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A concept lattice-based expert opinion aggregation method for multi-attribute group decision-making with linguistic information

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ABSTRACT

During the multi-attribute group decision-making (MAGDM) processing, the individuals often hold different opinions about the alternatives. It is necessary to aggregate the different individual opinions into a unified group opinion. In the real world, experts sometimes use linguistic expressions to evaluate attributes in uncertain environments. To address the problem of reducing the information loss of expert opinion aggregation in MAGDM, this paper proposes a MAGDM approach based on linguistic concept lattices in the context of uncertain linguistic expression. A linguistic concept lattice for multi-expert linguistic formal context is first constructed based on linguistic truth-valued lattice implication algebra, which can express both comparable and incomparable linguistic information in the decision-making process. Different expert opinions are aggregated via the extent of fuzzy linguistic concepts, which can reduce information loss in the aggregation process. Second, meet-irreducible elements in the linguistic concept lattice are introduced to reduce the computational complexity of obtaining all fuzzy linguistic concepts in the decision-making process. the distance between the intents of different fuzzy linguistic concepts is considered to enhance the rationality of linguistic decision results. In addition, the expert's decision-making process for each alternative is visualized via linguistic concept lattices. Finally, the case study and comparative analysis illustrate the validity and rationality of the proposed approach in MAGDM with linguistic information.

1. Introduction

Decision-making as a common activity occurs regularly in our daily life. With the continuous development of society and economic changes, the decision-making environment is becoming highly complex. It is difficult for a single expert to obtain an optimal solution to a complex decision-making problem. Multi-attribute group decision-making (MAGDM) plays a crucial role as a valuable tool, where multiple experts collectively evaluate and rank alternatives from a set that encompasses diverse attributes. This methodology has demonstrated its effectiveness across a spectrum of research domains, such as investment decisions (Jiang & Hu, 2021), city construction (Meng et al., 2021), and company recruitment (Zhan et al., 2019). However, the MAGDM process encounters situations where experts grapple with the intricacies of objective assessments, hindering their ability to provide precise numerical information. In response, Zadeh (1965) introduced fuzzy sets to deal with fuzzy information. This seminal work triggered a surge of interest in fuzzy decision-making within the scholarly community (Xiao et al., 2022). Numerous studies have proposed various extended forms of fuzzy sets to be applied in MAGDM, including interval fuzzy sets (Garg, 2021), intuitionistic fuzzy sets (Zhang & Wang, 2023), interval-valued intuitionistic fuzzy sets (Wang & Wan, 2020), etc. Such innovations serve to refine the decision-making process, enabling it to effectively accommodate and navigate the complexities inherent in real-world scenarios.

In real-world MAGDM problems, experts frequently opt to convey their preferences using linguistic evaluations, a mode adept at accommodating vague and imprecise knowledge. Evaluative linguistic expressions (Novák, 2008) are used to characterize positions on an ordered scale in natural languages. For example, evaluative linguistic expressions such as "very high", "more or less low", and "low" are employed to assess the "innovation" of papers. Computing with words (CW) (Zadeh, 1996) is commonly employed to fuse linguistic information in MAGDM. Diverse models of linguistic representation have

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emerged, including linguistic term sets (Lin et al., 2021), linguistic truth-valued lattice implication algebra (LTV-LIA) (Xu et al., 2006), 2-tuple linguistic model (Herrera & Martínez, 2000), and type-2 fuzzy sets (Wu & Mendel, 2011). To improve the flexibility of preference expressions, some researchers focused on the construction of expressions in a closer way to human beings' cognitive process, and developed some complex linguistic expressions to elicit individuals' preferences. Hesitant fuzzy linguistic term set (HFLTS) (Rodriguez et al., 2011) and linguistic distribution (Zhang, Yu, et al., 2019) are becoming popular tools to model complex linguistic expressions and have been proposed to grapple with the intricacies of MAGDM problems imbued with linguistic information (Rodríguez et al., 2013; Wang, Jia, et al., 2023). For example, Wang et al. (2018b) provided a systematic overview of modeling techniques for complex linguistic representations in qualitative decision making. Wu et al. (2019) proposed flexible linguistic expressions and developed a novel approach to linguistic MAGDM. Since the same linguistic term has different meanings for different experts, Li et al. (2017) proposed a personalized individual semantic model. A notable contender, LTV-LIA, excels at concurrently handling linguistic information marked by both comparable and incomparable attributes. The integration of LTV-LIA within MAGDM stands as a meaningful approach to harmoniously fuse the multifaceted linguistic information advanced by experts.

In the MAGDM process, few experts in a group have the same opinion about alternatives. This creates the need to gather all the different expert opinions into one group opinion (Ben-Arieh & Chen, 2006; Hsu & Chen, 1996). The applications of linguistic representation model to MAGDM have evolved using various approaches that seek a group decision from the individual opinions (Mao et al., 2019; Wu et al., 2019). For example, to overcome the limitations of some existing 2-tuple linguistic models, Akram, Niaz, and Feng (2023) proposed 2-tuple linguistic Fermatean fuzzy Hamacher aggregation operators. Verma and Álvarez-Miranda (2023) proposed two new aggregation operations to aggregate 2-tuple linguistic Pythagorean fuzzy information. In the context of linguistic decision environments, the adaptation of technique for order preference by similarity to ideal solution (TOPSIS) necessitates the incorporation of linguistic aggregation operators, which serve to synthesize the intricate fabric of linguistic decision information (Pang et al., 2016). It is important to note, however, that the prevalent linguistic aggregation operators encounter challenges in effectively managing decision information marked by both comparable and incomparable attributes. Consequently, the utilization of such aggregation operators could inadvertently lead to the loss of information during the aggregation process.

To tackle this problem, formal concept analysis (FCA) (Ganter & Wille, 2012) provides a theoretical framework for designing and discovering concept hierarchies from a relational information system. FCA has been successfully applied in various research areas, including three-way decision (Pang et al., 2023), decision-making (Liu et al., 2019; Yang & Xu, 2010), and cognitive learning (Shi et al., 2021). Among those work, Yang and Xu (2010) constructed a linguistic truth-valued concept lattice and applied it to the decision-making process. Liu et al. (2019) proposed a fuzzy linguistic concept lattice based on linguistic term sets and applied it to teaching evaluation. To use the concept lattice in MAGDM with linguistic information, this paper further researches a linguistic concept lattice based on LTV-LIA, which can be utilized to deal with the fuzziness and uncertainty in MAGDM. Harnessing the concept's inherent capacity to aggregate expert evaluations through extent, this paper endeavors to introduce an innovative concept lattice-based approach to MAGDM. The proposed approach draws inspiration from the potential of FCA, aiming to fortify the decision-making process by seamlessly integrating and synthesizing expert evaluation information.

To sum up, although there are several proposals for the aggregation of expert opinions in MAGDM problems, there are still several issues that require further improvement:

1. In existing linguistic MAGDM approaches, different linguistic models are used to represent the linguistic evaluation information of experts (Akram, Bibi, & Deveci, 2023; Gou et al., 2017; Wang & Wang, 2022). These linguistic models can handle comparable linguistic information. However, when experts evaluate attributes of alternatives, numerous linguistic expressions such as "almost good", "rather bad", etc., are often not comparable. Therefore, we introduce LTV-LIA in our linguistic MAGDM approach to handle both comparable and incomparable linguistic information.
2. Despite the large amount of research dealing with linguistic MAGDM, there is still a need to improve methods for aggregating individual opinions into group opinions, especially when there exist comparable and incomparable linguistic expressions of these opinions. The existing LAAO and LIFFAA operators are the basic aggregation tools for aggregating LTV-LIA (Diao et al., 2022; Liu et al., 2020). However, these expert opinion aggregation methods use approximation operations resulting in information loss when using aggregation operators. Therefore, we introduce the concept lattice theory into the expert opinion aggregation method to reduce the information loss in the aggregation process.

In real-world scenarios, experts prefer to express their preferences by using linguistic information that is vague and incomparable, which is more in line with people's thinking patterns. This paper introduces a linguistic approach to MAGDM centered around linguistic concept lattices. The proposed approach serves to aggregate diverse linguistic insights provided by multiple experts. The efficacy of this proposed approach in tackling MAGDM quandaries embedded with linguistic information is demonstrated through a comprehensive case study and comparative analysis. The contributions of this paper are:

- The utilization of LTV-LIA for expressing expert linguistic evaluation information, enabling the concurrent handling of both comparable and incomparable linguistic information.
- The expert opinions are aggregated by obtaining all the fuzzy linguistic concepts corresponding to each alternative to reduce the information loss. On this basis, the proposed approach reduces the computational complexity of obtaining all fuzzy linguistic concepts by calculating the meet-irreducible elements in the linguistic concept lattice.
- A novel linguistic concept lattice-based MAGDM approach is employed. The proposed approach empowers the visualization of the decision-making process undertaken by all experts through the construction of linguistic concept lattices. This visual representation notably augments the interpretability of the MAGDM approach.

The remainder of the paper is organized as follows. Section 2 briefly recalls FCA and LTV-LIA. Section 3 proposes linguistic concept lattices based on LTV-LIA. Section 4 defines meet-irreducible elements in the linguistic concept lattice and discusses the distance between intents in fuzzy linguistic concepts. Section 5 proposes the MAGDM approach based on fuzzy linguistic concepts and gives the corresponding case study and comparative analysis. Finally, Section 6 concludes the paper and provides future work to be completed. An overall diagram of the paper is given in Fig. 1.

2. Preliminaries

This section briefly recalls FCA and LTV-LIA.

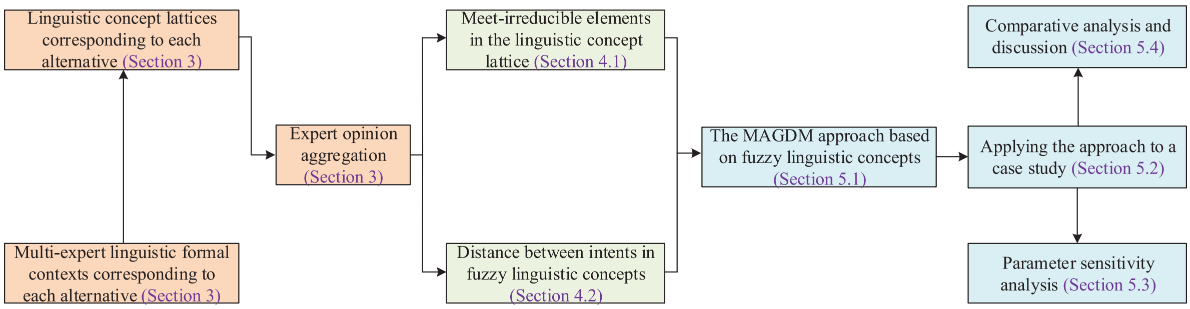


Fig. 1. The framework of the paper.

2.1. The basic of FCA

Definition 1 (Ganter & Wille, 2012). A formal context is a triple (G, M, I), where G is a non-empty finite set of objects, M is a non-empty finite set of attributes, and $I \subseteq G \times M$ is a binary relation between G and M. For $g \in G$ and $m \in M$, $(g, m) \in I$ means that the object g has the attribute m.

Definition 2 (Ganter & Wille, 2012). Let (G, M, I) be a formal context, for $X \subseteq G$ and $B \subseteq M$, two operators "↑" and "↓" can be defined as follows:

$$\begin{aligned} (\bullet)^{\uparrow} &: 2^G \to 2^M, \\ X^{\uparrow} &= \{m \mid m \in M,\ \forall g \in X,\ (g, m) \in I\}, \end{aligned} \tag{1}$$

$$\begin{aligned} (\bullet)^{\downarrow} &: 2^M \to 2^G, \\ B^{\downarrow} &= \{g \mid g \in G,\ \forall m \in B,\ (g, m) \in I\}. \end{aligned} \tag{2}$$

Definition 3 (Ganter & Wille, 2012). Let (G, M, I) be a formal context, for $X \subseteq G$ and $B \subseteq M$, if there exist $X^{\uparrow} = B$ and $X = B^{\downarrow}$, then a pair (X, B) is called a concept of (G, M, I). X and B are called the extent and intent of the concept (X, B), respectively.

2.2. The basic of LTV-LIA

Definition 4 (Xu et al., 2003). Let $(L, \vee, \wedge, ')$ be a bounded lattice with universal boundaries O and I respectively. For any $x, y, z \in L$, if mapping $\to : L \times L \to L$ satisfies:

1. $x \to (y \to z) = y \to (x \to z)$,
2. $x \to x = I$,
3. $x \to y = y' \to x'$,
4. If $x \to y = y \to x = I$, then $x = y$,
5. $(x \to y) \to y = (y \to x) \to x$,
6. $(x \vee y) \to z = (x \to z) \wedge (y \to z)$,
7. $(x \wedge y) \to z = (x \to z) \vee (y \to z)$,

then $(L, \vee, \wedge, ', \to)$ is called a lattice implication algebra.

Lattice implication algebra can handle both comparable and incomparable elements, to process linguistic information, Xu et al. (2006) proposed LTV-LIA.

Definition 5 (Xu et al., 2006). Denote $MT = \{c_1, c_2\}$, which is called as the set of meta linguistic truth values. The lattice implication algebra defined on the set of meta linguistic truth values is called a meta linguistic truth-valued lattice implication algebra, where $c_1 < c_2$. The operation "$'$" is defined as $c_1' = c_2$ and $c_2' = c_1$. The operation "→" is defined as

$$\begin{aligned} \to &: MT \times MT \longrightarrow MT, \\ & x \to y = x' \vee y. \end{aligned}$$

Definition 6 (Xu et al., 2006). Let $AD_n = \{h_k \mid k = 0, 1, \ldots, n$, n is an even number$\}$ be a set of hedges and $h_1 < h_2 < \cdots < h_n$. For $0 \le l, m \le n$, the operations are defined as follows:

$$h_l \vee h_m = h_{max\{l,m\}},$$
$$h_l \wedge h_m = h_{min\{l,m\}},$$
$$h_l' = h_{n-l},$$
$$h_l \to h_m = h_{min\{n, n-l+m\}}.$$

Then $(AD_n, \vee, \wedge, ', \to, h_0, h_n)$ is called a lattice implication algebra with hedges.

Definition 7 (Xu et al., 2006). Let $AD_n = \{h_k \mid k = 0, 1, \ldots, n$, n is an even number$\}$ be a set of hedges, $MT = \{c_1, c_2\}$ be the set of meta linguistic truth values, denoted $\mathcal{L}_{V(n\times2)} = AD_n \times MT$. Then $\mathcal{L}_{V(n\times2)} = (\mathcal{L}_{V(n\times2)}, \vee, \wedge, ', \to, (h_n, c_1), (h_n, c_2))$ is called a linguistic truth-valued lattice implication algebra.

The Hasse diagram of the LTV-LIA is shown in Fig. 2. The "∨" operation is defined as:

$$\begin{cases} (h_i, c_1) \vee (h_j, c_1) = (h_{min\{i,j\}}, c_1) \\ (h_i, c_1) \vee (h_j, c_2) = (h_{max\{j, n-i+1\}}, c_2) \\ (h_i, c_2) \vee (h_j, c_2) = (h_{max\{i,j\}}, c_2) \\ (h_i, c_2) \vee (h_j, c_1) = (h_{max\{i, n-j+1\}}, c_2) \end{cases}$$

The "∧" operation is defined as:

$$\begin{cases} (h_i, c_1) \wedge (h_j, c_1) = (h_{max\{i,j\}}, c_1) \\ (h_i, c_1) \wedge (h_j, c_2) = (h_{max\{i, n-j+1\}}, c_1) \\ (h_i, c_2) \wedge (h_j, c_2) = (h_{min\{i,j\}}, c_2) \\ (h_i, c_2) \wedge (h_j, c_1) = (h_{max\{j, n-i+1\}}, c_1) \end{cases}$$

The "→" operation is defined as:

$$\begin{cases} (h_i, c_2) \to (h_j, c_1) = (h_{max\{0, i+j-n\}}, c_1) \\ (h_i, c_1) \to (h_j, c_2) = (h_{min\{n, i+j\}}, c_2) \\ (h_i, c_2) \to (h_j, c_2) = (h_{min\{n, n-i+j\}}, c_2) \\ (h_i, c_1) \to (h_j, c_1) = (h_{min\{n, n-j+i\}}, c_2) \end{cases}$$

The "$'$" operation is defined as:

$$\begin{cases} (h_i, c_1)' = (h_i, c_2) \\ (h_j, c_2)' = (h_j, c_1) \end{cases}$$

The construction process of the lattice implication algebra uses $AD_n \times MT$. The construction process is divided into two parts, (1) constructing lattice implication algebras on AD_n and MT respectively, and (2) inducing lattice implication algebras on the set of linguistic truth values based on the lattice implication algebras on AD_n and MT.

In the LTV-LIA, let $MT = \{c_1 = \text{false}, c_2 = \text{true}\}$ be the meta linguistic truth-valued set. In the lattice implication algebra, the all hedges we consider can weaken the degree of truth or false to some extent, such as "roughly", "almost", "rather", "more or less", and so on.

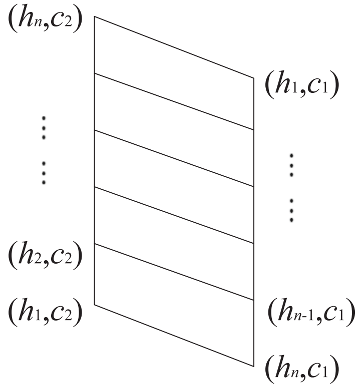


Fig. 2. Hasse diagram of $\mathcal{L}_{V(n\times 2)}$.

So the two chains in lattice implication algebra, namely the truth chain and the false chain, have gradually weakened the degrees of truth and false due to the hedges. With the different degrees of weakening, some weakened linguistic expressions are incomparable intuitively, such as "almost true" and "rather false".

3. Expert opinion aggregation based on linguistic concept lattice

Experts tend to rely on linguistic expressions to evaluate each alternative during the MAGDM process, given the intricate nature of objective matters and the subjective nature of human thinking. To aggregate information about the evaluation of different attributes by different experts, this section proposes linguistic concept lattices based on the LTV-LIA and aggregates the expert opinions through fuzzy linguistic concepts.

For a MAGDM problem, suppose that $U = \{x_1, x_2, \dots, x_o\}$ denotes the set of alternatives, $E = \{e_1, e_2, \dots, e_r\}$ denotes the set of experts and $A = \{a_1, a_2, \dots, a_p\}$ denotes the set of attributes.

When experts use evaluative linguistic expressions to describe attributes, evaluative linguistic predications are obtained, which have the following form syntactically

$$a \ \text{ is } \ \mathcal{D}, \tag{3}$$

where a is an attribute and $\mathcal{D}$ is an evaluative linguistic expression. In this paper, the LTV-LIA is used to represent evaluative linguistic expressions. For $(h_k, c_l)(k = 1, 2, \dots, n; l = 1, 2) \in \mathcal{L}_{V(n\times 2)}$, the converted evaluative linguistic predications can be represented by

$$a \ \text{ is } \ (h_k, c_l), \tag{4}$$

the set of $\mathbb{A}$ can be represented by the set of evaluative linguistic predications $\mathbb{A} = \{\langle a \ \text{ is } \ (h_k, c_l)\rangle | (h_k, c_l) \in \mathcal{L}_{V(n\times 2)}, a \in A\}$. Based on the above discussion, we provide the definition of multi-expert linguistic formal context.

Definition 8. A multi-expert linguistic formal context is a triple $(E, \mathbb{A}, \mathbb{I})$, where $E = \{e_1, e_2, \dots, e_r\}$ is a set of experts, $A = \{a_1, a_2, \dots, a_p\}$ is a set of attributes, $\mathbb{A} = \{\langle a \ \text{ is } \ (h_k, c_l)\rangle | (h_k, c_l) \in \mathcal{L}_{V(n\times 2)}, a \in A\}$ is a set of evaluative linguistic predications, and $\mathbb{I} \subseteq E \times \mathbb{A}$ is a binary relation between E and $\mathbb{A}$. For $e \in E$ and $\langle a \ \text{ is } \ (h_k, c_l)\rangle \in \mathbb{A}$, $(e, \langle a \ \text{ is } \ (h_k, c_l)\rangle) \in \mathbb{I}$ means that expert e has the evaluative linguistic predication $\langle a \ \text{ is } \ (h_k, c_l)\rangle$.

To derive the definition of the linguistic concept lattice from $(E, \mathbb{A}, \mathbb{I})$, we provide the definition of induction operators "$\lhd$" and "$\rhd$".

Definition 9. Let $(E, \mathbb{A}, \mathbb{I})$ be a multi-expert linguistic formal context. For $W \subseteq E$ and $Y \subseteq \mathbb{A}$, two operators "$\lhd$" and "$\rhd$" can be defined as follows:

$$\begin{aligned} &(\bullet)^{\lhd} : 2^W \to 2^Y, \\ &W^{\lhd} = \{\langle a \ \text{ is } \ (h_k, c_l)\rangle \in \mathbb{A} | \forall e \in W, \\ &\quad (e, \langle a \ \text{ is } \ (h_k, c_l)\rangle) \in \mathbb{I}\}, \end{aligned} \tag{5}$$

$$\begin{aligned} &(\bullet)^{\rhd} : 2^Y \to 2^W, \\ &Y^{\rhd} = \{e \in E | \forall \langle a \ \text{ is } \ (h_k, c_l)\rangle \in Y, \\ &\quad (e, \langle a \ \text{ is } \ (h_k, c_l)\rangle) \in \mathbb{I}\}. \end{aligned} \tag{6}$$

Definition 10. Let $(E, \mathbb{A}, \mathbb{I})$ be a multi-expert linguistic formal context. For $W \subseteq E$ and $Y \subseteq \mathbb{A}$, if there exist $W^{\lhd} = Y$ and $Y^{\rhd} = W$, then a pair (W, Y) is called a fuzzy linguistic concept of $(E, \mathbb{A}, \mathbb{I})$. W and Y are called the extent and intent of the fuzzy linguistic concept (W, Y), respectively.

In $(E, \mathbb{A}, \mathbb{I})$, the set of all the fuzzy linguistic concepts is denoted as

$$L(E, \mathbb{A}, \mathbb{I}) = \{(W, Y) | W^{\lhd} = Y, Y^{\rhd} = W\}.$$

For $(W_1, Y_1), (W_2, Y_2) \in L(E, \mathbb{A}, \mathbb{I})$, the partial order "$\leq$" between fuzzy linguistic concept is denoted as

$$(W_1, Y_1) \leq (W_2, Y_2) \Leftrightarrow W_1 \subseteq W_2 (\Leftrightarrow Y_2 \subseteq Y_1), \tag{7}$$

then $(L(E, \mathbb{A}, \mathbb{I}), \leq)$ is a complete lattice, called a linguistic concept lattice of $(E, \mathbb{A}, \mathbb{I})$. The infimum and supremum can be defined as follows:

$$(W_1, Y_1) \vee (W_2, Y_2) = ((W_1 \cup W_2)^{\lhd\rhd}, Y_1 \cap Y_2),$$

$$(W_1, Y_1) \wedge (W_2, Y_2) = (W_1 \cap W_2, (Y_1 \cup Y_2)^{\rhd\lhd}).$$

Theorem 1. *Let $(E, \mathbb{A}, \mathbb{I})$ be a multi-expert linguistic formal context. For $W, W_1, W_2 \subseteq E$ and $Y, Y_1, Y_2 \subseteq \mathbb{A}$, the following properties hold.*

1. $W_1 \subseteq W_2 \Rightarrow W_2^{\lhd} \subseteq W_1^{\lhd}, Y_1 \subseteq Y_2 \Rightarrow Y_2^{\rhd} \subseteq Y_1^{\rhd}$;
2. $W \subseteq W^{\lhd\rhd}, Y \subseteq Y^{\rhd\lhd}$;
3. $W^{\lhd} = W^{\lhd\rhd\lhd}, Y^{\rhd} = Y^{\rhd\lhd\rhd}$;
4. $(W_1 \cup W_2)^{\lhd} = W_1^{\lhd} \cap W_2^{\lhd}, (Y_1 \cup Y_2)^{\rhd} = Y_1^{\rhd} \cap Y_2^{\rhd}$;
5. $(W_1 \cap W_2)^{\lhd} \supseteq W_1^{\lhd} \cup W_2^{\lhd}, (Y_1 \cap Y_2)^{\rhd} \supseteq Y_1^{\rhd} \cup Y_2^{\rhd}$;
6. *Both $(W^{\lhd\rhd}, W^{\lhd})$ and $(Y^{\rhd}, Y^{\rhd\lhd})$ are fuzzy linguistic concept.*

Proof.

1. Suppose $W_1 \subseteq W_2$. According to Definition 9, we have $W_1 = \bigcap_{e_i \in W_1} e_i^{\lhd}$ and $W_2 = \bigcap_{e_j \in W_2} e_j^{\lhd}$. $W_2^{\lhd} \subseteq W_1^{\lhd}$ is obtained. Therefore, $W_1 \subseteq W_2 \Rightarrow W_2^{\lhd} \subseteq W_1^{\lhd}$ holds. Similarly, we can prove $Y_1 \subseteq Y_2 \Rightarrow Y_2^{\rhd} \subseteq Y_1^{\rhd}$.
2. It can be proved by Definition 9.
3. According to properties 1 and 2, we have $W^{\lhd\rhd\lhd} \subseteq W^{\lhd}$. Suppose $W^{\lhd} = Y$. Then, $W^{\lhd} \subseteq W^{\lhd\rhd\lhd}$ is obtained. Hence, $W^{\lhd} = W^{\lhd\rhd\lhd}$ holds. Similarly, we can prove $Y^{\rhd} = Y^{\rhd\lhd\rhd}$.
4. According to Definition 9, we have $(W_1 \cup W_2)^{\lhd} = \bigcap_{e_i \in W_1 \cup W_2} e_i^{\lhd} = (\bigcap_{e_i \in W_1} e_i^{\lhd}) \cap (\bigcap_{e_i \in W_2} e_i^{\lhd}) = W_1^{\lhd} \cap W_2^{\lhd}$. Thus, $(W_1 \cup W_2)^{\lhd} = W_1^{\lhd} \cap W_2^{\lhd}$ holds. Similarly, we can prove $(Y_1 \cup Y_2)^{\rhd} = Y_1^{\rhd} \cap Y_2^{\rhd}$
5. The proof is similar to property 1.
6. According to Definition 9 and property 3, we have $W = W^{\lhd\rhd}$ and $W^{\lhd} = Y$. Therefore, $(W^{\lhd\rhd}, W^{\lhd})$ is a fuzzy linguistic concept.

Example 1. We consider the multi-expert linguistic formal context $(E, \mathbb{A}, \mathbb{I})$ of Table 1, where $E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ denotes the set of experts and $A = \{a_1, a_2, a_3\}$ denotes the set of attributes.

Let $AD_3 = \{h_1 = roughly, h_2 = very, h_3 = extremely\}$ be the linguistic hedges set and $MT = \{c_1 = bad, c_2 = good\}$ be the meta linguistic truth-valued set. The 6-element LTV-LIA $\mathcal{L}_{V(3\times 2)} = (\mathcal{L}_{V(3\times 2)}, \vee, \wedge, \prime, \to, (h_3, c_1), (h_3, c_2))$ is obtained as shown in Fig. 3.

The set of evaluative linguistic predications $\mathbb{A} = \{\langle a_i \ \text{ is } \ (h_p, c_q)\rangle | (h_p, c_q) \in \mathcal{L}_{V(3\times 2)}, a_i \in A\}$ is obtained, and $\mathbb{I}$ is given in Table 1. The clarified multi-expert linguistic formal context $(E, \mathbb{A}, \mathbb{I})_c$ is obtained

Table 1
Multi-expert linguistic formal context $(E, \mathbb{A}, \mathbb{I})$.

E	a_1						a_2						a_3					
	EB	RG	VB	VG	RB	EG	EB	RG	VB	VG	RB	EG	EB	RG	VB	VG	RB	EG
e_1						×		×						×				
e_2			×				×							×				
e_3			×							×			×					
e_4	×											×			×			
e_5						×						×		×				
e_6					×				×								×	
e_7						×						×				×		

Note: EB: (h_3, c_1), RG: (h_1, c_2), VB: (h_2, c_1), VG: (h_2, c_2), RB: (h_1, c_1), EG: (h_3, c_2).

Table 2
Clarified multi-expert linguistic formal context $(E, \mathbb{A}, \mathbb{I})_c$.

E	a_1				a_2					a_3				
	EB	VB	RB	EG	EB	RG	VB	VG	EG	EB	RG	VB	VG	RB
e_1				×		×					×			
e_2		×			×						×			
e_3		×						×		×				
e_4	×								×			×		
e_5				×					×		×			
e_6			×				×							×
e_7				×					×				×	

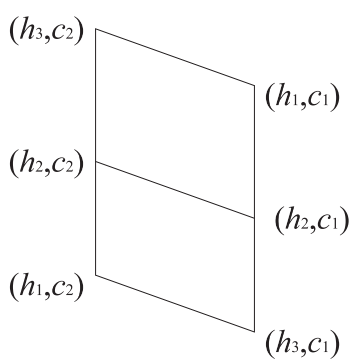


Fig. 3. Hasse diagram of $\mathcal{L}_{V(3\times 2)}$.

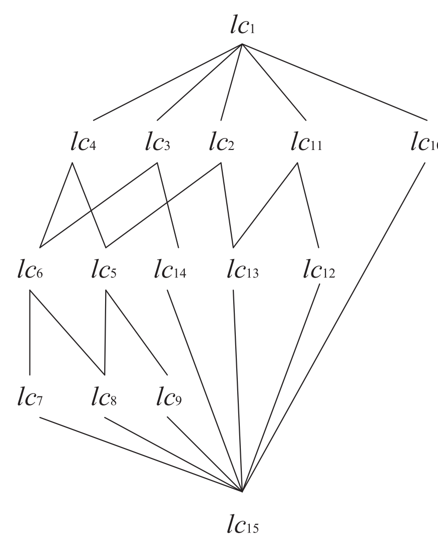


Fig. 4. Linguistic concept lattice $L(E, \mathbb{A}, \mathbb{I})$.

through the clarification method (Ganter & Wille, 2012), as shown in Table 2.

According to Definitions 9 and 10, all fuzzy linguistic concepts are obtained in $(E, \mathbb{A}, \mathbb{I})$ as follows. The linguistic concept lattice $L(E, \mathbb{A}, \mathbb{I})$ is depicted by Fig. 4.

$lc_1 : (E, \emptyset)$,

$lc_2 : (\{e_1, e_2, e_5\}, \{\langle a_3 \text{ is } (h_1, c_2)\rangle\})$,

$lc_3 : (\{e_4, e_5, e_7\}, \{\langle a_2 \text{ is } (h_3, c_2)\rangle\})$,

$lc_4 : (\{e_1, e_5, e_7\}, \{\langle a_1 \text{ is } (h_3, c_2)\rangle\})$,

$lc_5 : (\{e_1, e_5\}, \{\langle a_1 \text{ is } (h_3, c_2)\rangle, \langle a_3 \text{ is } (h_1, c_2)\rangle\})$,

$lc_6 : (\{e_5, e_7\}, \{\langle a_1 \text{ is } (h_3, c_2)\rangle, \langle a_2 \text{ is } (h_3, c_2)\rangle\})$,

$lc_7 : (\{e_7\}, \{\langle a_1 \text{ is } (h_3, c_2)\rangle, \langle a_2 \text{ is } (h_3, c_2)\rangle, \langle a_3 \text{ is } (h_2, c_2)\rangle\})$,

$lc_8 : (\{e_5\}, \{\langle a_1 \text{ is } (h_3, c_2)\rangle, \langle a_2 \text{ is } (h_3, c_2)\rangle, \langle a_3 \text{ is } (h_1, c_2)\rangle\})$,

$lc_9 : (\{e_1\}, \{\langle a_1 \text{ is } (h_3, c_2)\rangle, \langle a_2 \text{ is } (h_1, c_2)\rangle, \langle a_3 \text{ is } (h_1, c_2)\rangle\})$,

$lc_{10} : (\{e_6\}, \{\langle a_1 \text{ is } (h_1, c_1)\rangle, \langle a_2 \text{ is } (h_2, c_1)\rangle, \langle a_3 \text{ is } (h_1, c_1)\rangle\})$,

$lc_{11} : (\{e_2, e_3\}, \{\langle a_1 \text{ is } (h_2, c_1)\rangle\})$,

$lc_{12} : (\{e_3\}, \{\langle a_1 \text{ is } (h_1, c_1)\rangle, \langle a_2 \text{ is } (h_2, c_2)\rangle, \langle a_3 \text{ is } (h_3, c_1)\rangle\})$,

$lc_{13} : (\{e_2\}, \{\langle a_1 \text{ is } (h_1, c_1)\rangle, \langle a_2 \text{ is } (h_3, c_1)\rangle, \langle a_3 \text{ is } (h_1, c_2)\rangle\})$,

$lc_{14} : (\{e_4\}, \{\langle a_1 \text{ is } (h_3, c_1)\rangle, \langle a_2 \text{ is } (h_3, c_2)\rangle, \langle a_3 \text{ is } (h_2, c_1)\rangle\})$,

$lc_{15} : (\emptyset, \mathbb{A})$.

Taking the fuzzy linguistic concept lc_5 as an example, according to the extent of lc_5, experts e_1 and e_5 are aggregated. According to the extent of lc_5, the reason for such an aggregation result is that these two experts evaluated attribute a_1 of a specific alternative extremely good and evaluated attribute a_3 of a specific alternative roughly good.

4. Distance between intents under the fuzzy linguistic concept

4.1. Meet-irreducible elements in the linguistic concept lattice

During the aggregation of MAGDM, any fuzzy linguistic concept can be represented as the meet of some meet-irreducible elements in the linguistic concept lattice. Therefore, this subsection proposes meet-irreducible fuzzy linguistic concepts as concept knowledge spaces to avoid computing the entire linguistic concept lattice.

Definition 11. Let $(E, \mathbb{A}, \mathbb{I})$ be a multi-expert linguistic formal context. For any $e \in E$ and $\langle a \text{ is } (h_k, c_l)\rangle \in \mathbb{A}$, $(\langle a \text{ is } (h_k, c_l)\rangle^{\rhd}, \langle a \text{ is } (h_k, c_l)\rangle^{\rhd\lhd})$ and $(e^{\lhd\rhd}, e^{\lhd})$ are called evaluative linguistic predication concept and fuzzy expert concept, respectively.

Definition 12 (Davey & Priestley, 2002). Given a lattice $(L, \preceq)$, an element $w \in L$ verifying:

1. If L has a top element $\top$, then $w \neq \top$.
2. If $w = y \wedge z$, then $w = y$ or $w = z$, for all $y, z \in L$.

Then w is called meet-irreducible element of L.

Theorem 2 (Davey & Priestley, 2002). *Let $(L, \preceq)$ be a lattice, then every element in L is the meet of the meet-irreducible elements.*

Let $(E, \mathbb{A}, \mathbb{I})$ be a multi-expert linguistic formal context and write $\mathbb{A}_0 = \{\langle a \text{ is } (h_k, c_l)\rangle \in \mathbb{A} | (\langle a \text{ is } (h_k, c_l)\rangle^{\rhd}, \langle a \text{ is } (h_k, c_l)\rangle^{\rhd\lhd})$ is meet-irreducible element$\}$. For any $(W, Y) \in L(E, \mathbb{A}, \mathbb{I})$,

$$(Y, W) = \bigwedge_{\langle a \text{ is } (h_k, c_l)\rangle \in W \cap \mathbb{A}_0} (\langle a \text{ is } (h_k, c_l)\rangle^{\rhd}, \langle a \text{ is } (h_k, c_l)\rangle^{\rhd\lhd}),$$

where $(\langle a \text{ is } (h_k, c_l)\rangle^{\rhd}, \langle a \text{ is } (h_k, c_l)\rangle^{\rhd\lhd})$ is meet-irreducible element.

Lemma 1. *Let $(E, \mathbb{A}, \mathbb{I})$ be a multi-expert linguistic formal context. For any $\langle a \text{ is } (h_k, c_l)\rangle \in \mathbb{A}$, $(\langle a \text{ is } (h_k, c_l)\rangle^{\rhd}, \langle a \text{ is } (h_k, c_l)\rangle^{\rhd\lhd})$ is meet-irreducible element if and only if*

$$\{\langle b \text{ is } (h_p, c_q)\rangle \in \mathbb{A} | \langle a \text{ is } (h_k, c_l)\rangle^{\rhd} \subset \langle b \text{ is } (h_p, c_q)\rangle^{\rhd}\} = \emptyset$$

or

$$\langle a \text{ is } (h_k, c_l)\rangle^{\rhd} \subset \bigcap_{\langle a \text{ is } (h_k, c_l)\rangle^{\rhd} \subset \langle b \text{ is } (h_p, c_q)\rangle^{\rhd}} \langle b \text{ is } (h_p, c_q)\rangle^{\rhd}.$$

4.2. Intent distance under the fuzzy linguistic concept

This subsection proposes a method to calculate the intent distance of different fuzzy linguistic concepts.

We first provide the definition of distances for different evaluative linguistic expressions under the same attribute.

Definition 13. Let $(E, \mathbb{A}, \mathbb{I})$ be a multi-expert linguistic formal context. For any $\langle a \text{ is } (h_k, c_l)\rangle \in \mathbb{A}$, the distance between two different evaluative linguistic expressions (h_k, c_l) and (h_p, c_q) under a is defined as follows

$$d((h_k, c_l), (h_p, c_q)) = \begin{cases} |k - p|, & l = q, \\ |k - (n - p)|, & l \neq q. \end{cases} \tag{8}$$

Theorem 3. *Let $(E, \mathbb{A}, \mathbb{I})$ be a multi-expert linguistic formal context, (h_k, c_l), (h_u, c_v) and (h_p, c_q) be different evaluative linguistic expressions under the same attribute. The distance $d((h_k, c_l), (h_p, c_q))$ between (h_k, c_l) and (h_p, c_q) satisfies the following properties.*

1. $0 \leq d((h_k, c_l), (h_p, c_q)) \leq n$.
2. $d((h_k, c_l), (h_p, c_q)) = 0$ *if and only if* $(h_k, c_l) = (h_p, c_q)$.
3. $d((h_k, c_l), (h_p, c_q)) = d((h_p, c_q), (h_k, c_l))$.
4. $d((h_k, c_l), (h_p, c_q)) \leq d((h_k, c_l), (h_u, c_v)) + d((h_u, c_v), (h_p, c_q))$.

Proof.

1. According to Definition 7, (h_n, c_2) and (h_n, c_1) are the maximum and minimum elements in the LTV-LIA. According to Definition 13, the distance between any evaluative linguistic expression and itself is minimum, i.e.

   $$d((h_k, c_l), (h_k, c_l)) = |k - k| = 0.$$

   The distance between the minimum element (h_n, c_1) and the maximum element (h_n, c_2) is maximum, i.e.

   $$d((h_n, c_1), (h_n, c_2)) = |n - (n - n)| = n.$$

   Therefore, $0 \leq d((h_k, c_l), (h_p, c_q)) \leq n$ holds.
2. Necessity. Suppose $d((h_k, c_l), (h_p, c_q)) = 0$, then $k = p$ and $l = q$, i.e., $(h_k, c_l) = (h_p, c_q)$.
   Sufficiency. Suppose $(h_k, c_l) = (h_p, c_q)$, then

   $$d((h_k, c_l), (h_p, c_q)) = d((h_k, c_l), (h_k, c_l)) = |k - k| = 0.$$
3. According to Definition 13, when $l = q$, we have

   $$d((h_k, c_l), (h_p, c_q)) = |k - p|,$$

   $$d((h_p, c_q), (h_k, c_l)) = |p - k|.$$

   Since $|k - p| = |p - k|$, we can get $d((h_k, c_l), (h_p, c_q)) = d((h_p, c_q), (h_k, c_l))$. When $l \neq q$, we have

   $$d((h_k, c_l), (h_p, c_q)) = |k - (n - p)|,$$

   $$d((h_p, c_q), (h_k, c_l)) = |p - (n - k)|.$$

   Since $|k - (n - p)| = |p - (n - k)|$, we can get $d((h_k, c_l), (h_p, c_q)) = d((h_p, c_q), (h_k, c_l))$. Combining the above arguments, we obtain that $d((h_k, c_l), (h_p, c_q)) = d((h_p, c_q), (h_k, c_l))$.
4. **Case 1**: Suppose $v = q = l$, then we have

   $$\begin{aligned} d((h_k, c_l), (h_p, c_q)) &= |k - p| \\ &= |k - u + u - p| \\ &\leq |k - u| + |u - p|. \end{aligned}$$

   This implies that $d((h_k, c_l), (h_p, c_q)) \leq d((h_k, c_l), (h_u, c_v)) + d((h_u, c_v), (h_p, c_q))$.
   Case 2: Suppose $v = q \neq l$, then we have

   $$\begin{aligned} d((h_k, c_l), (h_p, c_q)) &= |k + p - n| \\ &= |k + u - n + p - u| \\ &\leq |k + u - n| + |u - p|. \end{aligned}$$

   This implies that $d((h_k, c_l), (h_p, c_q)) \leq d((h_k, c_l), (h_u, c_v)) + d((h_u, c_v), (h_p, c_q))$.
   Case 3: Suppose $v \neq q = l$, then we have

   $$\begin{aligned} d((h_k, c_l), (h_p, c_q)) &= |k - p| \\ &= |k - u + u - p| \\ &\leq |k + u - n| + |u + p - n|. \end{aligned}$$

   This implies that $d((h_k, c_l), (h_p, c_q)) \leq d((h_k, c_l), (h_u, c_v)) + d((h_u, c_v), (h_p, c_q))$.

Definition 14. Let $(E, \mathbb{A}, \mathbb{I})$ be a multi-expert linguistic formal context and $L(E, \mathbb{A}, \mathbb{I})$ be the linguistic concept lattice corresponding to $(E, \mathbb{A}, \mathbb{I})$. For any $(Y_b, W_b), (Y_g, W_g) \in L(E, \mathbb{A}, \mathbb{I})$, the distance between two intents W_b and W_g is defined as follows

$$d(W_b, W_g) = \frac{1}{t} \sum_{u=1}^{t} d_u((h_k, c_l), (h_p, c_q)), \tag{9}$$

where t represents the number of attributes in $(E, \mathbb{A}, \mathbb{I})$, (h_k, c_l) and (h_p, c_q) are different evaluative linguistic expressions under the same attribute.

Theorem 4. *Let $(E, \mathbb{A}, \mathbb{I})$ be a multi-expert linguistic formal context and $L(E, \mathbb{A}, \mathbb{I})$ be the linguistic concept lattice corresponding to $(E, \mathbb{A}, \mathbb{I})$. For any $(Y_b, W_b), (Y_c, W_c), (Y_g, W_g) \in L(E, \mathbb{A}, \mathbb{I})$, the distance $d(W_b, W_g)$ between W_b and W_g satisfies the following properties.*

1. $0 \leq d(W_b, W_g) \leq n$.
2. $d(W_b, W_g) = 0$ *if and only if* $W_b = W_g$.
3. $d(W_b, W_g) = d(W_g, W_b)$.
4. $d(W_b, W_g) \leq d(W_b, W_c) + d(W_c, W_g)$.

Proof. Theorem 4 is similarly provable to Theorem 3.

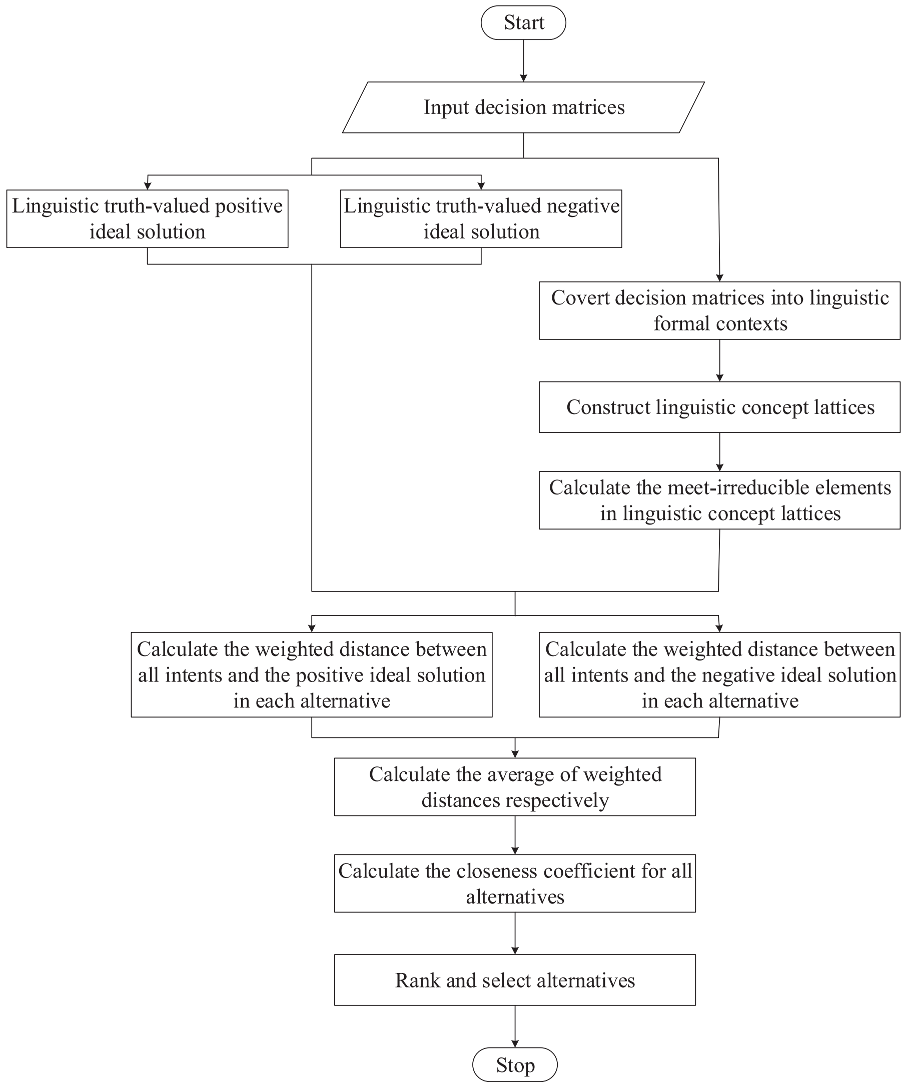


Fig. 5. The flow chart of the MAGDM approach based on fuzzy linguistic concepts.

5. An approach for MAGDM based on fuzzy linguistic concepts

This section proposes an approach for MAGDM based on meet-irreducible element in linguistic concept lattice.

5.1. Model construction

Considering a linguistic MAGDM problem, let $U = \{x_1, x_2, \ldots, x_o\}$ be a set of alternatives, $A = \{a_1, a_2, \ldots, a_p\}$ be a set of attributes, $E = \{e_1, e_2, \ldots, e_r\}$ be a set of experts, and $\mathcal{L}_{V(n\times 2)} = (\mathcal{L}_{V(n\times 2)}, \vee, \wedge, \prime, \to, (h_n, c_1), (h_n, c_2))$ be a LTV-LIA. Let $w = (w_1, w_2, \ldots, w_p)$ be a weight vector of attributes, where $w_\zeta > 0$ and $\sum_\zeta^p w_\zeta = 1$. Ω_b and Ω_c denote the sets of benefit attribute and cost attribute, respectively. The flow chart of the MAGDM approach based on fuzzy linguistic concepts is shown in Fig. 5. The steps of MAGDM can be described as follows.

Step 1: Given decision matrices $M^{(z)} = (m_{ij}^{(z)})_{o\times p} (1 \le z \le r)$ as follows

$$M^{(z)} = (m_{ij}^{(z)})_{o\times p} = \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1p} \\ m_{21} & m_{22} & \cdots & m_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ m_{o1} & m_{o2} & \cdots & m_{op} \end{bmatrix},$$

where $m_{ij}^{(z)} = (h_k, c_l) \in \mathcal{L}_{V(n\times 2)}$.

Step 2: Determine positive and negative ideal solutions for linguistic expressions.

1. Linguistic truth-valued positive ideal solution (LTV-PIS):

$$P^{(z)+} = (m_{i1}^{(z)+}, m_{i2}^{(z)+}, \ldots, m_{ip}^{(z)+}), \tag{10}$$

where

$$m_{ij}^{(z)+} = (max_i m_{ij}^{(z)}, j \in \Omega_b; min_i m_{ij}^{(z)}, j \in \Omega_c).$$

Then we have

$$P^{+} = (m_{i1}^{+}, m_{i2}^{+}, \ldots, m_{ip}^{+}), \tag{11}$$

where

$$m_{ij}^{+} = (max_z m_{ij}^{(z)+}, j \in \Omega_b; min_z m_{ij}^{(z)+}, j \in \Omega_c).$$

2. Linguistic truth-valued negative ideal solution (LTV-NIS):

$$P^{(z)-} = (m_{i1}^{(z)-}, m_{i2}^{(z)-}, \ldots, m_{ip}^{(z)-}), \tag{12}$$

where

$$m_{ij}^{(z)-} = (min_i m_{ij}^{(z)}, j \in \Omega_b; max_i m_{ij}^{(z)}, j \in \Omega_c).$$

Then we have

$$P^{-} = (m_{i1}^{-}, m_{i2}^{-}, \ldots, m_{ip}^{-}), \tag{13}$$

where

$$m_{ij}^{-} = (min_z m_{ij}^{(z)+}, j \in \Omega_b; max_z m_{ij}^{(z)+}, j \in \Omega_c).$$

Step 3: Convert r decision matrices $M^{(z)}$ into multi-expert linguistic formal contexts $(E,\mathbb{A},\mathbb{I})^s(1\le s\le o)$ corresponding to each alternative according to the method given in Section 3.

Step 4: Construct linguistic concept lattices $L(E,\mathbb{A},\mathbb{I})^s$ corresponding to $(E,\mathbb{A},\mathbb{I})^s$ according to concept induction operators "$\lhd$" and "$\rhd$".

Step 5: Calculate κ meet-irreducible elements and their intents $W_\Phi(\Phi=1,2,\ldots,\kappa)$ in $L(E,\mathbb{A},\mathbb{I})^s$ corresponding to each alternative according to the method given in Section 4.1.

Step 6: Calculate the weighted distance $d_w(W_\Phi,P^+)^s$ between W_Φ and the LTV-PIS P^+, respectively. Obtain the average value $d_w(W,P^+)^s_{avg}$ of $d_w(W_\Phi,P^+)^s$ for each alternative,

$$d_w(W_\Phi,P^+)^s=\frac{1}{t}\sum_{u=1}^{t}w_u d_u((h_k,c_l),(h_p,c_q)), \tag{14}$$

$$d_w(W,P^+)^s_{avg}=\frac{1}{\kappa}\sum_{\Phi=1}^{\kappa}d_w(W_\Phi,P^+), \tag{15}$$

where t represents the number of attributes in $(E,\mathbb{A},\mathbb{I})$, (h_k,c_l) and (h_p,c_q) are different evaluative linguistic expressions under the same attribute.

Step 7: Calculate the weighted distance $d_w(W_\Phi,P^-)^s$ between W_Φ and the LTV-NIS P^-, respectively. Obtain the average value $d_w(W,P^-)^s_{avg}$ of $d_w(W_\Phi,P^-)^s$ for each alternative,

$$d_w(W_\Phi,P^-)^s=\frac{1}{t}\sum_{u=1}^{t}w_u d_u((h_k,c_l),(h_p,c_q)), \tag{16}$$

$$d_w(W,P^-)^s_{avg}=\frac{1}{\kappa}\sum_{\Phi=1}^{\kappa}d_w(W_\Phi,P^-). \tag{17}$$

Step 8: Calculate the closeness coefficient $C(x_s)$ for the alternative x_s,

$$C(x_s)=\frac{d_w(W,P^-)^s_{avg}}{d_w(W,P^+)^s_{avg}+d_w(W,P^-)^s_{avg}}. \tag{18}$$

Step 9: The alternatives $x_s(s=1,2,\ldots,o)$ are ranked according to the order of the closeness coefficient $C(x_s)$ from largest to smallest.

5.2. Case study

To illustrate the practicality of our proposed approach, we provide a concrete example adapted from previous works by Pang et al. (2016), Parreiras et al. (2010).

Suppose the board of directors of a company will plan the development of large projects for the following five years. In order to prioritize and make the best decision which projects $x_s(s=1,2,3)$ is the best, five members $e_z(z=1,2,3,4,5)$ of the board make decisions on all projects based on the four attributes $a_j(j=1,2,3,4)$. The weights of all four attributes are set to $w=0.25$. The meanings of each of these four attributes are represented below:

- a_1: Financial perspective,
- a_2: The customer satisfaction,
- a_3: Internal business process perspective,
- a_4: Learning and growth perspective.

Since LTV-LIA can handle both comparable and incomparable linguistic evaluation information expressed by experts in MAGDM, we use 6-element LTV-LIA $\mathcal{L}_{V(3\times2)}$ in this paper. Table 3 shows the fundamental linguistic scale, and Fig. 3 shows a Hasse diagram of $\mathcal{L}_{V(3\times2)}$.

Based on the evaluation of the different projects by the board members, their original decision matrices are as follows.

$$M^{(1)}=\begin{bmatrix}(h_2,c_1)&(h_2,c_2)&(h_2,c_2)&(h_1,c_1)\\(h_2,c_1)&(h_2,c_1)&(h_1,c_2)&(h_2,c_1)\\(h_2,c_2)&(h_2,c_1)&(h_2,c_1)&(h_2,c_2)\end{bmatrix},$$

$$M^{(2)}=\begin{bmatrix}(h_2,c_2)&(h_1,c_2)&(h_2,c_2)&(h_1,c_1)\\(h_2,c_1)&(h_2,c_1)&(h_3,c_1)&(h_2,c_1)\\(h_2,c_2)&(h_2,c_1)&(h_1,c_1)&(h_2,c_2)\end{bmatrix},$$

Table 3
6-element linguistic truth-valued fundamental scale.

Scale	Meaning
(h_3,c_2)	Extremely high (EH)
(h_1,c_1)	Roughly low (RL)
(h_2,c_2)	Very high (VH)
(h_2,c_1)	Very low (VL)
(h_1,c_2)	Roughly high (RH)
(h_3,c_1)	Extremely low (EL)

$$M^{(3)}=\begin{bmatrix}(h_2,c_2)&(h_2,c_2)&(h_2,c_2)&(h_2,c_1)\\(h_1,c_1)&(h_1,c_2)&(h_2,c_2)&(h_2,c_2)\\(h_2,c_1)&(h_2,c_1)&(h_2,c_2)&(h_3,c_2)\end{bmatrix},$$

$$M^{(4)}=\begin{bmatrix}(h_2,c_2)&(h_2,c_2)&(h_2,c_2)&(h_2,c_1)\\(h_2,c_1)&(h_2,c_2)&(h_2,c_1)&(h_2,c_1)\\(h_2,c_1)&(h_2,c_2)&(h_2,c_1)&(h_2,c_2)\end{bmatrix},$$

$$M^{(5)}=\begin{bmatrix}(h_2,c_1)&(h_2,c_2)&(h_2,c_1)&(h_1,c_1)\\(h_2,c_1)&(h_2,c_1)&(h_1,c_2)&(h_2,c_1)\\(h_2,c_1)&(h_2,c_2)&(h_1,c_2)&(h_2,c_2)\end{bmatrix}.$$

All four attributes in this decision problem are efficiency attributes. The LTV-PIS of each decision matrix can be calculated by Eq. (10) as follows

$$P^{(1)+}=((h_2,c_2),(h_2,c_2),(h_2,c_2),(h_1,c_1)),$$

$$P^{(2)+}=((h_2,c_2),(h_2,c_1),(h_1,c_1),(h_1,c_1)),$$

$$P^{(3)+}=((h_1,c_1),(h_2,c_2),(h_2,c_2),(h_3,c_2)),$$

$$P^{(4)+}=((h_2,c_2),(h_2,c_2),(h_2,c_2),(h_2,c_2)),$$

$$P^{(5)+}=((h_2,c_1),(h_2,c_2),(h_2,c_1),(h_1,c_1)).$$

The LTV-PIS is determined by Eq. (11) as follows

$$P^+=((h_1,c_1),(h_2,c_2),(h_1,c_1),(h_3,c_2)).$$

Similarly, the LTV-NIS of each decision matrix can be calculated by Eq. (12) as follows

$$P^{(1)-}=((h_2,c_1),(h_2,c_1),(h_1,c_2),(h_2,c_1)),$$

$$P^{(2)-}=((h_2,c_1),(h_1,c_2),(h_3,c_1),(h_2,c_1)),$$

$$P^{(3)-}=((h_2,c_1),(h_1,c_2),(h_2,c_2),(h_2,c_1)),$$

$$P^{(4)-}=((h_2,c_1),(h_2,c_2),(h_2,c_1),(h_2,c_1)),$$

$$P^{(5)-}=((h_2,c_1),(h_2,c_1),(h_1,c_2),(h_2,c_1)).$$

The LTV-NIS is determined by Eq. (13) as follows

$$P^-=((h_2,c_1),(h_1,c_2),(h_3,c_1),(h_2,c_1)).$$

The five decision matrices are transformed into three multi-expert linguistic formal contexts $((E,\mathbb{A},\mathbb{I})^1$, $(E,\mathbb{A},\mathbb{I})^2$, $(E,\mathbb{A},\mathbb{I})^3)$ as shown in Tables 4–6. Linguistic evaluation information on the project is collected from all experts through the multi-expert linguistic formal context corresponding to each project.

To aggregate different members' opinions and visualize the project decision-making process through fuzzy linguistic concepts, the corresponding linguistic concept lattices $(L(E,\mathbb{A},\mathbb{I})^1$, $L(E,\mathbb{A},\mathbb{I})^2$, $L(E,\mathbb{A},\mathbb{I})^3)$ are constructed according to the multi-expert linguistic formal contexts as shown in Fig. 6. The concepts in the purple circles are the meet-irreducible elements in the linguistic concept lattice. All fuzzy linguistic concepts and meet-irreducible elements contained in the linguistic concept lattices are shown in Tables 7–9.

Table 4
Multi-expert linguistic formal context $(E, \mathbb{A}, \mathbb{I})^1$ corresponding to project x_1.

E	a_1						a_2						a_3						a_4					
	EH	RL	VH	VL	RH	EL	EH	RL	VH	VL	RH	EL	EH	RL	VH	VL	RH	EL	EH	RL	VH	VL	RH	EL
e_1				×					×						×					×				
e_2			×								×				×					×				
e_3			×						×						×							×		
e_4			×						×						×							×		
e_5				×					×							×				×				

Table 5
Multi-expert linguistic formal context $(E, \mathbb{A}, \mathbb{I})^2$ corresponding to project x_2.

E	a_1						a_2						a_3						a_4					
	EH	RL	VH	VL	RH	EL	EH	RL	VH	VL	RH	EL	EH	RL	VH	VL	RH	EL	EH	RL	VH	VL	RH	EL
e_1				×						×							×				×			
e_2				×						×								×				×		
e_3		×									×				×						×			
e_4				×					×							×						×		
e_5				×						×							×					×		

Table 6
Multi-expert linguistic formal context $(E, \mathbb{A}, \mathbb{I})^3$ corresponding to project x_3.

E	a_1						a_2						a_3						a_4					
	EH	RL	VH	VL	RH	EL	EH	RL	VH	VL	RH	EL	EH	RL	VH	VL	RH	EL	EH	RL	VH	VL	RH	EL
e_1			×							×						×					×			
e_2			×							×				×							×			
e_3				×						×					×				×					
e_4				×					×							×					×			
e_5				×					×								×				×			

Table 7
All fuzzy linguistic concepts contained in $L(E, \mathbb{A}, \mathbb{I})^1$.

Index	Fuzzy linguistic concept	Meet-irreducible element
lc_1	$(E, \emptyset)$	×
lc_2	$(\{e_1, e_2, e_5\}, \{\langle a_4 \text{ is } (h_1, c_1)\rangle\})$	✓
lc_3	$(\{e_1, e_2, e_3, e_4\}, \{\langle a_3 \text{ is } (h_2, c_2)\rangle\})$	✓
lc_4	$(\{e_1, e_2\}, \{\langle a_3 \text{ is } (h_2, c_2)\rangle, \langle a_4 \text{ is } (h_1, c_1)\rangle\})$	×
lc_5	$(\{e_1, e_3, e_4, e_5\}, \{\langle a_2 \text{ is } (h_2, c_2)\rangle\})$	✓
lc_6	$(\{e_1, e_3, e_4\}, \{\langle a_2 \text{ is } (h_2, c_2)\rangle, \langle a_3 \text{ is } (h_2, c_2)\rangle\})$	×
lc_7	$(\{e_1, e_5\}, \{\langle a_1 \text{ is } (h_2, c_1)\rangle, \langle a_2 \text{ is } (h_2, c_2)\rangle, \langle a_4 \text{ is } (h_1, c_1)\rangle\})$	×
lc_8	$(\{e_5\}, \{\langle a_1 \text{ is } (h_2, c_1)\rangle, \langle a_2 \text{ is } (h_2, c_2)\rangle, \langle a_3 \text{ is } (h_2, c_1)\rangle, \langle a_4 \text{ is } (h_1, c_1)\rangle\})$	✓
lc_9	$(\{e_1\}, \{\langle a_1 \text{ is } (h_2, c_1)\rangle, \langle a_2 \text{ is } (h_2, c_2)\rangle, \langle a_3 \text{ is } (h_2, c_2)\rangle, \langle a_4 \text{ is } (h_1, c_1)\rangle\})$	×
lc_{10}	$(\{e_2, e_3, e_4\}, \{\langle a_1 \text{ is } (h_2, c_2)\rangle, \langle a_3 \text{ is } (h_2, c_2)\rangle\})$	✓
lc_{11}	$(\{e_2\}, \{\langle a_1 \text{ is } (h_2, c_2)\rangle, \langle a_2 \text{ is } (h_1, c_2)\rangle, \langle a_3 \text{ is } (h_2, c_2)\rangle, \langle a_4 \text{ is } (h_1, c_1)\rangle\})$	×
lc_{12}	$(\{e_3, e_4\}, \{\langle a_1 \text{ is } (h_2, c_2)\rangle, \langle a_2 \text{ is } (h_2, c_2)\rangle, \langle a_3 \text{ is } (h_2, c_2)\rangle, \langle a_4 \text{ is } (h_2, c_1)\rangle\})$	×
lc_{13}	$(\emptyset, \mathbb{A})$	×

Table 8
All fuzzy linguistic concepts contained in $L(E, \mathbb{A}, \mathbb{I})^2$.

Index	Fuzzy linguistic concept	Meet-irreducible element
lc_1	$(E, \emptyset)$	×
lc_2	$(\{e_1, e_3\}, \{\langle a_4 \text{ is } (h_2, c_2)\rangle\})$	✓
lc_3	$(\{e_1, e_2, e_4, e_5\}, \{\langle a_1 \text{ is } (h_2, c_1)\rangle\})$	✓
lc_4	$(\{e_2, e_4, e_5\}, \{\langle a_1 \text{ is } (h_2, c_1)\rangle, \langle a_4 \text{ is } (h_2, c_1)\rangle\})$	✓
lc_5	$(\{e_1, e_2, e_5\}, \{\langle a_1 \text{ is } (h_2, c_1)\rangle, \langle a_2 \text{ is } (h_2, c_1)\rangle\})$	✓
lc_6	$(\{e_2, e_5\}, \{\langle a_1 \text{ is } (h_2, c_1)\rangle, \langle a_2 \text{ is } (h_2, c_1)\rangle, \langle a_4 \text{ is } (h_2, c_1)\rangle\})$	×
lc_7	$(\{e_2\}, \{\langle a_1 \text{ is } (h_2, c_1)\rangle, \langle a_2 \text{ is } (h_2, c_1)\rangle, \langle a_3 \text{ is } (h_3, c_1)\rangle, \langle a_4 \text{ is } (h_2, c_1)\rangle\})$	✓
lc_8	$(\{e_1, e_5\}, \{\langle a_1 \text{ is } (h_2, c_1)\rangle, \langle a_2 \text{ is } (h_2, c_1)\rangle, \langle a_3 \text{ is } (h_1, c_2)\rangle\})$	✓
lc_9	$(\{e_5\}, \{\langle a_1 \text{ is } (h_2, c_1)\rangle, \langle a_2 \text{ is } (h_2, c_1)\rangle, \langle a_3 \text{ is } (h_1, c_2)\rangle, \langle a_4 \text{ is } (h_2, c_1)\rangle\})$	×
lc_{10}	$(\{e_1\}, \{\langle a_1 \text{ is } (h_2, c_1)\rangle, \langle a_2 \text{ is } (h_2, c_1)\rangle, \langle a_3 \text{ is } (h_1, c_2)\rangle, \langle a_4 \text{ is } (h_2, c_2)\rangle\})$	×
lc_{11}	$(\{e_4\}, \{\langle a_1 \text{ is } (h_2, c_1)\rangle, \langle a_2 \text{ is } (h_2, c_2)\rangle, \langle a_3 \text{ is } (h_2, c_1)\rangle, \langle a_4 \text{ is } (h_2, c_1)\rangle\})$	✓
lc_{12}	$(\{e_3\}, \{\langle a_1 \text{ is } (h_1, c_1)\rangle, \langle a_2 \text{ is } (h_1, c_2)\rangle, \langle a_3 \text{ is } (h_2, c_2)\rangle, \langle a_4 \text{ is } (h_2, c_2)\rangle\})$	✓
lc_{13}	$(\emptyset, \mathbb{A})$	×

Table 9
All fuzzy linguistic concepts contained in $L(E,\mathbb{A},\mathbb{I})^3$.

Index	Fuzzy linguistic concept	Meet-irreducible element
lc_1	$(E,\emptyset)$	×
lc_2	$(\{e_1,e_2,e_4,e_5\},\{\langle a_4 \text{ is } (h_2,c_2)\rangle\})$	✓
lc_3	$(\{e_1,e_4\},\{\langle a_3 \text{ is } (h_2,c_1)\rangle,\langle a_4 \text{ is } (h_2,c_2)\rangle\})$	✓
lc_4	$(\{e_1,e_2,e_3\},\{\langle a_2 \text{ is } (h_2,c_1)\rangle\})$	✓
lc_5	$(\{e_3,e_4,e_5\},\{\langle a_1 \text{ is } (h_2,c_1)\rangle\})$	✓
lc_6	$(\{e_3\},\{\langle a_1 \text{ is } (h_2,c_1)\rangle,\langle a_2 \text{ is } (h_2,c_1)\rangle,\langle a_3 \text{ is } (h_2,c_2)\rangle,\langle a_4 \text{ is } (h_3,c_2)\rangle\})$	×
lc_7	$(\{e_4,e_5\},\{\langle a_1 \text{ is } (h_2,c_1)\rangle,\langle a_2 \text{ is } (h_2,c_2)\rangle,\langle a_4 \text{ is } (h_2,c_2)\rangle\})$	×
lc_8	$(\{e_5\},\{\langle a_1 \text{ is } (h_2,c_1)\rangle,\langle a_2 \text{ is } (h_2,c_2)\rangle,\langle a_3 \text{ is } (h_1,c_2)\rangle,\langle a_4 \text{ is } (h_2,c_2)\rangle\})$	✓
lc_9	$(\{e_4\},\{\langle a_1 \text{ is } (h_2,c_1)\rangle,\langle a_2 \text{ is } (h_2,c_2)\rangle,\langle a_3 \text{ is } (h_2,c_1)\rangle,\langle a_4 \text{ is } (h_2,c_2)\rangle\})$	×
lc_{10}	$(\{e_1,e_2\},\{\langle a_1 \text{ is } (h_2,c_2)\rangle,\langle a_2 \text{ is } (h_2,c_1)\rangle,\langle a_4 \text{ is } (h_2,c_2)\rangle\})$	×
lc_{11}	$(\{e_1\},\{\langle a_1 \text{ is } (h_2,c_2)\rangle,\langle a_2 \text{ is } (h_2,c_1)\rangle,\langle a_3 \text{ is } (h_2,c_1)\rangle,\langle a_4 \text{ is } (h_2,c_2)\rangle\})$	×
lc_{12}	$(\{e_2\},\{\langle a_1 \text{ is } (h_2,c_2)\rangle,\langle a_2 \text{ is } (h_2,c_1)\rangle,\langle a_3 \text{ is } (h_1,c_1)\rangle,\langle a_4 \text{ is } (h_2,c_2)\rangle\})$	✓
lc_{13}	$(\emptyset,\mathbb{A})$	×

Table 10
The distances between the intents of the meet-irreducible elements contained in $L(E,\mathbb{A},\mathbb{I})^1$ and $P^+(P^-)$.

Index	Intent	$d_w(W,P^+)^1$	$d_w(W,P^-)^1$
lc_2	$\{\langle a_4 \text{ is } (h_1,c_1)\rangle\}$	1	1
lc_3	$\{\langle a_3 \text{ is } (h_2,c_2)\rangle\}$	1	2
lc_5	$\{\langle a_2 \text{ is } (h_2,c_2)\rangle\}$	0	1
lc_8	$\{\langle a_1 \text{ is } (h_2,c_1)\rangle,\langle a_2 \text{ is } (h_2,c_2)\rangle,\langle a_3 \text{ is } (h_2,c_1)\rangle,\langle a_4 \text{ is } (h_1,c_1)\rangle\}$	0.75	0.75
lc_{10}	$\{\langle a_1 \text{ is } (h_2,c_2)\rangle,\langle a_3 \text{ is } (h_2,c_2)\rangle\}$	0	1.5

Table 11
The distances between the intents of the meet-irreducible elements contained in $L(E,\mathbb{A},\mathbb{I})^2$ and $P^+(P^-)$.

Index	Intent	$d_w(W,P^+)^2$	$d_w(W,P^-)^2$
lc_2	$\{\langle a_4 \text{ is } (h_2,c_2)\rangle\}$	1	1
lc_3	$\{\langle a_1 \text{ is } (h_2,c_1)\rangle\}$	1	0
lc_4	$\{\langle a_1 \text{ is } (h_2,c_1)\rangle,\langle a_4 \text{ is } (h_2,c_1)\rangle\}$	1.5	0
lc_5	$\{\langle a_1 \text{ is } (h_2,c_1)\rangle,\langle a_2 \text{ is } (h_2,c_1)\rangle\}$	1	0
lc_7	$\{\langle a_1 \text{ is } (h_2,c_1)\rangle,\langle a_2 \text{ is } (h_2,c_1)\rangle,\langle a_3 \text{ is } (h_3,c_1)\rangle,\langle a_4 \text{ is } (h_2,c_1)\rangle\}$	1.5	0
lc_8	$\{\langle a_1 \text{ is } (h_2,c_1)\rangle,\langle a_2 \text{ is } (h_2,c_1)\rangle,\langle a_3 \text{ is } (h_1,c_2)\rangle\}$	1	0.33
lc_{11}	$\{\langle a_1 \text{ is } (h_2,c_1)\rangle,\langle a_2 \text{ is } (h_2,c_2)\rangle,\langle a_3 \text{ is } (h_2,c_1)\rangle,\langle a_4 \text{ is } (h_2,c_1)\rangle\}$	1	0.5
lc_{12}	$\{\langle a_1 \text{ is } (h_1,c_1)\rangle,\langle a_2 \text{ is } (h_1,c_2)\rangle,\langle a_3 \text{ is } (h_2,c_2)\rangle,\langle a_4 \text{ is } (h_2,c_2)\rangle\}$	0.5	1

Table 12
The distances between the intents of the meet-irreducible elements contained in $L(E,\mathbb{A},\mathbb{I})^3$ and $P^+(P^-)$.

Index	Intent	$d_w(W,P^+)^3$	$d_w(W,P^-)^3$
lc_2	$\{\langle a_4 \text{ is } (h_2,c_2)\rangle\}$	1	1
lc_3	$\{\langle a_3 \text{ is } (h_2,c_1)\rangle,\langle a_4 \text{ is } (h_2,c_2)\rangle\}$	1	1
lc_4	$\{\langle a_2 \text{ is } (h_2,c_1)\rangle\}$	1	0
lc_5	$\{\langle a_1 \text{ is } (h_2,c_1)\rangle\}$	1	0
lc_8	$\{\langle a_1 \text{ is } (h_2,c_1)\rangle,\langle a_2 \text{ is } (h_2,c_2)\rangle,\langle a_3 \text{ is } (h_1,c_2)\rangle,\langle a_4 \text{ is } (h_2,c_2)\rangle\}$	0.75	0.75
lc_{12}	$\{\langle a_1 \text{ is } (h_2,c_2)\rangle,\langle a_2 \text{ is } (h_2,c_1)\rangle,\langle a_3 \text{ is } (h_1,c_1)\rangle,\langle a_4 \text{ is } (h_2,c_2)\rangle\}$	0.5	1

According to Eqs. (14) and (16), the distances between the intents of the meet-irreducible elements contained in the linguistic concept lattice and the LTV-PIS (LTV-NIS) are calculated as shown in Tables 10–12, respectively.

The average value of the distances between the intents and the LTV-PIS is as follows.

$$d_w(W,P^+)^1_{avg} = 0.3500,$$

$$d_w(W,P^+)^2_{avg} = 1.0625,$$

$$d_w(W,P^+)^3_{avg} = 0.8750.$$

The average value of the distances between the intents and the LTV-NIS is as follows.

$$d(W,P^-)^1_{avg} = 1.2500,$$

$$d(W,P^-)^2_{avg} = 0.3537,$$

$$d(W,P^-)^3_{avg} = 0.6250.$$

Note that since the range of distances between the evaluative linguistic expressions (h_k,c_l) and (h_p,c_q) can be expressed as

$$0 \le d((h_k,c_l),(h_p,c_q)) \le n.$$

Therefore, the distance between the intents of the meet-irreducible elements contained in the linguistic concept lattice and the LTV-PIS (LTV-NIS) satisfies

$$0 \le d(W,P^+) \le pn, 0 \le d(W,P^-) \le pn.$$

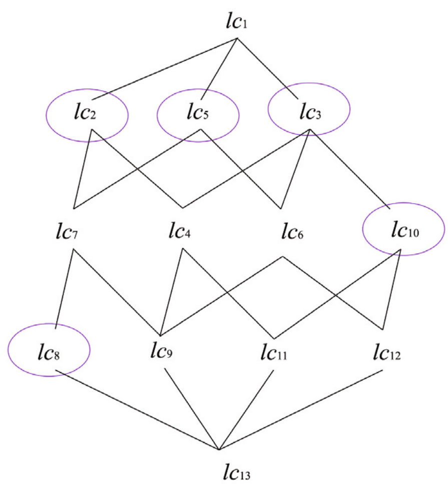
The closeness coefficient $C(x_s)$ for each project can be obtained according to Eq. (18) as follows.

$$C(x_1) = 0.7813, C(x_2) = 0.2497, C(x_3) = 0.4166.$$

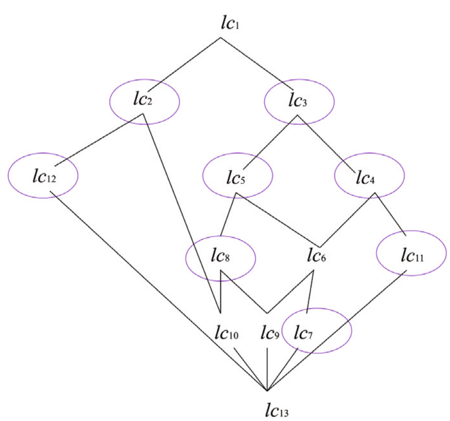
By comparing the closeness coefficient of all projects $x_s (s = 1,2,3)$, the priority of the projects can be obtained as $x_1 \succ x_3 \succ x_2$. Therefore, the project x_1 should be selected as the optimal alternative.

5.3. Parameter sensitivity analysis

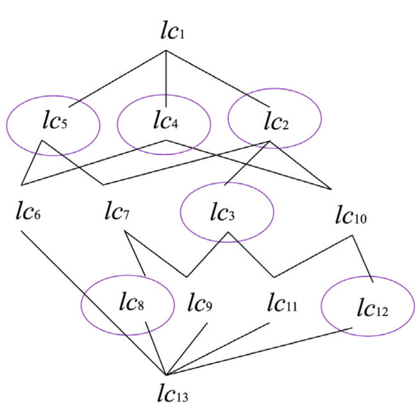
To consider the possibility of different DMs presenting personalized individual semantics (Li et al., 2017, 2018; Liang et al., 2020; Zhang, Li,



(a) Linguistic concept lattice $L(E, \mathbb{A}, \mathbb{I})^1$



(b) Linguistic concept lattice $L(E, \mathbb{A}, \mathbb{I})^2$



(c) Linguistic concept lattice $L(E, \mathbb{A}, \mathbb{I})^3$

Fig. 6. Linguistic concept lattices constructed from multi-expert linguistic formal contexts.

et al., 2019) in a linguistic context, inspired by Li et al. (2017), Pang et al. (2023), we introduce a hyperparameter called fuzzy linguistic-valued trust degree λ to capture the different semantics of different DMs. In this subsection, for the enterprise project selection problem, we explore the fuzzy linguistic-valued trust degree λ to analyze the decision-making results of this case study and the construction of linguistic concept lattices.

In the following, we add the fuzzy linguistic-valued trust degree λ in Step 3 and set the hyperparameter λ as $\lambda = \{(h_3, c_1), (h_1, c_2), (h_2, c_1), (h_2, c_2), (h_1, c_1), (h_3, c_2)\}$. According to our proposed method, the ranking results based on different linguistic expressions of λ are obtained and represented by Fig. 7.

From Fig. 7, it is clear that the ranking results of the alternatives do not change when the two fuzzy linguistic-valued trust degrees is not comparable. As λ gradually increases, the ranking results of all alternatives change. Specifically, as λ changes from (h_2, c_1) to (h_2, c_2), x_1 decreases from the first to the second ranking and x_3 increases from the second to the first ranking. As λ changes from (h_1, c_1) to (h_3, c_2), x_3 is ranked the same as x_2.

We analyze the effect of fuzzy linguistic-valued trust degree on the construction of linguistic concept lattices in the decision-making process as shown in Table 13.

As listed in Table 13, when two fuzzy linguistic-valued trust degree λ are the same, the linguistic concept lattice corresponding to each alternative is the same. The structure of the linguistic concept lattice is gradually simplified as λ keeps increasing, which indicates an increase in linguistic granularity, reflecting the uncertainty characterizing the linguistic preferences of different experts.

5.4. Comparative analysis and discussion

In this subsection, we analyze our proposed approach in comparison with existing approaches and compare the complexity of different approaches. The former is to illustrate the effectiveness of the proposed approach, while the latter is to illustrate the advantages of the proposed model in reducing computational complexity.

5.4.1. A comparison analysis with existing MAGDM approaches

We compare the proposed approach with the existing MAGDM approaches as shown in Table 14. The ranking of alternatives is consistent with the ranking results obtained by the approach proposed by Pang et al. (2016). This shows the correctness and validity of our proposed approach.

As listed in Table 14, we can draw the following conclusions on four dimensions.

- PIS and NIS: In calculating PIS and NIS, Pang's approach (Pang et al., 2016) uses virtual linguistic terms (Liao et al., 2014) for calculation, and the obtained results have no actual semantics. Xu's approach (Xu & Zhang, 2013) uses the HFLTS-based TOPSIS method for decision making. In order to make all HFLTSs have the same number of linguistic expressions, we extend the HFLTSs with relatively few linguistic expressions. The extended linguistic expressions are the smallest linguistic expressions in the original HFLTS, which in a certain way will change the linguistic preferences of the experts. The Fu's approach (Fu et al., 2023) makes decisions based on the TOPSIS method of hesitant fuzzy β-covering rough set models, which can deal with hesitant fuzzy information without needing additional information outside the dataset. However, the approach can only deal with numerical information and cannot deal with the uncertainty of the linguistic expression itself to get the optimal ranking result. The proposed approach uses evaluative linguistic expressions for calculation. The LTV-LIA can handle both comparable and incomparable linguistic information. The evaluative linguistic expressions are more interpretable than the virtual linguistic terms.
- The distance (similarity) between each alternative and the PIS (NIS): After aggregating all expert information, Pang's approach (Pang et al., 2016) and Xu's approach (Xu & Zhang, 2013) directly calculate the distance between each alternative and the

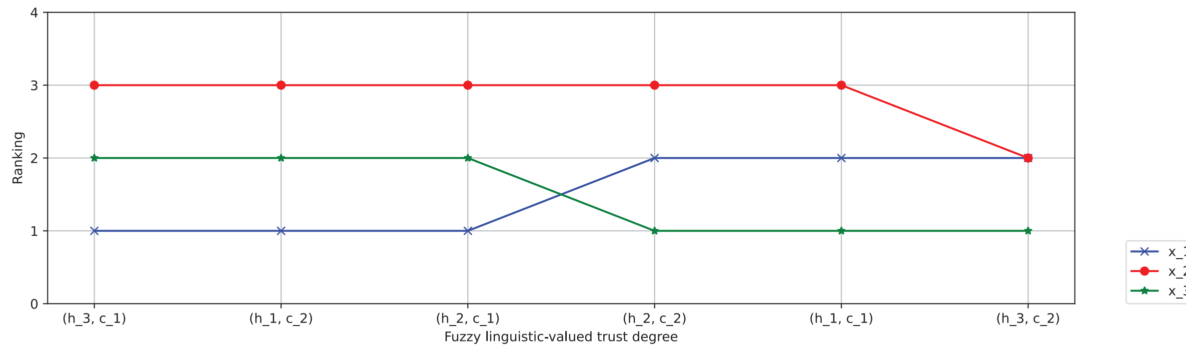
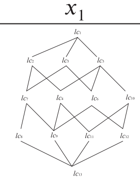
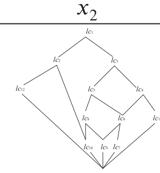
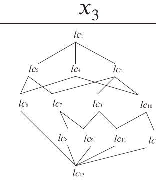
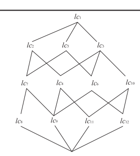
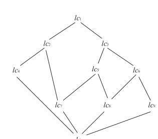
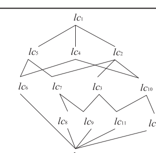
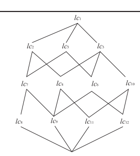
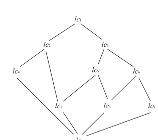
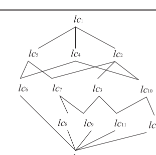
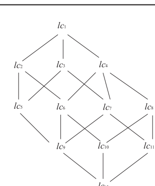
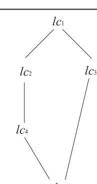
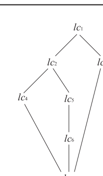
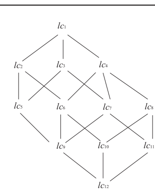
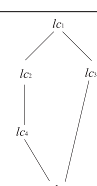
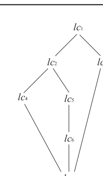


Fig. 7. The ranking results with different fuzzy linguistic-valued trust degree λ.

Table 13
The effect of λ on the construction of linguistic concept lattices.

λ	x_1	x_2	x_3
(h_3, c_1)			
(h_1, c_2)			
(h_2, c_1)			
(h_2, c_2)			
(h_1, c_1)			
(h_3, c_2)			

PIS (NIS). Fu's approach (Fu et al., 2023) proposes hesitation fuzzy similarity, which can only calculate the similarity between each alternative and the PIS (NIS), and the obtained similarity calculation results are hesitation fuzzy elements.

Table 14
Quantitative comparison of MAGDM approaches.

Dimension	Pang's approach (Pang et al., 2016)	Xu's approach (Xu & Zhang, 2013)	Fu's approach (Fu et al., 2023)	Our approach
PIS	$P^{+}=(\{s_{2.4},s_{1.6},s_0\},\{s_{3.2},s_1,s_{0.5}\},\{s_{3.2},s_{1.32},s_{0.99}\},\{s_{3.2},s_{1.2},s_0\})$	$P^{+}=(\{s_5,s_3,s_3\},\{s_4,s_3,s_3\},\{s_5,s_4,s_3\},\{s_6,s_4,s_4\})$	$P^{+}=(\{0.5,0.3,0.3\},\{0.4,0.3,0.3\},\{0.5,0.4,0.3\},\{0.6,0.4,0.4\})$	$P^{+}=((h_1,c_1),(h_2,c_2),(h_1,c_1),(h_3,c_2))$
NIS	$P^{-}=(\{s_{1.8},s_1,s_0\},\{s_{1.5},s_{0.4},s_0\},\{s_1,s_{0.6},s_0\},\{s_{2.4},s_{0.8},s_0\})$	$P^{-}=(\{s_4,s_3,s_3\},\{s_4,s_2,s_2\},\{s_3,s_2,s_1\},\{s_4,s_3,s_3\})$	$P^{-}=(\{0.4,0.3,0.3\},\{0.4,0.2,0.2\},\{0.3,0.2,0.1\},\{0.4,0.3,0.3\})$	$P^{-}=((h_2,c_1),(h_1,c_2),(h_3,c_1),(h_2,c_1))$
The distance (similarity) between each alternative and the PIS	$d(x_1,P^{+})=0.479$, $d(x_2,P^{+})=0.993$, $d(x_3,P^{+})=0.608$	$d(x_1,P^{+})=0.843$, $d(x_2,P^{+})=1.337$, $d(x_3,P^{+})=0.065$	$s(x_1,P^{+})=\{1,1,0.7459\}$, $s(x_2,P^{+})=\{1,1,1\}$, $s(x_3,P^{+})=\{1,1,1\}$	–
The distance (similarity) between each alternative and the NIS	$d(x_1,P^{-})=0.935$, $d(x_2,P^{-})=0.225$, $d(x_3,P^{-})=0.623$	$d(x_1,P^{-})=0.745$, $d(x_2,P^{-})=0.175$, $d(x_3,P^{-})=1.382$	$s(x_1,P^{-})=\{1,1,1\}$, $s(x_2,P^{-})=\{1,1,1\}$, $s(x_3,P^{-})=\{1,1,1\}$	–
The average of the distances between the intents of all fuzzy linguistic concepts corresponding to each alternative and the PIS	–	–	–	$d(W,P^{+})^{1}_{avg}=0.3500$, $d(W,P^{+})^{2}_{avg}=1.0625$, $d(W,P^{+})^{3}_{avg}=0.8750$
The average of the distances between the intents of all fuzzy linguistic concepts corresponding to each alternative and the NIS	–	–	–	$d(W,P^{-})^{1}_{avg}=1.2500$, $d(W,P^{-})^{2}_{avg}=0.3537$, $d(W,P^{-})^{3}_{avg}=0.6250$
Closeness coefficient for each alternative	$C(x_1)=0$, $C(x_2)=-1.8$, $C(x_3)=-0.6$	$C(x_1)=-12.43$, $C(x_2)=-20.4$, $C(x_3)=0$	$C(x_1)=1$, $C(x_2)=1$, $C(x_3)=1$	$C(x_1)=0.7813$, $C(x_2)=0.2497$, $C(x_3)=0.4166$
Ranking of alternatives	$x_1 \succ x_3 \succ x_2$	$x_3 \succ x_1 \succ x_2$	$x_1 \approx x_3 \approx x_2$	$x_1 \succ x_3 \succ x_2$

The proposed approach does not require direct calculation of the distance between the alternative and the PIS (NIS). Since the fuzzy linguistic concepts corresponding to each alternative can aggregate common information from different experts' opinions, the proposed approach can obtain the ranking results by calculating the distance between the intents of the fuzzy linguistic concepts corresponding to each alternative and the PIS (NIS).

- Closeness coefficient for each alternative: Pang's method (Pang et al., 2016) and Xu's approach (Xu & Zhang, 2013) calculate the closeness coefficient of each alternative by considering the distance between each alternative and the PIS (NIS) together. Fu's approach (Fu et al., 2023) calculates the closeness coefficient of each alternative by considering the similarity between each alternative and the PIS (NIS). The proposed approach calculates the closeness coefficient by obtaining the meet-irreducible elements in the linguistic concept lattice and integrating the distance between the intent of the fuzzy linguistic concept and the PIS (NIS).
- Ranking results of alternatives: The ranking results of our proposed method and the other three MAGDM approaches are not exactly the same, which is because different MAGDM approaches have different ranking principles. The ranking result of Xu's approach (Xu & Zhang, 2013) is $x_3 \succ x_1 \succ x_2$, which is different from the ranking result of our proposed approach since the comparable and incomparable information between linguistic expressions cannot be reflected when using HFLTSs, which leads to different decision results. In addition, different HFLTSs have different numbers of linguistic terms, and we will change the original information when expanding the linguistic terms. The reason Fu's approach (Fu et al., 2023) cannot rank these three alternatives is that Fu's approach can handle expert opinions in a hesitant fuzzy environment, but cannot handle uncertainty in linguistic expressions. Our proposed approach and Pang's approach (Pang et al., 2016) obtain the same ranking of alternatives and the same optimal alternative x_1. Compared with Fu's approach and Xu's approach, our proposed method is more effective and feasible.

5.4.2. A comparison analysis with expert opinion aggregation methods

To validate the effectiveness of our proposed expert opinion aggregation method, we consider different linguistic aggregation operators, i.e., the PLWA operator (Pang et al., 2016), the LTV-LIAWAA operator (Diao et al., 2022), Min_upper operator (Rodriguez et al., 2011) and the LIFFAA operator (Liu et al., 2020). We apply our proposed linguistic concept lattice-based expert opinion aggregation method and the above linguistic aggregation operators to the case study, and the ranking results of the corresponding alternatives are shown in Table 15.

Table 15
Comparison of alternative ranking results for different aggregation methods.

Aggregation method	x_1	x_2	x_3	The optimal project
PLWA	1	3	2	x_1
LTV-LIAWAA	2	3	1	x_3
Min_upper	2	3	1	x_3
LIFFAA	2	3	1	x_3
Our proposed method (Case 1)[a]	1	3	2	x_1
Our proposed method (Case 2)[b]	1	3	2	x_1

[a] Considering all fuzzy linguistic concepts.
[b] Considering meet-irreducible elements in a linguistic concept lattice.

As listed in Table 15, our proposed aggregation method has the same ranking results as the PLWA operator, thus proving the effectiveness of the proposed aggregation method. The comparison between our proposed aggregation method and other linguistic aggregation methods is as follows:

1. The LTV-LIAWAA and the LIFFAA are two LTV-LIA-based linguistic aggregation operators that differ from the ranking results of the alternatives to our proposed aggregation method. Similar to LTV-LIAWAA and LIFFAA, our proposed method uses LTV-LIA to represent experts' linguistic evaluation information. There exists a loss of information in obtaining the ranking results of the alternatives since LTV-LIAWAA and LIFFAA use rounding $round(\cdot)$ and integrating $INT(\cdot)$ in aggregating the different expert opinions, respectively. Our proposed method aggregates the common information of expert opinions by forming different fuzzy linguistic concepts without involving approximation operations.
2. The ranking results of the alternatives obtained by considering all fuzzy linguistic concepts corresponding to each alternative

Table 16
The running time (s) of the second part.

λ	Case 1[a]			Case 2[b]		
	x_1	x_2	x_3	x_1	x_2	x_3
(h_3, c_1)	0.6388	0.6173	0.4711	0.2984	0.3278	0.3894
(h_1, c_2)	0.8541	0.5872	0.5169	0.3098	0.3987	0.4192
(h_2, c_1)	0.4582	0.6648	0.4523	0.3869	0.4258	0.3212
(h_2, c_2)	1.0642	0.7963	0.8211	0.2741	0.1109	0.4306
(h_1, c_1)	1.0585	0.8431	0.7742	0.3186	0.1682	0.2874
(h_3, c_2)	0.8803	0.6919	0.6811	<0.01	<0.01	<0.01

[a] Considering all fuzzy linguistic concepts.
[b] Considering meet-irreducible elements in a linguistic concept lattice.

(Case 1) and the meet-irreducible elements of the linguistic concept lattice corresponding to each alternative (Case 2) are the same. This indicates that using meet-irreducible elements in the linguistic concept lattice reduces computational complexity while reducing information loss.

3. Min_upper is a linguistic aggregation operator based on HFLTS, and its ranking results for the alternatives are different from the ranking results of our proposed aggregation method. The adoption of Min_upper needs to fully utilize the original linguistic information provided by the experts and thus may produce distorted decision results. Min_upper needs to apply the upper bound of each HFLTS and obtain the minimum linguistic terms for the attribute set of each alternative when aggregating the expert opinions, which causes the problem of information loss. Our proposed method not only considers comparable and incomparable linguistic information, but also eliminates the need for approximation operations on linguistic expressions.

5.4.3. Time complexity analysis

The proposed approach is divided into three parts. The first part is data preprocessing, which spends lots of time and we need to process data manually. This step is a preparation for ranking alternatives. We will analyze the complexity of our proposed approach in terms of the construction of the linguistic concept lattice and the ranking of alternatives in second part and third part.

Suppose that there are r experts and p attributes in the MAGDM problem and that the linguistic representation is modeled as a $2n$-element LTV-LIA. For each alternative, the time complexity of obtaining all fuzzy linguistic concepts is $O(2^{2np})$ when the meet-irreducible elements of the linguistic concept lattice are not considered. The time complexity of obtaining all fuzzy linguistic concepts is $O(2n \cdot p \cdot r)$ if only the meet-irreducible elements of the linguistic concept lattice are considered. In case study, $r = 5$, $p = 4$, the linguistic representation model used is 6-element LTV-LIA, and the computing time of the second part is shown in Table 16.

As listed in Table 16, the running time required to consider the meet-irreducible elements in the linguistic concept lattice corresponding to each alternative is shorter than the running time required to consider all fuzzy linguistic concepts corresponding to each alternative.

In the third part, it is assumed that there are a total of o alternatives, the number of all fuzzy linguistic concepts corresponding to each alternative is c, and the number of meet-irreducible elements in the linguistic concept lattice corresponding to each alternative is κ. The time complexity of ranking alternatives is $O(o \cdot p \cdot \kappa)$ when only the meet-irreducible elements in each alternative's corresponding linguistic concept lattice are considered. Considering all the fuzzy linguistic concepts corresponding to each alternative, the time complexity of ranking alternatives is $O(o{\cdot}p{\cdot}c)$. Since $\kappa \leq c$, we have $o{\cdot}p{\cdot}\kappa \leq o{\cdot}p{\cdot}c$. As a result, the time complexity of using all fuzzy linguistic concept ranking alternatives is higher than the time complexity of ranking alternatives using the meet-irreducible elements in the linguistic concept lattice.

The comparison of time complexity with other MAGDM approaches is as follows:

1. The time complexity of Fu's approach (Fu et al., 2023) in ranking alternatives is $O(o^2 + op)$. In constructing the optimal decision object H^+ and the worst decision object H^- respectively, Fu's approach needs to compute the upper and lower approximations of H^+ and H^-, which results in high computational complexity because Fu's approach is based on the hesitant fuzzy β-covering rough set. Our proposed approach does not need to find the optimal and worst decision objects. Our proposed approach only needs to compute LTV-PIS and LTV-NIS based on LTV-LIA.
2. The time complexity required by Peng's approach (Peng et al., 2022) in ranking alternatives is $O(o^2 p)$. In the MAGDM process, Peng's approach needs to set the similarity threshold L and compute the L-level probabilistic similarity class. Our proposed approach does not need to set a similarity threshold. It can obtain public information about expert opinions through the extents of fuzzy linguistic concepts and provide semantic interpretations of expert opinion aggregation based on the intents of fuzzy linguistic concepts.
3. Wang's approach (Wang, Zhan, et al., 2023) requires the time complexity of $O(o^3 p)$ in ranking the alternatives. The main computational complexity of the approach comes from obtaining a priori probability tolerance dominance classes for each alternative. Our proposed approach does not require prior probability tolerant dominance relations to deal with the binary relationships between the evaluation values. After obtaining the decision matrices, our proposed approach handles the relationship between each alternative and attribute by converting the decision matrices into multi-expert linguistic formal contexts.

5.5. Further discussion of the effectiveness of the proposed approach

Table 17 further shows the difference between our proposed approach and existing MAGDM approaches. As listed in Table 17, existing MAGDM approaches are cited to illustrate the strength of our proposed approach on two aspects.

(1) *Differences in linguistic representation models*

1. Most of the existing methods are based on linguistic term sets when dealing with the linguistic evaluation information of experts in MAGDM problems. Herrera's approach (Herrera & Martínez, 2000) reduces the information loss in obtaining ranking results by expanding linguistic terms into a 2-tuple linguistic model. Xu's approach (Xu & Zhang, 2013) considers the situation where an expert would hesitate between several linguistic terms when evaluating alternatives. Rao's approach (Rao et al., 2022) and Akram's approach (Akram, Bibi, & Deveci, 2023) convert the linguistic variables into dual uncertain Z-number and 2-tuple linguistic Fermatean fuzzy sets, respectively. The essence of these approaches is to apply linguistic symbolic models to linguistic evaluation information. The linguistic symbolic model cannot handle the ambiguity of linguistic expressions.
2. Fan's approach (Fan et al., 2022) uses flexible linguistic expressions to represent expert evaluation information, which can effectively deal with the ambiguity of linguistic expressions. Garg's approach (Garg & Kumar, 2019) and Meng's approach (Meng et al., 2016) use linguistic interval-valued Atanassov intuitionistic fuzzy set and linguistic interval hesitant fuzzy set, respectively, to represent the expert's linguistic evaluation information. These two approaches can effectively reflect the uncertainty and inconsistency of experts in the decision-making process. However, the above approaches have difficulty in dealing with the incomparable linguistic knowledge prevalent in natural languages.

Table 17
Qualitative comparison of MAGDM approaches.

Approach	Linguistic representation model	Visualization of the MAGDM process
Fan et al. (2022)	Flexible linguistic expression	No
Herrera and Martínez (2000)	2-tuple linguistic model	No
Rao et al. (2022)	Dual uncertain Z-number	No
Akram, Bibi, and Deveci (2023)	2-tuple linguistic Fermatean fuzzy set	No
Xu and Zhang (2013)	Hesitant fuzzy linguistic term set	No
Gou et al. (2017)	Double hierarchy hesitant fuzzy linguistic term set	No
Garg and Kumar (2019)	Linguistic interval-valued Atanassov intuitionistic fuzzy set	No
Meng et al. (2016)	Linguistic interval hesitant fuzzy set	No
Wang and Wang (2022)	Linguistic term with weakened hedge	No
Our approach	Linguistic truth-valued lattice implication algebra	Yes

3. By taking advantage of lattice implication algebra, one can better perform decision-making with incomparable elements. Therefore, our proposed approach uses lattice implication algebra, which is applied to represent imprecise information and deal with both comparable and incomparable linguistic information.

Compared with the double hierarchy hesitant fuzzy linguistic term set (DHHFLTS) (Gou et al., 2017) and linguistic term with weakened hedge (LTWH) (Wang & Wang, 2022; Wang et al., 2018a), the LTV-LIA is an algebra model with linguistic terms based on a logical algebraic structure with the following advantages.

1. For a LTWH, it begins with a linguistic term modified by a weakened hedge. DHHFLTS allows for a more accurate and comprehensive description of the hesitancy of linguistic information by means of a dual hierarchy of linguistic terms. In LTV-LIA, hedges are used to weaken the true or false degree. There are incomparable linguistic expressions in the true and false chains in the lattice implication algebra.
2. Unlike LTWH and DHHFLTS, in LTV-LIA, the semantics of linguistic expressions is embodied in the algebraic structure, making linguistic expressions processed in the logic system not only symbolic but also have the semantic properties of natural language.

(2) *Visualization of the MAGDM process*
For each alternative, the proposed approach can visualize the decision process of all experts by constructing a linguistic concept lattice, which improves the interpretability of the decision approach.

5.6. Managerial insights

When a company's board of directors selects large-scale projects to develop over the next five years, it usually weighs the pros and cons of implementing each project from different aspects to determine the strategic direction of the company's stage-by-stage positioning and goals. Based on the results of the calculations and the decision-making process, we can conclude the following recommendations to the company's board of directors:

1. In this case study, the best project is x_1. As shown by the multi-expert linguistic formal context $(E, \mathbb{A}, \mathbb{I})^1$, most experts rated x_1 better in the perspective of learning and growth as well as customer satisfaction. This indicates that the company will need to implement project x_1 to ensure high customer satisfaction while continuing to learn and revise the project to adapt to market trends.
2. When making a selection of projects, the board of directors can rank projects that consider a combination of attributes and provide the company with a wider range of choices to select the right project based on the risk preferences of the directors.
3. The company should provide an apparent reason for ranking different projects so that all employees know why a specific project was chosen as the strategic plan for the next five years. Analyzing the strengths and weaknesses of different projects will also help to synthesize the strengths of different projects to come up with new projects that are more comprehensive and effective. In contrast to most existing MAGDM methods, our proposed approach can provide the board of directors with reasons for choosing a project by visualizing the decision-making process through linguistic concept lattices.

6. Conclusions

Expert opinion aggregation and processing of linguistic evaluation information from experts play an important role in MAGDM. In this paper, we propose a novel method for dealing with MAGDM in a linguistic environment based on the linguistic concept lattice. Based on the lattice implication algebra, LTV-LIA is used to represent the linguistic evaluation information of experts, and a new distance measure based on LTV-LIA is proposed. The comparative analysis demonstrates that the proposed approach can effectively improve the interpretability in the decision-making process and reduce the information loss arising from the aggregation of expert opinion. In terms of MAGDM, the main advantages of the proposed approach can be described as follows:

1. The proposed approach represents experts as objects and linguistic evaluation information as attributes, and the adoption of LTV-LIA-based multi-expert linguistic formal context facilitates the expression of different linguistic preferences of experts for the same alternative.
2. Since experts come from different fields and have different background knowledge, they may have different opinions about the uncertainty of a decision problem. Considering that the use of aggregation operators to aggregate individual opinions into group opinions causes certain information loss, the proposed approach aggregates expert opinions through fuzzy linguistic concepts, and comparative analyses with existing expert opinion aggregation methods show that the proposed approach reduces the information loss in the aggregation process.
3. Due to the high computational complexity of obtaining all fuzzy linguistic concepts, the proposed approach introduces meet-irreducible elements in the linguistic concept lattice and applies them to the decision-making process, and the time complexity analysis demonstrates that the proposed approach can effectively reduce the computational complexity.
4. The construction of a linguistic concept lattice corresponding to each alternative visualizes the partial order relationships inherent in fuzzy linguistic concepts and improves the interpretability of the proposed approach in the decision-making process.
5. The problem of subjective uncertainty arising from time constraints and experts' domain-specific limitations can be effectively addressed through the calculation of distances between the intent of fuzzy linguistic concepts and the PIS (NIS), thereby

testing for inconsistency between expert preference information and distinct criteria.

As avenues for future exploration, two prominent areas have been identified. Firstly, a pivotal inquiry pertains to devising methodologies that streamline the computation of meet-irreducible elements within the linguistic concept lattice, particularly when confronted with numerous fuzzy linguistic concepts, thus enabling their application to larger-scale MAGDM scenarios. Secondly, while our current study has laid a foundation for MAGDM using linguistic concept lattices, the subsequent phase necessitates a comprehensive investigation into the determination and application of attributes and expert weights within the context of large-scale MAGDM scenarios.

CRediT authorship contribution statement

Kuo Pang: Methodology, Conceptualization, Validation, Formal analysis, Investigation, Writing – original draft. **Luis Martínez:** Methodology, Conceptualization, Software, Supervision, Writing – review & editing. **Nan Li:** Visualization, Validation, Formal analysis, Writing – review & editing. **Jun Liu:** Conceptualization, Formal analysis, Supervision, Writing – review & editing. **Li Zou:** Formal analysis, Validation, Supervision, Writing – review & editing. **Mingyu Lu:** Conceptualization, Supervision, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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