Several Novel Aggregation Functions for PHFS and Their Application to MCGDM

Zhensong Chen
School of Civil Engineering
Wuhan University
Wuhan, China
zschen@whu.edu.cn

Jianpeng Chang
School of Business Planning
Chongqing Technology and Business University
Chongqing, China
jpchang@ctbu.edu.cn

Xuan Zhang
School of Civil Engineering
Wuhan University
Wuhan, China
xuanzhang@whu.edu.cn

Xiaolu Liu
School of Civil Engineering
Wuhan University
Wuhan, China
lxl00074@whu.edu.cn

Min Li
School of Civil Engineering
Wuhan University
Wuhan, China
lm6224@whu.edu.cn

Luis Martínez
Department of Computer Science
Univerisity of Jaén
Jaén, Spain
martin@ujaen.es

***Abstract*—Proportional hesitant fuzzy set (PHFS) is characterized with several membership degrees and their affiliated proportions. It is a well-established extension of the classic notion of hesitant fuzzy set (HFS). This paper aims at developing proportional hesitant fuzzy weighted averaging (PHFWA) operator and its generalized form (the GPHFWA operator), and applying the GPHFWA operator to establish a multiple criteria group decision making (MCGDM) model to address the collective decision-making problem given that the representation information is HFS-based inputs. Eventually, a case study is conducted to illustrate the effectiveness and feasibility of the established MCGDM method.**

***Index Terms*—proportional hesitant fuzzy set, generalized proportional hesitant fuzzy aggregation functions, multiple criteria group decision making, collective evaluation opinions**

I. Introduction

The notion of fuzzy set (FS) is referred to as a grade of elements attached to a continuum of membership grades within the interval [0,1], which is determined through employing membership functions. Since its original proposal by Zadeh [1], FS has attracted appropriable attention from the theory and practical world in view of its outstanding capability of modelling uncertainty. At present, focusing on relieving its limitation that the membership functions allocate only a solo membership degree to the considered elements, a group of literature have generalized FS to type-2 FS [2], intuitionistic FS [3][4], and hesitant FS [5], etc. Among them, hesitant fuzzy set (HFS) is coined to model this case where a decision maker (DM) hesitates among a set of potentially-emerged membership degrees and assigns all the possible degrees to the element (Case 1). In addition, HFS can also be used to model the scenario where there exist several DMs and each gives a different membership degree (Case 2). The two hesitant patterns occur frequently in practical decision-making settings. Thus, HFS has drawn the attention from a lot of researchers and practitioners since its introduction [6]. The previous theoretical researches on HFS majorly focus on the diverse extensions of HFS, information measurements for HFS, and the aggregation functions for the gathering of HFSs [7].

At present, many scholars have put forward various extensions of HFS with the target to depict the uncertainty from different perspectives [7], such as interval-valued hesitant FS [8], hesitant fuzzy linguistic terms set [9]-[12], proportional hesitant fuzzy set (PHFS) [13]. With regard to information measures, the most existing related literature aim at the establishment of distance operations, correlation coefficients, entropy and cross-entropy measures. Moreover, some academics put their core researches on the HFS aggregation operators to facilitate the decision-making efforts, for instance, the proposals of hesitant fuzzy averaging-type [14], the hesitant fuzzy geometric-type [14], hesitant fuzzy power-type [15], hesitant fuzzy geometric Bonferroni-type [16] aggregation operators.

Among various extensions of HFS, PHFS, which is coined by Xiong et al. in [13], is featured with the possible membership function regarding set-based component to a predefined set and their affiliated proportions and performs better in modelling hesitancy and fuzziness owing to the introduction of a new dimension of information. In PHFS, the membership degrees stand for the collective evaluation opinions, and the associated proportions denote the collective preferences of the decision team. Initially, PHFS is used to model the Case 2 in which individual assessment information expressed with FS is fused into a collective opinion with the expression of PHFS[13]. Actually, under hesitant fuzzy context, each DM are intended to output his or her arguments in the form of FS or HFS, and we hold that PHFS can uncertain collective argument which is amalgamated from individual preference opinions expressed with HFS or FS. In the paper [13], Xiong et al. have developed a proportional hesitant normalized Hamming distance, proposed a method for proportional hesitant fuzzy elements (PHFEs) comparison, and constructed a multiple criteria group decision making (MCGDM) approach supported by PHFSs. Nevertheless, existing studies have not paid any attention to the development of aggregation operators for PHFSs, which motivates us to fill the gap. The aims of this paper are to construct averaging operators for PHFSs and to construct an MCGDM method built on the averaging operators for PHFSs. Before achieving these aims, we should give the operational laws for PHFEs. Finally, we conduct a case study aiming at validating the feasibility and effectiveness of the established model.

This paper sets forth as the following structure. In Section II, several essential concepts and extensions of PHFSs that are of significant importance to this work will be reviewed. Section III gives the operational rules for PHFEs. The proportional hesitant fuzzy weighted averaging (PHFWA) operator and its generalized version, the GPHFWA operator, are put forward in Section IV. And the

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GPHFWA is used to develop an MCGDM model under the context of PHFSs in Section V. A case study is conducted in Section VI to validate the practicality of the developed model. Section VII draws the conclusions of the current article.

II. Preliminaries

This part provides several fundamental concepts of PHFSs to lay the foundations of this study.

Definition 1. [13] Assuming that X is a reference set, the PHFS E defined over X is indicated mathematically as below:

$$E=\{\langle x,T_E(x)\rangle | x\in X\}=\{\langle x,(h_E(x),p_E(x))\rangle | x\in X\},$$

where

(a) $h_E(x)=\{\gamma_1,\gamma_2,\cdots,\gamma_n\}$ is a series of values belonging to the interval of $[0,1]$. It implies that n genres of possible membership degrees regarding the element x to the set E; and

(b) $p_E(x)=\{\tau_1,\tau_2,\cdots,\tau_n\}$ is a set of values belonging to $[0,1]$, in which $\tau_i(i=1,2,\cdots,n)$ indicates the proportion vector with respect to the membership degree $\gamma_i(i=1,2,\cdots,n)$, where $\sum_{i=1}^{n}\tau_i=1$.

In order to ease the implementation, Xiong et al. [13] denoted $T=T_E(x)$ as a proportional hesitant fuzzy element (PHFE). They further put forward the score and deviation functions for PHFE.

Definition 2. [13] Let T be a PHFE on the reference set X. The score function of T is given as

$$s(T)=\sum_{(\gamma,\tau)\in T}\gamma\bullet\tau,$$

and the deviation calculation on T is defined by

$$t(T)=\sum_{(\gamma,\tau)\in\beta}\tau(\gamma-s(T))^2.$$

Combing with the aforementioned definitions on PHFEs, Xiong et al. [13] defined a comparison rule on PHFEs as below.

Definition 3. [13] Let T_1 and T_2 be two PHFEs on the reference set X.

(1) if $s(T_1)>s(T_2)$, we obtain $T_1>T_2$;

(2) if $s(T_1)=s(T_2)$ and $t(T_1)<t(T_2)$, we get $T_1>T_2$;

(3) if $s(T_1)=s(T_2)$, $t(T_1)=t(T_2)$,

(a) and $d(\{T_1\},\bigcup)=d(\{T_2\},\bigcup)$, this implies $T_1=T_2$;

(b) and $d(\{T_1\},\bigcup)<d(\{T_2\},\bigcup)$, this derives that $T_1>T_2$.

where $\bigcup$ represent the full PHFS and $d(A,B)$ has been utilized for calculating the distance measurements between arbitrary two PHFSs which has been developed in the work of Xiong et al. [13] .

III. Operational Laws for PHFEs

Let T_1 and T_2 be two PHFEs on the reference set X. It's assumed that the two PHFEs have different reference sets. On the basis of the operation laws defined on HFSs[14], the operation rules for PHFEs are given by this section in the view of probability. Owing to the limited space of the paper, this section only gives the conclusions and omits the proving process.

Definition 4. Let T, T_1 and T_2 be three PHFEs on X, then

(1) $T^{\lambda}=\bigcup_{(\gamma,\tau)\in T}\{(\gamma^{\lambda},\tau)\},\lambda>0$;

(2) $\lambda T=\bigcup_{(\gamma,\tau)\in T}\{(1-(1-\gamma)^{\lambda},\tau)\},\lambda>0$;

(3) $T_1\oplus T_2=\bigcup_{(\gamma_1,\tau_1)\in T_1,(\gamma_2,\tau_2)\in T_2}\{(\gamma_1+\gamma_2-\gamma_1\gamma_2,\tau_1\tau_2)\}$;

(4) $T_1\otimes T_2=\bigcup_{(\gamma_1,\tau_1)\in T_1,(\gamma_2,\tau_2)\in T_2}\{(\gamma_1\gamma_2,\tau_1\tau_2)\}$.

Theorem 1. *For three PHFEs T, T_1 and T_2 on X. The following equations can be obtained:*

(1) $(T^C)^{\lambda}=(\lambda T)^C,\lambda>0$;

(2) $\lambda T^C=(T^{\lambda})^C,\lambda>0$;

(3) $T_1^C\oplus T_2^C=(T_1\otimes T_2)^C$;

(4) $T_1^C\otimes T_2^C=(T_1\oplus T_2)^C$.

Theorem 2. *Let $T_j(j=1,2,\cdots,n)$ be a group of PHFEs. Moreover, $\boldsymbol{\omega}=(\omega_1,\omega_2,\cdots,\omega_n)^{\mathrm{T}}$ represents the weighting vector (WC) regarding $T_j(j=1,2,\cdots,n)$, where $\omega_j\in[0,1]$ and $\sum_{j=1}^{n}\omega_j=1$, $\lambda>0$. And equations in the following can be established:*

(1) $\oplus_{j=1}^{n}(\omega_j T_j^C)=(\otimes_{j=1}^{n}T_j^{\omega_j})^C$;

(2) $\otimes_{j=1}^{n}(T_j^C)^{\omega_j}=(\oplus_{j=1}^{n}\omega_j T_j)^C$;

(3) $\left(\oplus_{j=1}^{n}\left(\omega_j(T_j^C)^{\lambda}\right)\right)^{1/\lambda}=\left(\frac{1}{\lambda}\left(\otimes_{j=1}^{n}(\lambda T_j)^{\omega_j}\right)\right)^C$;

(4) $\frac{1}{\lambda}\left(\otimes_{j=1}^{n}(\lambda T_j^C)^{\omega_j}\right)=\left(\left(\oplus_{j=1}^{n}(\omega_j T_j^{\lambda})\right)^{1/\lambda}\right)^C$.

IV. Novel Averaging Operators for PHFEs

This section gives two averaging operators for PHFEs depending on the operational laws given in section III.

Definition 5. Let $E=\{T_1,T_2,\cdots,T_n\}$ be n PHFEs on X, and suppose Θ is a function with set E, $\Theta:[0,1]^n\to[0,1]$, we have

$$\Theta_E=\bigcup_{(\gamma,\tau)\in\{T_1\times T_2\times\cdots\times T_n\}}\{\Theta(\gamma,\tau)\}.$$

Definition 6. Let $T_j\,(j=1,2,\cdots,n)$ be an assemblage of PHFEs. Then a PHFWA operator defined on $PH^n\to PH$ can be given as below:

$$\text{PHFWA}(T_1,T_2,\cdots,T_n)=\overset{n}{\underset{j=1}{\oplus}}(\omega_j T_j),$$

where $\boldsymbol{\omega}=(\omega_1,\omega_2,\cdots,\omega_n)^{\mathrm{T}}$ is the WC of $T_j\,(j=1,2,\cdots,n)$.

In particular, PHFWA reduces to the proportional hesitant fuzzy averaging (PHFA) operator if all arguments are assigned with equal weights, that is:

$$\text{PHFA}(T_1,T_2,\cdots,T_n)=\overset{n}{\underset{j=1}{\oplus}}\left(\frac{1}{n}T_j\right)$$

Theorem 3. *Let $T_j\,(j=1,2,\cdots,n)$ be a group of PHFEs, $\boldsymbol{\omega}=(\omega_1,\omega_2,\cdots,\omega_n)^{\mathrm{T}}$ be the WC of $T_j\,(j=1,2,\cdots,n)$. Using PHFWA gives the collective value with $T_j\,(j=1,2,\cdots,n)$, which is also a PHFE and*

$$\text{PHFWA}(T_1,T_2,\cdots,T_n)=$$
$$\bigcup_{(\gamma_1,\tau_1)\in T_1,(\gamma_2,\tau_2)\in T_2,\cdots,(\gamma_n,\tau_n)\in T_n}\left\{\left(1-\prod_{j=1}^{n}(1-\gamma_j)^{\omega_j},\prod_{j=1}^{n}\tau_j\right)\right\},$$

in which $0\le\omega_j\le1$, $\sum_{j=1}^{n}\omega_j=1$.

The proving process is omitted due to the limited space.

Theorem 4. *Let $T_j\,(j=1,2,\cdots,n)$ be a group of PHFEs, $\boldsymbol{\omega}=(1/n,1/n,\cdots,1/n)^{\mathrm{T}}$ be the corresponding weighting vector. The collective value obtained by the PHFA remains to be a PHFE, and*

$$\text{PHFA}(T_1,T_2,\cdots,T_n)=$$
$$\bigcup_{(h_1,p_1)\in T_1,(h_2,p_2)\in T_2,\cdots,(h_n,p_n)\in T_n}\left\{\left(1-\prod_{j=1}^{n}(1-h_j)^{1/n},\prod_{j=1}^{n}p_j\right)\right\}.$$

Theorem 5. (Commutativity) *Let $T_j\,(j=1,2,\cdots,n)$ be a group of PHFEs. Supposing that $T_i'(i=1,2,\cdots,n)$ is any permutation of $T_j\,(j=1,2,\cdots,n)$, then we have*

$$\text{PHFA}(T_1,T_2,\cdots,T_n)=\text{PHFA}(T_1',T_2',\cdots,T_n').$$

Nevertheless, PHFA operator fails to satisfy the property of idempotence, boundary as well as monotonicity.

Definition 7. Let $T_j\,(j=1,2,\cdots,n)$ be an assemblage of PHFEs. Then the GPHFWA operator defined mathematically as $PH^n\to PH$ can be given as follows:

$$\text{GPHFWA}_\lambda(T_1,T_2,\cdots,T_n)=\left(\overset{n}{\underset{j=1}{\oplus}}(\omega_j T_j^\lambda)\right)^{1/\lambda},$$

where $\boldsymbol{\omega}=(\omega_1,\omega_2,\cdots,\omega_n)^{\mathrm{T}}$ denotes the WC with regard to $T_j\,(j=1,2,\cdots,n)$.

In particular, then GPHFWA degenerates to the PHFWA given that $\lambda=1$.

Theorem 6. *Let $T_j\,(j=1,2,\cdots,n)$ be a group of PHFEs, and $\boldsymbol{\omega}=(\omega_1,\omega_2,\cdots,\omega_n)^{\mathrm{T}}$ be the WC of $T_j\,(j=1,2,\cdots,n)$. For any $\lambda>0$, the collective outcome obtained by the GPHFWA remains to be a PHFE, and*

$$\text{GPHFWA}_\lambda(T_1,T_2,\cdots,T_n)=$$
$$\bigcup_{(h_1,p_1)\in T_1,\cdots,(h_n,p_n)\in T_n}\left\{\left(\left(1-\prod_{j=1}^{n}(1-h_j^\lambda)^{\omega_j}\right)^{1/\lambda},\prod_{j=1}^{n}p_j\right)\right\},$$

in which $0\le\omega_j\le1$, $\sum_{j=1}^{n}\omega_j=1$, *and* $\lambda>0$.

V. Multiple Criteria Group Decision Making Model Based on PHFSs

All subsequent notations will be used throughout the remaining sections.

- $A=\{A_1,\cdots,A_i,\cdots,A_m\}$: The alternative set under consideration containing m available alternatives, and $i\in M=\{1,2,\cdots,m\}$
- $C=\{C_1,\cdots,C_j,\cdots,C_n\}$: The criterion set containing n independent criteria, and $j\in N=\{1,2,\cdots,n\}$
- $D=\{D_1,\cdots,D_k,\cdots,D_t\}$: The set of DMs consisting of t invited DMs, D_k is the *k*th expert, $k\in T=\{1,2,\cdots,t\}$
- $\boldsymbol{\delta}=(\delta_1,\cdots,\delta_k,\cdots,\delta_t)^T$: The weight vector assigned to the invited DMs, satisfying $\delta_k\in[0,1]$, $\sum_{k=1}^{t}\delta_k=1$, and $k=1,2,\cdots,t$
- $\boldsymbol{\omega}=(\omega_1,\omega_2,\cdots,\omega_n)^T$: The weighting vector assigned to the criteria set, such that $\omega_j\in[0,1]$, $\sum_{j=1}^{n}\omega_j=1$, and $j=1,2,\cdots,n$.
- N_b,N_c: A series of cost criteria and benefit criteria such that $N_b\cup N_c=N$
- s_{ij}^k: Assessment information provided by DM D_k for alternative A_i against criterion C_j, which is expressed in the form of HFS or FS.

Then, the MCGDM based on the GPHFWA operator can be constructed as the next steps:

Step 1. Transform the original individual decision matrix $\boldsymbol{S}^k=(s_{ij}^k)_{m\times n}\,(k\in T)$ into the normalized decision information $\boldsymbol{R}^k=(r_{ij}^k)_{m\times n}\,(k\in T)$, in which

$$r_{ij}^k = \begin{cases} s_{ij}^k, & C_j \in N_b; \\ \left(s_{ij}^k\right)^C, & C_j \in N_c. \end{cases}$$

Step 2. Fuse individual preference opinions into the collective preference opinion by generating PHFS. Given the individual arguments take the form of HFS or FS, we can determine the collective assessment information

$$r_{ij} = \left\{\left(h_{ij}^1, p_{ij}^1\right), \left(h_{ij}^2, p_{ij}^2\right), \cdots, \left(h_{ij}^{\#r_{ij}}, p_{ij}^{\#r_{ij}}\right)\right\},$$

in which $\left\{h_{ij}^1, h_{ij}^2, \cdots, h_{ij}^{\#r_{ij}}\right\}$ is the union of individual information, and $p_{ij}^l \left(l = 1, 2, \cdots, \#r_{ij}\right)$ represents the proportion vector associated with h_{ij}^l and can be calculated by

$$p_{ij}^l = \sum\nolimits_{D_k \in D'} \left(\delta_k / \#r_{ij}^k\right),$$

where D' denotes the collection of DMs whose assessment information contain h_{ij}^l. And δ_k means the significance with regard to the kth DM and $0 \le \delta_k \le 1$, $\lambda > 0$, $i \in M$;.

Then this paper can derive the collective decision matrix $\boldsymbol{R} = \left(r_{ij}\right)_{m \times n}$, noting that r_{ij} is a PFHS.

Step 3. Employ the GPHFWA operator to derive the overall evaluations on the performance towards alternative $r_i \left(i \in M\right)$

$$r_i = \mathrm{GPHFWA}_\lambda \left(r_{i1}, r_{i2}, \cdots, r_{in}\right)$$
$$= \bigcup_{(h_{i1}, p_{i1}) \in r_{i1}, \cdots, (h_{in}, p_{in}) \in r_{in}} \left\{\left(\left(1 - \prod_{j=1}^n \left(1 - h_{ij}^\lambda\right)^{\omega_j}\right)^{1/\lambda}, \prod_{j=1}^n p_{ij}\right)\right\}$$

in which ω_j denotes the significance regarding the jth attribute, where $\sum\nolimits_{j=1}^n \omega_j = 1$, $0 \le \omega_j \le 1$, $\lambda > 0$, $i \in M$;

Step 4. Sort $r_i \left(i \in M\right)$ according to the comparison method given by **Definitions 2** and **3**, and select the most appropriate alternative by the obtained orders.

VI. Case Study

In [15], Zhang quoted the case on a software selection problem which is originally presented by Wang and Lee [17]. This paper also uses this case to validate the effectiveness of the proposed techniques. Following this case, in order to select the best software package among four alternatives $A_i \left(i = 1, 2, 3, 4\right)$ to improve work productivity, a manager of the computer center build a decision team consisting of three experts $D_k \left(k = 1, 2, 3\right)$ with the WC $\boldsymbol{\delta} = \left(0.4, 0.3, 0.3\right)^T$. In the process of on-going decision-making, four criteria $C_k \left(k = 1, 2, 3, 4\right)$ are considered: costs of investment; contribution to organization performance; effort devoted to switching; and reliability of outsourcing. Among them, C_1 is a cost criterion and $C_j \left(j = 2, 3, 4\right)$ are benefit criteria. Besides, the four criteria are assigned with a WC $\boldsymbol{\omega} = \left(0.3, 0.25, 0.25, 0, 2\right)^T$. Meanwhile, the original individual preference opinions provided by DMs are also kept the same, which are denoted as $\boldsymbol{S}^k = \left(s_{ij}^k\right)_{m \times n} \left(k = 1, 2, 3\right)$ and presented in TABLE I.

TABLE I. The Original Assessment Information $\boldsymbol{S}^1$, $\boldsymbol{S}^2$, $\boldsymbol{S}^3$ Provided by DMs

		C_1	C_2	C_3	C_4
D_1	A_1	{0.7,0.6,0.5}	{0.4,0.3}	{0.5,0.4}	{0.6}
	A_2	{0.6,0.5}	{0.4,0.3,0.2}	{0.8}	{0.7,0.5}
	A_3	{0.8}	{0.2,0.1}	{0.6,0.4,0.3}	{0.8,0.7}
	A_4	{0.9}	{0.9,0.8}	{0.7}	{0.8,0.7,0.6}
D_2	A_1	{0.8,0.7,0.6}	{0.9,0.7}	{0.3,0.2}	{0.6,0.5,0.3}
	A_2	{0.3}	{0.6}	{0.7}	{0.4,0.3}
	A_3	{0.7,0.5}	{0.4,0.3}	{0.9,0.8,0.7}	{0.5,0.4}
	A_4	{0.9}	{0.5,0.3}	{0.5,0.4,0.3}	{0.8}
D_3	A_1	{0.5}	{0.4,0.3,0.2}	{0.7,0.6}	{0.7,0.6,0.5}
	A_2	{0.6,0.5}	{0.8,0.7}	{0.5,0.3,0.2}	{0.5,0.4}
	A_3	{0.8}	{0.5,0.3}	{0.9,0.8}	{0.9,0.8,0.7}
	A_4	{0.9,0.8}	{0.7}	{0.6}	{0.8,0.6}

TABLE II. The Normalized Assessment Information $\boldsymbol{R}^1$, $\boldsymbol{R}^2$, $\boldsymbol{R}^3$ Provided by DMs

		C_1	C_2	C_3	C_4
D_1	A_1	{0.5,0.4,0.3}	{0.4,0.3}	{0.5,0.4}	{0.6}
	A_2	{0.5,0.4}	{0.4,0.3,0.2}	{0.8}	{0.7,0.5}
	A_3	{0.2}	{0.2,0.1}	{0.6,0.4,0.3}	{0.8,0.7}
	A_4	{0.1}	{0.9,0.8}	{0.7}	{0.8,0.7,0.6}
D_2	A_1	{0.4,0.3,0.2}	{0.9,0.7}	{0.3,0.2}	{0.6,0.5,0.3}

TABLEII (continued)

		C_1	C_2	C_3	C_4
	A_2	{0.7}	{0.6}	{0.7}	{0.4,0.3}
	A_3	{0.5,0.3}	{0.4,0.3}	{0.9,0.8,0.7}	{0.5,0.4}
	A_4	{0.1}	{0.5,0.3}	{0.5,0.4,0.3}	{0.8}
D_3	A_1	{0.5}	{0.4,0.3,0.2}	{0.7,0.6}	{0.7,0.6,0.5}
	A_2	{0.5,0.4}	{0.8,0.7}	{0.5,0.3,0.2}	{0.5,0.4}
	A_3	{0.2}	{0.5,0.3}	{0.9,0.8}	{0.9,0.8,0.7}
	A_4	{0.2,0.1}	{0.7}	{0.6}	{0.8,0.6}

TABLE III. THE COLLECTIVE ASSESSMENT INFORMATION $\boldsymbol{R}$

	C_1	C_2	C_3	C_4
A_1	{(0.5,0.4,0.3,0.2), (0.433,0.233,0.233,0.1)}	{(0.9,0.7,0.4,0.3,0.2), (0.15,0.15,0.3,0.3,0.1)}	{(0.7,0.6,0.5,0.4,0.3,0.2), (0.15,0.15,0.2,0.2,0.15,0.15)}	{(0.7,0.6,0.5,0.3), (0.1,0.6,0.2,0.1)}
A_2	{(0.7,0.5,0.4), (0.3,0.35,0.35)}	{(0.8,0.7,0.6,0.4,0.3,0.2), (0.15,0.15,0.3,0.133,0.133,0.133)}	{(0.8,0.7,0.5,0.3,0.2), (0.4,0.3,0.1,0.1,0.1)}	{(0.7,0.5,0.4,0.3), (0.2,0.35,0.3,0.15)}
A_3	{(0.5,0.3,0.2), (0.15,0.15,0.7)}	{(0.5,0.4,0.3,0.2,0.1), (0.15,0.15,0.3,0.2,0.2)}	{(0.9,0.8,0.7,0.6,0.4,0.3), (0.25,0.25,0.1,0.133,0.133,0.133)}	{(0.9,0.8,0.7,0.5,0.4), (0.1,0.3,0.3,0.15,0.15)}
A_4	{(0.2,0.1), (0.15,0.85)}	{(0.9,0.8,0.7,0.5,0.3), (0.2,0.2,0.3,0.15,0.15)}	{(0.7,0.6,0.5,0.4,0.3), (0.4,0.3,0.1,0.1,0.1)}	{(0.8,0.7,0.6), (0.583,0.133,0.283)}

TABLE IV. SCORE AND RANKINGS OF EACH ALTERNATIVE OBTAINED WITH THE GPHFWA OPERATOR

	$GPHFWA_{1/10}$	$GPHFWA_{1/5}$	$GPHFWA_{1/2}$	$GPHFWA_1$	$GPHFWA_2$	$GPHFWA_5$	$GPHFWA_{10}$
A_1	0.476	0.477	0.481	0.489	0.504	0.544	0.584
A_2	0.559	0.560	0.564	0.571	0.585	0.621	0.658
A_3	0.489	0.493	0.505	0.525	0.561	0.635	0.690
A_4	0.527	0.532	0.547	0.570	0.607	0.666	0.707
Ranking	$A_2 > A_4 > A_3 > A_1$	$A_2 > A_4 > A_3 > A_1$	$A_2 > A_4 > A_3 > A_1$	$A_2 > A_4 > A_3 > A_1$	$A_4 > A_2 > A_3 > A_1$	$A_4 > A_3 > A_2 > A_1$	$A_4 > A_3 > A_2 > A_1$

TABLE V. THE COLLECTIVE ASSESSMENT INFORMATION OBTAINED WITH THE GHFWA OPERATOR

	C_1	C_2	C_3	C_4
A_1	(0.473,0.453,0.439,0.434,0.410,0.394,0.402,0.376,0.357)	(0.680,0.670,0.663,0.526,0.509,0.497,0.667,0.657,0.650,0.503,0.485,0.472)	(0.540,0.492,0.530,0.479,0.509,0.455,0.497,0.441)	(0.634,0.600,0.573,0.611,0.573,0.544,0.579,0.537,0.504)
A_2	(0.576,0.556,0.549,0.526)	(0.632,0.577,0.616,0.558,0.606,0.545)	(0.708,0.687,0.681)	(0.579,0.559,0.565,0.543,0.473,0.444,0.453,0.422)
A_3	(0.329,0.235)	(0.379,0.304,0.350,0.265,0.365,0.285,0.334,0.243)	(0.831,0.791,0.791,0.739,0.765,0.705,0.810,0.763,0.763,0.703,0.733,0.663,0.803,0.755,0.755,0.692,0.722,0.649)	(0.793,0.743,0.708,0.785,0.732,0.696,0.758,0.696,0.654,0.748,0.683,0.638)
A_4	(0.138,0.100)	(0.784,0.769,0.708,0.687)	(0.622,0.605,0.592)	(0.800,0.756,0.766,0.713,0.739,0.679)

TABLE VI. SCORE AND RANKINGS OF EACH ALTERNATIVE OBTAINED WITH THE GHFWA OPERATOR

	$GHFWA_{1/10}$	$GHFWA_{1/5}$	$GHFWA_{1/2}$	$GHFWA_1$	$GHFWA_2$	$GHFWA_5$	$GHFWA_{10}$
A_1	0.480	0.482	0.487	0.498	0.520	0.581	0.650
A_2	0.564	0.566	0.571	0.579	0.596	0.640	0.685
A_3	0.493	0.498	0.512	0.535	0.578	0.667	0.736
A_4	0.530	0.536	0.552	0.577	0.617	0.681	0.730
Ranking	$A_2 > A_4 > A_3 > A_1$	$A_2 > A_4 > A_3 > A_1$	$A_2 > A_4 > A_3 > A_1$	$A_2 > A_4 > A_3 > A_1$	$A_4 > A_2 > A_3 > A_1$	$A_4 > A_3 > A_2 > A_1$	$A_3 > A_4 > A_2 > A_1$

Then, the MCGDM model as we have described in detail is applied to identify the best software package.

Step 1. Convert the initial preference information $\boldsymbol{S}^k = \left(s_{ij}^k\right)_{4\times4} \left(k=1,2,3\right)$ into the normalized decision information $\boldsymbol{R}^k = \left(r_{ij}^k\right)_{4\times4} \left(k=1,2,3\right)$ that is provided in TABLE II. ;

Step 2. Synthesize the collective assessment information by generating PHFS from the individual evaluation opinions expressed with HFS or FS. The outcomes are depicted in TABLE III.

Step 3. Calculate the aggregated assessment $r_i\left(i=1,2,3,4\right)$ for alternatives $A_i\left(i=1,2,3,4\right)$ employing the GPHFWA operator with $\lambda=2$. Because of the space limitation, we do not list the detailed outcomes. Following **Definition 11**, we acquire the scores of $r_i\left(i=1,2,3,4\right)$,

$s\left(r_1\right)=0.504$; $s\left(r_2\right)=0.585$; $s\left(r_3\right)=0.561$; $s\left(r_4\right)=0.607$.

Step 4. Following **Definition 3**, this paper succeeds to sort the alternatives $A_i\left(i=1,2,3,4\right)$ as: $A_4 > A_2 > A_3 > A_1$. Then, A_4 is treated as the optimal alternative.

The results derive from the developed model is the same with those got by the model proposed in [15]. Thus, we can hold that the proposed model is feasible.

In this case, we assign different values to λ and apply the GPHFWA operator to conduct decision making. The values of score function and the final sorting results are shown in TABLE IV. According to TABLE IV. , it is apparent that the score function results produced using the GPHFWA operator increase with the values of λ , and different λ leads to different rankings and the DMs are capable to choose desired λ values in accordance with their preferences.

It is worth noting that the constructed MCGDM approach is built on the basis of the GPHFWA operator which is developed built on PHFS and the arithmetic averaging operator, and the approach proposed in [15] is constructed on the basis of the WGHFPA operator that is developed based on HFS and power averaging operator. Thus, we cannot directly compare the proposed MCGDM approach with the MCGDM approach proposed in [15] owing to the differences in the information representation form and the information gather format. In a bid to further validate the outstanding advantages regarding the discussed MCGDM method, a comparison analysis is conducted to compare the suggested MCGDM method with the approach based on GHFWA operator, which is proposed in [14] and is developed based on HFS and arithmetic averaging operator. The model used for comparison is similar to the proposed MCGDM model and the GHFWA operator is used to achieve the fusion of decision information on expert level and criteria level instead of **Steps 2** and **3** in the constructed MCGDM approach.

During the application of the approach based on GHFWA into the case described above, the input information remains the same and the GHFWA operator with $\lambda=2$ is utilized to derive the collective decision matrix that is presented in TABLE V. . After that, GHFWA with $\lambda=2$ is used once more to compute the overall evaluations of all the alternatives. And the score values of these overall assessment information are as below.

$s\left(r_1\right)=0.52$; $s\left(r_2\right)=0.596$; $s\left(r_3\right)=0.578$; $s\left(r_4\right)=0.617$.

Furthermore, we assign different values to λ and apply GHFWA to process the decision information. The final outcomes are described in TABLE VI. , in which the following findings are obvious.

(1) When λ is equal to 0.1, 0.2, 0.5, 1, 2 or 5, the results obtained with the GPHFWA operator is the same with their counterparts obtained with the GHFWA operator.

(2) When λ is equal to 10, the application of GPHFWA leads to the result that A_4 remains the best alternative, but the application of GHFWA gives the result that A_3 becomes the best.

Based on the above findings, we can further verify the feasibility of the established model and infer that the results obtained with GPHFWA is more robust than those obtained by GHFWA. The main difference between the results mainly results from the introduction of PHFS during the MCGDM process under hesitant fuzzy context. Through this comparative analysis, the main merits of PHFS can be summarized as follows:

(1) PHFS introduces proportional information and can effectively reduce the uncertainty of original assessment information which takes the form of HFS.

(2) During the MCGDM process, the method for generating proportional information is used to achieve the aggregation of individual assessment information on expert level. This type of information processing can effectively mitigate information distortion.

For example, the assessment information given by the three DMs on alternative A_1 against criteria C_1 is $\{0.5,0.4,0.3\}$, $\{0.4,0.3,0.2\}$ and $\{0.5\}$, respectively. The application of the proposed method gives the collective assessment information

$$\left\{\left(0.5,0.4,0.3,0.2\right),\left(0.433,0.233,0.233,0.1\right)\right\}.$$

While the utilization of GHFWA leads to the result

$$\left\{0.473,0.453,0.439,0.434,0.410,0.394,0.402,0.376,0.357\right\}.$$

Evidently, when PHFS is used, the proportional information on new dimension is generated from the original assessment information, and multi-dimensional information can effectively reduce uncertainty. Besides, the introduction of PHFS can reserve original information as much as possible and increase the reliable of decision results.

VII. Conclusions

The current paper has defined several renew operations for PFHEs and has put forward the PHFWA and the GPHFWA operators. Furthermore, an MCGDM model based on the GPHFWA operator has been developed, and a case study that applied the constructed MCGDM approach is conducted to illustrate the reasonability of the established approach. Our future efforts will be devoted to the extensions of the generalized extended Bonferroni mean (BM), the extended power average operator, and the extended geometric BM to accommodate PHFS environments attempting at modeling the heterogeneous interrelations among criterion or experts under the situation of MCGDM.

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