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Madhumangal Pal · Ronald R. Yager
Editors

Decision Making Under Uncertainty Via Optimization, Modelling, and Analysis

Springer

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I hereby dedicate this edited book to my esteemed teacher, Shri Sujit Kumar Kar, who imparted the invaluable lessons of education, knowledge, and wisdom.

—Laxminarayan Sahoo

I dedicate this edited book to my parents, my better half Smt. Swagata Dutta Senapati; and son, Ayush.

—Tapan Senapati

I dedicate this edited book to my parents, who impart the initial lessons of education.

—Madhumangal Pal

This edited book is dedicated to my wife, Rachel.

—Ronald R. Yager

Preface

With the use of the optimization technique, decision-makers may state that the best choice under consideration should be optimal. This optimization technique has an affinity to the scientific method or approaches that makes optimal to the decision-makers. The challenges posed by contemporary technology, which is also supported by cutting-edge scientific methodologies, call for a significant number of novel procedures and sophisticated analytical tools. It causes modeling and optimization to develop as a method for quickly selecting the best course of action from a wide range of scenarios and possibilities. Even if there are multiple divides of possible activities for which the decision-makers must make decisions in all aspects of daily life and they have come to realize that various issues can arise in various situations. Modeling and optimization are thus used everywhere, but both have a lot of same distinctive features. In addition, a common analytical technique can be utilized to identify the best answer to issues ranging under the identical umbrella to resolve real-world decision-making issues. As usual, in today's world, the demands of users and decision-makers, as well as the quantity of businesses, are what keep any industry competitive. Several businesses concentrate on creating and adding to highly reliable and dependable solutions. Additionally, it has become an impossible task for many manufacturing companies in the present day to determine the best strategies to manage corporate policies. Further, because of ongoing monetary liberalization, improved safety, rising consumer demands, rising globalization, sophisticated modern technical systems, and a constantly evolving business model, the manager of the company faces regular bouts of stress for sustaining the business systems. Moreover, due to uncontrollable factors of dynamic economic environment, the manager or decision-makers of the manufacturing company faces a lot of difficulties. Consequently, parameters relating to the decision-making may arising in terms of uncertainty or lack of proper information and uncertainty occurs at each step at the point of suitable decisions. During the last decades, various solution methodologies have been proposed to deal with the challenges about uncertainties. Decision-making under uncertainty via optimization, modeling, and analysis have provided their effectiveness to manage the difficulties and solving the real-world problems with imprecise parameters. In recent years, the landscape of medical treatment has undergone a profound transformation, driven by advances in machine learning and the rising importance of addressing uncertainty within the healthcare system. This edited book also explores the integration of smart city planning, waste management system, strategic decision-making in different industries, based on utility theory, game theory, heuristic approaches offering innovative solutions to modern challenges. Through meticulous modeling and optimization, it elucidates the applications of Markovian processes and their critical role in business sectors. The convergence of game theory and inventory management systems is examined, highlighting how strategic interactions can optimize supply chains and enhance efficiency. By leveraging these theoretical frameworks, the complexities of urban infrastructures are addressed, presenting new avenues for research and implementation. This comprehensive guide is intended to provide a robust foundation for professionals and researchers alike, seeking to navigate the intricacies of modern managerial decision-making through modeling and optimization. Emphasis is placed on the practical applications of these theories, ensuring relevance to real-world scenarios. The present edited book has been divided into 34 chapters. "Linguistic Z Numbers-Based FMEA of the Delivery of Stereotactic Body Radiation Therapy for Lung Cancer Treatment," has been highlighted in the first chapter. Through a Linguistic Z numbers-based Failure Mode and Effects Analysis (FMEA), this chapter investigates potential failure modes and their effects in delivering stereotactic body radiation therapy for lung cancer treatment. It offers insights into enhancing safety protocols and minimizing risks associated with this critical medical procedure. In the second chapter, entitled "A Novel Triangular Divergence Distance-Based TOPSIS Method for Interval-Valued Fermatean Fuzzy Group Decision-Making," an innovative approach to group decision-making is presented. The use of triangular divergence distance in the TOPSIS method for interval-valued Fermatean fuzzy data is explored. Advanced methods for making decisions are discussed in the third chapter via titled, "Archimedean Averaging Operator with Linear Programming Model for Generalized Orthopair Fuzzy Decision-Making." An in-depth analysis of the combination of the Archimedean averaging operator and a linear programming model is presented for managing fuzzy generalized orthopair data, offering a thorough method for addressing decisions in difficult situations. In the fourth chapter, titled "Mean Squared Error Utility for Fuzzy Preference Relations," the application of mean squared error utility in analyzing fuzzy preference relations is examined. The effectiveness of this method in enhancing decision-making accuracy within fuzzy systems is explored in depth. The fifth chapter deals with "MCDA/F-DEMATEL/ICTs Method Under Uncertainty in Mathematics Education: How to Make a Decision with Flipped, Gamified, and Sustainable Criteria." In this chapter, decision-making strategies under uncertainty in educational settings are examined. The integration of modern teaching approaches and sustainability is explored using advanced MCDA and F-DEMATEL methods. "Uncertainty Approaches for Spatial Data Management," has been addressed in the sixth chapter. Here, the handling of spatial data under uncertainty is explored. Various techniques for managing and mitigating uncertainties in spatial data are examined, providing insights into effective spatial data management practices. In the seventh chapter, "Evaluating Sustainable Production Barriers and Efficiency-Enhancing Techniques in Manufacturing Companies Using Bipolar Neutrosophic REF," the challenges to sustainable production have been examined. Efficiency-enhancing techniques are analyzed within the context of manufacturing companies. The application of Bipolar Neutrosophic REF in identifying and addressing these barriers is explored, providing valuable insights into achieving sustainability and efficiency in the manufacturing sector. In the eighth chapter, titled "Assessment of Smart City Performance Using Multi-Criteria Decision-Making Methods," the evaluation of smart city performance is examined. Through the utilization of various multi-criteria decision-making methods, this chapter investigates the effectiveness of smart city initiatives. By analyzing multiple criteria, it provides valuable insights into assessing and improving the performance of smart cities, contributing to the advancement of urban development strategies in the digital age. "Impact of Technological Innovation on Employment Under Intuitionistic Fuzzy Einstein Aggregation Information with Z-Numbers," has been addressed in the ninth chapter. Through the utilization of intuitionistic fuzzy Einstein aggregation information with Z-numbers, this chapter examines the complex dynamics of technological advancement and its implications for the workforce. It provides valuable insights into the evolving relationship between technology and employment, informing strategies for navigating the future of work. In the tenth chapter, entitled "Cubic Analytic Hierarchy Process with Application in Decision-Making," innovative decision-making methodologies are explored. The application of the Cubic Analytic Hierarchy Process (CAHP) is examined, offering a robust framework for making complex decisions. This chapter delves into the practical applications of CAHP across various domains, providing valuable insights into its effectiveness in facilitating informed and structured decision-making processes. In the eleventh chapter, entitled "Decision-Making Process Under Uncertain Domain of Pythagorean Fuzzy Sets Based on an Enhanced Similarity Operator," the complexities of decision-making under uncertainty have been covered. Enhanced similarity operators are applied to navigate the uncertain domain of Pythagorean fuzzy sets, enabling a nuanced approach to decision-making. This chapter gives significant insights into addressing uncertainty, boosting the effectiveness of decision-making processes in varied contexts. "Bipolar Fuzzy Generalized Dombi Aggregation Operators for Group Decision-Making" has been addressed in the twelfth chapter. In this chapter, innovative aggregation techniques are investigated for group decision-making scenarios. The utilization of bipolar fuzzy sets in generalized Dombi aggregation operators is explored, offering a robust framework for reconciling diverse viewpoints within group deliberations. This chapter sheds light on advanced methodologies to streamline the decision-making process in complex group settings. In the thirteenth chapter, provides "Utility-Theory-Based Product Selection: Strategic Decision-Making in Automotive Industry," where, strategic decision-making processes within the automotive sector are examined. The application of utility theory for product selection is explored, providing a framework for making informed choices. This chapter highlights the importance of aligning product selection with strategic goals to enhance competitiveness and sustainability in the automotive industry. The issues of managing inventory within multi-production periods are addressed in the fourteenth chapter, "Imperfect Production Inventory Under Multi-Production Cycle for Non-deteriorating Items with Carbon Tax and Green Investment." With the carbon tax and green investment methods, sustainability is emphasized. In the fifteenth chapter, headed "An EOQ Model of a Fresh Product Having Variable Demand Under Trade Credit Policy," the dynamics of inventory management for perishable goods are examined. The impact of variable demand on the Economic Order Quantity (EOQ) model is explored within the context of trade credit policies. This chapter offers insights into optimizing inventory levels to meet fluctuating demands effectively. "Inventory Model with Negative Exponential Demand and Instantaneous Replenishment Under Preservation Technology," has been discussed in the sixteenth chapter. The chapter explores how preservation technology influences inventory models with negative exponential demand and instantaneous replenishment. Through rigorous analysis, it offers insights into optimizing inventory levels while accounting for preservation requirements. This chapter contributes to enhancing efficiency in inventory management systems, particularly in industries where preservation of goods is paramount. "A Real Coded Heuristic Technique in Solving a Plant Location Optimization Problem with Interval-Valued Transportation Costs and Quantities," has been highlighted in the seventeenth chapter. In this chapter, advanced methodologies for plant location optimization are examined. The application of real coded heuristic techniques is explored to address challenges posed by interval-valued transportation costs and quantities. This chapter provides valuable insights into optimizing plant locations under uncertain transportation parameters. In the eighteenth chapter, outlined "Interval Quadratic Programming Problem with Interval-Valued Decision Variables," the complexities of quadratic programming are explored. The focus is on problems where decision variables are interval-valued. This chapter delves into techniques for solving such problems, offering insights into optimizing decision-making processes when faced with uncertainty in variable values. In the nineteenth chapter, entitled "A Case Study on Medical Company Selection in the Health Sector by Using an Integrated Fuzzy AHP-MOORA Method," the process of allocating reliability resources is investigated. The chapter explores methodologies for allocating reliability in systems under uncertainty, considering both optimistic and pessimistic scenarios. Through rigorous analysis, it aims to optimize reliability allocation strategies, ensuring robust performance in systems where uncertainty is prevalent. This chapter contributes to enhancing reliability engineering practices, particularly in contexts where uncertainty poses significant challenges. In the twentieth chapter, headlined "Novel Pythagorean Fuzzy Hamacher Aggregation Operator and Its Application to Green Supplier Selection in Pharmaceutical Industry," novel approaches to game theory are discussed. The use of heuristic strategies is examined for solving games with interval-valued payoffs and objectives. This chapter presents a novel perspective on solving difficult decision-making situations providing useful insights into the application of heuristics in game theory. The twenty-first chapter, provides "On the Maintenance Oversight of the Healthcare Sector Based on Artificial Intelligence." Through the utilization of an integrated fuzzy AHP-MOORA method, this chapter examines the systematic approach to evaluating and selecting medical companies, providing valuable insights into decision-making processes within the healthcare industry. In the twenty-second chapter, entitled "Early Diagnosis of Medical Images in Healthcare Management by Artificial Intelligence," the development and application of a novel aggregation operator are explored. The chapter investigates how this operator enhances the process of green supplier selection within the pharmaceutical industry. Through rigorous analysis, it offers insights into improving sustainability practices and supplier evaluation methodologies, contributing to greener supply chain management in the pharmaceutical sector. The twenty-third chapter, discusses "Comparing Metrics of Classification Algorithms in Sentiment Analysis: A Comparative Study of Logistic Regression and KNN Using Count Vectorizer." The utilization of artificial intelligence for maintenance oversight in healthcare is investigated in this chapter. Advanced AI techniques are examined for optimizing maintenance operations, ensuring the efficient functioning of healthcare facilities. This chapter offers insights into leveraging AI to enhance the reliability and performance of maintenance processes within the healthcare sector. "Healthcare Waste Disposal Location Selection Using q-Rung Orthopair Fuzzy MEREC—CoCoSo Technique—A Case Study," has been explored in the twenty-fourth chapter. The chapter investigates the potential of AI to facilitate early diagnosis through the analysis of medical images. By employing advanced algorithms and deep learning techniques, it aims to enhance medical decision-making processes, leading to more timely and accurate diagnoses for improved patient care and outcomes. In the twenty-fifth chapter, titled "Optimal Plastic Waste Recycling Technology Selection Using Picture Fuzzy SWARA–MARCOS Technique," the effectiveness of classification algorithms in sentiment analysis is examined. A comparative study between Logistic Regression and KNN is conducted, utilizing Count Vectorizer. This chapter offers valuable insights into the performance metrics of different algorithms, enhancing our understanding of their applicability in sentiment analysis tasks. The twenty-sixth chapter presents "MCDM PROMETHEE Method in Identifying Best Teacher Awardee." In this chapter, the selection process for healthcare waste disposal locations is examined. Utilizing the q-Rung Orthopair Fuzzy MEREC—CoCoSo technique, this chapter investigates optimal site selection methodologies. Through a detailed case study, it provides valuable insights into efficient healthcare waste management strategies, ensuring environmentally sound and sustainable disposal practices. In the twenty-seventh chapter, entitled "Evaluating Challenges in Smart Transportation with Grey Theory-Based Multi-criteria Decision-Making," the process of selecting optimal recycling technologies for plastic waste is investigated. The Picture Fuzzy SWARA—MARCOS technique is utilized to analyze and prioritize various recycling methods. This chapter offers insights into the systematic approach of choosing the most effective recycling technologies, contributing to sustainable waste management practices and environmental conservation efforts. The twenty-eighth chapter, provides "T-spherical Fuzzy Group Decision-Making Using Subjective and Objective Weights of Experts and Copula Aggregation Operators." Here, the process of identifying outstanding teachers is explored. The Multi-Criteria Decision-Making (MCDM) Promethee method is employed to assess and rank teacher candidates based on various criteria. This chapter provides insights into the systematic approach of selecting the best teacher awardees, contributing to the recognition and appreciation of exemplary educators within educational institutions. In the twenty-ninth chapter, titled "Chap "Effect of Internal Delay on the Dynamics of a Mean-Field Diffusive Coupled Oscillating System"," the complexities of smart transportation are analyzed. Utilizing a Grey Theory-based Multi-Criteria Decision-Making approach, this chapter delves into the assessment of challenges within smart transportation systems. Through rigorous analysis, it offers insights into addressing multifaceted issues such as traffic congestion, environmental sustainability, and technological integration. This chapter serves as a valuable resource for policymakers, urban planners, and transportation stakeholders seeking to enhance smart transportation initiatives. "Markovian Brand Switching Model for Long-Term Steady-State Market Shares: A Study on Toothpaste Market," has been investigated in the thirtieth chapter. The chapter investigates the integration of subjective and objective weights of experts in T-spherical fuzzy decision-making processes. Through the utilization of Copula aggregation operators, it aims to provide a comprehensive framework for effectively synthesizing diverse expert opinions in decision-making scenarios. The thirty-first chapter deals with "Root Hair Algorithm: A Swarm Intelligence Algorithm." Through rigorous analysis, the chapter investigates how delays within coupled oscillating systems influence their behavior. By examining the dynamics of mean field diffusive coupling, it sheds light on the intricate interplay between internal delays and system oscillations, contributing to a deeper understanding of complex dynamical systems. "Pythagorean Fuzzy Ordered Weighted Averaging Aggregation Operator Based on Appropriate Score Function and Their Application to Multi-criteria Decision-Making in IT Project Management," has been provided in the thirty-second chapter. In this chapter, the dynamics of market share in the toothpaste industry are investigated. Utilizing a Markovian approach, long-term steady-state market shares are analyzed. This chapter provides valuable insights into the factors influencing brand switching behavior among consumers, offering a comprehensive study of market dynamics within the toothpaste market. In the thirty-third chapter, entitled "33 Root Hair Algorithm: A Swarm Intelligence Algorithm," the exploration of swarm intelligence is undertaken. This chapter delves into the principles and applications of the Root Hair Algorithm, elucidating its effectiveness in solving optimization problems inspired by the collective behavior of root hairs. Through a comprehensive analysis, it offers insights into harnessing natural phenomena to design efficient optimization algorithms. Finally, "Pythagorean fuzzy ordered weighted averaging aggregation operator based on appropriate score function and their application to multi-criteria decision-making in IT Project Management" has been highlighted in Chapter 34. In this chapter, the Pythagorean fuzzy ordered weighted averaging (PFOWA) aggregation operator, utilizing an appropriate score function, is examined. Its application to multi-criteria decision-making in IT project management is explored. Emphasis is placed on the operator's effectiveness in handling uncertainty and improving decision-making quality in IT projects.

Researchers, students, engineers, managers of manufacturing firms, decision-makers, and anyone else interested in decision-making under uncertainty through modeling, analysis, and optimization can all benefit from this book. The editors extend their utmost gratitude to the authors from all over the world who contributed to this edited work under a variety of conditions. Special thanks go out to all the reviewers whose comments and suggestions have helped improve the quality of the chapters. Mr. Aninda Bose, Executive Editor (Interdisciplinary Applied Sciences, Computational Intelligence, Energy) at Springer Nature, London, is also thanked for his kind consideration and endless support for this edited book. It is hoped that the excitement felt in presenting this edited book will be shared by the reader and that it will be found beneficial.

Raiganj, India
April 2024

Laxminarayan Sahoo
Tapan Senapati
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Laxminarayan Sahoo
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Prof. Madhumangal Pal is currently a professor and head of Applied Mathematics and Director of IQAC, Vidyasagar University. He has received gold and silver medals from Vidyasagar University for ranking first and second in the M.Sc. and B.Sc.

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Prof. Ronald R. Yager has worked in the area of machine intelligence for over 25 years. He has published near about 1000 papers and more than 30 books in areas related to artificial intelligence, fuzzy sets, decision-making under uncertainty, and the fusion of information. He is among the world’s top 1% most highly cited researchers with over 95237 citations. He was the recipient of the IEEE Computational Intelligence Society’s highly prestigious Frank Rosenblatt Award in 2016. He was the recipient of the IEEE Systems, Man and Cybernetics Society 2018 Lotfi Zadeh Pioneer Award. He was also the recipient of the IEEE Computational Intelligence Society Pioneer Award in Fuzzy Systems. He received honorary doctorates from the Azerbaijan Technical University, the State University of Information Technologies, Sofia Bulgaria, and the Rostov on the Don University, Russia.

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Mean Squared Error Utility for Fuzzy Preference Relations

Diego García-Zamora and Luis Martínez

Abstract The use of Fuzzy Preference Relations (FPRs) is prevalent in expert-driven decision-making, and thus obtaining the overall performance of each alternative from an FPR is a key topic in the area. This chapter proposes the notion of Mean Squared Error (MSE) utility vector associated with an FPR, which is defined as the utility vector that determines the pairwise comparison matrix that is closest to the original FPR. We demonstrate that this utility vector can be analytically computed and is a valid utility function since it assigns higher utility values to higher rated alternatives. Furthermore, we show that the MSE utilities can be computed although the comparison matrix is elicited using a multiplicative scale, or even when just one row and one column of the matrix are given. The applicability of the MSE utility vectors is then shown in a Large-Scale Group Decision-Making problem, in which we use them to accelerate the computational time necessary to reach a consensus. Additionally, we provide a comparative analysis with other classic methods for deriving priorities from pairwise comparison matrices.

Keywords Converting matrix to vector representation · Utility values for pairwise comparison matrix · Analytical solution for Analytic Hierarchy Process (AHP) · Analytical solution for Best-Worst Method (BWM) · Computational efficiency for large-scale group decision-making (LSGDM)

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1 Introduction

The use of pairwise comparisons to model experts' opinions in decision-making problems is widely extended in the specialized literature [1] because human beings tend to perform better when eliciting preferences by pairwise comparisons among a set of alternatives, rather than comparing each alternative individually against all the others [2, 3]. Essentially, in the decision-making literature, the most widely used type of pairwise comparison matrices are Fuzzy Preference Relations (FPRs), which are given in the [0, 1] interval [4], and Multiplicative Preference relations (MPRs), often provided in a 1–9 Saaty's scale [5].

Pairwise comparisons may provide more precise modeling according to the human psyche than a single preference of one alternative versus all the others [2, 3]. However, some inconsistencies could emerge when they are elicited [6]. These inconsistencies are translated into contradictory evaluations that will negatively impact the decision process if they are not properly managed [6, 7]. When facing inconsistent preference relations, there are two main strategies. The first one is asking the expert to modify their preferences until they are consistent enough [7]. The second one consists of using an automatic mechanism that modifies the matrix until it is consistent [8].

Another aspect to be taken into account when using pairwise comparisons is the fact that ranking the alternatives from pairwise comparisons requires carrying out an additional step. In this regard, some of the most popular approaches are the computation of the dominance of each alternative [9], the eigenvalue algorithm [6], or the Best-Worst Method (BWM) [7]. The first of them provides a simple procedure to compute an overall performance for each alternative but neglects the inconsistencies of the pairwise comparison matrix. On the contrary, the other two approaches consider the consistency of the matrix, but they are computationally more complex: the eigenvalue method requires computing the eigenvalues and eigenvectors of the matrix, whereas BWM needs to solve an optimization problem.

Therefore, several researchers have attempted to solve both problems together by defining methods to derive priorities from pairwise comparison matrices that consider their consistency [10]. In this regard, a common approach is approximating the pairwise comparison matrix by the closest consistent pairwise comparison matrix using an optimization method. Even though such a methodology has led to different results, several aspects remain neglected.

The first one is the relationship between all of these proposals. Different researchers have used such an optimization approach to obtain priority values or weights from pairwise comparison matrices [11, 12]. However, the interaction of such priorities or weights with classical non-optimization-based methods such as AHP has not been studied yet [13].

Second, some methods assume that priority and weights are similar concepts and, consequently, there is no distinction in the methodology to compute them [5].

Third, it should be noticed that obtaining priorities or weights from a pairwise comparison matrix allows for the reduction of the dimensionality from n^2 to n. This fact is especially relevant when managing large volumes of data. Nevertheless, to guarantee that such a dimension reduction is robust, it is necessary to show that after performing computations with the priorities/weights, the resulting priorities/weights are still related to the pairwise comparison matrix that would be obtained if the same computations were performed on the original pairwise comparisons instead.

Therefore, this chapter proposes an alternative method based on the Mean Squared Error (MSE) to obtain an overall performance of each alternative from an FPR by considering its consistency. To do so, we analytically solve an optimization problem based on minimizing the MSE to derive a utility vector that *summarizes* the information within the FPR. Such a utility vector satisfies two main properties. On the one hand, it allows univocally constructing the consistent FPR which is closest to the original preference. On the other hand, such a vector, although it is defined as the solution of an optimization problem, can be computed through an algebraic expression that increasingly depends on the mean dominance of the alternatives. Furthermore, remapping the pairwise comparison matrix into such a utility vector implies a dimensionality reduction, and thus more efficiency in computations with them. Additionally, these utility values can also be obtained when the pairwise comparison matrix is given in a multiplicative scale [6] or BWM preferences [7]. In a parallel way, we also define the MSE weighting vector for FPRs, MPRs, and BWM preferences, which instead of utility values provide weights resembling the importance of the alternatives.

To illustrate the potential of the proposed method, the MSE utility vector is then applied to address to improve the computational time necessary to carry out a Consensus-Reaching Processes (CRPs) [14–16] for Large-Scale Group Decision-Making (LSGDM) [17, 18]. Furthermore, the performance of the MSE weighting vector is later compared with two classic methods to derive weights, namely the eigenvalue method usually applied in Analytic Hierarchy Process (AHP) [6], and classic Best-Worst Method (BWM) [7].

The remainder of this chapter is as follows. Section 2 introduces some basic notions about decision-making and pairwise comparison matrices. Section 3 analyzes first how additive and multiplicative consistency are related through a nonlinear rescaling of MPRs and FPRs. Furthermore, it provides a characterization of both additive and multiplicative consistency in terms of utility vectors. Later, we introduce a method to derive such utilities in the case in which the original preference relation is not fully consistent. In Sect. 4 the MSE utility vector is applied to LSGDM. Section 5 compares the MSE weighting vectors with other classic weight determination methods. Finally, Sect. 6 concludes the chapter.

2 Preliminaries: Pairwise Comparisons in Decision-Making

This section introduces the background related to decision-making with pairwise comparison matrices. In a decision-making problem, various alternatives are analyzed to select which one is the most suitable [19]. In this regard, the increasing complexity of decisions nowadays demands the consideration not only of data-driven methods but also of expert knowledge [17].

However, gathering the experts' opinions about the alternatives is not a simple task because when humans are asked to provide their preferences, they do not always reflect what they truly think [2, 3]. Therefore, in the field of decision-making, determining the most appropriate preference structure is a crucial and ongoing area of research. As such, understanding and selecting the most suitable preference structure continues to be a key topic in the decision-making literature [17].

To improve the quality of the decisions, the notion of a pairwise comparison matrix was introduced to better reflect experts' thoughts [4]. In contrast to rating the alternatives separately, when using pairwise comparisons, each expert provides how much he or she prefers an alternative to all the others by comparing each one individually. Once the expert has fulfilled the pairwise comparisons, an auxiliary method is applied to determine the priorities of each alternative. Among other popular methods with this purpose, we can highlight the well-known eigenvalue method [6], the BWM [7], or the dominance computation.

The most widely used types of pairwise comparison matrices are the FPRs and MPRs. Below, their corresponding definitions are introduced.

Definition 1 *(Multiplicative preference relation (MPR)* [6]*)* Let $n \in \mathbb{N}$. An MPR $Y = (y_{ij}) \in \mathcal{M}_{n\times n}(\mathbb{R})$ is a matrix satisfying $y_{ij} \cdot y_{ji} = 1 \;\; \forall\, 1 \leq i, j \leq n$. If $y_{ij} \in [\frac{1}{m}, m] \;\forall\, 1 \leq i, j \leq n$, where $m \in \mathbb{N} \setminus \{1\}$, we say that the MPR has been elicited in the scale m. Usually, $m = 9$, but it could be any other natural number greater than one. The set of all the MPRs of dimension n is denoted as

$$\mathbb{M}_n = \{Y \in \mathcal{M}_{n\times n}(\mathbb{R}) \;:\; y_{ij} \cdot y_{ji} = 1, y_{ij} > 0 \,\forall\, 1 \leq i, j \leq n\}$$

whereas the MPRs elicited in the scale m will be denoted as

$$\mathbb{M}_n^m = \{Y \in \mathcal{M}_{n\times n}(\mathbb{R}) \;:\; y_{ij} \cdot y_{ji} = 1, y_{ij} \in \left[\frac{1}{m}, m\right] \forall\, 1 \leq i, j \leq n\}.$$

Definition 2 *(Additive preference relation (APR)* [4]*)* Let $n \in \mathbb{N}$. An APR $X = (x_{ij}) \in \mathcal{M}_{n\times n}(\mathbb{R})$ is a matrix satisfying $x_{ij} + x_{ji} = 1 \;\; \forall\, 1 \leq i, j \leq n$. If $x_{ij} \in [\frac{1}{2} - m, \frac{1}{2} + m] \;\forall\, 1 \leq i, j \leq n$, where $m > 0$, we say that the APR has been elicited in the scale m. If $m = \frac{1}{2}$ the APR is known as an FPR. The set of all the APRs of dimension n is denoted as

$$\mathbb{A}_n = \{X \in \mathcal{M}_{n\times n}(\mathbb{R}) \;:\; x_{ij} + x_{ji} = 1 \,\forall\, 1 \leq i, j \leq n\},$$

whereas the APRs elicited in the scale m will be denoted as

$$\mathbb{A}_n^m = \{X \in \mathcal{M}_{n\times n}(\mathbb{R}) \;:\; x_{ij} + x_{ji} = 1, x_{ij} \in \left[\frac{1}{2} - m, \frac{1}{2} + m\right] \forall\, 1 \leq i, j \leq n\}.$$

Here, we have introduced the auxiliary notion of APR as a type of pairwise comparison that is slightly more general than an FPR. It should be highlighted that this distinction is only for the sake of simplicity in the notations. These APRs are not used in practice, but such a definition is very convenient to facilitate the reading of this proposal.

When considering pairwise comparisons, it is necessary to take their consistency into account [7, 10]. Intuitively, a pairwise comparison matrix is said to be consistent if it does not contain contradictory evaluations [6].

Definition 3 *(Consistent MPR* [6]*)* An MPR $Y \in \mathbb{M}_n$ is said to be consistent if it satisfies

$$y_{ij} = y_{ik} \cdot y_{kj} \; \forall \, 1 \leq i, j, k \leq n.$$

Definition 4 *(Consistent APR* [9]*)* An APR $X \in \mathbb{A}_n$ is said to be consistent if it satisfies

$$x_{ij} + \frac{1}{2} = x_{ik} + x_{kj} \; \forall \, 1 \leq i, j, k \leq n.$$

3 Mean Squared Error Utility

This section is devoted to introducing the idea of MSE Utility vector as the utility vector that allows reconstructing the consistent APR which is closest to a given FPR. First, we develop some useful results related to the consistency of pairwise comparison matrices. Subsequently, we derive the analytical expression of the MSE Utility vector. Finally, we show how to extend the idea of MSE Utility vector to manage MPRs and BWM preferences, which are a special type of incomplete MPRs.

3.1 On the Consistency of Preference Relations

In the previous section, we have provided definitions for MPRs and APRs which are slightly more general than the classic ones [4, 6]. However, the following result shows that APRs and MPRs are univocally related when their corresponding scales are fixed.

Proposition 1 *Let* $n \in \mathbb{N}$ *and consider* $m_1 \in \mathbb{N} \setminus \{1\}$, $m_2 > 0$. *Then the mapping* $\Phi : \mathbb{A}_n^{m_2} \to \mathbb{M}_n^{m_1}$ *defined by* $\Phi(X) = (\phi(x_{ij})) \, \forall \, X \in \mathbb{A}_n^m$, *where*

$$\phi : \left[\frac{1}{2} - m_2, \frac{1}{2} + m_2\right] \to \left[\frac{1}{m_1}, m_1\right]$$

is defined as

$$\phi(x) = m_1^{\frac{2x-1}{2m_2}} \; \forall \, x \in \left[\frac{1}{2} - m_2, \frac{1}{2} + m_2\right]$$

is a bijection whose inverse is $\Phi^{-1} : \mathbb{M}_n^{m_1} \to \mathbb{A}_n^{m_1}$ *defined by* $\Phi^{-1}(Y) = (\phi^{-1}(y_{ij}))\ \forall\, Y \in \mathbb{M}_n^{m}$*, where*

$$\phi^{-1} : \left[\frac{1}{m_1}, m_1\right] \to \left[\frac{1}{2} - m_2, \frac{1}{2} + m_2\right]$$

is defined as

$$\phi^{-1}(y) = \frac{1}{2} + m_2 \log_{m_1}(y)\ \forall\, y \in \left[\frac{1}{m_1}, m_1\right].$$

In addition, the mappings ϕ and ϕ^{-1} preserve the consistency.

Proposition 2 *Let* $n \in \mathbb{N}$ *and consider* $m_1 \in \mathbb{N} \setminus \{1\}$*,* $m_2 > 0$*. An APR* $X \in \mathbb{A}_n^{m_2}$ *is consistent if and only if the MPR* $\Phi(X) \in \mathbb{M}_n^{m_1}$ *is consistent.*

Proof By definition, X is (additively) consistent if, for any $1 \le i, j, k \le n$, the following holds:

$$\begin{aligned} & x_{ik} + x_{kj} = x_{ij} + \frac{1}{2} \\ \iff\ & 2x_{ik} + 2x_{kj} - 2 = 2x_{ij} - 1 \\ \iff\ & m_1^{\frac{2x_{ik}-1}{2m_2}}\, m_1^{\frac{2x_{kj}-1}{2m_2}} = m_1^{\frac{2x_{ij}-1}{2m_2}}, \end{aligned}$$

which is the (multiplicative) consistency for $\Phi(X)$. □

Proposition 2 guarantees that analyzing the consistency of an FPR is equivalent to studying the consistency of the MPR defined through the bijection Φ and vice versa. Therefore, in the rest of the chapter, we focus on the methods and proofs in the FPR case, i.e., $\mathbb{A}_n^{\frac{1}{2}}$, in order to obtain linearized results. However, all of them could be applied to the MPR scenario by considering the corresponding transformation through Φ^{-1} [12].

The following result shows that consistent FPRs store the same information as any of their rows, i.e., each consistent FPR is univocally determined by any of its rows.

Proposition 3 *Let* $n \in \mathbb{N}$ *and consider a consistent FPR* $X \in \mathbb{A}_n^{\frac{1}{2}}$*. Then, for any* $1 \le k \le n$*, the FPR* X *can be reconstructed from the utility vector* $u^k = (1 - x_{k1}, 1 - x_{k2}, ..., 1 - x_{kn})$ *by the equality*

$$x_{ij} := \frac{1}{2} + u_i^k - u_j^k\ \forall\, 1 \le i, j \le n.$$

Proof The consistency of X guarantees that

$$\begin{aligned} & x_{ij} = x_{ik} + x_{kj} - \frac{1}{2} \,\forall\, 1 \le i, j \le n \\ \iff & x_{ij} = 1 - x_{ki} + x_{kj} - \frac{1}{2} \,\forall\, 1 \le i, j \le n \\ \iff & x_{ij} = \frac{1}{2} - x_{ki} + x_{kj} \,\forall\, 1 \le i, j \le n \\ \iff & x_{ij} = \frac{1}{2} + (1 - x_{ki}) - (1 - x_{kj}) \,\forall\, 1 \le i, j \le n. \end{aligned}$$

□

Note that, if $x_{ki} > x_{kj}$, the j-th alternative is preferred to the i-th alternative, which motivates the definition of the utility vector u^k. As a consequence, we obtain a characterization for the consistency of FPRs.

Corollary 1 *Let $n \in \mathbb{N}$. An FPR $X \in \mathbb{A}_n^{\frac{1}{2}}$ is consistent if and only if there exists a vector $u \in \mathbb{R}^n$ such that*

$$x_{ij} := \frac{1}{2} + u_i - u_j \,\forall\, 1 \le i, j \le n.$$

In the same way, an MPR given in scale $m > 1$, $Y \in \mathbb{M}_n^m$ is consistent if and only if there exists a vector $u \in (\mathbb{R}^+)^n$ such that

$$Y_{ij} := \frac{u_i}{u_j} \,\forall\, 1 \le i, j \le n.$$

In other words, any consistent FPR can be simplified as a utility vector u that contains the same information as the original FPR and satisfies $u_i - u_j \in [-\frac{1}{2}, \frac{1}{2}] \,\forall\, 1 \le i, j \le n$. Note that the relationship described in Proposition 3 is not a bijection between the utility vectors belonging to $\mathbb{R}^n$ and the set of FPRs. The reason behind that is that given a utility vector $u \in \mathbb{R}^n$, it is not possible to guarantee that $x_{ij} := \frac{1}{2} + u_i - u_j \in [0, 1]$ for all i, j. However, the matrix thus defined is a consistent APR with very convenient properties.

Theorem 1 *Let us consider $n \in \mathbb{N}$ and the mapping $\mu : \mathbb{R}^n \to \mathcal{M}_{n\times n}(\mathbb{R})$ defined as*

$$(\mu(u))_{ij} = \frac{1}{2} + u_i - u_j \,\forall\, u \in \mathbb{R}^n.$$

Then, for any $u \in \mathbb{R}^n$, the matrix $\mu(u)$ is a consistent APR. In addition, the mapping μ satisfies the following properties:

- $\mu(u+v) = \mu(u) + \mu(v) - \frac{1}{2} 1_{n\times n}, \ \forall u, v \in \mathbb{R}^n$,
- $\mu(u + \alpha 1_{1\times n}) = \mu(u), \ \forall u \in \mathbb{R}^n, \alpha \in \mathbb{R}$,
- $\mu(\alpha u) = \alpha\mu(u) + (1-\alpha)\frac{1}{2}_{n\times n}, \ \forall u \in \mathbb{R}^n, \alpha \in \mathbb{R}$,
- $\mu(\alpha u + \beta v) = \alpha\mu(u) + \beta\mu(v) + (1-\alpha-\beta)\frac{1}{2}_{n\times n} \, \forall u, v \in \mathbb{R}^n, \alpha, \beta \in \mathbb{R}$,
- $\mu(\alpha u + \beta v) = \alpha\mu(u) + \beta\mu(v) \, \forall u, v \in \mathbb{R}^n, \alpha, \beta \in \mathbb{R}, \alpha + \beta = 1$.

Proof The additive reciprocity is a consequence of

$$\mu(u)_{ij} + \mu(u)_{ji} = (\frac{1}{2} + u_i - u_j) + (\frac{1}{2} + u_j - u_i) = 1 \, \forall \, 1 \le i, j \le n$$

and the consistency is given by

$$\begin{aligned}\mu(u)_{ik} + \mu(u)_{kj} - \frac{1}{2} &= (\frac{1}{2} + u_i - u_k) + (\frac{1}{2} + u_k - u_j) - \frac{1}{2} \\ &= \frac{1}{2} + u_i - u_j = \mu(u)_{ij} \, \forall \, 1 \le i, j, k \le n.\end{aligned}$$

The rest of the properties are also straightforward from the definition. □

To summarize, given a consistent FPR $X \in \mathbb{A}_n^{\frac{1}{2}}$, it is possible to obtain a utility vector $u \in \mathbb{R}^n$ such that $\mu(u) = X$. Furthermore, if $w_1, \ldots, w_K$ satisfy $w_k \ge 0$, $\sum_{k=1}^K w_k = 1$ and $X_1, \ldots, X_k$ are consistent FPRs, then it is possible to find $u^1, \ldots, u^K$ verifying $\mu(\sum_{k=1}^K w_k u^k) = \sum_{k=1}^K w_k \mu(u^k) = \sum_{k=1}^K w_k X_k$.

3.2 Mean Squared Error Utility for FPRs

We have shown that given a consistent FPR $X \in \mathbb{A}_n^{\frac{1}{2}}$, it is possible to *summarize* its information in a utility vector $u \in \mathbb{R}^n$ such that $\mu(u) = X$. However, obtaining a consistent FPR is nearly impossible when human DMs are required to provide their opinions. In this section, we develop a method to be able to *summarize* non-consistent FPRs as utility vectors.

To do so, for a given FPR $X \in \mathbb{A}_n^{\frac{1}{2}}$, which is not necessarily consistent, our aim is to find a utility vector $u \in \mathbb{R}^n$ whose associated APR through $\mu(u)$ is close to the original matrix X. In other words, we aim to find the utility vector that makes the differences

$$\left| x_{ij} - \frac{1}{2} - u_i + u_j \right|, 1 \le i, j \le n$$

as small as possible. To derive an analytical solution, which is essential to reduce computational costs, we consider an optimization problem consisting of minimizing the corresponding MSE. In addition, since the property $\mu(u + \alpha 1_{1\times n}) = \mu(u), \ \forall u \in \mathbb{R}^n, \alpha \in \mathbb{R}$ in Theorem 1 suggests that such a utility vector is not unique, we include the condition $\sum_{k=1}^n u_k = \sigma$ for some $\sigma \in \mathbb{R}$.

Theorem 2 *Let $X \in \mathbb{A}_n^{\frac{1}{2}}$ be an FPR and consider $\sigma \in \mathbb{R}$. Then, the utility vector $u \in \mathbb{R}^n$ defined as*

$$u_k = \frac{1}{2} + \frac{1}{n}(\sigma - \sum_{i=1}^{n} x_{ik})$$

is the solution to the optimization problem

$$\min_{u \in \mathbb{R}^n} \frac{1}{n^2} \xi_X(u)$$
$$s.t. \left\{ \sum_{k=1}^{n} u_k = \sigma \right.$$

where the function $\xi_X : \mathbb{R}^n \to \mathbb{R}$ is defined as

$$\xi_X(u) = \sum_{i=1}^{n} \sum_{j=1}^{n} (x_{ij} - \frac{1}{2} - u_i + u_j)^2 \, \forall\, u \in \mathbb{R}^n.$$

Proof Let us consider the function $f_{ij} : \mathbb{R}^n \to \mathbb{R}$ defined as $f_{ij}(u) = (x_{ij} - \frac{1}{2} - u_i + u_j)^2 \, \forall\, u \in \mathbb{R}^n$. Note that, for $1 \le k \le n$, the partial derivative $\partial_k f_{ij} := \frac{\partial f_{ij}}{\partial u_k}$ is given as

$$\partial_k f_{ij}(u) = \begin{cases} 0 & \text{if } i < k,\ j < k \\ 2(x_{ij} - \frac{1}{2} - u_i + u_j) & \text{if } i < k,\ j = k \\ 0 & \text{if } i < k,\ j > k \\ -2(x_{ij} - \frac{1}{2} - u_i + u_j) & \text{if } i = k,\ j < k \\ 0 & \text{if } i = k,\ j = k \\ -2(x_{ij} - \frac{1}{2} - u_i + u_j) & \text{if } i = k,\ j > k \\ 0 & \text{if } i > k,\ j < k \\ 2(x_{ij} - \frac{1}{2} - u_i + u_j) & \text{if } i > k,\ j = k \\ 0 & \text{if } i > k,\ j > k \end{cases}$$

$\forall\, u \in \mathbb{R}^n$. Consequently, $\partial_k \xi_X := \frac{\partial \xi_X}{\partial u_k}$ can be obtained as follows:

$$\begin{aligned} \partial_k \xi_X(u) &= 2 \sum_{\substack{i=1 \\ i \neq k}}^{n} \left(x_{ik} - \frac{1}{2} - u_i + u_k \right) - 2 \sum_{\substack{j=1 \\ j \neq k}}^{n} \left(x_{kj} - \frac{1}{2} - u_k + u_j \right) \\ &= 2 \sum_{\substack{i=1 \\ i \neq l}}^{n} \left(x_{ik} - \frac{1}{2} - u_i + u_k - x_{ki} + \frac{1}{2} + u_k - u_i \right) \\ &= 2 \sum_{\substack{i=1 \\ i \neq k}}^{n} (2x_{ik} - 1 - 2u_i + 2u_k) = 4 \sum_{i=1}^{n} \left(x_{ik} - \frac{1}{2} - u_i + u_k \right) \end{aligned}$$

$\forall\, u \in \mathbb{R}^n$. Now, let us consider the Lagrangian function $L : \mathbb{R}^{n+1} \to \mathbb{R}$ defined as

$$L(u, \lambda) = \xi_X(u) + 4\lambda \left(\sum_{i=1}^{n} u_i - \sigma \right) \forall (u, \lambda) \in \mathbb{R}^{n+1}.$$

After computing the corresponding partial derivatives, the resulting equations are as follows:

$$\begin{cases} \sum_{i=1}^{n} x_{ik} - \frac{n}{2} - \sum_{i=1}^{n} u_i + nu_k + \lambda = 0 \, \forall k = 1, 2, \ldots, n \\ \sum_{i=1}^{n} u_i = \sigma. \end{cases}$$

To solve it, note that if the first n equations are added, the following holds:

$$\sum_{k=1}^{n} \left(\sum_{i=1}^{n} x_{ik} - \frac{n}{2} - \sum_{i=1}^{n} u_i + nu_k + \lambda \right) = 0$$
$$\iff \sum_{k=1}^{n} \sum_{i=1}^{n} x_{ik} - \frac{n^2}{2} - n\sigma + n\sigma + n\lambda = 0 \iff \lambda = 0$$

in which the equations $\sum_{i=1}^{n} u_i = \sigma$ and $\sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij} = \frac{n^2}{2}$ have been used. Finally, we can clear the value of each u_k:

$$\sum_{i=1}^{n} x_{ik} - \frac{n}{2} - \sum_{i=1}^{n} u_i + nu_k + \lambda = 0$$
$$\iff \sum_{i=1}^{n} x_{ik} - \frac{n}{2} - \sigma + nu_k = 0$$
$$\iff u_k = \frac{1}{2} - \frac{1}{n} \sum_{i=1}^{n} x_{ik} + \frac{\sigma}{n}.$$

Since ξ_X is a convex continuous non-negative function in $\mathbb{R}^n$, then there exists a global minimum for it. In addition, due to the restriction $\sum_{i=1}^{n} u_i = \sigma$ stands for a hyperplane in $\mathbb{R}^n$, a global minimum of the set $\{\xi_X(u) \, : \, u \in \mathbb{R}^n, \sum_{i=1}^{n} u_i = \sigma\}$ also exists. Consequently, since u is the only solution to the Lagrangian system, it must be the global minimum for the aforementioned optimization problem. □

At this point, we define the MSE utility vector as the vector $u \in \mathbb{R}^n$ obtained in the previous theorem.

Definition 5 *[MSE Utility for FPRs]* Let $X \in \mathbb{A}_n^{\frac{1}{2}}$ be an FPR and consider $\sigma \in \mathbb{R}$. Then, the MSE utility vector for X and σ is the vector $u \in \mathbb{R}^n$ defined as

$$u_k = \frac{1}{2} + \frac{1}{n} \left(\sigma - \sum_{i=1}^{n} x_{ik} \right) \forall k = 1, 2, \ldots, n,$$

and the associated MSE weighting vector $W \in \mathbb{R}^n$ is defined as

$$W_k = \frac{\phi(u_k)}{\sum_{l=1}^n \phi(u_l)} = \frac{m^{\frac{2}{n}\sum_{i=1}^n x_{ki}}}{\sum_{l=1}^n \sum_{l=1}^n m^{\frac{2}{n}\sum_{i=1}^n x_{li}}} \, \forall k = 1, 2, \ldots, n,$$

where $m \in \mathbb{N} \setminus \{1\}$ stands for the scale.

The MSE utility vector $u \in \mathbb{R}^n$ has the following advantages:

- The APR $\mu(u)$ is the closest consistent APR to the FPR X such that $\sum_{i=1}^n u_i = \sigma$.
- Since $u_k = \frac{1}{n}\sum_{i=1}^n x_{ki} + \frac{\sigma}{n} - \frac{1}{2} \, \forall k = 1, 2, \ldots, n$, they can be easily computed from the dominance of each alternative, without going through an optimization problem.
- In addition, the higher the dominance of the $k-$th alternative, the higher the value u_k. This motivates the use of the word *utility* to name them [20].
- If positive utilities are required, then it suffices to choose

$$\sigma = n\left(\frac{1}{2} - \min_{k=1,\ldots,n}\{\frac{1}{n}\sum_{i=1}^n x_{ki}\}\right) \geq 0.$$

- The value $\xi_X(u)$ provides a measure of the consistency of the FPR X: if $\xi_X(u) = 0$, then X is fully consistent, and the higher the quantity $\xi_X(u)$, the higher the inconsistency of X.
- The associated MSE weighting vector satisfies $\sum_{k=1}^n W_k = 1$.

Regarding aggregations, the MSE utilities also have interesting behavior.

Corollary 2 *Let $X^1, \ldots, X^m \in \mathbb{A}_n^{\frac{1}{2}}$ be a family of FPRs and consider weights $w_1, \ldots, w_m \geq 0$ such that $\sum_{k=1}^m w_k = 1$. Then, for $\sigma \in \mathbb{R}$, the MSE utility vector u^c associated with the aggregated FPR $X^c := \sum_{k=1}^m w_k X^k$ satisfies*

- $u^c = \sum_{k=1}^m w_k u^k$,
- $\mu(u^c) = \sum_{k=1}^m w_k \mu(u^k)$,
- $\xi_{X^c}(u^c) \leq \sum_{k=1}^m w_k \xi_{X^k}(u^k)$,

where $u^1, \ldots, u^m$ stand for the MSE utilities of $X^1, \ldots, X^m$.

Proof The two first items are straightforward considering Theorems 1 and 2. To show the third one, note that

$$\begin{aligned}\xi_{X^c}(u^c) &= \sum_{i=1}^n \sum_{j=1}^n \left(x_{ij}^c - \frac{1}{2} - u_i^c + u_j^c\right)^2 \\ &= \sum_{i=1}^n \sum_{j=1}^n \left(\sum_{k=1}^m w_k\left(x_{ij}^k - \frac{1}{2} - u_i^k + u_j^k\right)\right)^2\end{aligned}$$

$$\leq \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{m} w_k \left(x_{ij}^k - \frac{1}{2} - u_i^k + u_j^k\right)^2$$
$$= \sum_{k=1}^{m} w_k \sum_{i=1}^{n}\sum_{j=1}^{n} \left(x_{ij}^k - \frac{1}{2} - u_i^k + u_j^k\right)^2$$
$$= \sum_{k=1}^{m} w_k \xi_{X^k}(u^k)$$

which has been derived from Jensen's inequality. □

This result guarantees that, in GDM problems [21], the collective MSE utility vector and the associated consistent APR can be computed by aggregating the individual MSE utility vectors, which could be very helpful in improving computational efficiency as we show in Sect. 4. Furthermore, the MSE when approximating the collective opinion is bounded according to the MSE associated with the individual FPRs.

In addition, the MSE Utility vector allows recovering the original FPR, provided it is consistent, using the mapping μ as follows.

Corollary 3 *Let $X \in \mathbb{A}_n^{\frac{1}{2}}$ be a consistent FPR and consider its MSE Utility vector $u \in \mathbb{R}^n$ for some $\sigma \in \mathbb{R}$. Then $\mu(u) = X$.*

Proof Since X is consistent, then $x_{ik} + x_{kj} = \frac{1}{2} + x_{ij}\ \forall i, j, k = 1, \ldots, n$. Therefore,

$$\mu(u)_{ij} = \frac{1}{2} + u_i - u_j$$
$$= \frac{1}{2} + \left(\frac{1}{2} + \frac{1}{n}\left(\sigma - \sum_{k=1}^{n} x_{ki}\right)\right) - \left(\frac{1}{2} + \frac{1}{n}\left(\sigma - \sum_{k=1}^{n} x_{kj}\right)\right)$$
$$= \frac{1}{2} + \frac{1}{n}\sum_{k=1}^{n}(x_{kj} - x_{ki}) = \frac{1}{2} + \frac{1}{n}\sum_{k=1}^{n}(x_{ik} + x_{kj} - 1)$$
$$= \frac{1}{2} + \frac{1}{n}\sum_{k=1}^{n}\left(x_{ij} - \frac{1}{2}\right) = x_{ij}\ \forall i, j = 1, \ldots, n.$$

□

Note that this capability of retrieving the original FPR from the MSE utility vector is essential to guarantee that no information is lost during the conversion process. In this sense, other proposals fail to satisfy this property. For instance, let us assume that the utility values for the alternatives are computed as the arithmetic mean of the corresponding row of the FPR. Then, it can be easily checked that it is not possible to recover the original FPR when using mappings such as $\hat{\mu} : \mathbb{R}^n \to \mathcal{M}_{n\times n}(\mathbb{R})$ defined as $(\hat{\mu}(u))_{ij} = \frac{1}{2}(1 + u_i - u_j)$, considered by Herrera-Viedma et al. [9]. In the same way, even though Xu et al. [22] also defined several methods to derive priority values from FPRs, they neglected how to reconstruct the original FPR from the obtained priority values. However, this is a key property of the MSE utility vector because it implies that the information contained in the original FPR can be obtained again, which is very helpful to improve the computational efficiency of processes that simultaneously deal with many FPRs as it is shown in Sect. 4.

3.3 Mean Squared Error Utility for MPRs

Here, the notion of MSE utility vector is adapted for pairwise comparisons given in a multiplicative scale.

Assume $Y \in \mathbb{M}_n^m$ is an MPR for some $m > 1$. Then, we can consider the FPR $X = \Phi^{-1}(Y)$, where $\Phi^{-1} : \mathbb{M}_n^{m_1} \to \mathbb{A}_n$, and compute the corresponding MSE utility vector $u \in \mathbb{R}^n$ as in Theorem 2. Therefore, for the sake of simplicity, we define the multiplicative MSE utilities as follows.

Definition 6 *[MSE utility for MPRs]* Let $Y \in \mathbb{M}_n^m$ be an MPR and consider $\sigma \in \mathbb{R}$. Then the MSE utility vector for Y and σ is the vector $u \in \mathbb{R}^n$ defined as

$$u_k = \frac{1}{2} + \frac{1}{n}\left(\sigma - \sum_{i=1}^{n} \phi^{-1}(y_{ik})\right) \forall k = 1, 2, \ldots, n,$$

and the associated MSE weighting vector $W \in \mathbb{R}^n$ is defined as

$$W_k = \frac{\phi(u_k)}{\sum_{l=1}^{n} \phi(u_l)} = \frac{m^{\frac{2}{n}\sum_{i=1}^{n}\phi^{-1}(y_{ki})}}{\sum_{l=1}^{n} m^{\frac{2}{n}\sum_{i=1}^{n}\phi^{-1}(y_{li})}} \forall k = 1, 2, \ldots, n.$$

Note that these multiplicative utilities determine the consistent MPR $\tilde{Y} = (\tilde{y}_{ij})$ defined as

$$\tilde{y}_{ij} = \frac{\phi(u_i)}{\phi(u_j)} = \frac{m^{\frac{2}{n}\sum_{k=1}^{n}\phi^{-1}(y_{ik}) + \frac{2\sigma}{n} - 2}}{m^{\frac{2}{n}\sum_{k=1}^{n}\phi^{-1}(y_{jk}) + \frac{2\sigma}{n} - 2}}$$
$$= m^{\frac{2}{n}\sum_{k=1}^{n}\phi^{-1}(y_{ik}) - \phi^{-1}(y_{jk})} \forall i, j = 1, 2, \ldots, n,$$

which does not actually depend on the value of σ. Furthermore, note that computing the normalized vector W is equivalent to computing the multiplicative MSE utilities for the value

$$\sigma = n - \frac{n}{2}\log_m\left(\sum_{l=1}^{n} m^{\frac{2}{n}\sum_{i=1}^{n} x_{li}}\right).$$

In addition, the multiplicative utility W constructed through this method is in line with the results obtained in [12] and minimizes the distance function $\Xi_Y : (\mathbb{R}^+)^n \to \mathbb{R}_0^+$ defined as

$$\Xi_Y(W) = \xi_X \circ \Phi^{-1}(W) = \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{1}{2}\log_m\left(y_{ij}\frac{W_j}{W_i}\right)\right)^2 \ \forall\, W_1, W_2, \ldots, W_n > 0,$$

which satisfies

$$\Xi_Y(W) = 0 \iff y_{ij} = \frac{W_i}{W_j}\ \forall\, i, j = 1, 2, \ldots, n \iff Y \text{ is fully consistent.}$$

Note that the weights computed by this procedure are also compatible with the results given by Xu et al. [8] when they are applied to completely consistent FPRs. Nevertheless, to address the non-consistent case, Xu et al. [8] suggested the use of mathematical optimization models. In this regard, the theoretical discussion developed in this section, which is based on Theorem 2, guarantees that the normalized MSE weighting vector is the analytical solution to the optimization problem described above, even when the FPRs are not consistent. Consequently, by using the MSE weighting vector it is possible to derive the priority weights for FPRs using simple arithmetic operations and skipping the numerical resolution of a complex mathematical programming model.

3.4 Mean Squared Error Utility in Best-Worst Method

The MSE utility vector can also be constructed if the only available pairwise comparisons are those corresponding to the best criteria with all the others, and all the criteria with the worst one. In other words, the MSE can be used to simplify the computations in the BWM because they do not require computationally solving any optimization problem.

If $n \in \mathbb{N}, n > 2$ stands for the number of criteria to compare, let us denote by $O = \{1, 2, 3, \ldots, n\} \setminus \{B, W\}$ the indexes corresponding to the criteria which are neither the best nor the worst. In addition, although it is not necessary, we assume here that the equality $x_{BW} = \frac{1}{2} + u_B - u_W$ holds, where x_{BW} is the pairwise comparison of the best criterion with the worst one and u_B, u_W are, respectively, the MSE utilities corresponding to the best and worst criteria.

Theorem 3 *Let $x_B = (x_{B1}, \ldots, x_{Bn})$ and $x_W = (x_{1W}, \ldots, x_{nW})$ be, respectively, the pairwise comparisons of the best criterion with all the others, and all the criteria with the worst one, and consider $\sigma \in \mathbb{R}$. Then, the utility vector $u \in \mathbb{R}^n$ defined as*

$$u_B = \frac{1}{2}(s + t)$$

$$u_W = \frac{1}{2}(s - t)$$
$$u_i = \frac{1}{2}(s - (x_{Bi} - x_{iW}))\,\forall\, i \in O$$

where $s := \frac{1}{n}(\sum_{i\in O}(x_{Bi} - x_{iW}) + 2\sigma) = u_B - u_W$ *and* $t := X_{BW} - \frac{1}{2} = u_B - u_W$ *is the solution to the optimization problem*

$$\min_{u\in\mathbb{R}^n} \xi_X(u)$$
$$s.t. \begin{cases} \sum_{i=1}^{n} u_k = \sigma \\ x_{BW} = \frac{1}{2} + u_B - u_W \end{cases}$$

where the function $\xi_X : \mathbb{R}^n \to \mathbb{R}$ *is defined as*

$$\xi_X(u) = \sum_{i\in O}\left(x_{Bi} - \frac{1}{2} - u_B + u_i\right)^2 + \left(x_{iW} - \frac{1}{2} - u_i + u_W\right)^2 \;\forall\, u \in \mathbb{R}^n.$$

Proof First, let us define the Lagrangian $L : \mathbb{R}^n \times \mathbb{R}^2 \to \mathbb{R}$ by

$$L(u, \lambda, \mu) = \xi_X(u) + \lambda\left(\sum_{i=1}^{n} u_i - \sigma\right)$$
$$+ \mu\left(x_{BW} - \frac{1}{2} - u_B + u_W\right) \forall\, (u, \lambda, \mu) \in \mathbb{R}^{n+2}.$$

The equations obtained from partial derivatives of this function are as follows:

$$\partial_B L(u, \lambda, \mu) = -2\sum_{i\in O}\left(x_{Bi} - \frac{1}{2} - u_B + u_i\right) - \mu + \lambda = 0$$
$$\partial_W L(u, \lambda, \mu) = 2\sum_{i\in O}\left(x_{iW} - \frac{1}{2} - u_i + u_W\right) + \mu + \lambda = 0$$
$$\partial_k L(u, \lambda, \mu) = 2(x_{Bi} - x_{iW} - u_B + 2u_i - u_W) + \lambda,\ \forall\, k \in O = 0$$
$$\partial_\lambda L(u, \lambda, \mu) = \sum_{i=1}^{n} u_i - \sigma = 0$$
$$\partial_\mu L(u, \lambda, \mu) = x_{BW} - \frac{1}{2} - u_B + u_W = 0.$$

The addition of the two first equations implies

$$\sum_{i\in O}(x_{Bi} - x_{iW} - u_B + 2u_i - u_W) = \lambda$$

which combined with the third equation guarantees $\lambda = 0$. At this point, the first and second equations can be rewritten as

$$\sum_{i \in O}\left(x_{Bi} - \frac{1}{2}\right) + \sigma - (n-1)u_B - u_W + \frac{\mu}{2}$$
$$\sum_{i \in O}\left(x_{iW} - \frac{1}{2}\right) - \sigma + u_B + (n-1)u_W + \frac{\mu}{2},$$

whose difference is

$$\sum_{i \in O}(x_{Bi} - x_{iW}) + 2\sigma - nu_b - nw_W = 0,$$

and then

$$u_b + u_w = \frac{1}{n}\left(\sum_{i \in O}(x_{Bi} - x_{iW}) + 2\sigma\right) = s.$$

Since $u_B - u_W = t$, the solution of the optimization problem can be computed by combining it with the third Lagrangian equation. □

Note that, since $t \in [0, \frac{1}{2}]$ and $x_{Bk} - x_{kW} \in [-t, t]\, \forall k \in O$, then

$$u_B - u_k = \frac{t}{2} + \frac{1}{2}(x_{Bk} - x_{kW}) \in [0, t]\, \forall k \in O$$
$$u_k - u_W = \frac{t}{2} - \frac{1}{2}(x_{Bk} - x_{kW}) \in [0, t]\, \forall k \in O$$
$$u_k - u_l = \frac{1}{2}((x_{Bj} - x_{jW}) - (x_{Bi} - x_{iW})) \in [-t, t]\, \forall k, l \in O,$$

and, consequently, the matrix $\mu(u)$ is an FPR. In addition, it follows that $u_i \in [u_W, u_B]$ and that u_i increases when the quantity $x_{iB} + x_{iW}$ increases $\forall i \in O$.

Note that the MSE utilities obtained in Theorem 3 are given in a linear scale, whereas the weights for BWM are provided for preferences elicited in a multiplicative scale $[1, m]$. Therefore, after using the bijection $\phi^{-1} : [\frac{1}{m}, m] \to [0, 1]$, the weights for the criteria can be obtained as

$$W_i = \frac{m^{2 \cdot u_i - 1}}{\sum_{k=1}^{n} m^{2 \cdot u_k - 1}}, i = 1, 2, \ldots, n$$

where $u \in \mathbb{R}^n$ is the MSE utility vector obtained in Theorem 3. Consequently, we can state the following definition.

Definition 7 *[MSE utility in BWM]* Let $x_B = (x_{B1}, \ldots, x_{Bn})$ and $x_W = (x_{1W}, \ldots, x_{nW})$ be, respectively, the pairwise comparisons in a scale where $m \in \mathbb{N} \setminus \{1\}$ of the best criterion with all the others, and all the criteria with the worst one, and consider $\sigma \in \mathbb{R}$. Then, the MSE utility vector for $x_B = (x_{B1}, \ldots, x_{Bn})$ and $x_W = (x_{1W}, \ldots, x_{nW})$ is the vector $u \in \mathbb{R}^n$ defined as

$$\begin{aligned} u_B &= \frac{1}{2}(s+t) \\ u_W &= \frac{1}{2}(s-t) \\ u_i &= \frac{1}{2}(s-(x_{Bi}-x_{iW}))\,\forall\, i \in O, \end{aligned}$$

where $s := \frac{1}{n}(\sum_{i \in O}(x_{Bi} - x_{iW}) + 2\sigma) = u_B - u_W$ and $t := X_{BW} - \frac{1}{2} = u_B - u_W$. Additionally, the MSE weighting vector for the criteria can be computed as $W \in \mathbb{R}^n$ defined as

$$W_i = \frac{m^{2 \cdot u_i - 1}}{\sum_{k=1}^{n} m^{2 \cdot u_k - 1}}, i = 1, 2, \ldots, n.$$

4 Application in E-Democracy

Even though the most immediate application of MSE utility vector could be ranking alternatives, here we prefer highlighting how they can be applied in LSGDM to reduce computational costs. Of note, the data used in this section has been randomly generated.

4.1 Illustrative Example

In a rapidly growing city, the government aims to decide the allocation of the annual budget for an infrastructure project. The project aims to construct a new transportation system to address the increasing traffic congestion and improve the overall transportation infrastructure. To do so, a preliminary analysis has revealed that the best options for this project are

- Alternative 1: High-Speed Railway System.
- Alternative 2: Road Expansion and Improvement.
- Alternative 3: Public Transit Enhancement.
- Alternative 4: Sustainable Transportation Initiatives.

In order to make a final decision, the government decides to carry out a survey on the population, which will evaluate different alternatives using FPRs. Subsequently, an automatic CRP will be carried out to smooth out the disagreements among them.

For the sake of simplicity, we consider a Minimum Cost Consensus (MCC) model [15, 23] to find the collectively agreed solution which is closest to the original preferences. Specifically, we consider the optimization problem:

$$\begin{aligned} &\min_{o \in \mathcal{M}_{m\times\hat{n}}} \sum_{k=1}^{m}\sum_{i=1}^{\hat{n}} |o_{ki} - o^0_{ki}| \\ &s.t. \begin{cases} g_i = \frac{1}{m}\sum_{k=1}^{m} o_{ki} \\ |o_{ki} - g_i| \leq \varepsilon \end{cases} \end{aligned} \tag{MCC}$$

where ε is a consensus threshold, $g \in \mathbb{R}^{\hat{n}}$ is the collective opinion, and $o, o^0 \in \mathcal{M}_{m\times\hat{n}}$ stand for the modified and initial opinions, respectively.

Note that, if FPRs were used directly in this optimization model, for each decision-maker it would be necessary to consider at least $n \times (n-1)/2$ variables, i.e., 6 variables for the case $n = 4$.

Therefore, in order to speed up the process, the MSE utility vector will be used to transform the FPRs into vectors before running the optimization process, reducing from $n \times (n-1)/2$ to only n variables. Concretely, if a decision-maker provides the approximated FPR

$$\begin{pmatrix} 0.5 & 0.06 & 0.44 & 0.51 \\ 0.93 & 0.5 & 0.51 & 0.37 \\ 0.55 & 0.48 & 0.5 & 0.42 \\ 0.48 & 0.62 & 0.57 & 0.5 \end{pmatrix}$$

we will consider instead the corresponding MSE Utility vector for $\sigma = 1$ computed as in Definition 5:

$$(0.13, 0.33, 0.24, 0.29).$$

Therefore, in order to make a consensual decision, we can run the above-mentioned (MCC) model using the parameter $\varepsilon = 0.1$ and the corresponding MSE utility vectors as inputs.

In this case, 2354 people answered the survey. Therefore, after computing the corresponding 4-dimensional MSE utility vectors, we run the model. The computational process finishes in $34s$ and the agreed opinion is the utility vector:

$$(0.247, 0.252, 0.249, 0.247),$$

which indicates that Alternative 2: Road Expansion and Improvement is the most preferred choice for the people in the city.

4.2 Time Cost Analysis

In the previous example, we have shown that computing the MSE utility vector reduces the number of variables considered in the CRP. Here, we provide additional statistical insight to compare the following procedures for solving a GDM problem:

- Use FPRs as inputs in the MCC model,
- Compute the MSE utility vectors for the FPRs and then run the MCC model.

To compare these two procedures we carry out Monte Carlo simulations. Concretely, we are interested in three variables. The first one is the computational time required to solve the MCC model using FPRs, the second one is the computational time required to solve the MCC model using the corresponding MSE utilities, and the third variable is the average distance between the collective FPR and the APR obtained from the collective utility vector using the mapping μ.

Let us consider a GDM problem in which m experts must reach an agreement regarding which alternative among n possibilities is the best one. In order to simplify the analysis, let us fix $\sigma = 1$ and $\varepsilon = 0.2$.

Note that in the first case, the input of the MCC model o^0 stands for the upper triangle of the FPR, i.e., $\hat{n} = \frac{n(n-1)}{2}$, whereas when using the MSE utilities associated with the corresponding FPR, the dimension decreases to $\hat{n} = n$ because o^0 is the MSE utility vector itself. In addition, the distance between the collective FPR obtained following the first procedure and the APR obtained from the second procedure can be measured as $d = \frac{2}{mn(n-1)} \sum_{k=1}^{m} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} |G_{ij} - \mu(u)_{ij}|$.

For some configurations of m and n, we have carried out both procedures 50 times using randomly generated FPRs. The obtained confidence intervals for the above-described variables are shown in Tables 1, 2, and 3.

We have also made the simulations in LSGDM scenarios with 2560 decision-makers. For 4, 7, and 9 alternatives, the confidence intervals for the time when using the first procedure are, respectively, [419.59, 497.91], [5449.09, 6439.70], and [19906.78, 22800.53] (expressed in seconds), whereas the corresponding confidence intervals for the time when using the second procedure are [210.92, 256.28], [1110.93, 1876.86], and [1814.02, 3222.98] (expressed in seconds too). In all these cases, the confidence intervals for the distances are, approximately, [0.005, 0.006].

Table 1 Confidence interval for computational time (in milliseconds) of the first procedure

n \ m	10	40	160	640
4	[1.99, 4.21]	[9.35, 11.45]	[57.15, 61.61]	[11820.75, 13139.53]
7	[7.54, 9.38]	[38.17, 41.39]	[4404.34, 4516.94]	[276656.0, 330088.16]
9	[13.51, 16.69]	[74.4, 78.84]	[13000.76, 13243.88]	[868444.3, 1046717.26]

Table 2 Confidence interval for computational time (in milliseconds) in the second procedure

n \ m	10	40	160	640
4	[1.87, 2.01]	[6.29, 7.79]	[39.78, 42.9]	[3887.2, 4779.32]
7	[2.48, 4.36]	[12.26, 15.58]	[80.48, 86.08]	[27081.37, 38392.87]
9	[3.34, 3.66]	[15.6, 19.2]	[111.25, 118.99]	[34956.62, 93454.62]

Table 3 Confidence interval for distance

n \ m	10	40	160	640
4	[0.072, 0.086]	[0.036, 0.043]	[0.017, 0.022]	[0.009, 0.011]
7	[0.086, 0.094]	[0.045, 0.05]	[0.022, 0.025]	[0.011, 0.012]
9	[0.087, 0.094]	[0.047, 0.051]	[0.023, 0.025]	[0.011, 0.012]

In conclusion, using the MSE utilities allows for considerably reducing the computational cost (around 10 times faster when using 9 alternatives and 2560 decision-makers and even faster for lesser numbers of experts). In addition, Theorem 1 guarantees that the MSE utilities corresponding to the collective FPR can be derived as the weighted sum of the MSE utilities corresponding to the individual opinions. Furthermore, the use of the MSE utilities in the consensus process implies that it is not necessary to include additional constraints in the optimization model that guarantee a certain consistency degree.

5 Comparative Analysis

5.1 Comparison with the Eigenvalue Method for MPRs

This section compares the use of the MSE utility vector for MPRs (Definition 6) with the eigenvalue method to compute priorities from an MPR, which is widely applied in AHP [6, 16, 24].

For the sake of clarity, let us consider the MPR, on a 1–9 Saaty's scale, given as

$$Y = \begin{pmatrix} 1 & 2 & 1/3 & 4 \\ 1/2 & 1 & 5 & 6 \\ 3 & 1/5 & 1 & 1/7 \\ 1/4 & 1/6 & 7 & 1 \end{pmatrix}.$$

For this matrix, the maximum eigenvalue is 6.903, and the corresponding normalized eigenvector is $eig = (0.272, 0.365, 0.156, 0.207)$. If we compute the MSE weighting vector instead, we obtain the vector $W = (0.283, 0.435, 0.12, 0.163)$. Note that both of them provide a similar ranking for the alternatives, i.e., $Alt_2 > Alt_1 > Alt_4 > Alt_3$. In addition, it is theoretically guaranteed (see Sect. 3) that the MSE weighting vector is the closest one to the original matrix Y. This can be easily checked by noticing that $\Xi_Y(eig) = 1.014$, whereas $\Xi_Y(W) = 0.956$.

Furthermore, if it is considered a completely consistent MPR both methods are guaranteed to provide exactly the same solution. In fact, for a consistent MPR such as

$$Y = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1/2 & 1 & 3/2 & 2 \\ 1/3 & 2/3 & 1 & 4/3 \\ 1/4 & 1/2 & 3/4 & 1 \end{pmatrix},$$

both the weights obtained from the eigenvalue method and the MSE coincide, i.e., $W = eig = (0.48, 0.24, 0.16, 0.12)$.

Despite it being widely used [5], the eigenvalue method has some drawbacks in comparison to the MSE weighting vector. On the one hand, it is necessary to compute both the eigenvalues and eigenvectors to apply the method. From the computational point of view, computing eigenvectors can be done in $O(n^3)$ [25], whereas the computation of the weighting vector associated with the MSE utility vector just requires $O(n)$.

On the other hand, if the original MPR Y is not completely consistent, the eigenvalue method could not be reliable [5]. However, with the MSE utility approach, we can guarantee that the obtained weights are as consistent as possible because they minimize the function Ξ_Y.

In addition, the rationale regarding the relationship between eigenvalues and the ranking of the evaluated alternatives could not be clear at first sight for researchers without a mathematical background. In this sense, the MSE utility vector is much more interpretable because it is computed from the dominance of each alternative, which guarantees that the more preferred one alternative is over the others, the higher its weight.

5.2 *Comparison with Classical BWM*

In order to illustrate the performance of the weights obtained from the MSE utilities for BWM preferences (Definition 7), let us compare them with the preferences obtained by using the original BWM [7].

Let us consider the inputs $x_B = (8, 2, 1)$ and $x_W = (1, 5, 8)$, where $B = 3$ and $W = 1$. After applying the original BWM, the optimal weights are $(0.07142, 0.3387, 0.5899)$. If we apply the method here described, after remapping the multiplicative preferences into the linear scale, the MSE utilities are

$u = (0.0620, 0.4028, 0.5352)$ and the corresponding MSE weights are $W = (0.0742, 0.3319, 0.5938)$, which are similar to the ones obtained using Rezaei's method [7] and do not require solving any optimization problem.

In addition, if both methods are applied to a fully consistent evaluation, the obtained weights are exactly the same. For example, for the consistent inputs $x_B = (2, 1, 9, 5)$ and $x_W = (9/2, 9, 1, 9/5)$, where $B = 2$ and $W = 3$, the obtained weights are, in both cases, $W = (0.2761, 0.5521, 0.0613, 0.1104)$.

Note that both the classical BWM and the MSE weighting vector produce a weighting vector that aims to minimize a certain consistency function. However, the objective function considered in classical BWM is not differentiable, and it is hard to give an analytical solution for it, which forces inevitably to solve the optimization model using numerical methods. On the contrary, the MSE weighting vector is the analytical and, consequently, the exact solution to its associated optimization problem. In addition, computing the MSE weighting vector requires $O(n)$, whereas solving the nonlinear programming problem of classical BWM may take much more than $O(n^2)$, depending on the solver and the linearization process [26].

6 Conclusions

This chapter has developed a method to derive priority values from pairwise comparisons based on the MSE. Such a method is based on the analytical resolution of the optimization problem consisting of finding the utility values that generate a certain consistent pairwise comparison matrix. The discussion highlighted several significant properties and benefits of this approach, which collectively enhance its practical application in decision-making processes.

The proposed method facilitates the computation of an overall performance score for each alternative by utilizing a pairwise comparison matrix. This feature enables seamless integration into established decision-making frameworks, such as the AHP, enhancing the method's applicability in traditional settings.

The MSE utilities represent the analytical solution to an optimization problem, ensuring accuracy and reliability in their results. These utilities allow the construction of a consistent pairwise comparison matrix that minimizes the MSE with respect to the original pairwise comparisons provided by decision-makers. This fact ensures that such utilities properly reflect decision-makers' opinions.

The computation of MSE utilities involves straightforward algebraic operations, bypassing the need for complex mathematical procedures such as the resolution of an optimization model. This simplicity makes the method particularly accessible to practitioners who may not have extensive training in advanced mathematics. Moreover, these utilities are derived based on the dominance of the alternatives, which enhances their interpretability and practical relevance in decision-making contexts.

Additionally, the MSE method offers significant computational efficiency, especially pertinent in LSGDM scenarios. By reducing the dimensionality of the data, the method streamlines the computational process, making it a valuable tool for handling extensive datasets efficiently and effectively.

The flexibility and robustness of the MSE utility discussed in this chapter suggest a broad potential for future research and application. The method's adaptability to different types of preference structures opens up new avenues for extending its utility across various decision-making contexts.

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