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Fuzzy extensions of PROMETHEE: Models of different complexity with different ranking methods and their comparison

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Abstract

Models of Fuzzy Multi-Criteria Decision Analysis (FMCDA) are based, as a rule, on different approaches to fuzzy extension of source MCDA methods. For this, simplified models are used to approximate the functions of fuzzy variables with propagation of parametric fuzzy numbers (FNs) through all calculations. In this paper, authors suggest a novel approach to fuzzy extension of MCDA methods, for PROMETHEE-I/II, through development of fuzzy PROMETHEE-I/II (FPOMETHEE-I/II) models of different complexity: in addition to simplified models, the standard fuzzy arithmetic (SFA), and transformation methods (TMs) are implemented for assessing functions of FNs corresponding to these models. For ranking of alternatives, two defuzzification based, and one pairwise comparison ranking methods are implemented within the developed models. Special attention is paid to analysis of the overestimation problem, which can occur when using SFA in the presence of dependent variables in corresponding expressions, and to "proper fuzzy extensions" of PROMETHEE-I/II (i.e., results of all functions of FNs within the model are in accordance with the extension principle) based on TMs and, for some models, on the SFA. One of the key goals of this contribution is comparison of the distinctions in ranking alternatives by different FPROMETHEE-II models. It is demonstrated by evaluating a large number of scenarios based on Monte Carlo simulation that the probability of distinction in ranking alternatives by "proper" and "approximated" FPROMETHEE-II models may be considered as significant for ranking multicriteria problems. Another goal of this paper is analysis of the correctness of FPROMETHEE-I/II models with respect to the basic MCDA axiom related to ranking of dominated and dominating alternatives. Authors demonstrate that the basic axiom can be violated, in the general case, by all developed FPROMETHEE-I/II models and suggest an approach to fix this problem.

Keywords: Fuzzy numbers; Ranking of fuzzy numbers; Standard fuzzy arithmetic; Overestimation problem; Transformation methods; Fuzzy MCDA; Fuzzy PROMETHEE; Violation of the basic FMCDA axiom

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1. Introduction

Classical methods of Multi-Criteria Decision Analysis (MCDA) [2,27,37] along with the methodology of their implementation and application [5] are a basis for subsequent extensions to corresponding models in the fuzzy environment and form a ground for this contribution in which we will focus on fuzzy extensions of PROMETHEE I/II MCDA methods.

Fuzzy MCDA (FMCDA) models [14,33,42,47] imply the use of fuzzy criteria values or/and fuzzy weight coefficients, implementing functions of fuzzy arguments, and ranking alternatives using one of the methods for ranking of Fuzzy Numbers (FNs). In the existing FMCDA models, instead of implementing the extension principle [64,55] for assessing functions of FNs, which is not effective when determining even simple arithmetic functions, various simplified methods have been implemented. For ranking alternatives, which are presented by fuzzy values of a generalized criterion (criteria), defuzzification based ranking methods are mainly used.

The objectives of this contribution, which form its novelty, are to explore *different fuzzy extensions* of PROMETHEE-I/II methods and *their comparison*, and evaluate the correctness of these methods in relation to the basic axiom of MCDA. The Basic Axiom (BA) for any MCDA/FMCDA model, M, is as follows: if alternative A dominates alternative B in Pareto, $A \succ_P B$, then A is not worse than B according to model M, $A \succeq_M B$.

The studies presented in this paper include:

- Development of the models of different complexity for fuzzy extensions of the classical methods PROMETHEE-I/II (Preference Ranking Organization METHod for Enrichment Evaluations) [6,9] with the use of:
  - Standard Fuzzy Arithmetic (SFA) for assessing functions of FNs based on corresponding operations with α-cuts considering all variables of expressions under consideration as independent ones [20,31,55];
  - Transformation Methods (TMs), which lead to proper assessing functions of FNs and are used when evaluated expressions have dependent variables [30,31];
  - Simplified approach to assessing functions of FNs based on the use of triangular FNs (TrFNs) and a consistent approximating the results of all functions of FNs by corresponding TrFNs [39];
  - Three methods for ranking of FNs, which are an integral part of the developed FPROMETHEE-I/II models: two defuzzification based ranking methods, Centroid Index/Yager-1 and Integral of Means/Yager-2 [53,57,56], and one pairwise comparison ranking method based on Yuan's fuzzy preference relation [54,63].
- Exploring the distinctions in ranking alternatives by FPROMETHEE-II models of different complexity with the use of Monte Carlo simulation;
- Analysis of the basic MCDA axiom violation by developed FPROMETHEE-I/II models.

In this paper, authors pay a special attention to the *overestimation problem* [31], which can occur when using SFA in the presence of dependent FNs in corresponding expressions, and to "proper" FPROMETHEE-I/II models (i.e., results of all functions of FNs within the model are in accordance with the extension principle) based on TMs and, for some models, on the SFA. Distinctions in ranking alternatives by "proper" FPROMETHEE-II models and those based on SFA and approximated approaches along with the use of different ranking methods are explored with the use of Monte Carlo simulation.

Authors prove that, in the general case, the basic MCDA/FMCDA axiom can be violated by all suggested FPROMETHEE-I/II models of different complexity. Additional requirement to fix this problem is suggested.

All computations in this work are based on the use of *DecernsMCDA* system [58] extended to FMCDA models by using SFA, approximate assessing functions of FNs (with triangular/trapezoidal FNs), and TMs methods with the use of different methods for ranking of FNs.

It should be pointed out, the suggested FPROMETHEE-I/II models are based on the *pure fuzzy approaches* (which can also be applied for fuzzy extensions of other MCDA methods), and defuzzification is used only on the stage of comparison of the output FNs when defuzzification based ranking methods (Centroid Index and Integral of Means) are implemented.

This paper is structured as follows. Section 2 revises the basic notions used in this paper. Fuzzy extension of PROMETHEE-I/II is presented in section 3. Violation of the basic axiom by different FPROMETHEE-I/II models along with the suggestion how to fix it is considered in section 4. Comparison of FPROMETHEE-II models with different complexity based on Monte Carlo simulation is considered in section 5. In section 6, the results of this contribution on development and application of the FMCDA models with different level of complexity as well as violation of the basic axiom are discussed. Eventually, section 7 concludes this paper.

2. Preliminaries

In this section, basic notions about Fuzzy Numbers (FNs), fuzzy ranking methods, the overestimation problem, and PROMETHEE-I/II method are reviewed.

2.1. Fuzzy numbers and fuzzy preference relations

The concept of $\alpha - cuts$ [20,38,55] is a basic one in the fuzzy sets theory and plays a key role in both definition and application of FNs.

Definition 1. Let Z be a fuzzy set on $\mathbb{X}$ with the membership function $\mu_Z : \mathbb{X} \to [0, 1]$, and $\alpha \in (0, 1]$. An α-cut of Z is defined as:

$$Z_\alpha = \{x \in \mathbb{X} : \mu_Z(x) \geq \alpha\}$$

A fuzzy set Z on $\mathbb{R}$ is bounded if there are real numbers a and b: $Supp(Z) = \{x : \mu_Z(x) > 0\} \subseteq [a, b]$. The following definition of FN is considered in this paper [38,49,55].

Definition 2. A fuzzy number Z is a normal bounded fuzzy set on $\mathbb{R}$ with the following property: for each $\alpha \in (0, 1]$, α-cut Z_α is a closed interval.

$\mathbb{F}$ denotes the set of FNs according to Definition 2.

As FN Z is bounded and taking into account the property of α-cuts [55], the closure of the set $supp(Z)$, $\overline{supp(Z)}$, is a closed interval (segment); denote it as $[c_1, c_2]$. FN Z is considered as positive, $Z > 0$, if $c_1 > 0$, and non-negative, $Z \geq 0$, if $c_1 \geq 0$.

In Definition 1, α-cuts, $Z_\alpha = [A_\alpha, B_\alpha]$, are given for $0 < \alpha \leq 1$. For $\alpha = 0$, consider the segment $[c_1, c_2]$ and set: $A_0 = c_1$ and $B_0 = c_2$. As membership function, $\mu_Z(x)$, is defined for all $x \in \mathbb{R}$, $0 \leq \mu_Z(c_1) \leq 1$ and $0 \leq \mu_Z(c_2) \leq 1$. Then, FN Z can be identified with the family of segments: $Z = \{[A_\alpha, B_\alpha],\ 0 \leq \alpha \leq 1\}$ (below, when an expression $Z = \{[A_\alpha, B_\alpha]\}$ is used and there are no additional specification on α, it is implied but not indicated directly, that α points to the whole segment $[0, 1]$).

Let $G \subseteq \mathbb{R}^n$, $f : G \to \mathbb{R}$ be a real function, and propagation of fuzziness by function $f(x_1, ..., x_n)$ is implemented, i.e., instead of real numbers x_i, FNs Z_i are used, and the membership function, $\mu_Z(z)$, of fuzzy quantity $Z = f(Z_1, ..., Z_n)$, is determined with application of the *extension principle* [64,39,55]:

$$\mu_Z(z) = \bigvee_{z=f(x_1,...,x_n)} \left(\bigwedge_{i=1,...,n} \mu_{Z_i}(x_i) \right); \tag{1}$$

and $\mu_Z(z) = 0$ if $f^{-1}(z) = \varnothing$;

Direct implementation of Eq. (1) for assessing functions of FNs is not effective even for simple functions, and in most cases the Standard Fuzzy Arithmetic (SFA) is implemented based on α-cuts for all FNs [20,31,55]. E.g., if $Z_i = \{[A^i_\alpha, B^i_\alpha]\}$, $Z_j = \{[A^j_\alpha, B^j_\alpha]\} \in \mathbb{F}$, then (in the general case, for independent FNs, see subsection 2.3),

$$Z_i - Z_j = \{[A^i_\alpha - B^j_\alpha, B^i_\alpha - A^j_\alpha]\}$$
$$Z_i Z_j = \{[A^i_\alpha A^j_\alpha, B^i_\alpha B^j_\alpha]\}\ (Z_i, Z_j \geq 0)$$

Definition 3. Fuzzy preference relation R is a binary fuzzy relation on $\mathbb{F} \times \mathbb{F}$: $R = ((Z_i, Z_j), \mu_R(Z_i, Z_j))$, where membership function $\mu_R(Z_i, Z_j)$ ($\in [0, 1]$) indicates the degree of preference of Z_i over Z_j.

One of the classes of binary fuzzy preference relations R, which are consistent regarding comparison of FNs and demanded in applications, is a class of reciprocal preference relations [46,63], defined by the expression:

$$\mu_R(Z_i, Z_j) + \mu_R(Z_j, Z_i) = 1 \tag{2}$$

For any $Z_i, Z_j \in \mathbb{F}$ and reciprocal fuzzy preference relation R, their fuzzy ranking is defined as:

$$Z_i \succ Z_j \text{ if } \mu_R(Z_i, Z_j) > 0.5,\ \ Z_i \sim Z_j \text{ if } \mu_R(Z_i, Z_j) = 0.5, Z_i \prec Z_j \text{ if } \mu_R(Z_i, Z_j) < 0.5. \tag{3}$$

The following notations are also used here within ranking by pairwise comparison methods:

$$\mu_{ij} = \mu_R(Z_i, Z_j) = \mu_R(Z_i \geq Z_j) = \mu_R(Z_j \leq Z_i). \tag{4}$$

Note that the symbols $\leq, \geq$, used here for notational purposes, are different from the symbols $\preceq, \succeq$, which are associated with ranking of FNs.

2.2. *Methods for ranking of fuzzy numbers*

The ranking of FNs is a key stage in all FMCDA models. Corresponding definitions and detailed reviews of the three main classes of ranking methods, defuzzification based, etalon/reference, and pairwise comparison ranking methods can be found in [34,53–57,63]. Below, the three wide-used ranking methods, which are implemented in this contribution, are revised.

1. Centroid Index, CI (or Yager-1) [53,56] is widely used *defuzzification* based ranking method and is represented by the following expression:
$$CI(Z) = \frac{\int xZ(x)dx}{\int Z(x)dx} \tag{5}$$
here, $Z(x) = \mu_Z(x)$.
Within this method, FNs, $Z_i,\ \ i = 1, ..., n$, are represented by corresponding real numbers $CI(Z_i)$ with their subsequent ranking: FN with higher value of CI has higher rank.
Centroid Index has evident association with mathematical expectation/mean value of random variable.
In terms of α-cuts, $CI(Z)$ for FN $Z = \{[A_\alpha, B_\alpha]\}$ can be presented by the formula:
$$CI(Z) = \int_0^1 (B_\alpha + A_\alpha)(B_\alpha - A_\alpha)/2S\, d\alpha, \tag{6}$$
here $S = S(Z)$ is the area under membership function $\mu_Z(x)$:
$$S = \int_{c_1}^{c_2} \mu_Z(x)dx = \int_0^1 (B_\alpha - A_\alpha)\, d\alpha. \tag{7}$$
For singleton FN, $Z = c$, $CI(Z) = c$ (in this case, formulas (5) and (6) are not used as area S under membership function equals 0).
2. Within Integral of Means, IM, (or Yager-2) ranking method [53,57], the following value for FN $Z = \{[A_\alpha, B_\alpha]\}$ is assessed:
$$IM(Z) = \int_0^1 (A_\alpha + B_\alpha)/2\, d\alpha. \tag{8}$$
FN Z with higher value of $IM(Z)$ has higher rank.
3. Yuan's ranking method (Y) is based on the Yuan's fuzzy preference relation [54,63] and belongs to the class of pairwise comparison ranking methods. A brief description of the Yuan's ranking method is presented below.
Let $Z_i = \{[A^i_\alpha, B^i_\alpha]\}$, $Z_j = \{[A^j_\alpha, B^j_\alpha]\} \in \mathbb{F}$ be two FNs and $Z_{ij} = Z_i - Z_j = \{[A_\alpha, B_\alpha]\}$. Within the *Yuan's fuzzy preference relation*, $R_Y = ((Z_i, Z_j), \mu_Y(Z_i, Z_j))$, the area, S^+_Y, is considered as a "distance" of the positive part of $Z_{ij} = \{[A_\alpha, B_\alpha]\}$ to the axis OY, which is computed as [62]:

$$S_Y^+(Z_{ij}) = \int_0^1 (B_\alpha \theta(B_\alpha) + A_\alpha \theta(A_\alpha)) d\alpha \tag{9}$$

here $\theta(x)$ is the Heaviside function:

$$\theta(x) = \{1,\ x \geq 0;\ 0,\ x < 0\} \tag{10}$$

The *total adjusted* area under FN Z_{ij} is assessed as [62]

$$S_Y(Z_{ij}) = S_Y^+(Z_{ij}) + S_Y^+(Z_{ji}) = \int_0^1 (|B_\alpha| + |A_\alpha|) d\alpha, \tag{11}$$

Definition 4. [63] Let $Z_i, Z_j \in \mathbb{F}$ be two FNs and $Z_{ij} = Z_i - Z_j$. The Yuan's fuzzy preference relation, $R_Y = ((Z_i, Z_j), \mu_Y(Z_i, Z_j))$, in which $\mu_{ij} = \mu_Y(Z_i, Z_j)$ represents the degree of preference of Z_i over Z_j, is defined as:

$$\mu_Y(Z_i, Z_j) = S_Y^+(Z_{ij})/S_Y(Z_{ij}) \ \text{ if } \ S_Y(Z_{ij}) > 0. \tag{12}$$

For singleton FNs, $Z_i = c_i, Z_j = c_j, c_{ij} = c_i - c_j$:

$$\mu_Y(Z_i, Z_j) = \{1,\ c_{ij} > 0;\ 0,\ c_{ij} < 0;\ 0.5,\ c_{ij} = 0\} \tag{13}$$

According to Definition 4, preference relation R_Y is reciprocal [63], and

$$\mu_Y(Z_1 \geq Z_2) = \mu_Y(Z_1 - Z_2 \geq 0) \ \text{ and } \ Z_1 \succeq_Y Z_2 \ \ iff \ \ (Z_1 - Z_2) \succeq_Y 0; \tag{14}$$

2.3. The overestimation problem

For development of different FPROMETHEE-I/II models in accordance with the goals of this paper (section 1), Standard Fuzzy Arithmetic (SFA) as well as Transformation Methods (TMs) (subsection 2.4) are implemented for assessing functions of FNs.

SFA is based on operations with α-cuts considering all variables of expression under estimation as independent ones [20,31,55]. Simplified computations with consistent propagation of TrFNs (or trapezoidal FNs, TpFNs) in all operations are based on the use of only two segments for each FN $Z_i = \{[A^i_\alpha, B^i_\alpha]\}$ under consideration: α-cut $[A^i_1, B^i_1]$ for $\alpha = 1$, and segment $[A^i_0, B^i_0]$ for $\alpha = 0$.

The use of SFA for assessing functions of FNs can lead, in the general case, to the *overestimation problem* [31], which can be shortly presented as follows.

Let W, A, and B be three non-negative not crisp FNs, and consider the two expressions for assessing FNs Z_O and Z_T based on SFA:

$$Z_O = WA - WB \tag{15}$$

$$Z_T = W(A - B) \tag{16}$$

Expressions (15) and (16) are fuzzy extensions of real functions f_1 and f_2 (17):

$$f_1(w, a, b) = wa - wb;\ \ f_2(w, a, b) = w(a - b) \tag{17}$$

Functions f_1 and f_2 are equivalent in the class of real numbers. However, implementation of SFA for assessing expressions (15) ($Z_1 = WA,\ Z_2 = WB,\ Z_O = Z_1 - Z_2$) and (16) ($Z = A - B,\ Z_T = WZ$) results in different FNs: $supp(Z_T) \subset supp(Z_O)$. The latest reflects the problem of overestimation when using SFA, Fig. 1 (in denotations Z_O and Z_T, O and T mean, correspondingly, Overestimation and Transformation).

The problem of overestimation [31] arises when there are dependent variables (subsection 2.4) in the fuzzy expression under consideration; e.g., in (15), FNs WA and WB are dependent ones. It should also be added, proper assessing Z_O (15) based on the extension principle (1) results in FN Z_T (16).

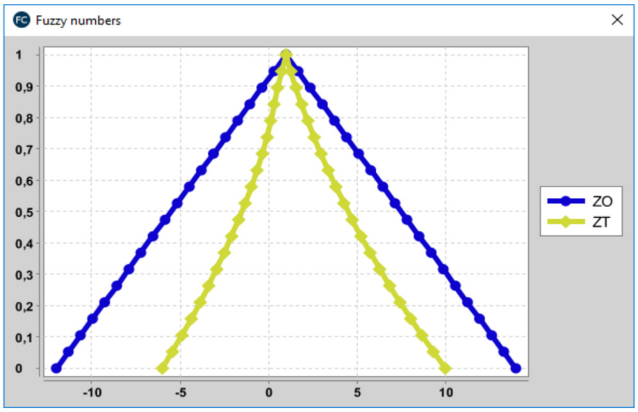

Fig. 1. Overestimation: FNs Z_O (15) and Z_T (16); W, A, B are Triangular FNs: $W = (0, 1, 2)$, $A = (3, 5, 7)$, $B = (2, 4, 6)$.

Remark 1. If instead of Eqs. (15), (16), FNs Z_1 and Z_2 are determined with the use of expressions

$$Z_1 = WA + WB, \quad Z_2 = W(A + B) \tag{18}$$

for positive FNs $W,\ A,\ B$, then $Z_1 = Z_2$ despite FN W occurs twice in expression for assessing Z_1. However, if some of FNs under consideration have both negative and positive points in their supports, Z_1 and Z_2 may differ in the general case. Features of the overestimation and its dependence on the use of arithmetic operations are discussed in [31].

Another (classical) example, which demonstrates the role of dependent FNs within fuzzy arithmetic, is presented below.

What is the FN $f(Z) = Z - Z$ for (not crisp) FN Z? Let $Z = (1, 2, 3)$ be the TrFN. According to direct implementation of α-cuts approach within SFA, $f(Z) = (1, 2, 3) - (1, 2, 3) = (-2, 0, 2)$. However, in fact, using such an approach, we assess FN $Z_3 = Z_1 - Z_2$, where $Z_1 = Z$ and $Z_2 = Z$, and Z_1 and Z_2 are considered as different (independent) FNs. The latest expression is the fuzzy extension of the real function $f(x_1, x_2) = x_1 - x_2$. However, if Z_1 and Z_2 are different denotations for *the same* FN Z (the same value/element of a model), then $f(Z)$ is the fuzzy extension of the function $g(x) = x - x$; hence, $f(Z) = Z - Z = 0$.

Taking into account this example, Definition 4 can be extended to the case $\mu_Y(Z, Z) = 0.5$, that is in agreement both with (13) ($Z_{ij} = 0$) and with the reciprocity property (2) for preference relation $\mu_Y(Z_i, Z_j)$.

2.4. Dependent fuzzy numbers and their ranking

In this subsection, the reason of overestimation problem and the features of ranking independent and dependent FNs are revised, and the use of SFA and TMs for developing FPROMETHEE-I/II models of different complexity is justified.

Let $f(X_1, ..., X_n)$ be a fuzzy extension of a real function $f(x_1, ..., x_n)$, and $X_i,\ i = 1, ..., n$, are FNs. The following definition of dependent FNs is considered here.

Definition 5. Fuzzy numbers, Z_1 and Z_2, are considered as dependent ones within a model/problem under study, if they are the results of functions, which have one or several common fuzzy arguments, i.e., $Z_1 = f_1(W_1, ..., W_k, X_1, ..., X_m)$ and $Z_2 = f_2(W_1, ..., W_k, Y_1, ..., Y_p)$, $k \geq 1,\ m \geq 0,\ p \geq 0$.

Arbitrarily selected FNs as well as FNs with missing links are considered to be independent.

Before discussing the example of ranking independent and dependent FNs, consider the following Lemma concerning Yuan's ranking method [62].

Lemma 1. *Let* $Z = \{[A_\alpha, B_\alpha]\} \in \mathbb{F}$. $Z \succeq_Y 0$ *iff* $y(Z) = \int_0^1 (B_\alpha + A_\alpha)d\alpha \geq 0$.

Despite IM and Y belong to different classes of ranking methods, the following Proposition was proved [60] based on Lemma 1.

Proposition 1. *Ranks of independent FNs by Integral of Means and Yuan's ranking methods coincide.*

Taking into account Proposition 1, the following statement was also proved in [60].

Proposition 2. *Ranking of dependent FNs by IM and Y ranking methods may differ.*

This proposition means that implementation of IM and Y ranking methods within an FMCDA can result in different ranking of alternatives, as values of the generalized criterion of a chosen FMCDA model are dependent FNs (see, e.g., Eqs. (30), (31), (38)).

The use of SFA for assessing expressions with dependent FNs can lead to the overestimation problem, subsection 2.3. To overcome the overestimation when the expression contains dependent variables, the Transformation Method(s) (TMs) can be used [30,31]. TMs are briefly revised below.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a real function, and propagation of fuzziness by a function $f(x_1, \ldots, x_n)$ is implemented, i.e., instead of real numbers x_i, FNs Z_i are used, and fuzzy extension function, $f(Z_1, ..., Z_n)$, is considered with application of the extension principle (1) [55,64]. The goal of TMs is a numerical determination of FN Z, $Z = f(Z_1, ..., Z_n)$, using an appropriate number M of α-cuts (for the same α), Z^i_α, for each FN $Z_i = \{[A^i_\alpha, B^i_\alpha]\}$.

1. If function $f(x_1, ..., x_n)$ is monotonic for each x_i, $i = 1, ..., n$, in segment $U_i = [A^i_0, B^i_0]$, that is, for differentiable functions, $\partial f/\partial x_i$ does not change its sign in the given segment U_i for fixed values of all other variables in corresponding segments, RTM (Reduced TM) is implemented: for each α-cut, Z^i_α, values $Y = f(X_1, ..., X_n)$ are determined for all combinations $\{X_1, ..., X_n\}$, where X_i is one of the marginal points of Z^i_α, i.e., $X_i \in \{A^i_\alpha, B^i_\alpha\}$, with subsequent assessing minimal and maximal values of Y to form α-cut $Z_\alpha = [A_\alpha, B_\alpha]$ of FN $Z = f(Z_1, ..., Z_n)$.
2. If function $f(x_1, ..., x_n)$ is not monotonic for each x_i in segment U_i, GTM (General TM) is used: for each α-cut, values $Y = f(X_1, ..., X_n)$ are determined for all combinations $\{X_1, ..., X_n\}$, where X_i is one of the N_α points in the segment $[A^i_\alpha, B^i_\alpha]$; for this, segment $[A^i_\alpha, B^i_\alpha]$ is subdivided for $N_\alpha - 1$ intervals, in accordance with a specific algorithm, by the points $C_1, ..., C_{N_\alpha - 2}$, i.e., $X_i \in \{A^i_\alpha, C_1, ..., C_{N_\alpha - 2}, B^i_\alpha\}$; then minimal and maximal values of Y for forming α-cut $Z_\alpha = [A_\alpha, B_\alpha]$ are found. Let's stress that RTM may also be used (as an approximation, using extended number of α-cuts) in this case too. However, in both cases 1 and 2, the special algorithm for organizing cycles for α-cuts from $\alpha = 1$ to $\alpha = 0$ is implemented.
3. In the general case, function $f(x_1, ..., x_n)$ can be monotonic for some variables x_i in segments $U_i, i = 1, ..., n_1$, and non-monotonic for other variables in their segments. In this case, to diminish number/time of computations, instead of GTM, an ETM (Extended TM) can be used: for "monotonic variables" x_i, RTM is used, for remaining variables, GTM is implemented.
4. In some cases, Problem Specific ETM (PSETM) can be used: the points $C_1, ..., C_k$, indicated in item 2 above, are chosen purposefully for each α-cut, taking into account the properties of function $f(x_1, ..., x_n)$ under consideration.

With the increase of the number of alpha-cuts M and correct implementation of TMs, the estimated fuzzy number $Z(M)$ tends to the proper value $Z = f(Z_1, ..., Z_n)$ in accordance with the extension principle. Taking into account this comment, the result of determining function of FNs with the use of TMs is called hereafter as proper assessment despite an approximation due to the use of a finite number of α-cuts during the computing of $Z(M)$.

Thus, approximate models (through propagating TrFNs for all computations with the use of α-cut for $\alpha = 1$ and corresponding segments for $\alpha = 0$), models based on SFA with M α-cuts ($M \geq 15$), and models with the use of more complicated (more resource and time consuming) computational algorithms based on TMs along with the three ranking methods form the set of FPROMETHEE-I/II models with different complexity, which have been stated in the title and in the introduction to this paper.

The use of SFA and TMs for implementation of FPROMETHEE-I/II models with ranking methods CI, IM, and Y is considered in section 3.

2.5. PROMETHEE-I/II methods and their fuzzy extensions

In this subsection, ordinary/classical PROMETHEE-I/II methods are revised, and their existing fuzzy extensions are briefly reviewed.

PROMETHEE-I/II [7,9] belong to the class of outranking MADM methods (Multi-Attribute Decision Making methods, when a finite number of predefined alternatives is considered). Among the PROMETHEE family [8,27], PROMETHEE-I/II methods are the most spreed. Different variants of PROMETHEEs methods along with the full list of PROMETHEEs' publications can be found in [2,6,27] and [44].

PROMETHEE-I/II methods can be presented through implementation of the following steps.

1. Forming the set of alternatives $A = \{a_i,\ i = 1, ..., n\}$ and the set of criteria $C = \{C_j,\ j = 1, ..., m\}$.
2. Forming the performance table with criteria values, c_{ij}, for alternative a_i and criterion C_j, $i = 1, ..., n$, $j = 1, ..., m$.
3. Assigning weight coefficients, w_j, for criteria C_j, $j = 1, ..., m$, based on one of the existing approaches [2,9]; weight coefficients are often normalized as follows:

$$0 < w_j < 1, \quad \sum_{j=1}^{m} w_j = 1 \tag{19}$$

4. Setting preference function for each criterion C_j, $f_j(x)$, $f_j(x) \in [0, 1]$, and evaluating the intensity of preference of alternative a_i over alternative a_k, $P_j(a_i, a_k)$, based on the difference of corresponding criterion values:

$$P_j(a_i, a_k) = f_j(c_{ij} - c_{kj}) \tag{20}$$

$j = 1, ..., m$; $i, k = 1, ..., n$. According to the properties of preference functions used in PROMETHEE, if $f_j(x) > 0$ then $f_j(-x) = 0$. Thus, according to (20), from $P_j(a_i, a_k) > 0$ follows $P_j(a_k, a_i) = 0$. There are at least six types of preference functions, which are used in PROMETHEE [2,9], and the most demanded of them in applications are linear functions with q, p thresholds.

5. Assessing the preference index, $P(a_i, a_k)$, which is the wighted average of the intensity of preference of alternative a_i over alternative a_k on all criteria:

$$P(a_i, a_k) = \sum_{j=1}^{m} w_j P_j(a_i, a_k) \tag{21}$$

6. Assessing positive, $\Phi^+(a_i)$, and negative, $\Phi^-(a_i)$, outranking flows for alternative a_i, $i = 1, ..., n$:

$$\Phi^+(a_i) = \sum_{k=1}^{n} P(a_i, a_k) \tag{22}$$

$$\Phi^-(a_i) = \sum_{k=1}^{n} P(a_k, a_i) \tag{23}$$

7. Ordering alternatives according to PROMETHEE-I method:
   - Alternative a_i (strongly) exceeds/outranks alternative a_k, $a_i \succ a_k$, iff the positive flow for a_i is not less than the positive flow for a_k AND the negative flow for a_i is not greater than the negative flow for a_k, i.e.:

   $$a_i \succ a_k \ \ iff \ \ \Phi^+(a_i) \geq \Phi^+(a_k) \ \ AND \ \ \Phi^-(a_i) \leq \Phi^-(a_k) \tag{24}$$

   wherein, at least one of the indicated above inequalities for flows Φ^+ or Φ^- is strong.
   - Alternative a_i is indifferent/equivalent to alternative a_k iff their positive and negative flows, correspondingly, are equal:

   $$a_i \sim a_k \ \ iff \ \ \Phi^+(a_i) = \Phi^+(a_k) \ \ AND \ \ \Phi^-(a_i) = \Phi^-(a_k) \tag{25}$$

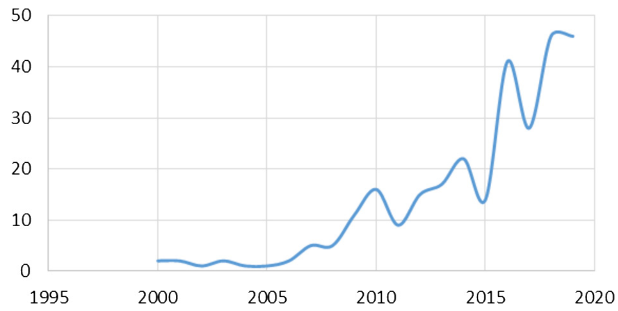

Fig. 2. Frequencies of fuzzy PROMETHEE publications with respect to years.

- If all three statements, $a_i \succ a_k$, $a_k \succ a_i$, and $a_i \sim a_k$, are wrong, in accordance with the definitions above, then alternatives a_i and a_k are considered as incomparable.

8. Ranking alternatives according to PROMETHEE-II method:
- Assessing the *net flow* for each alternative $a_i, \ i = 1, ..., n$:

$$\Phi(a_i) = \Phi^+(a_i) - \Phi^-(a_i) \tag{26}$$

- alternative with higher net flow has higher rank:

$$a_i \succ a_k \ \ iff \ \ \Phi(a_i) > \Phi(a_k), \ \ a_i \sim a_k \ \ iff \ \ \Phi(a_i) = \Phi(a_k) \tag{27}$$

PROMETHEE-I can result, in the general case, in partial order of alternatives when there are incomparable ones. PROMETHEE-II provides complete order of alternatives.

PROMETHEE has been extended to its fuzzy versions starting from the year 2000 [24,26]. Extension of an MCDA method, including PROMETHEE, to corresponding fuzzy model consists in substitution of all or some of the criteria values and weight coefficients by appropriate FNs (or linguistic variables), assessing the generalized criterion (or criteria, as for PROMETHEE-I) for each alternative through determining corresponding functions of FNs with subsequent ranking alternatives/FNs based on the use of a chosen ranking method. These questions are considered in details in section 3.

A literature review based on SCOPUS database yields 292 publications on fuzzy PROMETHEE (on Aug 1 2019). Frequencies of these publications with respect to years are illustrated in Fig. 2. Fuzzy PROMETHEE models are intensively used in various applied areas, and five the most popular among them are Computer science, Engineering, Mathematics, Decision sciences, Business and management [33].

Novel fuzzy sets are also actively used for creating corresponding versions of fuzzy PROMETHEE. In this direction, the following models have been developed: intuitionistic fuzzy PROMETHEE [3,43], hesitant fuzzy PROMETHEE [28,41], type-2 fuzzy PROMETHEE [11,15,16], and Pythagorean fuzzy PROMETHEE [17,18,65].

Advanced linguistic approaches, which implement heterogeneous frameworks by means of the fusion methods, including the use of 2-tuple linguistic values in pure linguistic PROMETHEE I/II models have also been developed and explored [13,19,23,29,32,45].

Despite developing new fuzzy PROMETHEE models based on the novel fuzzy sets, there are a range of problems concerning FPROMETHEE models with ordinary fuzzy sets, which require further deep research. Several problems in this direction, models of different complexity and their comparison (subsection 3.2 and section 5), and violation of the basic MCDA axiom by FPROMETHEE models (section 4), are explored in this contribution.

3. From PROMETHEE-I/II to fuzzy PROMETHEE-I/II

In this section, a general approach of transition from PROMETHEE-I/II to Fuzzy PROMETHEE-I/II (FPROMETHEE-I/II) models is presented and discussed.

Fig. 3. Linear preference function $f(x)$, FNs Z and $f(Z)$: $\mu_{f(Z)}(1) = 1$, $\mu_{f(Z)}(0) = 0.28$, $\lim_{x \to 1-} \mu_{f(Z)}(x) = 0.82$. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

3.1. Fuzzy extension of PROMETHEE-I/II

Extension of PROMETHEE-I/II to corresponding fuzzy PROMETHEE-I/II (FPROMETHEE-I/II) models can be presented through fuzzy extension of the steps 1-8 of PROMETHEE-I/II implementation, reviewed in subsection 2.5.

1. Forming the set of alternatives $A = \{a_i,\ i = 1, ..., n\}$ and the set of criteria $C = \{C_j,\ j = 1, ..., m\}$.
2. Setting criteria values, c_{ij}, for alternative i and criterion j, $i = 1, ..., n$, $j = 1, ..., m$, with the use of FNs, Z_{ij}, of any type (in accordance with Definition 2).
3. Assigning weight coefficient, w_j, for criterion C_j, $j = 1, ..., m$, using FNs of any type. In fuzzy models, weight coefficients w_j are often considered in the segment [0,1]: $supp(w_j) \in [0, 1]$, however, normalization (19) does not have place in the general case.
4. Setting a real preference functions, $f_j(x)$, for each criterion C_j, $f_j(x) \in [0, 1]$, as for PROMETHEE-I/II and evaluating the (fuzzy) intensity of preference of alternative a_i over alternative a_k, $P_j(a_i, a_k)$, based on the difference of corresponding fuzzy criterion values:

$$P_j(a_i, a_k) = f_j(Z_{ij} - Z_{kj}) \tag{28}$$

   $j = 1, ..., m;\ i, k = 1, ..., n$. It should be stressed here that preference function $f_j(x)$ is considered here as a continuous one, otherwise, in the general case, $f_j(Z)$ can be a fuzzy quantity, which is not a FN. Taking into account this comment, in this contribution continuous monotonically increasing/decreasing preference functions $f(x)$ for benefit/cost criteria with corresponding q, p thresholds are considered [2,9]. In the general case, both FNs, $P_j(a_i, a_k)$ and $P_j(a_k, a_j)$, can differ from zero. Function $f_j(Z)$ can result in upper semi-continuous FN, Fig. 3, which can be effectively implemented in assessing functions of FNs through the use of SFA or TMs. Fig. 3 demonstrates also that assessing functions of FNs through consistent implementation of TrFNs within approximate computational process can lead to significant differences for output results in comparison with the proper determination (see Section 5).
5. Assessing the (fuzzy) preference index, $P(a_i, a_k)$:

$$P(a_i, a_k) = \sum_{j=1}^{m} w_j P_j(a_i, a_k) = \sum_{j=1}^{m} w_j f_j(Z_{ij} - Z_{kj}) \tag{29}$$

6. Assessing positive, $\Phi^+(a_i)$, and negative, $\Phi^-(a_i)$, (fuzzy) outranking flows for alternative $a_i,\ i = 1, ..., n$:

$$\Phi^+(a_i) = \sum_{k=1}^{n} P(a_i, a_k) = \sum_{j=1}^{m} w_j(\sum_{k=1}^{n} f_j(Z_{ij} - Z_{kj})) \tag{30}$$

$$\Phi^-(a_i) = \sum_{k=1}^{n} P(a_k, a_i) = \sum_{j=1}^{m} w_j(\sum_{k=1}^{n} f_j(Z_{kj} - Z_{ij})) \tag{31}$$

7. Ordering alternatives according to FPROMETHEE-I method:
   - Alternative a_i exceeds/outranks alternative a_k according to a used model of FPROMETHEE-I with a ranking method R:

   $$a_i \succ_R a_k \ \ iff \ \ \Phi^+(a_i) \succeq_R \Phi^+(a_k) \ \ AND \ \ \Phi^-(a_i) \preceq_R \Phi^-(a_k) \tag{32}$$

   wherein, at least one of the indicated above inequalities for fuzzy flows Φ^+ or Φ^- is strong.
   In contrast to defuzzification ranking method I (CI and IM), when values $I(\Phi^+(a_i))$ and $I(\Phi^-(a_i))$ are computed and then compared according to (32), ordering FNs $\Phi^+(a_i) = \Phi_i^+$ and $\Phi^-(a_i) = \Phi_i^-$ by Yuan's/Y ranking method is based on assessing differences $\Phi_{il}^+ = \Phi_i^+ - \Phi_l^+$, and $\Phi_{il}^- = \Phi_i^- - \Phi_l^-,\ i, l = 1, ..., n$. The indicated values $\Phi_{il}^{+/-}$ can be computed with two different approaches using (30), (31):

   $$\Phi_{il}^{+,O} = (\sum_{j=1}^{m} w_j(\sum_{k=1}^{n} f_j(Z_{ij} - Z_{kj}))) - (\sum_{j=1}^{m} w_j(\sum_{k=1}^{n} f_j(Z_{lj} - Z_{kj}))) \tag{33}$$

   or

   $$\Phi_{il}^{+,T} = \sum_{j=1}^{m} w_j(\sum_{k=1}^{n} (f_j(Z_{ij} - Z_{kj}) - f_j(Z_{lj} - Z_{kj}))) \tag{34}$$

   and, correspondingly:

   $$\Phi_{il}^{-,O} = (\sum_{j=1}^{m} w_j(\sum_{k=1}^{n} f_j(Z_{kj} - Z_{ij}))) - (\sum_{j=1}^{m} w_j(\sum_{k=1}^{n} f_j(Z_{kj} - Z_{lj}))) \tag{35}$$

   or

   $$\Phi_{il}^{-,T} = \sum_{j=1}^{m} w_j(\sum_{k=1}^{n} (f_j(Z_{kj} - Z_{ij}) - f_j(Z_{kj} - Z_{lj}))) \tag{36}$$

   The meaning and difference of the O (Overestimation) and T (Transformation) models are discussed in subsections 2.3 and 2.4.
   - Alternative a_i is indifferent/equivalent to alternative a_k:

   $$a_i \sim_R a_k \ \ iff \ \ \Phi^+(a_i) \sim_R \Phi^+(a_k) \ \ AND \ \ \Phi^-(a_i) \sim_R \Phi^-(a_k) \tag{37}$$

   - If all three statements, $a_i \succ_R a_k$, $a_k \succ_R a_i$, and $a_i \sim_R a_k$, are wrong, in accordance with the definitions above, then alternatives a_i and a_k are considered as incomparable.
8. Ranking alternatives according to FPROMETHEE-II method:
   - Assessing *fuzzy net flow* for each alternative $a_i,\ i = 1, ..., n$:

   $$\Phi_i = \Phi(a_i) = \Phi_i^+ - \Phi_i^- \tag{38}$$

   According to (30), (31), Φ_i can be assessed with two approaches:

   $$\Phi_i^O = (\sum_{j=1}^{m} w_j(\sum_{k=1}^{n} f_j(Z_{ij} - Z_{kj}))) - (\sum_{j=1}^{m} w_j(\sum_{k=1}^{n} f_j(Z_{kj} - Z_{ij}))) \tag{39}$$

   or

$$\Phi_i^T = \sum_{j=1}^{m} w_j (\sum_{k=1}^{n} (f_j(Z_{ij} - Z_{kj}) - f_j(Z_{kj} - Z_{ij}))) \tag{40}$$

For ranking alternatives by Y method, FNs $\Phi_{il} = \Phi_i - \Phi_l$ are computed according to (38). The main two approaches for assessing Φ_{il} are as follows: direct determination of Φ_{il}^O as the difference of Φ_i^O and Φ_l^O computed according to (39):

$$\Phi_{il}^O = \Phi_i^O - \Phi_l^O \tag{41}$$

and, according to the second approach, based on the approaches similar to (34) and (40):

$$\Phi_{il}^T = \sum_{j=1}^{m} w_j (\sum_{k=1}^{n} (f_j(Z_{ij} - Z_{kj}) - f_j(Z_{kj} - Z_{ij}) - f_j(Z_{lj} - Z_{kj}) + f_j(Z_{kj} - Z_{lj}))) \tag{42}$$

- alternative with higher net flow (for a chosen ranking method R) has higher rank:

$$a_i \succ_R a_k \;\; iff \;\; \Phi(a_i) \succ_R \Phi(a_k), \;\; a_i \sim_R a_k \;\; iff \;\; \Phi(a_i) \sim_R \Phi(a_k) \tag{43}$$

The approaches to assessing generalized criteria $\Phi^+(a_i)$ (30), $\Phi^-(a_i)$ (31) for FPROMETHEE-I, and generalized criterion $\Phi(a_i)$ (38) for FPROMETHEE-II along with ordering alternatives based on the ranking methods under consideration are discussed in subsections 3.2.

3.2. FPROMETHEE-I/II: models with different complexity

Analysis of PROMETHEE Bibliographical Database [44] shows that there are no publications, where SFA or/and TMs are consistently implemented within an FPROMETHEE model for determining corresponding functions of FNs. Existing models are based on simplified computing functions of FNs with the use of Tr/TpFNs (or linguistic variables), and, in most cases, using CI ranking method, e.g., [4,13,21,25,40], and [48,50,51,66]. It should be stressed, in this contribution defuzzification is implemented only when comparing the output FNs within FPROMETHEE-I and FPROMETHEE-II with the use of defuzzification based ranking methods CI and IM.

In this subsection, different approaches for development of FPROMETHEE-I and FPROMETHEE-II models are considered based on the use of SFA and TMs and three ranking methods, CI, IM, and Y.

3.2.1. Different models for FPROMETHEE-I

For determining $\Phi^+(a_i)$ (30) and $\Phi^-(a_i)$ (31), the SFA may be implemented due to there is no overestimation when using SFA for these expressions (see also Remark 1).

1. Let I be ranking method CI or IM. The values $I(\Phi^+(a_i))$ and $I(\Phi^-(a_i))$, $i = 1, ..., n$, are assessed with subsequent implementing the decision rules (32) and (37) for ordering alternatives (in the general case, partial ordering). Corresponding models of FPROMETHEE-I are denoted hereafter as model FPISCI (that mens, FPROMETHEE-I with implementation of SFA for assessing all functions, and using CI ranking method) and model FPISIM.
2. The use of Y ranking method within FPROMETHEE-I can be based on determination of FNs $\Phi_{il}^{+,O}$ and $\Phi_{il}^{-,O}$, or $\Phi_{il}^{+,T}$ and $\Phi_{il}^{-,T}$ in accordance with (33)-(36).
   (a) In model with overestimation, FPIYO, SFA is implemented for assessing $\Phi_{il}^{+,O}$ (33) and $\Phi_{il}^{-,O}$ (35), and decision rules (32), (37), taking into account the property (14), are used for ordering alternatives.
   (b) Consider the approaches to assessing $\Phi_{il}^{+,T}$ (34) ($\Phi_{il}^{-,T}$, (36)). The basic part of Eq. (34) is the function

$$F_{jk}(Z_{ij}, Z_{lj}, Z_{kj}) = f_j(Z_{ij} - Z_{kj}) - f_j(Z_{lj} - Z_{kj}) \tag{44}$$

   This function can be considered as a fuzzy extension of the real function

$$F_j(x_1, x_2, z) = f_j(x_1 - z) - f_j(x_2 - z) \tag{45}$$

   Hereafter, to explore properties of different FPROMETHEE-I/II models, without loss of generality, benefit criteria are considered as an example; functions $f_j(x)$ are considered as linear or monotonically increasing

Gaussian preference functions with thresholds $q = q_j$ and $p = p_j$. Thus, preference function, f_j, is monotonically increasing in interval (q_j, p_j) and $f_j(x) = 0,\ x \le q_j,\ f_j(x) = 1,\ x \ge p_j$. Function $F_j(x_1, x_2, z)$ is monotonic in x_1 and x_2 and non-monotonic, in general case, in z. Thus, in accordance with subsection 2.4, item 3, ETM can be used to avoid overestimation when assessing (44) (RTM approach is implemented at each α-cut for FNs Z_{ij} and Z_{lj} along with GTM for Z_{kj}).

However, taking into account the features of preference functions $f_j(x)$, Problem Specific ETM (PSETM, subsection 2.4, item 4) can be used. For this, when determining function $F_{jk}(X_1, X_2, Z)$ (44) as the fuzzy extension of function $F_j(x_1, x_2, z)$ (45), in addition to the left and right marginal points of each α-cut, $Z_{L,\alpha},\ Z_{R,\alpha}$, the following points, $z^s_{rv},\ r = 1, 2,\ v = L, R,\ s = q, p$, are used provided that $z^s_{rv} \in (Z_{L,\alpha},\ Z_{R,\alpha})$: $z^s_{rv} = x_{rv} - s$ (here, x_{rv} is one of the marginal points of α-cut for FN X_r in accordance with RTM, subsection 2.4, item 1).

(c) Taking into account the property of function (45) and implementation of RTM for assessing left and right marginal points for α-cuts of FN $F_{jk} = F_{jk}(Z_{ij}, Z_{lj}, Z_{kj})$ (44), FN F_j as a part of Eq. (34) is assessed based on SFA:

$$F_j = \sum_{k=1}^{n} F_{jk} \tag{46}$$

(d) FN $\Phi^{+,T}_{il}$ (34) is determined according to (46) with the use of SFA by the expression

$$\Phi^{+,T}_{il} = \sum_{j=1}^{m} w_j F_j \tag{47}$$

(e) A similar approach is implemented for assessing $\Phi^{-,T}_{il}$ (36).

The described model for implementation of FPROMETHEE-I based on PSETM with ranking method Y is denoted as FPIYTE.

Summarizing the different approaches to estimating the function of FNs, the following models can be used within FPROMETHEE-I:

- Models FPISCI and FPISIM in accordance withe the item 1 of this subsection
- Model FPIYO according to item 2a
- Model FPIYTE based on the approaches presented in the items 2b-2e above
- FNs $\Phi^{+,T}_{il}$ (34) and $\Phi^{-,T}_{il}$ (36) can be estimated not only with the use of PSETM, but also approximated based on SFA and RTM; corresponding models are denoted as FPIYTO and FPIYTR.
- Consider also *simplified approach*, which is the most often used within FMCDA: weight coefficients and criteria values are considered as TrFNs (TpFNs can also be used based on such a simplified approach): all functions with TrFNs results in TrFNs. E.g., for monotonically non-decreasing function $f(x)$ and TrFN $Z = (a, b, c)$, FN $f(Z)$ is approximated by TrFN $(f(a), f(b), f(c))$. Arithmetic operations with TrFNs result in TrFNs and are implemented through standard (approximate) procedures with TrFNs. For this approach, implementation of functions (30) and (31) results in TrFNs with their subsequent ranking by CI or IM that forms models FPITrCI and FPITrIM.

3.2.2. Different models for FPROMETHEE-II

General approaches to extension of PROMETHEE-II to FPROMETHEE-II were suggested by Eqs. (38)-(43). Taking into account the models of subsection 3.2.1, the following *main models* of FPROMETHEE-II can be highlighted.

1. *Simplified approach* for implementation of FPROMETHEE-II based on TrFNs when assessing all functions in Eq. (39) forms the models FPIITrFNsCI and FPIITrFNsIM. The following abbreviations are used in these notations: FPII corresponds to FPROMETHEE-II; TrFNs means here that all functions within these models are approximated by TrFNs, CI and IM are corresponding ranking methods for ranking alternatives according to (43).

2. Implementation of ranking methods CI and IM to Eq. (39) forms models FPIISOCI and FPIISOIM of FPROMETHEE-II; S means implementation of SFA for determining functions of FNs without taking into account that there are dependent FNs in the expression that leads to overestimation (O) when assessing FNs Φ_i^O; CI and IM are ranking methods.
3. A basic part of Φ_i^T (40) is the expression

$$F_{jk} = F_j(Z_{ij}, Z_{kj}) = f_j(Z_{ij} - Z_{kj}) - f_j(Z_{kj} - Z_{ij}) \tag{48}$$

which can be considered as a fuzzy extension of function

$$F_j(x, z) = f_j(x - z) - f_j(z - x) \tag{49}$$

Function $f_j(x)$ is non-decreasing one (as benefit criteria are considered without loss of generality), then $F_j(x, z)$ is monotonic in x and monotonic in z. Therefore, for proper assessing (48) RTM can be used. Proper estimations of $F_{2,j}$ (50) and Φ_i^T (40)/(51) can be obtained using SFA:

$$F_{2,j} = \sum_{k=1}^{n} F_{jk} \tag{50}$$

and

$$\Phi_i^T = \sum_{j=1}^{m} w_j F_{2,k} \tag{51}$$

Implementation of CI and IM for ranking of FNs Φ_i^T (40)/(51) forms (proper) FPROMETHEE-II models FPIITRCI and FPIITRIM (R means here the use of RTM method).
4. The use of ranking method Y within FPROMETHEE-II based on SFA for assessing expressions (39) and (41) forms the model FPIIYO, which leads to overestimation (O) due to the presence of dependent FNs in corresponding expressions.
The model FPIIYTO is based on the use of SFA for determining (42).
5. Implementation of Y ranking method along with the proper assessing all functions of FNs requires implementation of a TM for determining Φ_{il}^T (42) for each pair of alternatives a_i and a_l, $i = 1, ..., n,\ l = 1, ..., n,\ i \neq l$. The basic part of expression (42) are functions F_{jk}^T:

$$F_{jk}^T = F_{jk}^T(Z_{ij}, Z_{lj}, Z_{kj}) = f_j(Z_{ij} - Z_{kj}) - f_j(Z_{kj} - Z_{ij}) - f_j(Z_{lj} - Z_{kj}) + f_j(Z_{kj} - Z_{lj}) \tag{52}$$

This function is a fuzzy extension of the following real function:

$$F_j^T(x_1, x_2, z) = f_j(x_1 - z) - f_j(z - x_1) - f_j(x_2 - z) + f_j(z - x_2) \tag{53}$$

Function (53) is monotonic in both x_1 and x_2 and non-monotonic in z in the general case. Thus, for proper determining $F_{jk}^T(X_1, X_2, Z) = F_{jk}^T(Z_{ij}, Z_{lj}, Z_{kj})$ (52), ETM can be used in this case: RTM is implemented for FNs Z_{ij} and Z_{lj}, and GTM is used for Z_{kj}.
However, taking into account the property of function $F_j^T(x_1, x_2, z)$ (53), Problem Specific ETM (PSETM) can be implemented (as it was done for proper estimating function (44)). Here, PSETM implies that apart from marginal points of α-cuts, $Z_{L,\alpha},\ Z_{R,\alpha}$, for FN $Z = Z_{kj}$, the following additional critical points, $z_{rv}^{st},\ r = 1, 2,\ v = L, R,\ s = q, p,\ t = 1, 2$, are used provided that $z_{rv}^{st} \in (Z_{L,\alpha},\ Z_{R,\alpha})$: $z_{rv}^{st} = x_{rv} + (-1)^t s$; here, x_{rv} is one of the marginal points of α-cut for FN X_r in accordance with RTM, subsection 2.4, item 1, $q = q_j,\ p = p_j$ are indifference and preference thresholds of preference function $f_j(x)$. Φ_{il}^T (42)/(55) is then assessed with the use of SFA:

$$F_j^T = \sum_{k=1}^{n} F_{jk}^T \tag{54}$$

and

$$\Phi_{il}^{T}=\sum_{j=1}^{m} w_j F_j^{T} \quad (55)$$

The described model can be denoted as FPIIYTE. Implementation of RTM instead of PSETM for approximation of (42) forms a model FPIIYTR.

Thus, the following models of FPROMETHEE-II have been considered in this contribution:

- Model FPIITrFNsCI, FPIITrFNsIM (in accordance with the item 1 of this subsection)
- Models FPIISOCI and FPIISOIM (item 2)
- Models FPIITRCI and FPIITRIM (item 3)
- Models FPIIYO and FPIIYTO (item 4)
- Models FPIIYTR and FPIIYTE (item 5).

4. Violation of the basic axiom by FPROMETHEE-I/II models

In this section, the basic axiom (BA) for FMCDA models is formulated and analyzed for FPROMETHEE-I/II as an example. Violation of BA by FPROMETHEE-I/II models is demonstrated in subsection 4.1. An approach to fix this problem to avoid violation of BA is presented in subsection 4.2.

4.1. Basic axiom for FMCDA and its violation by FPROMETHEE models

Justified use of MCDA methods is based on fulfillment of some requirements/axioms (hereafter the term axiom is used). E.g., for additive MAVT method, the axiom of *mutual difference independence* should have the place to ensure the existence of an additive multiattribute *measurable* value function [27]. At the same time, for each MCDA method *M*, the following *Basic Axiom* (BA) is assumed: if alternative a exceeds alternative b in Pareto, then, according to the decision rule of the method *M*, alternative a should be not worse than alternative b.

In MCDA/FMCDA, for the ranking or choice multicriteria problem, the decision rule for a given MCDA/FMCDA model consists of assessing the value of corresponding generalized criterion (criteria) for each alternative with subsequent ranking (or selecting the "best") of assessed values/alternatives by the used ranking method. E.g., for MAVT/FMAVT, the generalized criterion presents a generalized (overall) value, $V(a)$, of alternative a [37,59]; for TOPSIS/FTOPSIS [10,61], a coefficient of closeness, $CC(a)$, of alternative a is determined; for PROMETHEE-II/FPROMETHEE-II, it is the net flow, $\Phi(a)$, for alternative a; however, for PROMETHEE-I/FPROMETHEE-I, the decision rule consists of the two generalized criteria: the positive flow, $\Phi^+(a)$, and negative flow, $\Phi^-(a)$, for alternative a. It should be stressed the indicated generalized criteria are *benefit* ones, except criterion $\Phi^-(a)$, which is *cost* one.

Consider the following definitions adjusted for FMCDA models.

Definition 6. Let c_{ij} be a criterion value of alternative a_i for criterion j, $j=1,...,m$, $i=1,...,n$, and R is a method used for ranking of FNs. Alternative $a_i=(c_{i1},...,c_{im})$ dominates alternative $a_k=(a_{k1},...,a_{km})$ according to Pareto, $a_i \succ_{P(R)} a_k$, if a_i is at least as good as a_k for all criteria, i.e., $c_{ij} \succeq_R c_{kj}$ for all benefit criteria $\{j_b\}$ and $c_{ij} \preceq_R c_{kj}$ for all cost criteria $\{j_c\}$, and at least one of the indicated inequalities is strong.

The *Basic Axiom* for any FMCDA model *M* with the ranking method *R*, *M(R)*, can be formulated as follows.

Definition 7 *(Basic Axiom).* If for a given ranking method R alternative a dominates alternative b in Pareto, $a \succ_{P(R)} b$, then, according to the decision rule of an FMCDA model M(R), alternative a is not worse then alternative b: $a \succeq_{M(R)} b$.

For FPROMETHEE-I, the BA, Definition 7, implies the following conditions for positive and negative flows:

$$if\ \ a \succ_{P(R)} b\ \ then\ \ \Phi^+(a) \succeq_R \Phi^+(b)\ \ AND\ \ \Phi^-(a) \preceq_R \Phi^-(b) \quad (56)$$

Table 1
Performance table for FPROMETHEE-I/II models with ranking methods IM and Y; $q = 1,\ p = 5$.

	C_1	C_2	C_3
A_1	(0.5, 3.5, 4)	(1,1,1.75)	(0.5,1.5,1.5)
A_2	(1, 1, 8.25)	(0.75,1.25,1.5)	(1,1,2)
A_3	(0,2,6)	(1,1.5,2)	(1,1.5,2)
W	(0.05,0.325,0.6)	(0.6,0.7,0.8)	1

For PROMETHEE-II, fulfillment of the BA implies the following inequality for net flows:

$$if\ \ a \succ_{P(R)} b\ \ then\ \ \Phi(a) \succeq_R \Phi(b) \tag{57}$$

Before proceeding to exploring the BA for PROMETHEE-I/II, consider the axioms for fuzzy ranking methods, $A_1 - A_7$, [53,54] (with the denotations, adjusted to this paper: R is a method for ranking of FNs, $\mathbb{F}$ is the set of all FNs in accordance with Definition 2):

- A_1 (reflexivity): For an arbitrary finite subset $\mathbb{A}$ of $\mathbb{F}$ and FN $A \in \mathbb{A}$, $A \succeq_R A$;
- A_2 (antisymmetry): For an arbitrary finite subset $\mathbb{A}$ of $\mathbb{F}$ and $(A, B) \in \mathbb{A}^2$, if $A \succeq_R B$ and $B \succeq_R A$, then $A \sim_R B$;
- A_3 (transitivity): For an arbitrary finite subset $\mathbb{A}$ of $\mathbb{F}$ and $(A, B, C) \in \mathbb{A}^3$, if $A \succeq_R B$ and $B \succeq_R C$, then $A \succeq_R C$;
- A_4 (distinguishability): For an arbitrary finite subset $\mathbb{A}$ of $\mathbb{F}$ and $(A, B) \in \mathbb{A}^2$, if $\inf supp(A) > \sup supp(B)$, then $A \succeq_R B$;
- A_5 (absence of ranks reversal): Let $\mathbb{S}$ and $\mathbb{S}'$ be two arbitrary finite subsets of $\mathbb{F}$, in which method R can be applied; if A and B are in $\mathbb{S} \bigcap \mathbb{S}'$, then: $A \succ_R B$ on $\mathbb{S}'$ iff $A \succ_R B$ on $\mathbb{S}$;
- A_6: for any FNs $A,\ B$ and C in $\mathbb{F}$, if $A \succeq_R B$, then $A + C \succeq_R B + C$;
- A_7: for any FNs A, B and non-negative FN C in $\mathbb{F}$, if $A \succeq_R B$, then $AC \succeq_R BC$.

Ranking methods IM and Y satisfy the axiom A_6 and both do not satisfy the axiom A_7; ranking by CI does not satisfy the axioms A_6 and A_7 [53,54].

An extended analysis of axioms A_6 and A_7 violation by the three ranking methods, IM, CI, and Y, have been elaborated in [60]. One of the results from that research is presented below. Here, $\mathbb{F}_1$ is a subset of FNs Z in $\mathbb{F}$ with $supp(Z) \subseteq [0, 1]$.

Lemma 2. *(Violation of Axiom A_7 by ranking method Y) [60]. For any $\delta \in (0, 0.5)$ there exist FNs A, B, and C from the set $\mathbb{F}_1$ such that for Yuan's preference relation R_Y, $\mu_Y(B \geq A) \geq 1 - \delta$ and $\mu_Y(CA \geq CB) \geq 1 - \delta$.*

Violation of axioms A_6 and A_7 by demanded in applications ranking methods allowed to put forward a conjecture about violation of the BA by FPROMETHEE-I/II models. In [60], violation of the BA by FMAVT and FTOPSIS models was proved.

Violation of the BA has been proved for all FPROMETHEE-I/II models indicated in subsection 3.2. In this paper, violation is presented, as an example, for four FPROMETHEE-I models (FPITrCI, FPISCI, FPIYO, and FPIYTE) and for two FPROMETHEE-II models (FPIIYTO and FPIIYTE).

Consider the following input data for analysis of multi-criteria problems by FPROMETHEE-I/II models.

Below, $\Phi_i^{+/-} = \Phi^{+/-}(A_i)$.

Violation of BA for the chosen FPROMETHEE-I methods with the use of corresponding source data, Tables 1 and 2, is presented in Table 4: $A_1 \succ_{P(R)} A_2$ for three ranking methods R, Table 3, and alternative A_2 exceeds A_1 according to FPROMETHEE-I model M: $A_2 \succ_M A_1$.

For selected models of FPROMETHEE-II, violation of BA is presented by the following inequalities:
for FPIIYTO: $\mu_Y(\Phi_1 \leq \Phi_2) = 0.531$, and for FPIIYTE: $\mu_Y(\Phi_1 \leq \Phi_2) = 0.55869$.

When computing functions of FNs for indicated FPROMETHEE-I/II models, number of α-cuts $N_\alpha = 15$ was used. To demonstrate that the output results do not depend on the number of α-cuts, N_α, the assessments with different numbers N_α were computed. The results of the BA violation by the most complex model, FPIIYTE, are presented below:

Table 2
Performance table for FPROMETHEE-I/II models with ranking methods CI; $q = 3.5$, $p = 8$; for FPITrCI model: $q = 3$, $p = 10$.

	C_1	C_2	C_3
A_1	(2, 8, 9)	(0.75,1.25,1.5)	(1,1,1.75)
A_2	(2, 3, 13.75)	(1,1,1.5)	(0.75,1.5,1.5)
A_3	(0,2,6)	(1,1.5,2)	(1,1.5,2)
W	(0.05,0.325,0.6)	(0.6,0.7,0.8)	1

Table 3
Domination of alternative A_1 under A_2 in Pareto, $A_1 \succ_{P(R)} A_2$, for different ranking methods R.

	CI	IM	Y
$c_{11} \succ c_{21}$	$6.33 > 6.25$	$2.875 > 2.813$	0.51
$c_{12} \succeq c_{22}$	$1.167 = 1.167$	$1.188 = 1.188$	0.5
$c_{13} \succeq c_{23}$	1.25=1.25	1.25=1.25	0.5

Table 4
Violation of Basic Axiom for selected models of FPROMETHEE-I.

	FPITrCI	FPISCI	FPIYO	FPIYTE
$\Phi_1^+ \prec \Phi_2^+$	$0.1571 < 0.208$	$0.1921 < 0.2223$	0.5652	0.5613
$\Phi_1^- \succ \Phi_2^-$	$0.113 > 0.0728$	$0.1289 > 0.07921$	0.501	0.5124

$$\mu_Y(\Phi_1 \leq \Phi_2) =: \; 0.558088 \; for \; N_\alpha = 10; \; 0.558338 \; for \; N_\alpha = 20; \; 0.558416 \; for \; N_\alpha = 40$$

Additional assessments of $\mu_Y(\Phi_1 \leq \Phi_2)$ for FPIIYTE with the number of α-cuts $41 \leq N_\alpha \leq 55$ demonstrate that value $\mu_Y(\Phi_1 \leq \Phi_2)$ is in interval (0.55836, 0.558423).

Thus, the following statement has been proved taking into account the results presented above.

Proposition 3. *In the general case, when there are no restrictions on fuzzy criteria values and fuzzy weight coefficients, the basic axiom for FMCDA can be violated by FPROMETHEE-I/II models.*

4.2. Can any additional requirement fix violation of the basic axiom by FPROMETHEE-I/II models?

Correct use of (classical) MCDA methods demands not only fulfillment of the corresponding basic axiom for a generalized criterion, but also (e.g., for MAVT/MAUT) some additional requirements (axioms) [27,37]. It seems natural that a justified application of MCDA models in uncertain/fuzzy environment should be accompanied, in general, with some additional requirements.

In classical MCDA (except MAUT, where distributed/random criteria values may be considered with subsequent transition to expected utility for generalized criterion), the used criteria values are well distinguishable (e.g., $5.4 > 3.0$, $0.35 > 0.2$). Within FMCDA, the input criteria values with intuitively non-distinguishable FNs, as a rule, are not used. Distinguishability is also presented in one of the forms by axiom A_4 for ranking methods [53], revised in subsection 4.1.

An approach to distinguishability concept for FNs was suggested in [62] and elaborated in [60].

Definition 8. [62] Two FNs, $Z_i = \{[A_\alpha^i, B_\alpha^i]\} \in \mathbb{F}$, $i = 1, 2$, are distinguishable if one of them, say Z_1, is weakly on the left of another one: $Z_1 \trianglelefteq Z_2$, i.e.,

$$A_\alpha^1 \leq A_\alpha^2 \; and \; B_\alpha^1 \leq B_\alpha^2 \quad \forall \alpha \in [0, 1] \tag{58}$$

The set $\mathbf{Z} = \{Z_1, ..., Z_n\}$ is a set of distinguishable FNs (DFNs), if any two of them are distinguishable.

Below, each of two FNs, Z_i and Z_j, $i \neq j$, Z_i and W, are considered as independent. The following properties of DFNs have the place [60].

Lemma 3. *If $Z_1 \trianglelefteq Z_2$ then $Z_1 \preceq Z_2$ according to CI, IM, and Y ranking methods.*

Lemma 4. *Let Z_i, $i = 1, ..., 4$, Z, and W, be FNs, $W \geq 0$, and $Z_1 \trianglelefteq Z_2$, $Z_3 \trianglelefteq Z_4$, then $Z_1 + Z_3 \trianglelefteq Z_2 + Z_4$, $Z_1 - Z_4 \trianglelefteq Z_2 - Z_3$, $Z - Z_1 \trianglerighteq Z - Z_2$, (for $Z_i \geq 0$) $WZ_1 \trianglelefteq WZ_2$, and (for $Z_i > 0$) $Z_1/Z_4 \trianglelefteq Z_2/Z_3$.*

Lemma 5. *Let $Z_1 \trianglelefteq Z_2$, $f_1(x)$ and $f_2(x)$ be continuous monotonically non-decreasing and monotonically non-increasing real functions correspondingly. Then, $f_1(Z_1) \trianglelefteq f_1(Z_2)$ and $f_2(Z_1) \trianglerighteq f_2(Z_2)$.*

Lemma 6. *Let FNs W and $Z = \{[A_\alpha, B_\alpha]\} \in \mathbb{F}$, $W \geq 0$, $B_\alpha \geq 0$ and $A_\alpha + B_\alpha \geq 0$, $\alpha \in [0, 1]$, then $WZ \succeq_Y 0$.*

Lemma 7. *If $Z_i \succeq_Y 0$, $i = 1, 2$, then $Z_1 + Z_2 \succeq_Y 0$.*

Lemma 1 along with Lemmas 3-7 allows to prove the following statement.

Proposition 4. *If for a multicriteria problem, for each criterion, admissible criterion values form a set of distinguishable FNs, the Basic Axiom is not violated by FPROMETHEE-I/II models.*

Proof. Among the models of FPROMETHEE-I, subsection 3.2.1, consider here models with different approaches to fuzzy extension of PROMETHEE-I to FPROMETHEE-I: models FPISCI and FPISIM with proper assessing positive and negative flows for alternatives a_i, $i = 1, ..., n$, $\Phi^+(a_i)$, $\Phi^-(a_i)$ with CI/IM ranking method, and models with the use of Y ranking method: FPIYO, which leads to overestimation of the output FNs (flows), and model FPIYTE. Models FPIIYO and FPIIYTE are considered as an example to prove this proposition for FPROMETHEE-II.

Let criterion values Z_{ij}, $i = 1, ..., n$, are DFNs for each criterion j, $j = 1, ..., m$, and alternative a_1 dominates a_2 according to Pareto for ranking method R: $a_1 \succeq_{P(R)} a_2$. Without loss of generality, consider all criteria as benefit ones. Then, $Z_{1j} \trianglerighteq Z_{2j}$, $j = 1, ..., m$.

1. FPROMETHEE-I: *Models FPISCI and FPISIM*. According to Eq. (28), and taking into account Lemmas 4 and 5, $P_j(a_1, a_k) = f_j(Z_{1j} - Z_{kj}) \trianglerighteq P_j(a_2, a_k) = f_j(Z_{2j} - Z_{kj})$, as $Z_{1j} - Z_{kj} \trianglerighteq Z_{2j} - Z_{kj}$, and function $f_j(x)$ is monotonically non-decreasing, $j = 1, ...m$, $k = 1, ..., n$. Then, from (29) and Lemma 4: $P(a_1, a_k) \trianglerighteq P(a_2, a_k)$, $P(a_k, a_1) \trianglelefteq P(a_k, a_2)$ and, from (30), (31), and Lemma 4, $\Phi_1^+ \trianglerighteq \Phi_2^+$ and $\Phi_1^- \trianglelefteq \Phi_2^-$, here $\Phi_1^+ = \Phi^+(a_1)$, $\Phi_1^- = \Phi^-(a_1)$. From the two last expressions and Lemma 3: $\Phi_1^+ \succeq_R \Phi_2^+$ and $\Phi_1^- \preceq_R \Phi_2^-$, where ranking method R is CI or IM. Thus, $a_1 \succeq_{M(R)} a_2$, where model M(R) is FPISCI or FPISIM.

2. FPROMETHEE-I: *Model FPIYO*. According to item 1 above, from $Z_{1j} \trianglerighteq Z_{2j}$ follows $\Phi_1^+ \trianglerighteq \Phi_2^+$ and $\Phi_1^- \trianglelefteq \Phi_2^-$. Then, from Lemma 3: $\Phi_1^+ \succeq_Y \Phi_2^+$ and $\Phi_1^- \preceq_Y \Phi_2^-$. The latest means, in accordance with decision rule (32), $a_1 \succeq_M a_2$ for model M=FPIYO.

3. FPROMETHEE-I: *Model FPIYTE*. Implementing Y ranking method implies assessing differences $\Phi_{il}^{+,T}$ (34) and $\Phi_{il}^{-,T}$ (36). Consider below $\Phi_{12}^{+,T}$:

$$\Phi_{12}^{+,T} = \sum_{j=1}^{m} w_j \left(\sum_{k=1}^{n} (f_j(Z_{1j} - Z_{kj}) - f_j(Z_{2j} - Z_{kj}))\right) \tag{59}$$

To implement Extended TM (ETM) for assessing $\Phi_{12}^{+,T}$, consider a basic part (60) of the expression (59) with omitted index j, which is a fuzzy extension of a real function (45)

$$F(Z_1, Z_2, Z_k) = f(Z_1 - Z_k) - f(Z_2 - Z_k) \tag{60}$$

According to subsection 3.2.1, item 2, implementation of ETM to (60) (as well as to (59)) consists in the use of Reduced TM (RTM) to FNs Z_1 and Z_2, and General TM (GTM) to FN Z_k.

Let $Z_i = \{[A_\alpha^i, B_\alpha^i]\}$, $i = 1, 2, k$. According to subsection 2.4, implementing TM to expression (60) for the given FNs consists in finding the marginal points of real function (45), $x_1 \in [A_\alpha^1, B_\alpha^1]$, $x_2 \in [A_\alpha^2, B_\alpha^2]$, $z \in [A_\alpha^k, B_\alpha^k]$, to form the α-cut of FN $F = F(Z_1, Z_2, Z_k)$. For fixed $\alpha \in [0, 1]$, and taking into account that f is monotonically non-decreasing (for benefit criterion), the right and left marginal points, F_R and F_L, of (60) are determined as follows:

$$F_R = max_z(F_R(z) = f(B^1 - z) - f(A^2 - z)) \tag{61}$$

$$F_L = min_z(F_L(z) = f(A^1 - z) - f(B^2 - z)) \tag{62}$$

for $z \in [A^k, B^k]$. As $Z_{1j} \trianglerighteq Z_{2j}$, $B^1 \geq B^2 \geq A^2$, thus, as (non-negative) function $f(x)$ is monotonically non-decreasing, $F_R(z) \geq 0$. Correspondingly,

$$-F_L(z) = f(B^2 - z) - f(A^1 - z) \tag{63}$$

From (61) and (63), as $B^1 \geq B^2$ and $A^1 \geq A^2$, then $F_R(z) \geq -F_L(z)$ and $max F_R(z) = F_R \geq max(-F_L(z)) = -min F_L(z) = -F_L$; then $F_R + F_L \geq 0$.

We proved that for FN $F = \{[F_{L,\alpha}, F_{R,\alpha}]\}$ (60), $F_{L,\alpha} + F_{R,\alpha} \geq 0$, $\alpha \in [0, 1]$. Thus, taking into account Lemma 1, $F \succeq_Y 0$. Based on this result, the sequential use of Lemmas 7 and 6 for expression (59) allows to prove $\Phi_{12}^{+,T} \succeq_Y 0$; the latest means, according to (14), $\Phi_1^+ \succeq_Y \Phi_2^+$. Similar approach is used to prove that $\Phi_1^- \preceq_Y \Phi_2^-$. Thus, $a_1 \succeq_M a_2$ according to model M=FPIYTE.

4. FPROMETHEE-II: *Model FPIIYO*. According to item 1 above, from $Z_{1j} \trianglerighteq Z_{2j}$ follows $\Phi_1^+ \trianglerighteq \Phi_2^+$ and $\Phi_1^- \trianglelefteq \Phi_2^-$. Thus, according to Lemma 4, $\Phi_1 = \Phi_1^+ - \Phi_1^- \trianglerighteq \Phi_2 = \Phi_2^+ - \Phi_2^-$, and, from Lemma 3: $\Phi_1 \succeq_Y \Phi_2$. This means, $a_1 \succeq_M a_2$ according to model M=FPIIYO.

5. FPROMETHEE-II: *Model FPIIYTE*. This model with Y ranking method along with the proper assessing functions of FNs is based on determining difference of net flows Φ_1 and Φ_2 for alternatives a_1 and a_2 with the use of expression (42): $\Phi_{12}^T = \Phi_1 - \Phi_2$,

$$\Phi_{12}^T = \sum_{j=1}^{m} w_j (\sum_{k=1}^{n} (f_j(Z_{1j} - Z_{kj}) - f_j(Z_{kj} - Z_{1j}) - f_j(Z_{2j} - Z_{kj}) + f_j(Z_{kj} - Z_{2j}))) \tag{64}$$

Implementation of ETM for determining Φ_{12}^T is similar to model FPIYTE, item 3 above. For assessing basic components of (64) for fixed j and α-cut, the right and left margin point of function (53), F_R and F_L, are assessed as follows:

$$F_R = max(F_R(z) = f(B^1 - z) - f(z - B^1) - f(A^2 - z) + f(z - A^2)) \tag{65}$$

$$F_L = min(F_L(z) = f(A^1 - z) - f(z - A^1) - f(B^2 - z) + f(z - B^2)) \tag{66}$$

As function $f(x)$ is monotonically non-decreasing, it can be proved, similarly to Eqs. (61), (62), and (63) that $F_R + F_L \geq 0$ (for each $\alpha \in [0, 1]$). Based on Lemmas 1, 7, and 6, as in item 3 above for model FPIYTE, we prove that for expression (64), $\Phi_{12}^T \succeq_Y 0$ and, according to (14), $\Phi_1 \succeq_Y \Phi_2$. The latter means alternative a_1 exceeds alternative a_2 according to model M=FPIIYTE: $a_1 \succeq_M a_2$.

Thus, we proved that for different models FPROMETHEE-I/II in the case of using distinguishable criterion values for each criterion of an FMCDA problem the BA is not violated. □

Remark 2. The use of fuzzy linguistic approaches within FPROMETHEE-I/II with setting criteria values based on a linguistic scale for each criterion with subsequent implementation of linguistic terms as FNs does not violate the basic axiom due to fuzzy linguistic terms are distinguishable FNs.

5. Comparison of different FPROMETHEE-II models

In this section, different FPROMETHEE-II models are compared by using Monte Carlo simulation. For this, ranking alternatives by FPROMETHEE-II models with different levels of complexity, presented in subsection 3.2.2, are implemented at each Monte Carlo iteration, which forms a scenario for subsequent analysis. The following models are used within this analysis as an example:

- models with proper assessment of all functions: FPIITRIM, FPIITRIC, and FPIIYTE

- models, where overestimation has the place: FPIISOIM and FPIISOCI; taking into account Eqs. (39), (41) and Proposition 1, rankings by models FPIISOIM and FPIIYO coincide, therefore only model FPIISOIM is considered below; and
- models with approximate computation based on TrFNs: FPIITrFNsIM and FPIITrFNsCI.

Below, one of the indicated models, M_0, is considered as the basic model, and ranking alternatives by some of other FPROMETHEE-II model, $M_p,\ p = 1, ..., p_0$, indicated above, are compared with ranking by the model M_0. The following approach for comparing different FPROMETHEE-II models is implemented.

1. FMCDA problems with $n = K$ alternatives and $m = K$ criteria are analyzed, $K = 4,\ 6$; all criteria are considered as benefit ones.
2. Monte Carlo simulation is used to form scenarios of FMCDA problems for subsequent evaluation in accordance with the following steps.
   2.1. Random number generator with *uniform distribution in segment* [0,1] is used;
   2.2. At each Monte Carlo iteration, weight coefficients and criteria values are generated to form a scenario for evaluating by M_0 and M_p models; hereinafter, $p \in \{1, ..., p_0\}$;
   2.3. Weight coefficient $w_1 = 1$ is assigned for criterion C_1; to set the weight coefficient for each of other criterion C_k, in the general case, three points are generated, which form the TrFNs as a weight coefficient w_k, $k = 2, ..., K$;
   2.4. Criterion value $c_{ik},\ i, k = 1, ..., K$, is set as TrFN through generating, in the general case, 3 points;
   2.5. For generating symmetric (isosceles) TrFNs for weight coefficients and criteria values, two points are generated, and the third point is taken in the middle;
   2.6. A scenario with criteria values and weight coefficients are also generated with the use of linguistic variables (TrFNs) of a standard seven-term set [1,64] (using the discrete 7-points uniform distribution).
   2.7. Parameters q_j and p_j of preference function $f_j(x)$, $q_j < p_j$, are generated at each iteration for criterion $C_j,\ j = 1, ..., K$, as two (different) points;
3. Ranks of alternatives for formed scenario are evaluated by models M_0 and M_p;
4. The two or three groups of ranks $\{r\}$ for alternatives are considered: $g_1 = \{1\}$, $g_2 = \{1, ..., K_1\}$, and $g_3 = \{1, ..., K\}$, $K_1 \leq K$; if $K_1 = K$, two gropes of ranks, g_1 and g_2, are considered.
   4.1. The number of distinctions in ranking alternatives by models M_0 and M_p after implementation of iteration $(t + 1)$ in the group $g,\ g = g_1,\ g_2,\ g_3$, $DIS_{gp}(t + 1)$ $(DIS_{gp}(0) = 0)$, is assessed as follows. Let $i(r; M_l)$ be the number of alternative with the rank $r,\ r = 1, ..., K$, according to model $M_l,\ l = 0,\ p$. If at least for one $r \in g,\ i(r, M_0) \neq i(r, M_p)$, then $d_{gp}(t + 1) = 1$, otherwise, $d_{gp}(t + 1) = 0$.
   4.2. Eventually, the output result for iteration $(t + 1)$ is $DIS_{gp}(t + 1) = DIS_{gp}(t) + d_{gp}(t + 1),\ g = g_1, g_2, g_3$.
5. Process of simulation is over when the number of iterations reaches maximum of the predefined value: $t + 1 = N_{max} = N_2 = 5000$; intermediate results are also displayed for number of iteration $N_1 = 1000$.

Remark 3. Degenerate cases can occur when several alternatives have the same rank (probability of such a case is close to 0). This can also have the place if alternatives a_i and a_j with close values of the generalized criterion, $V(a)$ $(= \Phi(a))$, are considered as indifferent: $|V(a_i) - V(a_j)| < \epsilon$ for a predefined small $\epsilon > 0$.

There are several approaches to deal with such cases taking into account the goal of this simulation (analysis of distinctions between models M_0 and M_p). One of the approaches has been implemented in this contribution: if ranks of the same set of alternatives coincide for the group g_1, then $d_{g_1,p}(t + 1) = 0$, otherwise, $d_{g_1,p}(t + 1) = 1$; if for the group $g,\ g = g_2, g_3$, ranks of the same set of alternatives also coincide according to item 4.1 above, $d_{g,p}(t + 1) = 0$, otherwise, $d_{g,p}(t + 1) = 1$.

The distinctions in ranking alternatives by models $M_p,\ p \in \{1, ..., p_0\}$, in comparison with ranking by basic model M_0 are presented below; for all scenarios, basic number of α-cuts M=15 is used.

According to computations, the distinctions presented in Table 6 significantly exceed corresponding distinctions (except for the first column) when input values are linguistic terms (item 2.6), Table 5.

Table 5
Relative frequency of distinctions (%) in ranking alternatives by indicated models in comparison with the basic model FPIITRIM for ranks 1/(1-4), $K = 4$, when using *Linguistic Variables* (item 2.6).

	FPIISOIM	FPIITrFNsCI	FPIITrFNsIM	FPIIYTE
N_1=1000	0.4/2.7	6/15	4.6/12.7	2.3/8.2
N_2=5000	0.43/1.5	5.84/17.2	3.7/11.4	2.2/8.3

Table 6
Relative frequency of distinctions (%) in ranking alternatives by indicated models in comparison with the basic model FPIITRIM for ranks 1/(1-4), $K = 4$, when using *symmetric TrFNs* (item 2.5).

	FPIISOIM	FPIITrFNsCI	FPIITrFNsIM	FPIIYTE
N_1=1000	0.4/1.8	12.8/32.6	7.8/22.3	4.6/13
N_2=5000	0.4/1.4	11.3/32.54	8.68/25.66	4/12.5

Table 7
Relative frequency of distinctions (%) in ranking alternatives by indicated pairs of models for ranks 1/(1-4), $K = 4$, when using *Linguistic Variables* (item 2.6).

	FPIITRCI-FPIITRIM	FPIISOCI-FPIISOIM
N_1=1000	5.6/14.4	5.1/13.8

Table 8
Relative frequency of distinctions (%) in ranking alternatives by indicated pairs of models for ranks 1/(1-4), $K = 4$, when using *symmetric TrFNs* (item 2.5).

	FPIITRCI-FPIITRIM	FPIISOCI-FPIISOIM
N_1=1000	7/20.7	7.4/21.5

Table 9
Relative frequency of distinctions (%) in ranking alternatives by indicated models in comparison with basic model FPIITRIM for ranks (1-4), $K = 4$, when using *symmetric TrFNs/non-symmetric TrFNs* (items 2.3, 2.4, and 2.5).

	FPIISOIM	FPIITrFNsCI	FPIITrFNsIM
N_2=5000	1.4/1.5	32.5/48	25.7/39

Models FPIITRIM and FPIIYTE with an approach to proper assessing functions of FNs based on RTM, differ by ranking methods, correspondingly IM and Y. Tables 5 and 6 demonstrate that ranking by IM and Y of four dependent FNs, produced by the generalized criterion of FPROMETHEE-II (40), can distinct up to 13%.

The most significant distinctions when comparing with the basic model FPIITRIM have the place for approximate models, FPIITrFNsIM and FPIITrFNsCI – up to 33% for ranking problems and 12% for choice problems, Table 6. To analyze distinctions when ranking alternatives by CI and IM, additional comparisons of proper models, FPIITRCI and FPIITRIM, and models with the use of SFA, FPIISOCI and FPIISOIM, have been implemented, Tables 7, 8. These tables also confirm significant differences in ranking FNs/alternatives (up to 21%) by IM and CI ranking methods.

The distinctions in ranking alternatives by basic model M_0=FPIITRIM in comparison with indicated ones when using symmetric and non-symmetric input FNs are presented in Table 9. According to this table, distinctions when using non-symmetric input TrFNs exceed ones for symmetric TrFNs.

Table 10
Relative frequency of distinctions (%) in ranking alternatives by indicated models in comparison with the basic model FPIITRIM for ranks 1/(1-3)/(1-6) when using *Linguistic Variables* (item 2.6).

	FPIISOIM	FPIITrFNsCI	FPIITrFNsIM
N_1=1000	0.8/3.3/5	6.2/25.2/36.6	4.4/17.4/26
N_2=5000	0.8/3.6/5.9	6.7/25.1/36.9	4.4/17/25.95

Table 11
Relative frequency of distinctions (%) in ranking alternatives by indicated models in comparison with the basic model FPIITRIM for ranks 1/(1-3)/(1-6) when using *symmetric TrFNs* (item 2.5).

	FPIISOIM	FPIITrFNsCI	FPIITrFNsIM
N_1=1000	0.6/3.9/5.9	11.4/43.9/59.4	9/35.6/49.1
N_2=5000	0.88/4/6.2	12.6/43.4/59.6	9.2/34/48

Table 12
Relative frequency of distinctions (%) in ranking alternatives by indicated models in comparison with the basic model FPIITRIM for ranks 1 and (1-4) when using *symmetric TrFNs* (item 2.5) with different numbers of α-cuts M, 15/30/40, and number of Monte Carlo iteration $N_2 = 5000$.

	FPIISOIM	FPIITrFNsCI	FPIITrFNsIM
Rank 1	0.52/0.5/0.5	11.6/11.6/11.6	8.4/8.4/8.38
Ranks (1-4)	1.36/1.34/1.38	31.8/31.6/31.6	25.2/25.1/25.1

Additional evaluation of scenarios with 6 criteria and 6 alternatives ($K = 6$) have been implemented to analyze dependence of distinctions on the dimension of MCDA problems. Corresponding results for three FPROMETHEE-II models are presented in Tables 10 and 11.

The subject of this section is, first of all, exploring the level of distinctions when ranking alternatives by different FPROMETHEE-II models. According to Tables 5-11, the largest distinctions in ranking alternatives when comparing the basic model, FPIITRIM, with the other models, listed in the table, has the place for FPIITrFNsCI model (approximate assessing functions of FNs based on TrFNs and using ranking method CI): up to 60% for FMCDA problems with 6 alternatives, and up to 33% in the case of 4 alternatives (when input values are symmetric TrFNs); distinctions for the choice problem can exceed 10%. For other models, the relative frequency (statistical assessment of the probability) that a choice problem results in different alternatives is less then 0.1; such a probability does not exceed 0.07 if input values are linguistic variables (item 2.6).

For ranking problem, distinctions in ranking alternatives (in comparison with basic model FPIITRIM) grow with increasing dimension of the task, Tables 5, 6, and 10, 11, and can reach 30-60%. At the same time, for linguistic scenarios with the small numbers of criteria and alternatives (not more than six) corresponding distinctions can be up to 25-37%.

It should also be stressed here that for FMCDA problem of small dimension (number of alternatives and does not exceed 6) distinctions in ranking alternatives by two “related” methods, FPIITRIM and FPIISOIM may be considered as insignificant (when using symmetric TrFNs as input values) for choice problems and acceptable for ranking problems (correspondingly, probability of distinctions is about 0.01 and 0.06);

Assessments of all the scenarios in this section are based on the number of α-cuts $M = 15$. Additional assessments of scenarios for the numbers of α-cuts $M = 15,\ 30,\ 40$ for models with minimal, FPIISOIM, and maximal, FPIITrFNsCI and FPIITrFNsIM, distinctions in comparison with the basic model, FPIITRIM, demonstrate insignificant differences for output values, Table 12.

6. Discussion

In contrast to existing works on comparison of different MCDA methods, e.g., [12,22,35,36,52], in this contribution different FMCDA models of fuzzy extension for *the specific MCDA models*, PROMETHEE-I/II, are explored and compared. A natural question in such a situation is: "How different are these models concerning ranking alternatives?" One of the approaches to analysis of this problem is to explore the level of distinctions in ranking alternatives by the developed models. This approach is implemented in section 5.

According to Tables 5-11, the relative frequency (probability) that the outcome within a choice problem according to basic model differs from corresponding outcome by other FPROMETHEE-II models can be considered as acceptable or insignificant in fuzzy environment both for linguistic and symmetric TrFNs as the input data. Some exceptions here are models with an approximate estimation of functions of FNs with the use of TrFNs, FPIITrFNsCI and FPIITrFNsIM: for these models, differences within the choice problem in comparison with the (proper) FPIITRIM model can reach up to 13%.

At the same time, the probability of distinctions within the ranking problem should be considered as significant (13-50%) in the general case and increasing with augmentation of the dimension of FMCDA problems. Small distinctions (for a small numbers of criteria and alternatives) for both choice and ranking problems for the basic model, FPIITRIM, and FPIISOIM (Tables 5-11, 0.4-6.2%) is due to overestimation problem that has the place when using FPIISOIM model.

The second problem within application of FPROMETHEE models is as follows: taking into account the level of complexity and distinctions in ranking alternatives, which of the model(s) can be recommended for multicriteria decision analysis of the real case studies in the fuzzy environment?

To the moment, as a rule, simplified FMCDA models are implemented based on operations with Tr/TpFNs (subsection 3.2). These models (e.g., FPIITrFNsCI, FPIITrFNsIM), are easily implemented, unlike complex models with more accurate (taking into account the extension principle) assessing functions of FNs. Models with an approach to proper assessing functions of FNs, e.g., FPIITRIM and FPIIYTE, may be considered as a consistent fuzzy extension of the FPROMETHEE-I/II method with the use of Integral of Means and Yuan's ranking methods. However, in contrast to simplified models, proper models, especially FPIIYTE, require implementation of complex algorithms and are resource/time consuming.

When using different FMCDA models, e.g., FTOPSIS and FPROMETHEE-II, the distinctions in ranking alternatives may be justified by *conceptually different models for decision-making*. However, in the case of variety of fuzzy extension models for one basic (e.g., PROMETHEE-II) model, input information as well as requirements from decision maker(s) and experts may be considered as the same, however, the "data processing motor" is different for each model, which leads to distinctions in output results and can affect the appropriate decision making. E.g., proper models, FPIITRIM and FPIITRCI, differ only by ranking methods (IM and CI), at the same time, distinctions of output results for these models within the choice/ranking problems can exceed 7/20%, Tables 7, 8.

At the moment, without further research in this direction, development and usage of FMCDA models, which are based on the same MCDA method, is under conception of "*presumption of model adequacy*".

There is also the third problem, which is not explored in this paper but may be considered as important one for correct and reasonable examining the distinctions in ranking alternatives, presented in Tables 5-11: the *level* or the *measure* of distinctions in ranking alternatives by different methods. E.g., if for models M_1 and M_2, ranks of alternatives $A_i,\ i = 1, 2, 3$, are correspondingly as follows: $A_1 \succ A_2 \succ A_3$ and $A_2 \succ A_1 \succ A_3$, however, distinction between generalized criteria values for alternatives A_1 and A_2 is negligible according to the ranking method(s) of models M_1 and/or M_2. The approaches to exploring this problem can be based on the use of linguistic methods (e.g., distinguishes can be negligible, medium, or substantial), or/and on FRAA (Fuzzy Rank Acceptability Analysis) concept [62]), which provides a *degree of confidence/fuzzy measure* for the rank assigned to each alternative and can be computed by using different fuzzy preference relations.

Another problem explored in this contribution is violation of the Basic Axiom (BA) by FPROMETHEE-I/II models. It means there exist two alternatives, a_1 and a_2, such that a_1 exceeds a_2 in Pareto, $a_1 \succ_P a_2$, and a_2 exceeds a_1 according to FMCDA model M, $a_2 \succ_M a_1$ (Proposition 3). However, the use of distinguishable criteria values within FPROMETHEE, that is also a natural approach to the use of FNs in applications, does not result in violation of the BA (Proposition 4).

In classical MCDA, the justified use of some MCDA methods is accompanied by the fulfillment of additional requirements [37,27]. Regarding FPROMETHEE-I/II, existing additional requirements for founded application of FMCDA models may also be considered as a natural within the FMCDA theory.

At the moment, authors do not assert that the use of distinguishable criteria values (distinguishable FNs) is the only approach to avoid violation of the BA. This problem need further research. However, authors put the conjecture that the use of distinguishable FNs as criteria values allows to avoid the violation of BA for all discrete FMCDA models.

7. Conclusions

In this contribution, the following results are presented for the first time:

- FPOMETHEE-I/II models of different complexity have been developed based on implementation of approximate and proper (close to proper with increase of the number of α-cuts) assessing functions of FNs along with several methods for ranking of FNs;
- Distinctions in ranking alternatives by different FPOMETHEE-II models have been explored based on Monte Carlo simulating multicriteria problems with fuzzy criteria values and fuzzy weight coefficients. It has been demonstrated that within multicriteria ranking problems the distinctions may be considered as significant;
- Violation of the basic axiom has been demonstrated for different FPROMETHEE-I/II models;
- An approach to fix violation of the basic axiom through the use of distinguishable FNs for criteria values has been suggested.

Further research of the problems discussed in this paper will include the exploration of fuzzy models of different complexity for various MCDA methods, including deeper analysis of the degree of distinctions in ranking alternatives using linguistic and other approaches.

Another direction of research is violation of the basic axiom by different FMCDA models and approaches to fix this problem.

Authors consider the indicated problems and corresponding study as important both with fundamental and applied points of view.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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