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Some Construction Methods for Pseudo-Overlaps and Pseudo-Groupings and Their Application in Group Decision Making

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Abstract: In many real-world scenarios, the importance of different factors may vary, making commutativity an unreasonable assumption for aggregation functions, such as overlaps or groupings. To address this issue, researchers have introduced pseudo-overlaps and pseudo-groupings as their corresponding non-commutative generalizations. In this paper, we explore various construction methods for obtaining pseudo-overlaps and pseudo-groupings using overlaps, groupings, fuzzy negations, convex sums, and Riemannian integration. We then show the applicability of these construction methods in a multi-criteria group decision-making problem, where the importance of both the considered criteria and the experts vary. Our results highlight the usefulness of pseudo-overlaps and pseudo-groupings as a non-commutative alternative to overlaps and groupings.

Keywords: pseudo-overlap; pseudo-grouping; group decision making

MSC: 90B50; 68U35

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1. Introduction

Aggregation operators play a crucial role in various fields, including decision theory [1], information fusion [2], and fuzzy inference systems [3]. These operators aim to combine multiple numerical values into a single representative value [4]. To achieve this objective, aggregation operators are typically defined as increasing functions that satisfy certain boundary conditions. Monotonicity is an essential property in decision-making problems, ensuring that an increase in one criterion does not result in a decrease in the overall score. The boundary conditions of these functions reflect the idea that minimal (or maximal) inputs are aggregated into the minimal (or maximal) output of the scale we are working with.

The concept of overlap functions and their associated grouping functions was introduced by Bustince et al. [5,6]. Overlap functions measure the degree of certainty to which an object belongs simultaneously to two classes while grouping functions quantify the degree to which the same object belongs to any of the considered classes. These functions have found applications in tasks involving a maximal lack of information and fuzzy preference modeling and have been extensively studied in multi-attribute decision-making [7], rule-based classification [8], and image processing [9].

However, a key assumption in the existing definitions of overlap and grouping functions is commutativity [10]. In real-world applications, such as decision-making, criteria or experts often possess varying importance, rendering the assumption of commutativity unreasonable [11–13]. To address this limitation, researchers have introduced pseudo-overlaps [14] and pseudo-groupings [10] as non-commutative generalizations of overlaps and groupings, respectively.

This paper focuses on the development of new construction methods for pseudo-overlaps and pseudo-groupings. First, we present a recursive approach to obtaining proper pseudo-overlaps and pseudo-groupings from overlaps and groupings, which is especially convenient for improving computational efficiency [15,16]. Furthermore, we introduce a construction method that relates pseudo-overlaps and pseudo-groupings via fuzzy negations, broadening their applicability to n-dimensional operators [8]. Additionally, we demonstrate that convex sums of pseudo-overlaps or pseudo-groupings yield, respectively, pseudo-overlaps and pseudo-groupings. This result showcases the closure property of pseudo-overlaps and pseudo-groupings under convex sums. Afterward, we propose methods for generating pseudo-overlaps and pseudo-groupings using integration, which is particularly applicable to problem domains in physics, engineering, and computing and can be modeled using these operators, especially those involving the solution of differential equations [17]. Finally, we present a case study on a multi-criteria group decision-making problem that benefits from non-commutative aggregation. We solve the problem using examples of proper pseudo-overlaps and pseudo-groupings obtained through the construction methods outlined in this paper, allowing for a comparative analysis of their behavior. Consequently, the methods introduced in this paper aim at addressing the following challenges:

- Non-commutative generalizations: In real-world scenarios, it is common to address problems in which the involved factors have different levels of importance, rendering the assumption of commutativity invalid. Traditional overlap and grouping functions, which rely on commutativity, may not accurately capture the dynamics and complexities of such situations. The construction methods for pseudo-overlaps and pseudo-groupings provide non-commutative generalizations that can better represent and handle cases where factors have varying levels of importance.
- Comprehensive construction approaches: The construction methods outlined in the paper offer comprehensive approaches to obtaining pseudo-overlaps and pseudo-groupings. They provide systematic guidelines and algorithms for deriving these operators from existing overlaps, groupings, fuzzy negations, convex sums, and even integration. By offering diverse construction methods, the paper ensures that researchers and practitioners have a range of tools at their disposal to generate appropriate pseudo-overlaps and pseudo-groupings based on their specific requirements and problem domains.

The remainder of this paper is structured as follows. Section 2 establishes the necessary background on pseudo-overlaps and pseudo-grouping theory. In Section 3, we introduce several methods for generating pseudo-groupings and pseudo-overlap functions. Section 4 presents a case study in non-associative decision-making, Section 5 develops a comparative analysis, and Section 6 concludes the manuscript.

2. Literature Review

2.1. Pseudo Overlaps and Pseudo Groupings on $[0,1]^n$

This section develops some basic notions necessary to understand the proposal. First, we provide the general definition of an n-ary aggregation function. Then, we introduce overlaps and groupings as particular cases of aggregations and pseudo-overlaps and pseudo-groupings as their non-commutative generalizations. We also include some concrete examples to illustrate the theoretical definitions.

Aggregation functions allow combining multiple values into a single result [4]. The formal definition is as follows.

Definition 1 ([4]). *A function $A : [0,1]^n \longrightarrow [0,1]$ is called an n-ary aggregation function if it satisfies the following:*

(**A1**) *A is increasing in each coordinate: For each $i \in \{1, \ldots, n\}$, if $x_i \leq y$ then $A(x_1, \ldots, x_n) \leq A(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n)$;*

(**A2**) *A satisfies the following boundary conditions: $A(0, \ldots, 0) = 0$ and $A(1, \ldots 1) = 1$.*

Overlap functions are commutative aggregation functions. They were originally introduced by Bustince et al. [5] to measure the degree of overlap between two objects. The study of overlap functions remains relevant and has been extended by several researchers [8,18,19]. For instance, Paiva et al. [20] generalized the concept and analyzed it on a specific algebraic structure. Wang et al. [21] introduced a new construction method that reduces the number of arguments of generalizations of overlap and grouping functions.

Definition 2 ([5]). *A mapping $O : [0,1]^n \longrightarrow [0,1]$ is called an overlap function if it satisfies:*

(**O1**) *$O(x_1, \ldots, x_n) = O(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$ for all permutations $\sigma : \{1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$ and for all tuples $(x_1, \ldots, x_n) \in [0,1]^n$;*

(**O2**) *$O(x_1, \ldots, x_n) = 0$ if $x_1 \cdot \ldots \cdot x_n = 0$;*

(**O3**) *$O(x_1, \ldots, x_n) = 1$ if $x_1 \cdot \ldots \cdot x_n = 1$;*

(**O4**) *O is increasing in each variable;*

(**O5**) *O is continuous.*

Bustince et al. [6] introduced grouping functions as a dual notion to overlap functions, and these have been applied in decision-making problems to assess the evidence supporting any of the comparable alternatives.

Definition 3 ([6]). *A mapping $G : [0,1]^n \longrightarrow [0,1]$ is called a grouping function if it satisfies:*

(**G1**) *$G(x_1, \ldots, x_n) = G(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$ for all permutations $\sigma : \{1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$ and for all tuples $(x_1, \ldots, x_n) \in [0,1]^n$;*

(**G2**) *$G(x_1, \ldots, x_n) = 0$ if $x_1 = \ldots = x_n = 0$;*

(**G3**) *$G(x_1, \ldots, x_n) = 1$ if there exists $x_i \in [0,1]$ such that $x_i = 1$ and $1 \leq i \leq n$;*

(**G4**) *G is increasing in each variable;*

(**G5**) *G is continuous.*

In recent independent works, Batista [14] and Zhang and Liang [10] have removed the commutativity requirement from the properties of overlap and grouping functions, leading to pseudo-overlap and pseudo-grouping functions. These functions have been studied in terms of their related properties and applications. Furthermore, Liang and Zhang [10,22] have generalized the notion of interval-valued and n-dimensional pseudo-overlap functions.

Definition 4 ([10]). *A function $PO : [0,1]^n \longrightarrow [0,1]$ is said to be an n-ary pseudo-overlap function if, for each $x_1, \ldots, x_n \in [0,1]$, it satisfies:*

(**PO1**) *$PO(x_1, \ldots, x_n) = 0$ if and only if $\prod_{i=1}^{n} x_i = 0$;*

(**PO2**) *$PO(x_1, \ldots, x_n) = 1$ if and only if $\prod_{i=1}^{n} x_i = 1$;*

(**PO3**) *PO is increasing in each variable. For each $i \in \{1, \ldots, n\}$ and $y, x_1, \ldots, x_n$, if $x_i \leq y$, then*

$$PO(x_1, \ldots, x_n) \leq PO(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n);$$

(**PO4**) *PO is continuous.*

Example 1. *The functions $PO_{prod}^{r_1,\ldots,r_n}, PO_{mM}^{r_1,\ldots,r_n}, PO_{Mm}^{r_1,\ldots,r_n} : [0,1]^n \longrightarrow [0,1]$ such that for all $\vec{x} \in [0,1]^n$ are given by:*

1. $PO_{prod}^{r_1,\ldots,r_n}(\vec{x}) = \prod_{i=1}^{n} (x_i)^{r_i}$, *with $r_i > 0$ for each $i \in \{1, \ldots, n\}$;*

2. $PO_{mM}^{r_1,\ldots,r_n}(\vec{x}) = \min_{i\in\{1,\ldots,n\}}\{x_i\} \cdot \max_{i\in\{1,\ldots,n\}}\{x_i^{r_i}\}$, *with integers* $r_i \geq 0$ *for each* $i \in \{1,\ldots,n\}$;
3. $PO_{Mm}^{r_1,\ldots,r_n}(\vec{x}) = \max_{i\in\{1,\ldots,n\}}\{x_i\} \cdot \min_{i\in\{1,\ldots,n\}}\{x_i^{r_i}\}$, *with integers* $r_i > 0$ *for each* $i \in \{1,\ldots,n\}$.

Definition 5 ([10]). *A function* $PG : [0,1]^n \longrightarrow [0,1]$ *is said to be an n-ary pseudo-grouping function if, for each* $x_1,\ldots,x_n \in [0,1]$, *it satisfies:*

1. $PG(x_1,\ldots,x_n) = 0$ *if only if* $x_i = 0$ *for all* $i \in \{1,\ldots,n\}$;
2. $PG(x_1,\ldots,x_n) = 1$ *if only if* $x_i = 1$ *for some* $i \in \{1,\ldots,n\}$;
3. *PG is increasing in each variable. For each* $i \in \{1,\ldots,n\}$ *and* $y, x_1,\ldots,x_n$, *if* $x_i \leq y$, *then*
$$PG(x_1,\ldots,x_n) \leq PG(x_1,\ldots,x_{i-1},y,x_{i+1},\ldots,x_n);$$

(**PG1**) *PG is continuous.*

Example 2. *The functions* $PG_{prod}^{r_1,\ldots,r_n}, PG_{mM}^{r_1,\ldots,r_n}, PG_{Mm}^{r_1,\ldots,r_n} : [0,1]^n \longrightarrow [0,1]$ *such that for all* $\vec{x} \in [0,1]^n$ *are given by:*

1. $PG^{r_1,\ldots,r_n}(\vec{x}) = 1 - \prod_{i=1}^{n}(1-x_i)^{r_i}$, *with* $r_i > 0$ *for each* $i \in \{1,\ldots,n\}$;
2. $PG_{mM}^{r_1,\ldots,r_n}(\vec{x}) = 1 - \min_{i\in\{1,\ldots,n\}}\{1-x_i\} \cdot \max_{i\in\{1,\ldots,n\}}\{1-x_i^{r_i}\}$, *with integers* $r_i > 0$ *for each* $i \in \{1,\ldots,n\}$;
3. $PG_{Mn}^{r_1,\ldots,r_n}(\vec{x}) = 1 - \max_{i\in\{1,\ldots,n\}}\{1-x_i\} \cdot \min_{i\in\{1,\ldots,n\}}\{1-x_i^{r_i}\}$, *with integers* $r_i > 0$ *for each* $i \in \{1,\ldots,n\}$.

Clearly, every overlap (grouping) function is a pseudo-overlap (-grouping). A pseudo-overlap (-grouping) is said to be *proper* if it is not commutative. Since in many practical situations, commutativity is an undesired property, researchers have investigated non-commutative fuzzy operators and discussed their applications [23–26]. In this regard, pseudo-overlap and pseudo-grouping functions have proven useful in multi-attribute decision-making, fuzzy mathematical morphology, and image processing [10].

It should be highlighted that overlaps, groupings, pseudo-overlaps, and pseudo-groupings are aggregation functions in the sense of Definition 1. In other words, all of them are bounded (their codomain is the interval $[0,1]$), monotonous (see, for instance, (**PO3**) and 5), and satisfy the corresponding boundary condition (see, for example, (**O2**), (**O3**), (**G2**), and (**G3**)).

2.2. Multi-Criteria Group Decision Making

Decision making is a ubiquitous process that permeates every aspect of human life [27]. From simple everyday choices to complex organizational strategies, decision making shapes our actions, determines outcomes, and plays a pivotal role in personal and professional success [28]. The ability to make sound decisions is essential for individuals, groups, and societies as a whole. Decision making involves the selection of one course of action among several alternatives, driven by a desired goal or outcome. Understanding the intricacies of decision making and developing effective strategies to navigate its complexities is a subject of great interest to researchers, practitioners, and policymakers across diverse fields.

When evaluating various alternatives $A = \{A_1, A_2, \ldots A_r\}$ based on different criteria $C = \{C_1, C_2, \ldots, C_n\}$, a decision problem can be classified as a multi-criteria decision-making problem [29]. Furthermore, when a group of experts $E = \{E_1, E_2, \ldots, E_m\}$ is asked to evaluate the possible alternatives $A = \{A_1, A_2, \ldots A_r\}$ according to different criteria, the decision problem is known as an MCGDM problem [30]. MCGDM is a fundamental process that underpins various aspects of human interactions and organizational functioning [31]. Whether it involves a team of professionals making critical business decisions or a group of policymakers deliberating on public matters, the ability to effectively harness the collective intelligence of individuals is essential [32].

In recent years, researchers and practitioners have recognized the importance of integrating aggregation operators to enhance the accuracy and effectiveness of group decision-making processes [33]. Aggregation operators, such as weighted averages (WAs) [34] or ordered WAs (OWAs) [35], play a vital role in combining individual preferences, judgments, and evaluations to arrive at a collective decision [36].

Note that using different aggregation operators provide different approaches for modeling an MCGDM problem, each one of them with its own advantages and shortcomings [37]. For instance, whereas the arithmetic mean is simple, it neglects the weights of each expert or criterion in the aggregation. On the contrary, the weighted average takes such weights into account, but then it is necessary to use a proper weight allocation mechanism. Consequently, research on aggregation operators is essential for MCGDM, because aggregation operators may offer new perspectives for facilitating consensus building, addressing uncertainty and vagueness, considering group dynamics and relationships, or even developing decision support systems [38].

It should be remarked that, generally speaking, MCGDM processes are non-commutative from the point of view of the aggregation operators because the order in which the inputs are considered in the decisions is important and changing the order can result in different outcomes [39]. This phenomenon is common in situations where the decision needs to be made by a group of experts who have different levels of knowledge or expertise in the decision-making domain or when the attributes to be evaluated have different levels of importance [40]. In this sense, non-commutative aggregation operators such as pseudo-overlaps and pseudo-groupings are crucial to achieve optimal outcomes in complex and uncertain situations.

3. Construction Methods for Pseudo-Overlap and Pseudo-Grouping Functions

This section studies several construction methods related to pseudo-overlaps and pseudo-groupings.

3.1. Obtaining Proper Pseudo-Overlaps and Pseudo-Groupings from Overlaps and Groupings

As established in [41,42], construction methods for bivariate aggregations by means of unary functions, such as additive or multiplicative generators, reduce the computational complexity. Therefore, here, we show how to obtain proper pseudo-overlaps (pseudo-groupings) from overlaps (groupings).

Theorem 1. *Consider $m \in \mathbb{N}$ and let the map $O : [0,1]^2 \longrightarrow [0,1]$ be an overlap function. Then, the maps $O_L^{[m]}, O_R^{[m]} : [0,1]^n \longrightarrow [0,1]$, recursively defined as*

$$O_L^{[m]}(x_1,\ldots,x_n) = \begin{cases} O(x_1,x_2), & \text{if } m = 1 \\ O(O(x_1,x_2),x_3), & \text{if } m = 2 \\ O(O_L^{[m-1]}(x_1,\ldots,x_{n-1}),x_n), & \text{if } m \geq 3 \end{cases}$$

and

$$O_R^{[m]}(x_1,\ldots,x_n) = \begin{cases} O(x_1,x_2), & \text{if } m = 1 \\ O(x_1,O(x_2,x_3)), & \text{if } m = 2 \\ O(x_1,O_R^{[m-1]}(x_2,\ldots,x_n)), & \text{if } m \geq 3. \end{cases}$$

are two pseudo-overlap functions. Moreover, for any $m \geq 2$, $O_L^{[m]}$ and $O_R^{[m]}$, they are proper pseudo-overlaps if and only if O is non-associative.

Proof. Obviously, if $m = 1$, then $O_L^{[1]} = O_R^{[1]} = O(x_1,x_2)$. For $m \geq 2$, the proof follows by induction on m. Indeed, if $m = 2$, then $O_L^{[2]}(x_1,x_2,x_3) = O(O(x_1,x_2),x_3)$ and $O_R^{[2]}(x_1,x_2,x_3) = O(x_1,O(x_2,x_3))$. Therefore, we have:

(PO1): If $x_i = 0$ for some $i \in \{1,2,3\}$, then because O satisfies **(O2)**, one has $O_L^{[2]}(x_1,x_2,x_3) =$

$O(O(x_1,x_2),x_3) = 0$ and $O_R^{[2]}(x_1,x_2,x_3) = O(x_1,O(x_2,x_3)) = 0$. Reciprocally, if $O_L^{[2]}(x_1,x_2,x_3) = 0$ and $O_R^{[2]}(x_1,x_2,x_3) = 0$, then $O(O(x_1,x_2),x_3) = 0$ and $O(x_1,O(x_2,x_3)) = 0$, from which we conclude that $x_i = 0$ for some $i \in \{1,2,3\}$.
(PO2): Similar to the previous item.
(PO3): Suppose, without loss of generality, that $x_2 \leq z$ for some $z \in [0,1]$. Then, by **(O4)**, it follows that $O_L^{[2]}(x_1,x_2,x_3) = O(O(x_1,x_2),x_3) \leq O(O(x_1,z),x_3) = O_L^{[2]}(x_1,z,x_3)$ and $O_R^{[2]}(x_1,x_2,x_3) = O(x_1,O(x_2,x_3)) \leq O(x_1,O(z,x_3)) = O_R^{[2]}(x_1,z,x_3)$.
(PO4): Since O is continuous, for any increasing sequence $(\vec{x}_k)_{k\in\mathbb{N}} \in [0,1]^3$ in which $\vec{x}_k = (x_{1k},x_{2k},x_{3k})$, we have

$$\begin{aligned} O_L^{[2]}\left(\lim_{k\to\infty} x_{1k}, \lim_{k\to\infty} x_{2k}, \lim_{k\to\infty} x_{3k}\right) &= O\left(O\left(\lim_{k\to\infty} x_{1k}, \lim_{k\to\infty} x_{2k}\right), \lim_{k\to\infty} x_{3k}\right) \\ &= O\left(\lim_{k\to\infty} O(x_{1k},x_{2k}), \lim_{k\to\infty} x_{3k}\right) \\ &= \lim_{k\to\infty} O(O(x_{1k},x_{2k}),x_{3k}) \\ &= \lim_{k\to\infty} O_L^{[2]}(x_{1k},x_{2k},x_{3k}) \end{aligned}$$

and

$$\begin{aligned} O_R^{[2]}\left(\lim_{k\to\infty} x_{1k}, \lim_{k\to\infty} x_{2k}, \lim_{k\to\infty} x_{3k}\right) &= O\left(\lim_{k\to\infty} x_{1k}, O\left(\lim_{k\to\infty} x_{2k}, \lim_{k\to\infty} x_{3k}\right)\right) \\ &= O\left(\lim_{k\to\infty} x_{1k}, \lim_{k\to\infty} O(x_{2k},x_{3k})\right) \\ &= \lim_{k\to\infty} O(x_{1k},O(x_{2k},x_{3k})) \\ &= \lim_{k\to\infty} O_R^{[2]}(x_{1k},x_{2k},x_{3k}). \end{aligned}$$

Thus, $O_L^{[2]}$ and $O_R^{[2]}$ are two pseudo-overlap functions. Now, as an induction hypothesis, suppose that $O_L^{[m]}$ and $O_R^{[m]}$ are pseudo-overlaps for all $m \in \mathbb{N}$ such that $2 \leq m \leq p$. Let us show that $O_L^{[p+1]}$ and $O_R^{[p+1]}$ are also pseudo-overlaps. In fact, $O_L^{[p+1]}(x_1,\ldots,x_{p+2}) = 0$ and $O_R^{[p+1]}(x_1,\ldots,x_{p+2}) = 0$ if and only if $O(O_L^{[p]}(x_1,\ldots,x_{p+1}),x_{p+2}) = 0$, and $O(x_1,O_R^{[p]}$ $(x_2,\ldots,x_{p+2})) = 0$ if and only if $x_i = 0$ for some $i \in \{1,\ldots,p+2\}$. Thus, Property **(PO1)** is satisfied. Similarly, we prove Property **(PO2)**. As for Property **(PO3)**, suppose without loss of generality that $x_2 \leq z$ for some $z \in [0,1]$. Then, by **(O4)**, it follows that

$$\begin{aligned} O_L^{[p+1]}(x_1,x_2,\ldots,x_{p+2}) &= O(O_L^{[p]}(x_1,x_2,\ldots,x_{p+1}),x_{p+2}) \\ &\leq O(O_L^{[p]}(x_1,z,\ldots,x_{p+1}),x_{p+2}) \\ &= O_L^{[p+1]}(x_1,z,\ldots,x_{p+2}) \end{aligned}$$

and, with the same reasoning,

$$\begin{aligned} O_R^{[p+1]}(x_1,x_2,\ldots,x_{p+2}) &= O(x_1,O_R^{[p]}(x_2,\ldots,x_{p+2})) \\ &\leq O(x_1,O_R^{[p]}(z,\ldots,x_{p+2})) \\ &= O_R^{[p+1]}(x_1,z,\ldots,x_{p+2}). \end{aligned}$$

For Property **(PO4)**, since O is continuous, for any increasing sequence $(\vec{x}_k)_{k\in\mathbb{N}} \in [0,1]^{p+2}$ in which $\vec{x}_k = (x_{1k},\ldots,x_{(p+2)k})$, we have $O_L^{[p+1]}\left(\lim_{k\to\infty} x_{1k},\ldots,\lim_{k\to\infty} x_{(p+2)k}\right) = \lim_{k\to\infty} O_L^{[p+1]}\left(x_{1k},\ldots,x_{(p+2)k}\right)$ and $O_R^{[p+1]}\left(\lim_{k\to\infty} x_{1k},\ldots,\lim_{k\to\infty} x_{(p+2)k}\right) = \lim_{k\to\infty} O_R^{[p+1]}\left(x_{1k},\ldots,x_{(p+2)k}\right)$.

Therefore, $O_L^{[m]}$ and $O_R^{[m]}$ are two pseudo-overlap functions. Moreover, if $O_L^{[m]}$ is proper for any $m \geq 2$, in particular, for some $u, v, w \in [0,1]$, we have that

$$\begin{aligned} O(O(u,v),w) &= O_L^{[2]}(u,v,w) \\ &\neq O_L^{[2]}(w,v,u) \\ &= O(O(w,v),u) \\ &= O(u,O(v,w)). \end{aligned}$$

If $O_R^{[m]}$ is proper, the result is similar. Therefore, in any case, it follows that O is non-associative. Conversely, if O is non-associative, then, with inductive reasoning for any $m \geq 2$, we conclude that $O_L^{[m]}(r_1, r_2, \ldots, r_n) = O(O_L^{[m-1]}(r_1, r_2, \ldots, r_{n-1}), r_n) \neq O(O_L^{[m-1]}$ $(r_2, \ldots, r_{n-1}, r_n), r_1) = O_L^{[m]}(r_2, \ldots, r_n, r_1)$ and $O_R^{[m]}(t_1, t_2, \ldots, t_n) = O(t_1, O_R^{[m-1]}(t_2, \ldots, t_n)) \neq O(t_2, O_R^{[m-1]}(t_1, t_3 \ldots, t_n)) = O_L^{[m]}(t_2, t_1, \ldots, t_n)$ for some $r_i, t_i \in [0,1]$, where $i = \{1, \ldots, n\}$. Therefore, $O_L^{[m]}$ and $O_R^{[m]}$ are proper. □

Theorem 2. *Consider $m \in \mathbb{N}$, and let the map $G : [0,1]^2 \longrightarrow [0,1]$ be a grouping function. Then, the maps $G_L^{[m]}, G_R^{[m]} : [0,1]^n \longrightarrow [0,1]$, recursively defined as*

$$G_L^{[m]}(x_1, \ldots, x_n) = \begin{cases} G(x_1, x_2), & \text{if } m = 1 \\ G(G(x_1, x_2), x_3), & \text{if } m = 2 \\ G(G_L^{[m-1]}(x_1, \ldots, x_{n-1}), x_n), & \text{if } m \geq 3 \end{cases}$$

and

$$G_R^{[m]}(x_1, \ldots, x_n) = \begin{cases} G(x_1, x_2), & \text{if } m = 1 \\ G(x_1, G(x_2, x_3)), & \text{if } m = 2 \\ G(x_1, G_R^{[m-1]}(x_2, \ldots, x_n)), & \text{if } m \geq 3. \end{cases}$$

, are two pseudo-grouping functions. Moreover, for any $m \geq 2$, $G_L^{[m]}$ and $G_R^{[m]}$ are proper pseudo-groupings if and only if G is non-associative.

Proof. Similar to Theorem 1. □

Example 3. *The following examples are considered:*

1. *Consider the bivariate overlap function $O(x,y) = x^2y^2$. Then, since O is non-associative, $O_L^{[m]}(x_1, \ldots, , x_n) = x_1^{2^m} \cdot \prod_{i=2}^{n} x_i^{2^{m-i+2}}$ and $O_R^{[m]}(x_1, \ldots, , x_n) = x_1^2 \cdot \prod_{i=2}^{n} x_i^{2^{i-1}}$ are two proper pseudo-overlap functions.*
2. *Consider the bivariate overlap function $O_{mp}(x,y) = \min(x^p, y^p)$, where $p > 0$ e $p \neq 1$. Then, since O is non-associative, $O_L^{[m]}(x_1, \ldots, , x_n) = \min_{i \in \{3, \ldots, n\}} \left(\min(x_1^{p^m}, x_2^{p^m}), x_i^{p^{m-i+2}} \right)$ and $O_R^{[m]}(x_1, \ldots, , x_n) = \min_{i \in \{1, \ldots, m-1\}} \left(x_i^{p^i}, \min(x_{n-1}^{p^m}, x_n^{p^m}) \right)$ are two proper pseudo-overlap functions.*
3. *Consider the bivariate grouping function $G(x,y) = 1 - (1-x)^2(1-y)^2$. Then, since G is non-associative, $G_L^{[m]}(x_1, \ldots, , x_n) = 1 - (1-x_1)^{2^m} \cdot \prod_{i=2}^{n} (1-x_i)^{2^{m-i+2}}$ and $G_R^{[m]}(x_1, \ldots, , x_n)$ $= 1 - (1-x_1)^2 \cdot \prod_{i=2}^{n} (1-x_i)^{2^{i-1}}$ are two proper pseudo-grouping functions.*
4. *Consider the bivariate grouping function $G_{Mp}(x,y) = \max(x^p, y^p)$, where $p > 0$ e $p \neq 1$. Then, since G is non-associative, $G_L^{[m]}(x_1, \ldots, , x_n) = \max_{i \in \{3, \ldots, n\}} \left(\max(x_1^{p^m}, x_2^{p^m}), x_i^{p^{m-i+2}} \right)$*

and $G_R^{[m]}(x_1, \ldots, , x_n) = \max_{i \in \{1,\ldots,m-1\}} \left(x_i^{p^i}, \max(x_{n-1}^{p^m}, x_n^{p^m}) \right)$ *are two proper pseudo-grouping functions.*

3.2. Relationship between Pseudo-Overlap and Pseudo-Grouping Functions

Let us recall that a univariate function $N : [0,1] \longrightarrow [0,1]$ is called a fuzzy negation if it is non-increasing and such that $N(1) = 0$ and $N(0) = 1$. Further, it is said to be strict if it is strictly decreasing and continuous. Addditionally, N is called strong if $N \circ N = id_{[0,1]}$ [43].

The following results reveal that pseudo-overlap and pseudo-grouping functions are dual with respect to a strict fuzzy negation.

Theorem 3. *Consider* $N : [0,1] \longrightarrow [0,1]$ *a strict fuzzy negation and a mapping* $PO : [0,1]^n \longrightarrow [0,1]$. *The following statements are equivalent:*

(i) *The mapping PO is a pseudo-overlap function.*

(ii) *There exists a pseudo-grouping function PG such that for each* $x_1, \ldots, x_n \in [0,1]$,

$$PO(x_1, \ldots, x_n) = N^{-1}(PG(N(x_1), \ldots, N(x_n))). \tag{1}$$

The pair of functions (PO, PG) *will be shortly called N-dual functions.*

Proof. $(i) \Longrightarrow (ii)$. Let us show that PG is a pseudo-grouping function. In fact, if $x_1 = \ldots = x_n = 0$, then $N(x_1) = \ldots = N(x_n) = 1$, and so $PO(N(x_1), \ldots, N(x_n)) = PO(1, \ldots, 1) = 1$. Therefore, $PG(0, \ldots, 0) = N^{-1}(PO(N(0), \ldots, N(0))) = N^{-1}(PO(1, \ldots, 1)) = N^{-1}(1) = 0$. On the other hand, if $PG(x_1, \ldots, x_n) = 0$, then $N(PO(N(x_1), \ldots, N(x_n))) = 0$ if $PO(N(x_1), \ldots, N(x_n)) = 1$ if $N(x_1) = \ldots = N(x_n) = 1$ iff $x_1 = \ldots = x_n = 0$. Hence, PG satisfies 5. Similarly, it is shown that PG satisfies 5. Now consider $x_i, y_i \in [0,1]$ such that $x_i \leq y_i$ for each $i \in \{1, \ldots, n\}$. Then, $N(y_i) \leq N(x_i)$ and $PO(N(y_1), \ldots, N(y_n)) \leq PO(N(x_1), \ldots, N(x_n))$. Hence, $N(PO(N(x_1), \ldots, N(x_n))) \leq N(PO(N(y_1), \ldots, N(y_n)))$. Then, $PG(x_1, \ldots, x_n) \leq PG(y_1, \ldots, y_n)$. Thus, PG is increasing in each variable. Therefore, 5 is satisfied. Property **(PG1)** follows from the composition of continuous functions. Hence, PG is a pseudo-grouping function. Now, because N is strict, $PG(1, \ldots, x_n) = N^{-1}(PO(0, \ldots, N(x_n)) = N^{-1}(0) = 1$. Similarly, we have $PO(0, \ldots, x_n) = N^{-1}(PG(1, \ldots, N(x_n)) = N^{-1}(1) = 0$. Therefore, $PG(N(x_1), \ldots, N(x_n)) = N^{-1}(PO(N(x_1), \ldots, N(x_n)))$ if $PO(x_1, \ldots, x_n) = N^{-1}(PG(N(x_1), \ldots, N(x_n)))$. Hence, there exists a pseudo-grouping function that satisfies Equation (1).

$(ii) \Longrightarrow (i)$. Conversely, assume that there is a pseudo-grouping function PG that satisfies, for all $x, y \in [0,1]$, Equation (1). It should be shown that PO is a pseudo-overlap function. Indeed, PO satisfies **(PO1)** because for each $x_1, \ldots, x_n \in [0,1]$, $x_1 \cdot \ldots \cdot x_n = 0$ if there exists $x_i = 0$, where $1 \leq i \leq n$, if there exists $N(x_i) = N(0) = 1$ if $PG(N(x_1), \ldots, N(x_n)) = 1$ if $N(PG(N(x_1), \ldots, N(x_n))) = N(1) = 0$ if $PO(x_1, \ldots, x_n) = 0$. Similarly, it is shown that PO satisfies **(PO2)**. Moreover, for any $i \in \{1, \ldots, n\}$, $x_i, y_i \in [0,1]$ and $x_i \leq y_i$, if $N(y_i) \leq N(x_i)$ if $PG(N(y_1), \ldots, N(y_n)) \leq PG(N(x_1), \ldots, N(x_n))$ if $N(PG(N(x), N(y))) \leq N(PG(N(x), N(z)))$ if $PO(x_1, \ldots, x_n) \leq PO(x, z)$. Thus, we conclude that PO is increasing in each variable. Similarly, it is concluded that PO is increasing in the first place. Therefore, **(PO3)** is satisfied. Property **(PO4)** is obviously satisfied. □

Corollary 1. *Consider* $N : [0,1] \longrightarrow [0,1]$ *a strict fuzzy negation and a mapping* $PG : [0,1]^2 \longrightarrow [0,1]$. *The following statements are equivalent:*

(i) *The mapping PG is a pseudo-grouping function;*

(ii) *There exists a pseudo-overlap function PO such that for each* $x, y \in [0,1]$,

$$PG(x_1, \ldots, x_n) = N^{-1}(PO(N(x_1), \ldots, N(x_n))).$$

Remark 1. *Notice that we can conclude the same results of Theorem 3 and Corollary 1 for N being a strong negation since every strong negation is a strict negation [43].*

Another general result can be seen in the following theorems.

Theorem 4. *Let* $PO : [0,1]^n \longrightarrow [0,1]$ *be a pseudo-overlap and let N be a strict negation. Then, the map* $\widetilde{PO} : [0,1]^n \longrightarrow [0,1]$, *given by*

$$\widetilde{PO}(x_1, \ldots, x_n) = \frac{PO(x_1, \ldots, x_n)}{PO(x_1, \ldots, x_n) + N(PO(x_1, \ldots, x_n))},$$

is also a pseudo-overlap.

Proof. **(PO1)**: If $x_i = 0$ for some $i \in \{1, \ldots, n\}$, then, by Property **(PO1)**, $PO(x_1, \ldots, x_n) = 0$, and so $N(PO(x_1, \ldots, x_n)) = N(0) = 1$. Therefore, we have $\widetilde{PO}(x_1, \ldots, x_n) = \frac{0}{0+1} = 0$. If $\widetilde{PO}(x_1, \ldots, x_n) = 0$ then $PO(x_1, \ldots, x_n) = 0$ and so $x_i = 0$ for some $i \in \{1, \ldots, n\}$.
(PO2): Analogous to the previous item.
(PO3): Let $x_i, y \in [0,1]$ for all $i \in \{1, \ldots, n\}$. Suppose, without loss of generality, that $x_1 \leq y$. Then, the following sequence of inequalities is true:

1. $PO(x_1, \ldots, x_n) \leq PO(y, \ldots, x_n)$;
2. $N(PO(y, \ldots, x_n)) \leq N(PO(x_1, \ldots, x_n))$;
3. $PO(x_1, \ldots, x_n) \cdot N(PO(y, \ldots, x_n)) \leq PO(y, \ldots, x_n) \cdot N(PO(x_1, \ldots, x_n))$.

Thus, from the sequence of previous items we have

$$PO(x_1, \ldots, x_n) \cdot PO(y, \ldots, x_n) + PO(x_1, \ldots, x_n) \cdot N(PO(y, \ldots, x_n)) \leq PO(x_1, \ldots, x_n) \cdot PO(y, \ldots, x_n) + PO(y, \ldots, x_n) \cdot N(PO(x_1, \ldots, x_n))$$

which we can simplify to

$$PO(x_1, \ldots, x_n) \cdot (PO(y, \ldots, x_n) + N(PO(y, \ldots, x_n))) \leq PO(y, \ldots, x_n) \cdot (PO(x_1, \ldots, x_n) + N(PO(x_1, \ldots, x_n)))$$

and so

$$\frac{PO(x_1, \ldots, x_n)}{PO(x_1, \ldots, x_n) + N(PO(x_1, \ldots, x_n))} \leq \frac{PO(y, \ldots, x_n)}{PO(y, \ldots, x_n) + N(PO(y, \ldots, x_n))}.$$

Therefore, $\widetilde{PO}$ is increasing in each variable.
(PO4): Since $PO(x_1, \ldots, x_n) + N(PO(x_1, \ldots, x_n)) \neq 0$ for all $\vec{x} \in [0,1]^n$ and, moreover, N and PO are continuous, the continuity of $\widetilde{PO}$ follows from the fact that the quotient and sum of continuous functions result in a continuous function. □

Theorem 5. *Let* $PG : [0,1]^n \longrightarrow [0,1]$ *be a pseudo-grouping and let N be a strict negation. Then, the map* $\widetilde{PG} : [0,1]^n \longrightarrow [0,1]$, *given by*

$$\widetilde{PG}(x_1, \ldots, x_n) = \frac{PG(x_1, \ldots, x_n)}{PG(x_1, \ldots, x_n) + N(PG(x_1, \ldots, x_n))}$$

is also a pseudo-grouping.

Proof. This follows directly from Corollary 1 and Theorem 4. □

In the sequence, we show more construction methods for pseudo-overlap (pseudo-grouping) functions where some useful properties must be satisfied.

Example 4. *The following examples are considered:*

1. *The sinus induced pseudo-overlap* $\mathcal{S}(x_1,\ldots,x_n) = \sin\left(\frac{\pi}{2}\prod_{i=1}^n x_i^{p_i}\right)$ *where* $p_i > 0$ *for each* $i \in \{1,\ldots,n\}$ *and the strong negation* $N_{\cos} : [0,1] \longrightarrow [0,1]$ *given by* $N_{\cos}(x) = \frac{1}{2}(1+\cos(\pi x))$ *together determine the pseudo-overlap function*

$$\widetilde{PO}(x_1,\ldots,x_n) = \frac{\sin\left(\frac{\pi}{2}\prod\limits_{i=1}^n x_i^{p_i}\right)}{\sin\left(\frac{\pi}{2}\prod\limits_{i=1}^n x_i^{p_i}\right) + \frac{1}{2}\left(1+\cos\left(\pi\sin\left(\frac{\pi}{2}\prod\limits_{i=1}^n x_i^{p_i}\right)\right)\right)}.$$

2. *Since the map* $\mathcal{M}(x_1,\ldots,x_n) = \sqrt[n]{\prod_{i=1}^n x_i^{w_i}}$ *is an n-ary pseudo-overlap function and* $N : [0,1] \longrightarrow [0,1]$ *given by* $N(x) = 1 - x^2$ *is a strict negation, they together determine the pseudo-overlap function*

$$\widetilde{PO}(x_1,\ldots,x_n) = \frac{\sqrt[n]{\prod\limits_{i=1}^n x_i^{w_i}}}{\sqrt[n]{\prod\limits_{i=1}^n x_i^{w_i}} + 1 - \sqrt[n]{\prod\limits_{i=1}^n x_i^{2w_i}}}.$$

3. *The pseudo-grouping* $PG^{r_1,\ldots,r_n}(\vec{x}) = 1 - \prod\limits_{i=1}^n (1-x_i)^{r_i}$ *with* $r_i > 0$ *for each* $i \in \{1,\ldots,n\}$ *and the strong negation* $N : [0,1] \longrightarrow [0,1]$ *given by* $N(x) = \sqrt{3x^2+1} - 2x$ *together determine the pseudo-grouping function*

$$\widetilde{PG}(x_1,\ldots,x_n) = \frac{1 - \prod\limits_{i=1}^n (1-x_i)^{r_i}}{1 - \prod\limits_{i=1}^n (1-x_i)^{r_i} + \sqrt{3\left(1 - \prod\limits_{i=1}^n (1-x_i)^{r_i}\right)^2 + 1} - 2\left(1 - \prod\limits_{i=1}^n (1-x_i)^{r_i}\right)}.$$

4. *The pseudo-grouping* $PG(\vec{x}) = \max\limits_{i\in\{1,\ldots,n\}}\{x_i^{r_i}\}$*, with integers* $r_i > 0$ *for each* $i \in \{1,\ldots,n\}$ *and the strong negation* $N : [0,1] \longrightarrow [0,1]$ *given by* $N(x) = \sqrt{-\frac{3}{4}x^2+1} - \frac{1}{2}x$ *together determine the pseudo-grouping function*

$$\widetilde{PG}(x_1,\ldots,x_n) = \frac{\max\limits_{i\in\{1,\ldots,n\}}\{x_i^{r_i}\}}{\max\limits_{i\in\{1,\ldots,n\}}\{x_i^{r_i}\} + \sqrt{-\frac{3}{4}\left(\max\limits_{i\in\{1,\ldots,n\}}\{x_i^{r_i}\}\right)^2 + 1} - \frac{1}{2}\left(\max\limits_{i\in\{1,\ldots,n\}}\{x_i^{r_i}\}\right)}$$

3.3. Convex Sum of Pseudo-Overlaps and Pseudo-Groupings

The next result provides $n!$ distinct ways to obtain proper pseudo-overlaps (pseudo-groupings).

Theorem 6 (Convex sum of PO's)**.** *Let* $\sigma : \{1,\ldots,n\} \longrightarrow \{1,\ldots,n\}$ *be a permutation, let the mappings* $PO_1,\ldots,PO_n : [0,1]^n \longrightarrow [0,1]$ *be pseudo-overlap functions, and let* $w_1,\ldots,w_n$ *be nonnegative weights with* $\sum\limits_{i=1}^n w_i = 1$*. Then, the convex sum*

$$\mathcal{PO}(x_{\sigma(1)},\ldots,x_{\sigma(n)}) = \sum_{i=1}^n w_i \cdot PO_i(x_{\sigma(1)},\ldots,x_{\sigma(n)})$$

is also a pseudo-overlap function.

Proof. This proof is straightforward. □

Theorem 7 (Convex sum of PG's)**.** *Let $\sigma : \{1,\ldots,n\} \longrightarrow \{1,\ldots,n\}$ be a permutation, let the mappings $PG_1,\ldots,PG_n : [0,1]^n \longrightarrow [0,1]$ be pseudo-grouping functions, and let $w_1,\ldots,w_n$ be nonnegative weights with $\sum_{i=1}^{n} w_i = 1$. Then, the convex sum*

$$\mathcal{PG}(x_{\sigma(1)},\ldots,x_{\sigma(n)}) = \sum_{i=1}^{n} w_i \cdot PG_i(x_{\sigma(1)},\ldots,x_{\sigma(n)})$$

is also a pseudo-grouping function.

Proof. Straightforward. □

3.4. Obtaining Pseudo-Overlaps and Pseudo-Groupings via Riemann Integration

The Riemann integration process provides also n! different ways to obtain pseudo-overlaps and pseudo-groupings, as we will see below.

Theorem 8. *Let $\sigma : \{1,\ldots,n\} \longrightarrow \{1,\ldots,n\}$ be a permutation and let $PO : [0,1]^n \longrightarrow [0,1]$ be a pseudo-overlap function. If*

$$h = \int_{[0,1]^n} PO(u_1,\ldots,u_n)du_1\ldots du_n$$

, then the map $\mathcal{PO} : [0,1]^n \longrightarrow [0,1]$ given by

$$\mathcal{PO}(x_{\sigma(1)},\ldots,x_{\sigma(n)}) = \frac{1}{h}\int_0^{x_{\sigma(n)}} \cdots \int_0^{x_{\sigma(1)}} PO(u_1,\ldots,u_n)du_1\ldots du_n$$

is also a pseudo-overlap function.

Proof. Let $PO : [0,1]^n \longrightarrow [0,1]$ be a pseudo-overlap. Since PO is continuous and defined over the compact $[0,1]^2$, and $h \neq 0$, it follows that $\mathcal{PO}$ is well-defined and uniformly continuous. Therefore, **(PO4)** is satisfied. Moreover, since increasing monotonicity is a basic property of the Riemann integral, it follows that **(PO3)** is also satisfied. It also follows from the strict monotonicity of Riemann integrals that if $x_{\sigma(i)} > 0$ for all $i \in \{1,\ldots,n\}$, then

$$\mathcal{PO}(x_{\sigma(1)},\ldots,x_{\sigma(n)}) = \frac{1}{h}\int_0^{x_{\sigma(n)}} \cdots \int_0^{x_{\sigma(1)}} PO(u_1,\ldots,u_n)du_1\ldots du_n > 0$$

, and so, Property **(PO1)** is satisfied by the contrapositive. Finally, if $x_{\sigma(i)} = 1$ for all $i \in \{1,\ldots,n\}$ then

$$\begin{aligned}\mathcal{PO}(1,\ldots,1) &= \frac{1}{h}\int_0^1 \cdots \int_0^1 PO(u_1,\ldots,u_n)du_1\ldots\, du_n \\ &= \frac{1}{h}\cdot h \\ &= 1.\end{aligned}$$

Therefore, Property **(PO2)** is satisfied. □

Theorem 9. *Let $\sigma : \{1,\ldots,n\} \longrightarrow \{1,\ldots,n\}$ be a permutation and let $PG : [0,1]^n \longrightarrow [0,1]$ be a pseudo-grouping function. If*

$$h = \int \cdots \int_{[0,1]^n} (1 - PG(1-u_1,\ldots,1-u_n))du_1\ldots du_n$$

, then the map $\mathcal{PG} : [0,1]^n \longrightarrow [0,1]$ given by

$$\mathcal{PG}(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = 1 - \frac{1}{h}\int_0^{1-x_{\sigma(n)}} \cdots \int_0^{1-x_{\sigma(1)}} (1 - PG(1-u_1, \ldots, 1-u_n))du_1 \ldots du_n$$

is also a pseudo-grouping function.

Proof. As a special case of the 1 corollary, the proof it follows is based on considerations similar to Theorem 8. □

Example 5. *The following examples are considered:*

1. *Let $PO : [0,1]^4 \longrightarrow [0,1]$ be such that $PO(u_1, u_2, u_3, u_4) = u_1^2 u_2^3 u_3^4 u_4^5$. Then, $h = \frac{1}{360}$ and*

$$\begin{aligned} \mathcal{PO}(x,y,z,w) &= \frac{1}{h}\int_0^w \int_0^z \int_0^y \int_0^x PO(u_1,u_2,u_3,u_4)du_1 du_2 du_3\, du_4 \\ &= 360 \cdot \int_0^w \int_0^z \int_0^y \int_0^x u_1^2 u_2^3 u_3^4 u_4^5 du_1 du_2 du_3\, du_4 \\ &= x^3 y^4 z^5 w^6. \end{aligned}$$

On the other hand,

$$\begin{aligned} \mathcal{PO}(w,z,x,y) &= \frac{1}{h}\int_0^y \int_0^x \int_0^z \int_0^w PO(u_1,u_2,u_3,u_4)du_1 du_2 du_3\, du_4 \\ &= 360 \cdot \int_0^y \int_0^x \int_0^z \int_0^w u_1^2 u_2^3 u_3^4 u_4^5 du_1 du_2 du_3\, du_4 \\ &= x^6 y^5 z^4 w^3. \end{aligned}$$

More generally, if $PO : [0,1]^n \longrightarrow [0,1]$ is such that $PO(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = \prod_{i=1}^{n} x_{\sigma(i)}^{r_i}$, where r_i is a positive integer, then

$$\mathcal{PO}(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = \frac{1}{h}\int_0^{x_{\sigma(n)}} \cdots \int_0^{x_{\sigma(1)}} PO(u_1, \ldots, u_n)du_1 \ldots du_n = \prod_{i=1}^{n} x_{\sigma(i)}^{r_i+1}.$$

2. *Let $PG : [0,1]^4 \longrightarrow [0,1]$ be such that $PG(u_1,u_2,u_3,u_4) = 1 - (1-u_1)^2(1-u_2)^3(1-u_3)^4(1-u_4)^5$. Then, $h = \frac{1}{360}$ and*

$$\begin{aligned} &\mathcal{PG}(x,y,z,w) = \\ = \; & 1 - \frac{1}{h}\int_0^{1-w} \int_0^{1-z} \int_0^{1-y} \int_0^{1-x} 1 - PG(1-u_1, 1-u_2, 1-u_3, 1-u_4)du_1 du_2 du_3\, du_4 \\ = \; & 1 - 360 \cdot \int_0^{1-w} \int_0^{1-z} \int_0^{1-y} \int_0^{1-x} u_1^2 u_2^3 u_3^4 u_4^5 du_1 du_2 du_3\, du_4 \\ = \; & 1 - (1-x)^3(1-y)^4(1-z)^5(1-w)^6. \end{aligned}$$

On the other hand,

$$\begin{aligned} &\mathcal{PG}(w,z,x,y) = \\ = \; & 1 - \frac{1}{h}\int_0^{1-y} \int_0^{1-x} \int_0^{1-z} \int_0^{1-w} 1 - PG(1-u_1, 1-u_2, 1-u_3, 1-u_4)du_1 du_2 du_3\, du_4 \\ = \; & 1 - 360 \cdot \int_0^{1-y} \int_0^{1-x} \int_0^{1-z} \int_0^{1-w} u_1^2 u_2^3 u_3^4 u_4^5 du_1 du_2 du_3\, du_4 \\ = \; & 1 - (1-x)^6(1-y)^5(1-z)^4(1-w)^3. \end{aligned}$$

More generally, if $PG : [0,1]^n \longrightarrow [0,1]$ *is such that* $PG(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = 1 - \prod_{i=1}^{n}(1 - x_{\sigma(i)})^{r_i}$, *where* r_i *is a positive integer, then*

$$\begin{aligned} \mathcal{PG}(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) &= 1 - \frac{1}{h}\int_0^{1-x_{\sigma(n)}} \cdots \int_0^{1-x_{\sigma(1)}} PG(1-u_1, \ldots, 1-u_n)du_1 \ldots du_n \\ &= 1 - \prod_{i=1}^{n}(1 - x_{\sigma(i)})^{r_i+1}. \end{aligned}$$

4. Illustrative Example

In various real-world scenarios, such as personal, social, work, or business contexts, it is necessary to take into account different aspects to select among different alternatives [39]. However, decision-making can become more intricate when the available information is uncertain or ambiguous, making it essential to incorporate the perspectives of a group of experts to consider multiple viewpoints [44]. When addressing a multi-criteria group decision-making problem, it must be noticed that the commutative of the aggregations must be dropped if either the considered criteria have different levels of importance or the experts belong to different levels of expertise. This section presents a case study that aims to illustrate the implications of pseudo-overlaps and pseudo-groupings in a decision-making real-world scenario: the winner selection process for a TV music contest.

The TV music contest in question involves three types of decision-makers: a professional jury, a popular jury, and the public vote. The professional jury is comprised of five experts in the music industry who have a wealth of experience and knowledge in evaluating performances. The popular jury consists of 50 people who attended the show on-site, while the public vote is open to anyone who wishes to participate via an online voting platform. Since each group is assumed to have different levels of commitment to the success of the show, the professional jury is given a weight of 45%, whereas the popular jury and the public vote are weighted as 35% and 20%, respectively, i.e., $u = (0.45, 0.35, 0.20)$.

In addition, all three decision-making groups evaluate the contestants based on four criteria: voice, tune, lyrics, and staging, whose respective weight vectors in the decision process are $v = (0.30, 0.40, 0.15, 0.35)$. After a counting round, the obtained ratings for the last three finalists are shown in Table 1.

To decide the winner of the contest, here, we define a decision method based on proper pseudo-overlaps and pseudo-groupings. On the one hand, in this kind of situation, where multiple groups of experts (with different weights) provide judgments, the use of a pseudo-overlap can be highly effective for aggregating their opinions. In this sense, a pseudo-overlap can quantify how well a contestant meets the requirements of all three groups of experts simultaneously. Therefore, let us consider the weight vector u for the groups of experts and the pseudo-overlap defined as Item 4 of Example 4 to aggregate the different judgments for each alternative:

$$\widetilde{PO}(x_1, x_2, x_3) = \frac{\sqrt[3]{\prod_{i=1}^{3} x_i^{u_i}}}{\sqrt[3]{\prod_{i=1}^{3} x_i^{u_i}} + 1 - \sqrt[3]{\prod_{i=1}^{3} x_i^{2u_i}}}$$

Table 1. Average opinions for the finalists on a 1–10 scale.

		Finalist 1		
	Voice	Tune	Lyrics	Staging
Prof. Jury	6/10	7/10	9/10	7/10
Pop. Jury	2/10	7/10	5/10	9/10
Public	2/10	9/10	6/10	2/10
		Finalist 2		
	Voice	Tune	Lyrics	Staging
Prof. Jury	1/10	9/10	9/10	9/10
Pop. Jury	3/10	7/10	9/10	2/10
Public	9/10	4/10	6/10	4/10
		Finalist 3		
	Voice	Tune	Lyrics	Staging
Prof. Jury	7/10	8/10	1/10	9/10
Pop. Jury	2/10	9/10	2/10	7/10
Public	7/10	3/10	9/10	5/10

The normalized global scores for the finalists resulting from performing the corresponding computational processes are shown in Table 2.

Table 2. Aggregated performance for each finalist.

	Voice	**Tune**	**Lyrics**	**Staging**
Finalist 1	0.568	0.83	0.792	0.742
Finalist 2	0.494	0.808	0.889	0.651
Finalist 3	0.651	0.798	0.469	0.828

Similarly, to evaluate each finalist's performance based on multiple criteria, a pseudo-grouping that measures the degree of certainty that the alternative meets at least one of the criteria would be beneficial. Then, if we consider the weighting vector for criteria v and the pseudo-grouping defined in Example 4,

$$\widetilde{PG}(x_1, x_2, x_3, x_4, x_5) = \frac{\max\limits_{i\in\{1,\ldots,5\}} \{x_i^{v_i}\}}{\max\limits_{i\in\{1,\ldots,5\}} \{x_i^{v_i}\} + \sqrt{1 - \frac{3}{4}\left(\max\limits_{i\in\{1,\ldots,5\}} \{x_i^{v_i}\}\right)^2 - \frac{1}{2}\left(\max\limits_{i\in\{1,\ldots,5\}} \{x_i^{v_i}\}\right)}},$$

we obtain the global final scores for each finalist, which are shown in Table 3.

Table 3. Final score for each finalist.

Finalist 1	0.713
Finalist 2	0.715
Finalist 3	0.705

Consequently, the winner of the music contest should be Finalist 2.

5. Comparative Analysis

Here, we develop a comparative analysis to assess how the results obtained using the proposed aggregation operators compare to those obtained with alternative approaches. By considering multiple aggregation operators, we provide a comprehensive evaluation that captures a broader perspective and enables us to make informed recommendations for practical applications.

First, we compare our method to the case where commutative aggregation operators are used. Specifically, instead of employing a proper pseudo-overlap, we consider the minimum operator to obtain the aggregated performance of each finalist. The minimum also represents an "and"-like aggregation, and the resulting value aims at measuring the degree to which all the groups are satisfied simultaneously. The aggregated performance for each finalist, in this case, is presented in Table 4.

Table 4. Aggregated performance for each finalist when using the minimum operator.

	Voice	**Tune**	**Lyrics**	**Staging**
Finalist 1	0.2	0.7	0.5	0.2
Finalist 2	0.1	0.4	0.6	0.2
Finalist 3	0.2	0.3	0.1	0.5

Note that in this case, the assumption of commutativity implies that the weights of the groups are not considered in the aggregation, thus neglecting their importance.

To compute the overall score of each finalist, instead of using a pseudo-grouping, we use the commutative or-like aggregation operator defined by the maximum. This operator allows modeling the degree to which one finalist highlights at one of the skills. The results are shown in Table 5.

Table 5. Final score for each finalist when using the maximum operator.

Finalist 1	0.7
Finalist 2	0.6
Finalist 3	0.5

Again, the assumption of a commutative aggregation operator does not allow considering the weight of each criterion in the aggregation. As a consequence, the final ranking, in which the winner would be Finalist 1, differs from the one obtained when considering pseudo-overlaps and pseudo-groupings, but it is not very reliable because the importance of the group of experts and the criteria are not considered in the aggregation, despite the fact that they are essential in the decision-making problem.

Additionally, we study the resolution of the same problem when instead of pseudo-overlaps and pseudo-groupings, we use the WA operator to compute the aggregated performance of each finalist. For the weighting vector u, we obtain the results displayed in Table 6.

Table 6. Aggregated performance for each finalist using the weighted average.

	Voice	**Tune**	**Lyrics**	**Staging**
Finalist 1	0.38	0.74	0.7	0.67
Finalist 2	0.33	0.73	0.84	0.555
Finalist 3	0.525	0.735	0.295	0.75

Subsequently, we apply the WA operator to compute the final score for each finalist by considering the weighting vector for the criteria v (see Table 7).

Table 7. Final score for each finalist when using the weighted average.

Finalist 1	0.75
Finalist 2	0.711
Finalist 3	0.758

In this case, the winner of the contest would be Finalist 3. Even though the WA operator succeeds at accounting for the importance of the considered groups and criteria, it cannot model the and-like and or-like behaviors of pseudo-overlaps and pseudo-groupings, respectively, which are key to accounting for the degree to which all the groups are satisfied or a participant is particularly good at one skill.

6. Conclusions

In this paper, we have introduced and developed various construction methods for non-commutative aggregation operators, specifically, pseudo-overlaps and pseudo-groupings. First, we have shown how pseudo-overlaps and pseudo-groupings can be obtained from the classical overlaps and groupings, respectively. Additionally, we have analyzed the relation between pseudo-overlaps and pseudo-groupings via fuzzy negations and shown that convex sums of pseudo-overlaps or pseudo-groupings are, respectively, pseudo-overlaps and pseudo-groupings. Moreover, we have proposed some results related to generating pseudo-overlaps and pseudo-groupings using integration.

To illustrate the practical relevance of the proposed methods, we have presented a case study of a multi-criteria group decision-making problem. The problem was solved using some of the proper pseudo-overlaps and pseudo-groupings constructed in this paper. Our results demonstrate that pseudo-overlaps and pseudo-groupings can provide useful and effective tools for decision-making in real-world applications, where commutativity may not be a reasonable assumption.

In conclusion, the methods presented in this paper offer new opportunities for decision-making and information fusion in real-world scenarios, where the commutativity assumption is not satisfied. Future research may explore further properties, such as Lipschitzianity, homogeneity or idempotency, and applications of pseudo-overlaps and pseudo-groupings into other real-world problems.

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