494	4	3	1	5	2
	0.2	0.443	0.489	0.573	0.442	0.513	4	3	1	5	2
	0.3	0.543	0.571	0.606	0.556	0.543	5	2	1	3	4
	0.4	0.627	0.642	0.637	0.646	0.576	4	2	3	1	5
	0.5	0.698	0.702	0.665	0.717	0.611	3	2	4	1	5
	0.6	0.756	0.752	0.692	0.772	0.645	2	3	4	1	5
	0.7	0.803	0.795	0.717	0.815	0.677	2	3	4	1	5
	0.8	0.842	0.830	0.739	0.847	0.708	2	3	4	1	5
	0.9	0.874	0.858	0.759	0.869	0.736	1	3	4	2	5
	1	0.900	0.872	0.777	0.878	0.759	1	3	4	2	5
Application of GPHFOWG with $\lambda = 1$	0	0.257	0.307	0.505	0.203	0.495	4	3	1	5	2
	0.1	0.291	0.361	0.532	0.253	0.488	4	3	1	5	2
	0.2	0.346	0.417	0.558	0.314	0.499	4	3	1	5	2
	0.3	0.407	0.474	0.584	0.381	0.516	4	3	1	5	2
	0.4	0.475	0.534	0.611	0.456	0.538	4	3	1	5	2
	0.5	0.548	0.595	0.638	0.536	0.564	4	2	1	5	3
	0.6	0.625	0.658	0.665	0.622	0.595	3	2	1	4	5
	0.7	0.705	0.722	0.693	0.708	0.630	3	1	4	2	5
	0.8	0.784	0.784	0.721	0.790	0.670	3	2	4	1	5
	0.9	0.853	0.840	0.748	0.853	0.716	1	3	4	2	5
	1	0.900	0.872	0.777	0.878	0.759	1	3	4	2	5

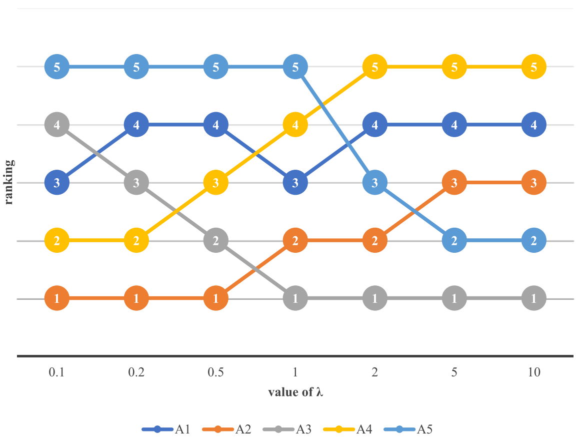


FIGURE 6. Results obtained by the GPHFOWG operator with $AC = 0.6$ and different values of λ.

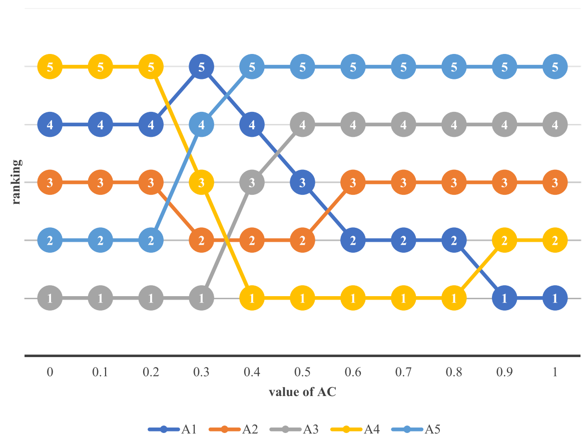


FIGURE 7. Results obtained by the GPHFOWA operator with $\lambda = 1$ and different values of AC.

on the GHFWA and GHFWG operators and consists of the following steps:

Step 1-1′: Normalize the evaluation information in the same way as in **Step 1** of **Model 1**. The original assessment information $\{S^k = (s_{ij}^k)_{m\times n} | k \in T\}$ is converted into the normalized assessment information $\{R^k = (r_{ij}^k)_{m\times n} | k \in T\}$ where r_{ij}^k is a HFE.

Step 1-2′: Fuse individual assessment information $\{R^k = (r_{ij}^k)_{m\times n} | k \in T\}$ into collective assessment informa-

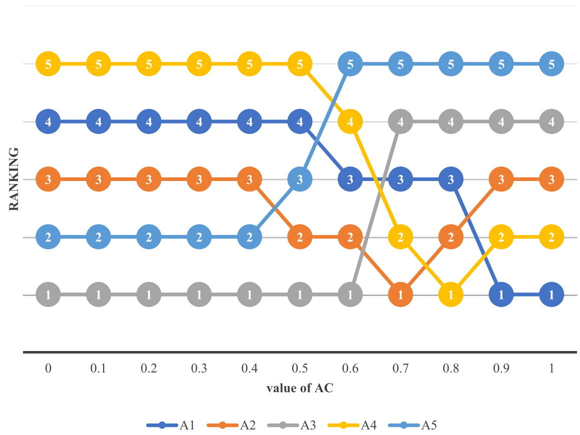


FIGURE 8. Results obtained by the GPHFOWG operator with $\lambda = 1$ and different values of AC.

tion $R = \left(r_{ij}\right)_{m\times n}$ by using the GHFWA operator,

$$r_{ij} = GPHFWA_\lambda\left(r_{ij}^1, r_{ij}^2, \ldots, r_{ij}^t\right) = \cup_{\gamma_{ij}^1\in r_{ij}^1,\ldots,\gamma_{ij}^t\in r_{ij}^t}\left\{\left(1-\prod_{j=1}^{n}\left[1-\left(\gamma_{ij}^k\right)^\lambda\right]^{\varphi_k}\right)^{1/\lambda}\right\},$$

or the GHFWG operator,

$$r_{ij} = GPHFWG_\lambda\left(r_{ij}^1, r_{ij}^2, \ldots, r_{ij}^t\right) = \cup_{\gamma_{ij}^1\in r_{ij}^1,\ldots,\gamma_{ij}^t\in r_{ij}^t}\left\{1-\left(1-\prod_{j=1}^{n}\left[1-\left(1-\gamma_{ij}^k\right)^\lambda\right]^{\varphi_k}\right)^{1/\lambda}\right\},$$

where φ_k is the weight of the kth DM, which is provided in advance, $\sum_{k=1}^{t}\varphi_k = 1, 0 \le \varphi_k \le 1,\ \lambda > 0, i \in M, j \in N, k \in T$.

Step 1-3′: Determine the weights of criteria in accordance with **Step 3** of **Model 1** and represent the results as $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$.

Step 1-4′: For each alternative, aggregate assessment information for all criteria into overall assessment information by using the GHFWA operator,

$$r_i = GHFWA_\lambda\left(r_{i1}, r_{i2}, \ldots, r_{in}\right) = \cup_{\gamma_{i1}\in r_{i1},\ldots,\gamma_{in}\in r_{in}}\left\{\left(1-\prod_{j=1}^{n}\left[1-\left(\gamma_{ij}\right)^\lambda\right]^{\omega_j}\right)^{1/\lambda}\right\},$$

or the GHFWG operator,

$$r_i = GHFWG_\lambda\left(r_{i1}, r_{i2}, \ldots, r_{in}\right) = \cup_{\gamma_{i1}\in r_{i1},\ldots,\gamma_{in}\in r_{in}}\left\{\left(1-\left(1-\prod_{j=1}^{n}\left[1-\left(1-\gamma_{ij}\right)^\lambda\right]^{\omega_j}\right)^{1/\lambda}\right)\right\},$$

where ω_j is the weight of the kth criterion, $\sum_{j=1}^{n}\omega_j^k = 1$, $0 \le \omega_j \le 1,\ \lambda > 0, i \in M, j \in N$.

TABLE 10. Comparison of results obtained by aggregation operators based on HFSs and PHFSs.

Aggregation operators (AC, AC_i)	Scores of alternatives					Rankings of alternatives					Aggregation operators (AC_j)	Scores of alternatives					Rankings of alternatives				
	A_1	A_2	A_3	A_4	A_5	A_1	A_2	A_3	A_4	A_5		A_1	A_2	A_3	A_4	A_5	A_1	A_2	A_3	A_4	A_5
GHFWA	0.680	0.675	0.682	0.753	0.697	4	5	3	1	2	GPHFWA	0.680	0.675	0.682	0.753	0.593	3	4	2	1	5
GHFWG	0.524	0.670	0.387	0.230	0.064	2	1	3	4	5	GPHFWG	0.524	0.565	0.653	0.581	0.550	5	3	1	2	4
GHFOWA (0.6,0.6)	0.767	0.767	0.714	0.784	0.764	3	2	5	1	4	GPHFOWA(0.6)	0.756	0.752	0.692	0.772	0.645	4	2	3	1	5
GHFOWA(0.4,0.6)	0.752	0.746	0.685	0.764	0.751	2	4	5	1	3											
GHFOWA(0.6,0.4)	0.640	0.657	0.651	0.660	0.683	5	3	4	2	1	GPHFOWA(0.4)	0.627	0.642	0.637	0.646	0.576	4	2	3	1	5
GHFOWA(0.4,0.4)	0.620	0.631	0.628	0.634	0.666	5	3	4	2	1											
GHFOWG(0.6,0.6)	0.636	0.670	0.679	0.631	0.706	4	3	2	5	1	GPHFOWG(0.6)	0.367	0.435	0.579	0.345	0.503	4	3	1	5	2
GHFOWG(0.4,0.6)	0.615	0.644	0.656	0.607	0.689	4	3	2	5	1											
GHFOWG(0.6,0.4)	0.485	0.542	0.621	0.464	0.638	4	3	2	5	1	GPHFOWG(0.4)	0.317	0.386	0.549	0.290	0.484	4	3	1	5	2
GHFOWG(0.4,0.4)	0.462	0.516	0.600	0.439	0.618	4	3	2	5	1											

Step 1-5′: Rank the r_i $(i \in M)$ in descending order by using **Definitions 7** and **8** and select the best alternative according to the ranking.

Next, we apply the **Model 1′** to the EVB-supplier-selection problem described above. During the process, the input information for **Model 1′**, including the original assessment information provided by DMs, the weights of DMs, and the incomplete preference information given by the decision team for the importance of criteria, remains the same with the input information of **Model 1** to ensure comparability. The results obtained by using the GHFWA or GHFWG operators with $\lambda = 1$ are given in Table 10. The counterparts derived by using the GPHFWA or GPHFWG operators with $\lambda = 1$ are also given in Table 10.

- When the GHFWA operator is applied to the case study, A_4 becomes the best supplier, followed by A_5. When the GPHFWA operator is used, A_4 remains the best supplier, followed by A_3. The main difference is that the ranking of A_5 slips from 2 in the ranking obtained by the GHFWA operator to 5 in the ranking derived by the GPHFWA operator. Comparing the original assessment information of A_3 and A_5 in Table 4, we find that A_5 only outperforms A_3 on criteria C_1 and C_7, but A_3 outperforms A_5 on criteria C_2, C_3, C_4, C_5, and C_6. The final decision of selecting A_3 as the best supplier shows that the GHFWA operator is preferable to the GPHFWA operator.
- When the GHFWG operator is used, A_2 becomes the best supplier, followed by A_1. However, the application of the GPHFWG operator leads to the result that A_3 is the best supplier, followed by A_4. Remarkable differences exist between the results obtained by the GHFWG and GPHFWG operators. Comparing A_2 with A_3 with the help of the original assessment information in Table 4, we see that A_2 scores lower on criteria C_2 and C_5, which are beyond the scope of tolerance, but A_3 has more balanced scores on all the criteria, which is the main reason why the OEM prefers A_3 to the other alternatives. Thus, we come to a similar conclusion that the GPHFWG operator outperforms the GHFWG operator in this case study.

2) GPHFOWA VS GHFOWA AND GPHFOWG VS GHFOWG

In this section, we construct another MCGDM model, called **Model 2′** and that is based on the GHFOWA or GHFOWG operator, by replacing the GPHFOWA operator or the GPHFOWG operator with the GHFOWA operator or the GHFOWG operator, respectively. In addition, note that, unlike the weights in the weighted means, which represent the relative importance of inputs, the weights in OWA functions are associated with ordered positions, meaning that the weight ω_i reflects the importance of the *ith* ordered position. Thus, instead of the original weighting vector $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_t)^T$ that indicates the relative degree of importance of DMs, we specify a new type of expert weight for the execution of OWA functions. **Model 2′** can be described as follows:

Step 2-1′: Transform the original assessment information $\{S^k = (s_{ij}^k)_{m\times n} | k \in T\}$ into the normalized assessment information $\{R^k = (r_{ij}^k)_{m\times n} | k \in T\}$, where r_{ij}^k is a HFE.

Step 2-2′: Determine the weights of DMs to facilitate the execution of the GHFOWA operator or the GHFOWG operator. We let the OEM give a value to measure their AC value, with $AC \in [0, 1]$, for the performance of the decision team, and then introduce the AC into Method [M2] to determine the order weights $(\varphi_1, \varphi_2, \ldots, \varphi_t)$ for aggregating individual arguments on the expert level.

Step 2-3′: Fuse individual assessment information $\{R^k = (r_{ij}^k)_{m\times n} | k \in T\}$ into the collective assessment information $R = (r_{ij})_{m\times n}$ by using the GHFOWA operator,

$$r_{ij} = GHFOWA_\lambda\left(r_{ij}^1, r_{ij}^2, \ldots, r_{ij}^t\right) = \cup_{\gamma_{ij}^{\delta(1)} \in r_{ij}^{\delta(1)}, \ldots, \gamma_{ij}^{\delta(t)} \in r_{ij}^{\delta(t)}} \left\{\left(1 - \prod_{k=1}^{t}\left[1 - \left(\gamma_{ij}^{\delta(k)}\right)^\lambda\right]^{\varphi_k}\right)^{1/\lambda}\right\},$$

or the GHFOWG operator,

$$r_{ij} = GPHFWG_\lambda\left(r_{ij}^1, r_{ij}^2, \ldots, r_{ij}^t\right) = \cup_{\gamma_{ij}^{\delta(1)} \in r_{ij}^{\delta(1)}, \ldots, \gamma_{ij}^{\delta(t)} \in r_{ij}^{\delta(t)}} \times \left\{1 - \left(1 - \prod_{j=1}^{n}\left[1 - \left(1 - \gamma_{ij}^{\delta(k)}\right)^\lambda\right]^{\varphi_k}\right)^{1/\lambda}\right\},$$

where φ_k is the weight of the kth ordered position in $(r_{ij}^{\delta(1)}, r_{ij}^{\delta(2)}, \ldots, r_{ij}^{\delta(t)})$, $(r_{ij}^{\delta(1)}, r_{ij}^{\delta(2)}, \ldots, r_{ij}^{\delta(t)})$ is the reordering of $(r_{ij}^1, r_{ij}^2, \ldots, r_{ij}^t)$ satisfying $r_{ij}^{\delta(1)} \geq r_{ij}^{\delta(2)} \geq \cdots \geq r_{ij}^{\delta(t)}$, and $\sum_{k=1}^{t} \varphi_k = 1, 0 \leq \varphi_k \leq 1,\ \lambda > 0, i \in M, j \in N, k \in T$.

Step 2-4′: Determine the weights of criteria in accordance with **Step 2-3** of **Model 2** and represent the results as $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$.

Step 2-5′: Synthesize the overall assessment information $\{r_i | i \in M\}$ for criterion level by using the GHFOWA operator,

$$r_i = GHFOWA_\lambda\left(r_{i\delta(1)}, r_{i\delta(2)}, \ldots, r_{i\delta(n)}\right) = \cup_{\gamma_{i\delta(1)} \in r_{i\delta(1)}, \ldots, \gamma_{i\delta(n)} \in r_{in}} \left\{\left(1 - \prod_{j=1}^{n}\left[1 - \left(\gamma_{i\delta(j)}\right)^\lambda\right]^{\omega_j}\right)^{1/\lambda}\right\},$$

or the GHFOWG operator,

$$r_i = GHFOWG_\lambda\left(r_{i\delta(1)}, r_{i\delta(2)}, \ldots, r_{i\delta(n)}\right) = \cup_{\gamma_{i\delta(1)} \in r_{i\delta(1)}, \ldots, \gamma_{i\delta(n)} \in r_{in}} \times \left\{\left(1 - \left(1 - \prod_{j=1}^{n}\left[1 - \left(1 - \gamma_{i\delta(j)}\right)^\lambda\right]^{\omega_j}\right)^{1/\lambda}\right\},$$

where $(r_{i\delta(1)}, r_{i\delta(2)}, \ldots, r_{i\delta(n)})$ is the rearrangement of $(r_{i1}, r_{i2}, \ldots, r_{in})$ satisfying $r_{i\delta(1)} \geq r_{i\delta(2)} \geq \cdots \geq r_{i\delta(n)}$, $r_{i\delta(j)}$ is the jth largest in $(r_{i\delta(1)}, r_{i\delta(2)}, \ldots, r_{i\delta(n)})$, and ω_j denotes the weight of the kth position in $(r_{i\delta(1)}, r_{i\delta(2)}, \ldots, r_{i\delta(n)})$ with the following conditions satisfied: $\sum_{j=1}^{n} \omega_j^k = 1, 0 \leq \omega_j \leq 1, \lambda > 0, i \in M, j \in N$.

Step 2-6′: Rank $r_i\,(i \in M)$ in descending order by using **Definitions 7** and **8** and select the best alternative according to the ranking.

Next, we use **Model 2′** to address the EVB-supplier-selection problem for this case. During the process, to compare the results from various angles, we assign 0.6 and 0.4 to the AC based on the performance of the decision team, and also assign the AC $AC_i\,(i \in M)$ on each alternative 0.6 and 0.4. In addition, λ is set to unity. Four experiments are needed for the GHFOWA and GHFOWG operators. The results are given in Table 10. Note that, during the application of the GPHFOWA and GPHFOWG operators, the AC value $AC_i\,(i \in M)$ for all alternatives should also be set to 0.6 and 0.4 to ensure comparability. The results obtained by using the GPHFOWA and GPHFOWG operators are presented in Table 10.

- For the GHFOWA operator, if $AC_i = 0.6\,(i \in M)$, A_4 is the best supplier; if $AC_i = 0.4\,(i \in M)$, A_5 becomes the best supplier. For the GPHFOWA operator, A_4 remains the best supplier independent of whether $AC_i = 0.6$ *or* $0.4\,(i \in M)$. Thus, we infer that the results obtained by the GPHFOWA operator are more robust than those derived by the GHFOWA operator.
- When using the GHFOWG operator with $AC_i = 0.6$ *or* $0.4\,(i \in M)$, A_5 remains the best supplier, followed by A_3. When using the GPHFOWG operator with $AC_i = 0.6$ *or* $0.4\,(i \in M)$, A_3 remains the best supplier, followed by A_5. We cannot differentiate between the robustness of the two operators. However, as mentioned above, the results obtained by the GPHFOWG operator are preferable to those obtained by the GHFOWG operator.

Based on a comparative analysis, we find that the aggregation operators based on PHFSs, including GPHFWA, GPHFWG, GPHFOWA, and GPHFOWG, outperform the aggregation operators based on HFSs including GHFWA, GHFWG, GHFOWA, and GHFOWG, which indicates that PHFSs perform better than HFSs for uncertain MCGDM problems. The main reasons for this result can be summarized as follows:

(1) Compared with HFSs, PHFSs introduce a new proportional dimension of information that is mined from the original assessment information to reduce the uncertainty associated with the original assessment information. Thus, MCGDM based on PHFSs can effectively improve the reliability of decision results.

(2) Instead of aggregation operators, a different type of information processing is introduced during the aggregation process on the expert level, which is described in **Step 1-2** of **Model 1**. This information fusion aims to generate proportional information while reserving original information as much as possible. Thus, it can mitigate the risk of information distortion. In addition, it is easy to operate and has a high degree of interpretability. For example, the five DMs give the original assessment information for the performance of alternative A_1 with respect to quality (criterion C_2) in this case study,

$$\{0.8, 0.7\},\ \{0.7\},\ \{0.8\},\ \{0.8, 0.7, 0.6\},\ \{0.8, 0.7\}.$$

By using **Step 1-2** of **Model 1**, we can synthesize the collective assessment information of A_1 for C_2:

$$\{(0.8, 0.533), (0.7, 0.433), (0.6, 0.034)\},$$

for which the proportional information has been integrated with the DM weights. The application of **Step 1-2′** of **Model 1′** leads to the result

$$\{0.727,\ 0.734,\ 0.743,\ 0.745,\ 0.75,\ 0.753,\ 0.76,\ 0.76,\ 0.7688,\ 0.77,\ 0.774,\ 0.783\}.$$

Evidently, the new type of information fusion on the expert level is preferable.

(3) The aggregation operators based on HFSs and PHFSs necessitate all possible combinations of individual arguments. In the context of MCGDM, the aggregation operators based on PHFSs only need to be executed once, but the aggregation operators based on HFSs need to be executed twice, easily leading to information distortion and the combinatorial-explosion problem in combinatorics. Thus, the processing of the proposed MCGDM model based on PHFSs requires less computing time and is more efficient.

D. DISCUSSION

This section summarizes the conclusions obtained by the case study.

(1) The case study further verifies the theorems given in Sec. 3; namely, that the score determined by the GPHFWA or GHPFOWA operator increases with increasing λ, the score obtained by the GPHFWG or GPHFOWG operator decreases with increasing λ, the scores obtained by the GPHFWA operator are always greater than those obtained by the GPHFWG operator, and the scores derived by the GPHFOWA operator are always greater than those obtained by the GPHFOWG operator.

(2) Based on this case study, we find that the GPHFWA and GPHFOWA operators have a higher level of compensation than the GPHFWG and GPHFOWG operators, respectively. But the results obtained by the GPHFWG and GPHFOWG operators are preferable. In addition, the compensation level for the GPHFWA and GPHFOWA operators increases with increasing λ, whereas the compensation level for the GPHFWA and GPHFOWA operators decrease with increasing λ.

(3) Attitudinal character, which can be characterized by the ORness value and used to guide the aggregation process, has a remarkable impact on the final ranking

of alternatives. In fact, the main advantage of OWA functions over traditional averaging operators is that the former can flexibly provide an aggregation operator ranging between min and max by incorporating the AC of DMs. In addition, the compensation level of the GPHFOWA and GPHFOWG operators increases with increasing AC. Furthermore, OWA operators can flexibly model different degrees of compensation with the help of the ORness measure. When ORness takes the value of zero, OWA reduces to the max operator, which indicates full compensation between criteria. When ORness takes the value of unity, OWA reduces to the min operator, which indicates no compensation between criteria. A positive linear relationship exists between ORness measure and compensation level.

(4) Based on a comparative analysis, we conclude that the introduction of PHFSs into the MCGDM process under a hesitant, uncertain context is preferable because the addition of a new information dimension reduces the uncertainty. In addition, instead of aggregation operators, we propose a method to fuse individual assessment information on the expert level and, simultaneously, transform from FSs or HFSs to PHFSs. The fusion of information by this method outperforms its counterpart involving aggregation operators.

Based on these conclusions, we give the following prerequisites for applying each PHFS-based aggregation operator:

(1) When DMs output assessment information with no AC and have clear preferences on criteria regardless of whether the preference information is complete or incomplete, the GPHFWA or GPHFWG operator can be applied to fuse assessment information during the MCGDM process in the hesitant fuzzy context. For GPHFWA and GPHFWG operators, if limited compensation exists between criteria, or if DMs hope to mitigate the influence of outliers, the GPHFWG operator is more preferable.

(2) When DMs output the hesitant fuzzy assessment information under optimistic or pessimistic conditions and have no preferences for criteria, or if DMs hope to flexibly choose aggregation operators ranging from the minimum to the maximum by specifying a parameter, the GPHFOWA or GPHFOWG operator can be used. Similarly, if limited compensation exists between criteria, or if DMs hope to mitigate the influence of outliers, the GPHFOWG operator is more preferable.

VI. CONCLUSION

By introducing a new proportional dimension, PHFSs offer outstanding advantages for modeling uncertainty. In this paper, we restrict our attention to expanding PHFS theory in terms of information fusion, constructing two MCGDM models involving PHFS-based aggregation techniques and exploring applications to which the two models may be applied. The three main contributions of this paper are summarized below.

(1) We present some basic operations on PHFSs, develop a series of aggregation operators for PHFSs, and validate their properties and relationships. These aggregation operators include the PHFWA, PHFWG, PHFOWA, and the PHFOWG operators and their generalized forms. The introduction of these aggregation operators lays the theoretical foundation for the application of PHFSs.

(2) We construct two MCGDM models, one of which is based on the GPHFWA or GPHFWG operator, and the other on the GPHFOWA or GPHFOWG operator. For both models, we provide two methods based on the maximum entropy principle to determine the weights of criteria. In addition, we propose a method to transform FSs or HFSs into PHFSs. The two proposed models are effective and practical techniques for dealing with MCGDM problems in a hesitant fuzzy context and can serve to bridge between theory and practice for PHFSs.

(3) We present a practical case study involving EVB supplier selection as an example of an application of PHFSs. In this case study, we demonstrate the effectiveness and feasibility of the proposed MCGDM models, explore the compensation characteristics and the applicability of the PHFS-based aggregation operators, and validate the significant advantages of PHFS through a comparative analysis.

Overall, PHFS deals well with MCGDM problems in a hesitant fuzzy context, and we propose a series of techniques including aggregation operators, MCGDM models, a method to transform FSs or HFSs to PHFSs, and methods to determine criterion weights to explore applications of PHFSs. The results provide a useful reference when dealing with MCGDM problems in a hesitant fuzzy context. However, this work also has some limitations, including the expansion of PHFS-based aggregation operators, such as (geometric) Bonferroni means [28], [64], power aggregation operators [65], the proof of compensation characteristics of PHFS-based aggregation operators, and the impact of consensus-reaching problems in MCGDM [66], [67]. These omissions give the main directions for future research.

REFERENCES

[1] L. A. Zadeh, "Fuzzy sets," *Inf. Control*, vol. 8, no. 3, pp. 338–353, Jun. 1965.

[2] L. A. Zadeh, "The concept of a linguistic variable and its application to approximate reasoning—I," *Inf. Sci.*, vol. 8, no. 3, pp. 199–249, 1975.

[3] K. T. Atanassov, "Intuitionistic fuzzy sets," *Fuzzy Sets Syst.*, vol. 20, pp. 87–96, Aug. 1986.

[4] R. R. Yager, "Pythagorean membership grades in multicriteria decision making," *IEEE Trans. Fuzzy Syst.*, vol. 22, no. 4, pp. 958–965, Aug. 2014.

[5] V. Torra, "Hesitant fuzzy sets," *Int. J. Intell. Syst.*, vol. 25, no. 6, pp. 529–539, 2010.

[6] D. Wu and J. M. Mendel, "Similarity measures for closed general type-2 fuzzy sets: Overview, comparisons, and a geometric approach," *IEEE Trans. Fuzzy Syst.*, vol. 27, no. 3, pp. 515–526, Mar. 2019.

[7] Z.-S. Chen, Y. Yang, X.-J. Wang, K.-S. Chin, and K.-L. Tsui, "Fostering linguistic decision-making under uncertainty: A proportional interval type-2 hesitant fuzzy TOPSIS approach based on Hamacher aggregation operators and andness optimization models," *Inf. Sci.*, vol. 500, pp. 229–258, Oct. 2019.

[8] M. S. A. Khan and S. Abdullah, "Interval-valued Pythagorean fuzzy GRA method for multiple-attribute decision making with incomplete weight information," *Int. J. Intell. Syst.*, vol. 33, no. 8, pp. 1689–1716, Aug. 2018.

[9] M. S. A. Khan, S. Abdullah, M. Y. Ali, I. Hussain, and M. Farooq, "Extension of TOPSIS method base on Choquet integral under interval-valued Pythagorean fuzzy environment," *J. Intell. Fuzzy Syst.*, vol. 34, no. 1, pp. 267–282, 2018.

[10] M. S. A. Khan, S. Abdullah, and P. Lui, "Gray method for multiple attribute decision making with incomplete weight information under the pythagorean fuzzy setting," *J. Intell. Syst.*, 2018, doi: 10.1515/jisys-2018-0099.

[11] L. Wang, R. M. Rodríguez-Domínguez, Y.-M. Wang, "A dynamic multi-attribute group emergency decision making method considering experts' hesitation," *Int. J. Comput. Intell. Syst.*, vol. 11, no. 1, pp. 163–182, 2018.

[12] R. M. Rodríguez, L. Martínez, V. Torra, Z. S. Xu, and F. Herrera, "Hesitant fuzzy sets: State of the art and future directions," *Int. J. Intell. Syst.*, vol. 29, no. 6, pp. 495–524, 2014.

[13] R. M. Rodríguez, B. Bedregal, H. Bustince, Y. C. Dong, B. Farhadinia, C. Kahraman, L. Martínez, V. Torra, Y. J. Xu, Z. S. Xu, and F. Herrera, "A position and perspective analysis of hesitant fuzzy sets on information fusion in decision making. Towards high quality progress," *Inf. Fusion*, vol. 29, pp. 89–97, May 2016.

[14] Y. Liu, R. M. Rodríguez, J. C. R. Alcantud, K. Qin, and L. Martínez, "Hesitant linguistic expression soft sets: Application to group decision making," *Comput. Ind. Eng.*, vol. 136, pp. 575–590, Oct. 2019.

[15] S. Ç. Onar, G. Büyüközkan, B. Öztayşi, and C. Kahraman, "A new hesitant fuzzy QFD approach: An application to computer workstation selection," *Appl. Soft Comput.*, vol. 46, pp. 1–16, Sep. 2016.

[16] C. Kahraman, S. Ç. Onar, and B. Öztayşi, "B2C marketplace prioritization using hesitant fuzzy linguistic AHP," *Int. J. Fuzzy Syst.*, vol. 20, no. 7, pp. 2202–2215, 2018.

[17] H. Garg, "Hesitant Pythagorean fuzzy Maclaurin symmetric mean operators and its applications to multiattribute decision-making process," *Int. J. Intell. Syst.*, vol. 34, no. 4, pp. 601–626, 2019.

[18] R. Arora and H. Garg, "A robust correlation coefficient measure of dual hesitant fuzzy soft sets and their application in decision making," *Eng. Appl. Artif. Intell.*, vol. 72, pp. 80–92, Jun. 2018.

[19] B. Farhadinia, "Information measures for hesitant fuzzy sets and interval-valued hesitant fuzzy sets," *Inf. Sci.*, vol. 240, pp. 129–144, Aug. 2013.

[20] M. S. A. Khan, S. Abdullah, A. Ali, N. Siddiqui, and F. Amin, "Pythagorean hesitant fuzzy sets and their application to group decision making with incomplete weight information," *J. Intell. Fuzzy Syst.*, vol. 33, no. 6, pp. 3971–3985, Nov. 2017.

[21] M. S. A. Khan, A. Ali, S. Abdullah, F. Amin, and F. Hussain, "New extension of TOPSIS method based on Pythagorean hesitant fuzzy sets with incomplete weight information," *J. Intell. Fuzzy Syst.*, vol. 35, no. 5, pp. 5435–5448, Nov. 2018.

[22] M. S. A. Khan, S. Abdullah, A. Ali, F. Amin, and F. Hussain, "Pythagorean hesitant fuzzy Choquet integral aggregation operators and their application to multi-attribute decision-making," *Soft Comput.*, vol. 23, no. 1, pp. 251–267, Jan. 2019.

[23] R. M. Rodríguez, L. Martínez, and F. Herrera, "Hesitant fuzzy linguistic term sets for decision making," *IEEE Trans. Fuzzy Syst.*, vol. 20, no. 1, pp. 109–119, Feb. 2011.

[24] S.-H. Xiong, Z.-S. Chen, and K.-S. Chin, "A novel MAGDM approach with proportional hesitant fuzzy sets," *Int. J. Comput. Intell. Syst.*, vol. 11, no. 1, pp. 256–271, 2018.

[25] Z. Xu and M. Xia, "Distance and similarity measures for hesitant fuzzy sets," *Inf. Sci.*, vol. 181, no. 11, pp. 2128–2138, 2011.

[26] M. M. Xia and Z. S. Xu, "Hesitant fuzzy information aggregation in decision making," *Int. J. Approx. Reason*, vol. 52, no. 3, pp. 395–407, 2011.

[27] Z. M. Zhang, "Hesitant fuzzy power aggregation operators and their application to multiple attribute group decision making," *Inf. Sci.*, vol. 234, pp. 150–181, Jun. 2013.

[28] B. Zhu, Z. Xu, and M. Xia, "Hesitant fuzzy geometric Bonferroni means," *Inf. Sci.*, vol. 205, pp. 72–85, Nov. 2012.

[29] Z.-S. Chen, K.-S. Chin, Y.-L. Li, and Y. Yang, "Proportional hesitant fuzzy linguistic term set for multiple criteria group decision making," *Inf. Sci.*, vol. 357, pp. 61–87, Aug. 2016.

[30] B. Zhu and Z. Xu, "Probability-hesitant fuzzy sets and the representation of preference relations," *Technol. Econ. Develop. Economy*, vol. 24, no. 3, pp. 1029–1040, Jan. 2018.

[31] Z. Zhang and C. Wu, "Weighted hesitant fuzzy sets and their application to multi-criteria decision making," *J. Adv. Math. Comput. Sci.*, vol. 4, no. 8, pp. 1091–1123, 2014.

[32] M. Grabisch, J.-L. Marichal, R. Mesiar, and E. Pap, *Aggregation Functions*, vol. 127. Cambridge, U.K.: Cambridge Univ. Press, 2009.

[33] G. Beliakov, H. B. Sola, and T. C. Sánchez, *A Practical Guide to Averaging Functions*. Cham, Switzerland: Springer, 2016.

[34] Z.-S. Chen, X. Zhang, R. M. Rodríguez, X.-J. Wang, and K.-S. Chin, "Heterogeneous interrelationships among attributes in multi-attribute decision-making: An empirical analysis," *Int. J. Comput. Intell. Syst.*, vol. 12, no. 2, pp. 984–997, 2019.

[35] Y. Yang, Z.-S. Chen, Y.-H. Chen, and K.-S. Chin, "Interval-valued Pythagorean fuzzy frank power aggregation operators based on an isomorphic Frank dual triple," *Int. J. Comput. Intell. Syst.*, vol. 11, no. 1, pp. 1091–1110, 2018.

[36] Z.-S. Chen, K.-S. Chin, Y.-L. Li, and Y. Yang, "On generalized extended Bonferroni means for decision making," *IEEE Trans. Fuzzy Syst.*, vol. 24, no. 6, pp. 1525–1543, Dec. 2016.

[37] H. Garg, "New exponential operational laws and their aggregation operators for interval-valued Pythagorean fuzzy multicriteria decision-making," *Int. J. Intell. Syst.*, vol. 33, no. 3, pp. 653–683, 2018.

[38] F. K. Gündoğdu and C. Kahraman, "A novel fuzzy TOPSIS method using emerging interval-valued spherical fuzzy sets," *Eng. Appl. Artif. Intell.*, vol. 85, pp. 307–323, Oct. 2019.

[39] Y. C. Dong, Y. T. Liu, H. M. Liang, F. Chiclana, and E. Herrera-Viedma, "Strategic weight manipulation in multiple attribute decision making," *Omega*, vol. 75, pp. 154–164, Mar. 2018.

[40] G. Baudry, C. Macharis, and T. Vallée, "Range-based multi-actor multi-criteria analysis: A combined method of multi-actor multi-criteria analysis and Monte Carlo simulation to support participatory decision making under uncertainty," *Eur. J. Oper. Res.*, vol. 264, no. 1, pp. 257–269, 2018.

[41] S.-M. Yu, J. Wang, J.-Q. Wang, and L. Li, "A multi-criteria decision-making model for hotel selection with linguistic distribution assessments," *Appl. Soft Comput.*, vol. 67, pp. 741–755, Jun. 2018.

[42] L. Jin, "Some properties and representation methods for ordered weighted averaging operators," *Fuzzy Sets Syst.*, vol. 261, pp. 60–86, Feb. 2015.

[43] Z.-S. Chen, C. Yu, K.-S. Chin, and L. Martínez, "An enhanced ordered weighted averaging operators generation algorithm with applications for multicriteria decision making," *Appl. Math. Model.*, vol. 71, pp. 467–490, Jul. 2019.

[44] H. Garg and K. Kumar, "Linguistic interval-valued atanassov intuitionistic fuzzy sets and their applications to group decision making problems," *IEEE Trans. Fuzzy Syst.*, vol. 27, no. 12, pp. 2302–2311, Dec. 2019.

[45] P. Liu and S.-M. Chen, "Multiattribute group decision making based on intuitionistic 2-tuple linguistic information," *Inf. Sci.*, vols. 430–431, pp. 599–619, Mar. 2018.

[46] H. Zhang, Y. Dong, I. Palomares-Carrascosa, and H. Zhou, "Failure mode and effect analysis in a linguistic context: A consensus-based multiattribute group decision-making approach," *IEEE Trans. Rel.*, vol. 68, no. 2, pp. 566–582, Jun. 2019.

[47] Z.-S. Chen, L. Martínez, J.-P. Chang, X.-J. Wang, S.-H. Xionge, and K.-S. Chin, "Sustainable building material selection: A QFD- and ELECTRE III-embedded hybrid MCGDM approach with consensus building," *Eng. Appl. Artif. Intell.*, vol. 85, pp. 783–807, Oct. 2019.

[48] H. Zhang, J. Xiao, I. Palomares, H. Liang, and Y. Dong, "Linguistic distribution-based optimization approach for large-scale GDM with comparative linguistic information. An application on the selection of wastewater disinfection technology," *IEEE Trans. Fuzzy Syst.*, to be published.

[49] L. Wang, Y.-M. Wang, and L. Martínez, "A group decision method based on prospect theory for emergency situations," *Inf. Sci.*, vol. 418, pp. 119–135, Dec. 2017.

[50] Z.-S. Chen, L. Martínez, K.-S. Chin, and K.-L. Tsui, "Two-stage aggregation paradigm for HFLTS possibility distributions: A hierarchical clustering perspective," *Expert Syst. Appl.*, vol. 104, pp. 43–66, Aug. 2018.

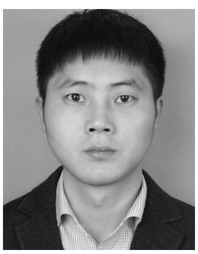
[51] M. S. A. Khan, S. Abdullah, and A. Ali, "Multiattribute group decision-making based on Pythagorean fuzzy Einstein prioritized aggregation operators," *Int. J. Intell. Syst.*, vol. 34, no. 5, pp. 1001–1033, 2019.

[52] M. S. A. Khan, "The pythagorean fuzzy Einstein Choquet integral operators and their application in group decision making," *Comput. Appl. Math.*, vol. 38, no. 3, p. 128, 2019.

[53] V. Torra and Y. Narukawa, "On hesitant fuzzy sets and decision," in *Proc. IEEE Int. Conf. Fuzzy Syst.*, Aug. 2009, pp. 1378–1382.

[54] Z. Xu, "Intuitionistic fuzzy multiattribute decision making: An interactive method," *IEEE Trans. Fuzzy Syst.*, vol. 20, no. 3, pp. 514–525, Jun. 2012.

[55] A. Awasthi, K. Govindan, and S. Gold, ''Multi-tier sustainable global supplier selection using a fuzzy AHP-VIKOR based approach,'' *Int. J. Prod. Econ.*, vol. 195, pp. 106–117, Jan. 2018.
[56] S. Gupta, U. Soni, and G. Kumar, ''Green supplier selection using multi-criterion decision making under fuzzy environment: A case study in automotive industry,'' *Comput. Ind. Eng.*, vol. 136, pp. 663–680, Oct. 2019.
[57] B. Nykvist and M. Nilsson, ''Rapidly falling costs of battery packs for electric vehicles,'' *Nature Climate Change*, vol. 5, no. 4, pp. 329–332, 2015.
[58] S. Mousakhani, S. Nazari-Shirkouhi, and A. Bozorgi-Amiri, ''A novel interval type-2 fuzzy evaluation model based group decision analysis for green supplier selection problems: A case study of battery industry,'' *J. Cleaner Prod.*, vol. 168, pp. 205–218, Dec. 2017.
[59] X. Gu, P. Ieromonachou, L. Zhou, and M.-L. Tseng, ''Developing pricing strategy to optimise total profits in an electric vehicle battery closed loop supply chain,'' *J. Cleaner Prod.*, vol. 203, pp. 376–385, Dec. 2018.
[60] M. A. Bushuev, ''Delivery performance improvement in two-stage supply chain,'' *Int. J. Prod. Econ.*, vol. 195, pp. 66–73, Jan. 2018.
[61] R. Alikhani, S. A. Torabi, and N. Altay, ''Strategic supplier selection under sustainability and risk criteria,'' *Int. J. Prod. Econ.*, vol. 208, pp. 69–82, Feb. 2019.
[62] A. Trautrims, B. L. MacCarthy, and C. Okade, ''Building an innovation-based supplier portfolio: The use of patent analysis in strategic supplier selection in the automotive sector,'' *Int. J. Prod. Econ.*, vol. 194, pp. 228–236, Dec. 2017.
[63] M. Formentini, L. M. Ellram, M. Boem, and G. Da Re, ''Finding true north: Design and implementation of a strategic sourcing framework,'' *Ind. Marketing Manage.*, vol. 77, pp. 182–197, Feb. 2019.
[64] Z.-S. Chen, K.-S. Chin, and K.-L. Tsui, ''Constructing the geometric Bonferroni mean from the generalized Bonferroni mean with several extensions to linguistic 2-tuples for decision-making,'' *Appl. Soft Comput.*, vol. 78, pp. 595–613, May 2019.
[65] S.-H. Xiong, Z.-S. Chen, J.-P. Chang, and K.-S. Chin, ''On extended power average operators for decision-making: A case study in emergency response plan selection of civil aviation,'' *Comput. Ind. Eng.*, vol. 130, pp. 258–271, Apr. 2019.
[66] Y. C. Dong, Q. Zha, H. Zhang, G. Kou, H. Fujita, F. Chiclana, and E. Herrera-Viedma, ''Consensus reaching in social network group decision making: Research paradigms and challenges,'' *Knowl. Based Syst.*, vol. 162, pp. 3–13, Dec. 2018.
[67] Z.-S. Chen, M. Xu, X.-J. Wang, K.-S. Chin, K.-L. Tsui, and L. Martínez, ''Individual semantics building for HFLTS possibility distribution with applications in domain-specific collaborative decision making,'' *IEEE Access*, vol. 6, pp. 78803–78828, 2018.



JIAN-PENG CHANG received the Ph.D. degree in traffic and transportation plan and management from the School of Transportation and Logistics, Southwest Jiaotong University, Chengdu, China.

He was a Research Assistant with the Department of Systems Engineering and Engineering Management, City University of Hong Kong. He is currently a Lecturer with the School of Business Planning, Chongqing Technology and Business University. His research results have been published in IEEE Access, *Information Sciences*, *Computers & Industrial Engineering*, *Journal of The China Railway Society*, *Computer Integrated Manufacturing Systems*, and *China Safety Science Journal*. His current research interests include group decision analysis and railway emergency management.

ZHEN-SONG CHEN received the Ph.D. degree in traffic and transportation plan and management from the School of Transportation and Logistics, Southwest Jiaotong University, Chengdu, China.

He was a Research Assistant and a Postdoctoral Researcher with the Department of Systems Engineering and Engineering Management, City University of Hong Kong. He was also a Senior Research Associate with the School of Data Science, City University of Hong Kong. He is currently an Associate Professor with the School of Civil Engineering, Wuhan University. His research results have been published in peer-reviewed journals, including the IEEE Transactions on Fuzzy Systems, IEEE Access, *Applied Mathematical Modelling*, *Engineering Applications of Artificial Intelligence*, *Information Sciences*, *Expert Systems with Applications*, *Applied Soft Computing*, *International Journal of Computational Intelligence Systems*, *International Journal of Intelligent Systems*, *International Journal of Information Technology & Decision Making*, *International Journal of Environmental Research and Public Health*, *Journal of Intelligent & Fuzzy Systems*, *International Journal of Fuzzy Systems*, and *Journal of Systems Engineering and Electronics*. His current research interests include preference modeling, aggregation theory, computing with words, and group decision analysis.

Dr. Chen received the Highest Honors Student Award of Southwest Jiaotong University, in 2015, the National Scholarship for Ph.D. Students Award granted by the Ministry of Education of China, in 2013, 2014, and 2015, the second prize in the National Postgraduate Mathematical Contest in Modeling Award granted by the China Academic Degrees and Graduate Education Development Center, in 2011, 2012, and 2013, the Outstanding Ph.D. Thesis Foundation Award granted by Southwest Jiaotong University, in 2013, the Tang Lixin Scholarship Award, and the Tang Lixin Fellowship Award granted by the Southwest Jiaotong University Education Foundation, in 2014 and 2015, respectively. He has been awarded the Outstanding Reviewer of *Information Fusion*, *Knowledge-based Systems*, and *Computer Integrated Manufacturing Systems*. He also received the 2016 Outstanding Ph.D. Thesis Award granted by Southwest Jiaotong University and the 2016 Outstanding Graduate of Sichuan Province Award granted by the Sichuan government of China. He serves as an Area Editor of the *International Journal of Computational Intelligence Systems* and an Associate Editor of the *Journal of Intelligent & Fuzzy Systems* and *Kybernetes*. Recently, he has been awarded the Outstanding Area Editor of *International Journal of Computational Intelligence Systems*.

XIAO-LU LIU received the bachelor's degree in engineering management from the School of Public Administration, Huazhong Agricultural University, Wuhan, China. She is currently a Graduate Student majoring in engineering management with the School of Civil Engineering, Wuhan University. Her current research interests include preference modeling, aggregation theory, computing with words, and group decision analysis.

She received the First Prize for Excellent Academic Achievement granted by Huazhong Agricultural University, in 2015 and 2016, the National Encouragement Scholarship granted by Huazhong Agricultural University, in 2016 and 2017, the First prize of "Glodon cup" Engineering Cost Skill Competition awarded by Glodon Company Limited and Huazhong Agricultural University jointly, in 2017. She was also awarded the title of Outstanding Undergraduate of Huazhong Agricultural University, in 2018.

WEN-TAO KONG received the Ph.D. degree from the School of Civil Engineering, Wuhan University, Wuhan, China.

He was a Visiting Scholar with New South Wales University, from 2016 to 2017. He is currently a Lecturer with the School of Civil Engineering, Wuhan University. His current research interests include civil engineering construction and management, intelligent building, engineering management, and building information modeling (BIM). His most recent research results have been published in the peer-reviewed journal, *International Journal of Environmental Research and Public Health*.

SHENG-HUA XIONG received the Ph.D. degree in system engineering from the School of Transportation and Logistics, Southwest Jiaotong University, Chengdu, China.

He was a Research Assistant with the Department of Management Sciences and the Department of Systems Engineering and Engineering Management, City University of Hong Kong. He is currently a Lecturer with the College of Civil Aviation Safety Engineering, Civil Aviation Flight University of China. His research results have been published in peer-reviewed journals, including *Information Sciences*, *Computers & Industrial Engineering*, *International Journal of Computational Intelligence Systems*, *International Journal of Information Technology & Decision Making*, and *Journal of Systems Engineering and Electronics*. His current research interests include preference modeling, aggregation theory, computing with words, and group decision analysis.

Dr. Xiong received the National Scholarship for Ph.D. Students Award granted by the Ministry of Education of China, in 2016, the National Scholarship for M.Sc. Students Award granted by the Ministry of Education of China, in 2013, the second prize in the National Postgraduate Mathematical Contest in Modeling Award granted by the China Academic Degrees and Graduate Education Development Center, in 2013, and the Distinguished Talents for Innovation Award of Ph.D. Students granted by Southwest Jiaotong University, in 2014.

LUIS MARTíNEZ received the M.Sc. and Ph.D. degrees in computer sciences from the University of Granada, Granada, Spain, in 1993 and 1999, respectively.

He is currently a Full Professor with the Computer Science Department, University of Jaén, Jaén, Spain. He is also a Visiting Professor with the University of Technology Sydney, University of Portsmouth (Isambard Kingdom Brunel Fellowship Scheme), and the Wuhan University of Technology (Chutian Scholar), a Guest Professor with the Southwest Jiaotong University, Chengdu, China, and a Honorable Professor with Xihua University, Chengdu. He has co-edited eleven journal special issues on fuzzy preference modeling, soft computing, linguistic decision making, and fuzzy sets theory. He has been a main researcher in 14 Research and Development projects. He has also published more than 100 articles in journals indexed by the SCI and more than 150 contributions in International Conferences related to his areas. His current research interests include decision making, fuzzy logic-based systems, computing with words and recommender systems.

Dr. Martínez is a member of the European Society for Fuzzy Logic and Technology. Eventually, he has been appointed as a Highly Cited Researcher 2017 in Computer Science, 2018 in Cross Field, and 2019 in Computer Science. He was a recipient of the IEEE TRANSACTIONS ON FUZZY SYSTEMS Outstanding Paper Award 2008 and 2012 (bestowed in 2011 and 2015, respectively). He is the Co-Editor-in-Chief of the *International Journal of Computational Intelligence Systems* and an Associate Editor of the journals, including the IEEE TRANSACTIONS ON FUZZY SYSTEMS, *Information Fusion*, the *International Journal of Fuzzy Systems*, and the *Journal of Intelligent & Fuzzy Systems*.