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Fuzzy dissimilarities and the fuzzy Choquet integral of triangular fuzzy numbers on $[0,1]$

A.F. Roldán López de Hierro [a,*], A. Cruz [b], R.H.N. Santiago [c], C. Roldán [a], D. García-Zamora [d], F. Neres [e], H. Bustince [f]

[a] Department of Statistics and Operations Research, University of Granada, Granada, Spain
[b] Instituto Metrópole Digital, Universidade Federal do Rio Grande do Norte, Natal, Rio Grande do Norte, Brazil
[c] Departamento de Informática e Matemática Aplicada, Universidade Federal do Rio Grande do Norte, Natal, Rio Grande do Norte, Brazil
[d] Department of Computing, University of Jaén, Jaén, Spain
[e] Departamento de Ciência e Tecnologia, Universidade Federal Rural do Semi-Árido, Caraúbas, Rio Grande do Norte, Brazil
[f] Departamento de Estadística, Informática y Matemáticas, Universidad Pública de Navarra, Pamplona, Navarra, Spain

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ABSTRACT

Having in mind the huge amount of data daily registered in the world, it is becoming increasingly important to summarize the information included in a data set. In Statistics and Computer Science, this task is successfully carried out by aggregation functions. One of the most widely applied methodologies of aggregating data is the Choquet integral. The main aim of this paper is to introduce an appropriate notion of Choquet integral in the context of fuzzy numbers. To do this, we face three challenges: the underlying uncertainty when handling fuzzy numbers, the way to order fuzzy numbers by appropriate binary relations and the way to compute the dissimilarity among fuzzy numbers. Illustrative examples are given by involving the α-order on the family of all triangular fuzzy numbers with support on $[0,1]$.

1. Introduction

Billions of data are collected every day in all sectors: economic, social, natural, etc. In practice, more data is generated than we can analyze with any computer, so it is impossible to analyze and transmit all this information. Faced with such an amount of information, we realize that knowledge is obtained by identifying the main characteristics of this huge amount of data (knowledge of the raw data in itself does not constitute knowledge). Therefore, one of the most interesting open problems in both Statistics and Computer Science is to find functions depending on a sample of real values that are able to summarize, in a single real number (representing all of them), the greatest amount of information contained in such a sample. This is the case of the mean, mode and median of a distribution, and more generally of any statistical measure. Such measures are very useful when it comes to examining the most important details contained in a dataset, especially when it is very large.

A function that summarizes a set of real data to a single real number is known as a *fusion function*. Fusion functions do not have to satisfy any specific algebraic condition. However, the processes that best adapt to human intuition are those that verify certain

* Corresponding author.
E-mail address: bustince@unavarra.es (H. Bustince).

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properties. For instance, non-decreasingness is a key property in many natural and economic processes. If we combine it with certain boundary conditions, we find the notion of *aggregation function*, which is being successfully applied in Computing. In this field of study, a remarkable notion also stands out because of its excellent results: the *Choquet (discrete) integral* (hereinafter, *CI*). Although it was initially introduced in order to carry out aggregation tasks (see [10]), its suitable properties have increased the interest of the scientific community towards this concept, and it can be found in a wide variety of applications of the CI in decision making, deep learning, image processing, etc. (see [28] and its corresponding references). In [17], it can be found a motivating application resulting from using the d-Choquet integral to build fuzzy similarity measures. This has led to successive generalizations of the notion of CI to adapt this concept to the requirements of the field of study (see [6,28]).

The main aim of this paper is to introduce an appropriate notion of CI in the context of fuzzy numbers. To do this, three main challenges must be faced.

- Firstly, fuzzy numbers are mathematical abstractions of the notion of real number that are useful for modeling values where uncertainty plays a prominent role, subject to certain ambiguities such as those arising from the subjectivity of personal opinions or measurements affected by the inaccuracy of measuring instruments. The use of this type of number is yielding excellent results in various scientific fields such as Computer Science or Artificial Intelligence, but it is associated with an additional difficulty in terms of their collection and operations.
- Secondly, the family of all fuzzy numbers of the real line, unlike real numbers, is not endowed with a globally accepted order extending the Euclidean order. Although several methodologies have been introduced for ranking fuzzy numbers, each algorithm yields counterintuitive results for humans. Ranking indices are computationally efficient but have the drawback of losing significant information when ordering fuzzy numbers, and they are also not true orders but rather preorders. Therefore, the text focuses on using preorders, specifically the recently introduced α-order (see [18]), which has good algebraic properties. Additionally, the concept of dissimilarity is replacing the difference between numbers in the extension of the concept of CI. In the fuzzy field, this concept is not well-developed, especially when referring to an arbitrary preorder. As a result, the text aims to introduce and provide non-trivial examples of this concept.
- Finally, successive extensions of CI have led to replace the difference by the notion of *dissimilarity* among real numbers (see [8]). In the fuzzy setting, this concept is not yet sufficiently studied, especially when it refers to an arbitrary preorder, so we will devote part of our effort to introduce this concept and to show some non-trivial examples.

This paper is organized as follows. In Section 2 we describe some necessary preliminaries to understand the contents of the paper. In Section 3 we present some properties of the *weight distribution functions* that are employed to apply the α-order. Section 4 is devoted to particularize the α-order to the family of triangular fuzzy numbers with support on $[0,1]$. The notion of *dissimilarity* of fuzzy numbers under an arbitrary binary relation is introduced and analyzed in Section 5. In order to present some examples of fuzzy dissimilarities, we start studying simple fuzzy dissimilarities taking values on crisp fuzzy numbers on Section 6. Finally, the notion of CI on fuzzy numbers is introduced and discussed in Section 7. The paper ends with some conclusions and comments for prospect works on Section 8.

2. Preliminaries

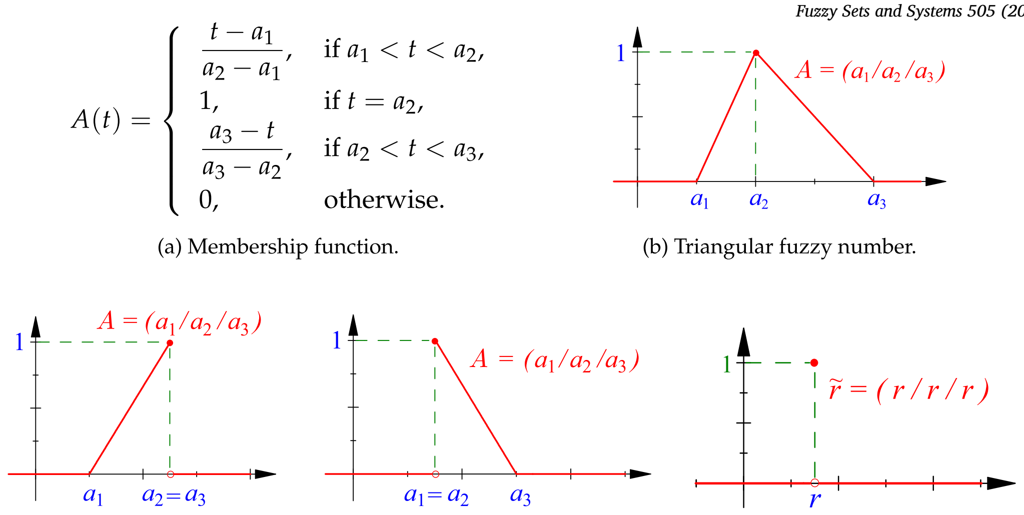
Some notations and basic results are introduced in this section. Through this manuscript, $\mathbb{N}=\{1,2,\dots\}$ denotes the set of all positive integers, and $\mathbb{R}$ will stand for the family of all real numbers. From now on, let $n\in\mathbb{N}$ be a natural number that will represent the number of coordinates we will handle. Let $J_n=\{1,2,\dots,n\}$ and let $J_n^0=\{0,1,2,\dots,n\}$. We denote $\mathbf{0}_n=(0,0,\dots,0)\in\mathbb{R}^n$ and $\mathbf{1}_n=(1,1,\dots,1)\in\mathbb{R}^n$. If no confusion is possible, we denote $\mathbf{0}_n$ simply by $\mathbf{0}$. Given two vectors $\mathbf{u}=(u_1,u_2,\dots,u_n),\mathbf{v}=(v_1,v_2,\dots,v_n)\in\mathbb{R}^n$, its *scalar product* is the real number $\langle\mathbf{u},\mathbf{v}\rangle=\sum_{i=1}^n u_i\,v_i\in\mathbb{R}$.

Henceforth let X be a non-empty set. We denote by X^n the Cartesian product of n copies of X, that is, $X^n=X\times X\times\overset{(n)}{\cdots}\times X$. For instance, $[0,1]^2=[0,1]\times[0,1]$.

2.1. Background on real functions

In this subsection we introduce some nomenclature about real functions of real variables. A function $f:D\subseteq\mathbb{R}\to[0,1]$ is *normal* if there is $t_0\in D$ such that $f(t_0)=1$ (that is, f has absolute maximum on D and it is 1). Given $n\in\mathbb{N}$, a function $T:[0,1]^n\to[0,1]$ is:

- *increasing* (or *non-decreasing*) if $T(t_1,t_2,\dots,t_n)\leq T\left(s_1,s_2,\dots,s_n\right)$ for all $(t_1,t_2,\dots,t_n),(s_1,s_2,\dots,s_n)\in[0,1]^n$ such that $t_j\leq s_j$ for all $j\in\{1,2,\dots,n\}$;
- *symmetric* (or *commutative*) if $T(t_{\sigma(1)},t_{\sigma(2)},\dots,t_{\sigma(n)})=T(t_1,t_2,\dots,t_n)$ for all $(t_1,t_2,\dots,t_n)\in[0,1]^n$ and all permutations σ of $\{1,2,\dots,n\}$;
- *continuous* if it is continuous with respect to the usual (Euclidean) topology;
- *0-dependent* if $T^{-1}(\{0\})=\{(0,0,\dots,0)\}$, that is, $T\left(t_1,t_2,\dots,t_n\right)=0\Leftrightarrow t_1=t_2=\dots=t_n=0$;
- *1-dependent* if $T^{-1}(\{1\})=\{(1,1,\dots,1)\}$, that is, $T\left(t_1,t_2,\dots,t_n\right)=1\Leftrightarrow t_1=t_2=\dots=t_n=1$.



$$A(t) = \begin{cases} \dfrac{t - a_1}{a_2 - a_1}, & \text{if } a_1 < t < a_2, \\ 1, & \text{if } t = a_2, \\ \dfrac{a_3 - t}{a_3 - a_2}, & \text{if } a_2 < t < a_3, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Membership function. (b) Triangular fuzzy number.

(c) Non right continuous. (d) Non left continuous. (e) Crisp FN.

Fig. 1. Membership functions and some classes of triangular fuzzy numbers. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

An (*n-ary*) *aggregation function on* $[0,1]$ (see [4,21]) is an increasing function $\mathcal{A} : [0,1]^n \to [0,1]$ such that $\mathcal{A}(\mathbf{0}_n) = 0$ and $\mathcal{A}(\mathbf{1}_n) = 1$. If $n = 1$, we agree that $\mathcal{A}(t) = t$ for each $t \in [0,1]$. Examples of aggregation functions of two variables are:

$$\text{Mean: } \overline{M}(t,s) = \frac{t+s}{2}; \quad \text{Minimum: } \min\{t,s\}; \quad \text{Maximum: } \max\{t,s\};$$

$$\text{Weighted mean: } \overline{W_\gamma M}(t,s) = (1-\gamma)t + \gamma s \quad (\text{where } \gamma \in (0,1)).$$

Notice that an aggregation function has not to be symmetric (the mean, the minimum and the maximum are symmetric, but the weighted mean is not when $\gamma \neq 0.5$).

2.2. Background on fuzzy sets and fuzzy numbers

Given a non-empty set X, a *fuzzy set* A *on* X is a set of pairs $\{ (x, \mu_A(x)) : x \in X \}$ such that $\mu_A(x) \in [0,1]$ for each $x \in X$. Each value $\mu_A(x)$ represents the membership degree in which the point x belongs to the fuzzy set A, and the function $\mu_A : X \to [0,1]$ is called the *membership function* of the fuzzy set A. For the sake of simplicity, we will identify each fuzzy set A with its corresponding membership function μ_A, and we will say that a *fuzzy set on* X is a mapping $A : X \to [0,1]$.

For each $\beta \in (0,1]$, the β*-level set* (or β*-cut*) of A is the set $A_\beta = \{ x \in X : A(x) \geq \beta \}$. The *kernel* (or *core*) *of* A is $\ker(A) = A_1$ and the *support of* A is the set $\operatorname{supp}(A) = \{ x \in X : A(x) > 0 \}$. If $\beta, \beta' \in (0,1]$ and $\beta \leq \beta'$, then $\ker(A) = A_1 \subseteq A_{\beta'} \subseteq A_\beta \subseteq \operatorname{supp}(A) \subseteq X$.

When X is endowed with a topology, it is usual to denote by $A_0 = \overline{\operatorname{supp}(A)}$ to the closure of the support of A in the considered topology. In such a case, A_0 is a closed subset of X and $\operatorname{supp}(A) \subseteq A_0$. Notice that, in general, the level sets of arbitrary fuzzy sets have not to be neither closed subsets nor open subsets of X. Henceforth, we are only interested on a particular kind of fuzzy sets of the real line $\mathbb{R}$ whose level sets are closed in the Euclidean topology of $\mathbb{R}$.

Recently, a debate has arisen about the notion of *fuzzy number of the real line* (see [2,22,7,11]), whose family is denoted by $\mathrm{FN}(\mathbb{R})$. In general, it must be a fuzzy set $A : \mathbb{R} \to [0,1]$ satisfying certain properties, but often distinct authors use different axioms: for instance, sometimes a fuzzy number must have compact support ([25,3]) and sometimes not; sometimes it must be normal (see [22]) and sometimes not; sometimes it must be a piecewise membership function ([9]) and sometimes not ([27]). However, we do not need to enter on this question because this manuscript is only interested on a concrete family of fuzzy numbers whose definition does not generate any controversy. A *triangular fuzzy number* is a fuzzy set $A : \mathbb{R} \to [0,1]$ of the real line for which there are three real numbers $a_1, a_2, a_3 \in \mathbb{R}$, with $a_1 \leq a_2 \leq a_3$, such that the membership function A is defined as in Fig. 1.a. The numbers a_1, a_2 and a_3 are known as the *corners* of the triangular fuzzy number, and the triangular fuzzy number is denoted by $A = (a_1/a_2/a_3)$. We will stand $\mathrm{TFN}(\mathbb{R})$ for the family of all triangular fuzzy numbers on $\mathbb{R}$. When $a_1 < a_2 < a_3$, the membership function of $(a_1/a_2/a_3)$ is continuous, and its graphic representation shows the shape of a triangle whose basis is the segment $[a_1, a_3]$ and whose vertex is placed at $t = a_2$ (see Fig. 1.b). However, when $a_1 = a_2$ or $a_2 = a_3$, the fuzzy number $(a_1/a_2/a_3)$ is not continuous at $t = a_2$ (see Figs. 1.c and 1.d). In the most simple case, if $r \in \mathbb{R}$ and $a_1 = a_2 = a_3 = r$, the fuzzy number $(r/r/r)$ is denoted by $\widetilde{r}$ (or by $\mathrm{cr}(r)$), it is known as *crisp fuzzy number* and its membership function $\widetilde{r} : \mathbb{R} \to [0,1]$ is defined by $\widetilde{r}(t) = 1$, if $t = r$, and $\widetilde{r}(t) = 0$, otherwise (see Fig. 1.e). The family of all crisp fuzzy numbers generalizes, to the fuzzy setting, the set of all real numbers, that is, $\mathbb{R} \equiv \{ \widetilde{r} : r \in \mathbb{R} \} \subset \mathrm{TFN}(\mathbb{R})$.

Given $\beta \in (0,1]$, the β-level set of $A = (a_1/a_2/a_3)$ is a closed and bounded real interval of the type $A_\beta = [\, \underline{a}_\beta, \overline{a}_\beta \,]$, whose extremes are given by:

$$\underline{a}_{\beta}=a_1+(a_2-a_1)\beta \quad \text{and} \quad \overline{a}_{\beta}=a_3-\left(a_3-a_2\right)\beta. \tag{1}$$

The equalities given in (1) also hold for $\beta=0$, where A_0 denotes the closure of the support of the fuzzy number, that is, $A_0=[a_1,a_3]$. The properties of the functions $\underline{a},\overline{a}:[0,1]\to\mathbb{R}$ (that determine the extremes of its corresponding β-level sets) can be found, for instance, in [14,16]. For convenience, we restrict our study to the family TFN of all triangular fuzzy numbers whose supports are included on $[0,1]$. Therefore, $(a_1/a_2/a_3)$ belongs to TFN if and only if $0\le a_1\le a_2\le a_3\le 1$. Hence $[0,1]\equiv\{\,\widetilde{r}:r\in[0,1]\,\}\subset \text{TFN}\subset \text{TFN}(\mathbb{R})\subset\text{FN}(\mathbb{R})$.

Based on its geometric shapes, there are other families of fuzzy numbers in which the reader can be interested: trapezoidal fuzzy numbers ([11]), parabolic fuzzy numbers ([15]), Gaussian fuzzy numbers ([24]), *LR*-fuzzy numbers ([11]), finite fuzzy numbers ([1]), discrete fuzzy numbers ([27,26]), etc.

2.3. Binary relations

A *binary relation* on X is a non-empty subset $\mathcal{R}\subseteq X\times X$. For simplicity, if $(x,y)\in\mathcal{R}$, we denote it by $x\precsim y$, and we will say that $\precsim$ is the binary relation. We will say that x and y are *$\precsim$-related* (or *$\precsim$-comparable*) if $x\precsim y$ or $y\precsim x$ (or both). If $x,y,z\in X$, the notation "$x\precsim y\precsim z$" will mean that $x\precsim y$ and $y\precsim z$. A binary relation $\precsim$ on X is:

- *reflexive* if $x\precsim x$ for all $x\in X$;
- *symmetric* if $x\precsim y$ implies that $y\precsim x$;
- *antisymmetric* if we can deduce $x=y$ when $x,y\in X$ satisfy $x\precsim y$ and $y\precsim x$;
- *transitive* if we can deduce $x\precsim z$ for each $x,y,z\in X$ satisfying $x\precsim y$ and $y\precsim z$;
- *total* (or *complete*) if $x\precsim y$ or $y\precsim x$ for all $x,y\in X$ (that is, each two points of X are $\precsim$-related).

A *partially ordered set* (a *poset*, for short) is a pair $(X,\precsim)$ where $\precsim$ is a reflexive, antisymmetric and transitive binary relation on X. A poset is *bounded* if there are $m,M\in X$ such that $m\precsim x\precsim M$ for all $x\in X$. In such a case, it will be denoted by $(X,\precsim,m,M)$. Notice that from $x\precsim y\precsim z$ we cannot deduce, in general, that $x\precsim z$ unless $\precsim$ is transitive.

Given a binary relation $\precsim$ on X, we will employ the notation $\prec$ and $\sim$ for the following two new binary relations on X:

$$x\prec y \quad \text{if "}x\precsim y\text{" holds but "}y\precsim x\text{" is false;} \quad x\sim y \quad \text{if both } x\precsim y \text{ and } y\precsim x. \tag{2}$$

Using the previous notation, notice that $x\precsim y$ if, and only if, either $x\prec y$ or $x\sim y$. Also notice that the binary relation $\prec$ is distinct than: "$x\precsim y$ and $x\neq y$".

Example 2.1. The following binary relations for triangular fuzzy numbers $A=(a_1/a_2/a_3), B=(b_1/b_2/b_3)\in\text{TFN}(\mathbb{R})$ were considered in [20]:

- $A\le_c B \quad$ if $\quad a_i\le b_i$ for each $i\in\{1,2,3\}$;
- $A\le_\Sigma B \quad$ if $\quad a_1+a_2+a_3\le b_1+b_2+b_3$.

In [22] it was introduced a methodology $\preccurlyeq$ for ranking fuzzy numbers verifying many properties that are according to human intuition. The binary relation $\le_c$ is a partial order on TFN($\mathbb{R}$), but it is not total; $\le_\Sigma$ is total and transitive, but it is not antisymmetric; finally, $\preccurlyeq$ is total, but it is neither transitive nor antisymmetric.

Every total binary relation is reflexive. If $\precsim$ is a total and transitive binary relation on X, then the binary relation $\sim$, defined as in (2), is an equivalence relation on X (that is, reflexive, symmetric and transitive). These properties and a comprehensive study of binary relations can be found on the classical text [12].

Throughout this manuscript, we will employ *total preorders* (also called *weak orders* in the literature), which are total (therefore, reflexive) and transitive binary relations.

2.4. Ranking fuzzy numbers

The set of all real numbers $\mathbb{R}$ is naturally endowed with the binary relation $\le$, that is, the usual order of real numbers. This binary relation is a total order on $\mathbb{R}$. However, the family FN($\mathbb{R}$) of all fuzzy numbers of $\mathbb{R}$ is not naturally endowed with a universally accepted methodology for ranking fuzzy numbers. The reason is simple: all procedures for ordering fuzzy numbers carry unavoidable shortcomings. Basically, there are two main procedures for ordering fuzzy numbers: ranking indices and fuzzy binary relations (that cannot be reduced to ranking indices).

Ranking processes have not to be necessarily considered on the whole set FN($\mathbb{R}$) because this set is very general. On the contrary, it is more usual to consider ranking procedures on a non-empty subset $\mathcal{S}$ of the family FN($\mathbb{R}$) of all fuzzy numbers on $\mathbb{R}$. In this case, the set $\mathcal{S}$ must contain, at least, the set of all crisp fuzzy numbers on $\mathbb{R}$ or on a subinterval $I\subset\mathbb{R}$.

The second above-mentioned way for ranking fuzzy numbers is based on a binary relation $\precsim$ on a subset $\mathcal{S}$ of FN($\mathbb{R}$) that satisfies certain properties. For instance, the reflexivity is usually required when the binary relation $\precsim$ is defined. Any reasonable way $\precsim$ for ranking fuzzy numbers must, at least, extend the real order, that is,

$$r,s\in\mathbb{R},\ \ \widetilde{r},\widetilde{s}\in\mathcal{S},\ \ r\leq s\ \ \Rightarrow\ \ \widetilde{r}\precsim\widetilde{s}. \tag{3}$$

However, it may be not entirely reasonable to impose the antisymmetric property to a binary relation on $\mathcal{S}$ because the symmetric triangular fuzzy numbers $(a-\delta/a/a+\delta)$ and $(a-\delta'/a/a+\delta')$, for $a\in\mathbb{R}$ and $\delta,\delta'\in(0,+\infty)$, are usually distinct but indistinguishable. In this case, it is more usual to employ the notation $A\sim B$ when $A,B\in\mathcal{S}$ satisfy $A\precsim B$ and $B\precsim A$. The binary relation $\sim$ establishes that the fuzzy numbers are not necessarily equal but are indistinguishable or, rather, equivalent. This is the reason why in this paper we are mainly interested on total and transitive binary relations and, accordingly, we will introduce the so-called "*α-order*" in Subsection 2.4.2. Before doing so, we discuss the most common method for ordering fuzzy numbers in practice. We point out that a total and transitive binary relation is often called a *preference relation* (see [19]).

2.4.1. Ranking indices

A *ranking index* is a procedure for ranking fuzzy numbers depending on a mapping $P:\mathcal{S}\to\mathbb{R}$ in the following way:

$$\begin{cases}\bullet\ \text{if}\quad P(A)<P(B),\quad\text{then}\quad A\prec B;\\ \bullet\ \text{if}\quad P(B)<P(A),\quad\text{then}\quad B\prec A;\\ \bullet\ \text{if}\quad P(B)=P(A),\quad\text{then}\quad A\sim B.\end{cases} \tag{4}$$

Many ranking indices were introduced in the past: expected value, median, center of gravity, ambiguity, credibilistic mean, possibilistic mean, etc. (a complete state of the art can be found on [5]). However, all of them have severe drawbacks, especially when they are compared to human intuition (see [2]).

2.4.2. The α-order for a class of fuzzy sets

The α-order, denoted by $\overset{\alpha}{\leq}$, is a binary relation that has been very recently introduced by Neres et al. in [18]. It allows to compare two fuzzy sets in the class C of all fuzzy sets $A\in\mathrm{FS}(\mathbb{R})$ such that A is normal, $\mathrm{supp}(A)$ is bounded, and for each $\beta\in(0,1]$, the β-cut A_β is a closed subset of $\mathbb{R}$ with a finite number of connected components. The class C is very extensive: it contains the family of all normal fuzzy numbers with compact support (that covers the subfamilies of all crisp fuzzy numbers and of all triangular fuzzy numbers whose support is included on $[0,1]$), but it also contains great classes of fuzzy sets that are not fuzzy numbers. The theoretical construction of the α-order is complicated, so we refer to [18] for details. Anyway, next we give the highlights of its definition.

A *membership degree vector* is a vector $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_n)\in(0,1]^n$, and a *weight vector* (or *weighting vector*) is $\omega=(\omega_1,\omega_2,\dots,\omega_n)\in[0,1]^n$ such that $\omega_1+\omega_2+\ldots+\omega_n=1$. Both vectors will be jointly considered, so they will have the same dimension (n). Let W_n be the family of all weight vectors in $[0,1]^n$. The *first* and the *second weight distributions* are two mappings $\vartheta,\varphi_\alpha:W_n\cup\{\mathbf{0}_n\}\to W_n\cup\{\mathbf{0}_n\}$ that reorder the components of a weight vector $\omega\in W_n\cup\{\mathbf{0}_n\}$ by increasing the number of components that are zero and preserving the sum equal to 1. These mappings can be iteratively applied, generating the compositions $\vartheta^k,\varphi_\alpha^k:W_n\cup\{\mathbf{0}_n\}\to W_n\cup\{\mathbf{0}_n\}$, where ϑ^0 is the identity on $W_n\cup\{\mathbf{0}_n\}$, $\vartheta^1=\vartheta$ and $\vartheta^{k+1}=\vartheta\circ\vartheta^k$ for all $k\in\mathbb{N}$ (similar definition for φ_α^k).

Following [18], an *aggregation function of the real line* is a function $\mathcal{A}:\cup_{m\in\mathbb{N}}\mathbb{R}^m\to\mathbb{R}$ such that: (1) for each $m\in\mathbb{N}$, the restriction $\mathcal{A}|_{\mathbb{R}^m}:\mathbb{R}^m\to\mathbb{R}$ is an increasing function, and (2) if $m=1$ then $\mathcal{A}(t)=t$ for each $t\in\mathbb{R}$. Notice that this notion extends the concept of m-ary aggregation function on $[0,1]$ introduced in Preliminaries because it does not impose the constraints $\mathcal{A}(\mathbf{0}_m)=0$ and $\mathcal{A}(\mathbf{1}_m)=1$.

Given $A\in C$ and $\beta\in(0,1]$, the β-cut A_β is a closed and bounded subset of $\mathbb{R}$ with a finite number of connected components. Such components must be non-empty, closed and bounded real subintervals of $\mathbb{R}$ and its collection is denoted by $\mathbb{I}_\beta^A=\{\,I_1^{A,\beta}=[a_1^{A,\beta},b_1^{A,\beta}],\dots,I_{r(A,\beta)}^{A,\beta}=[a_{r(A,\beta)}^{A,\beta},b_{r(A,\beta)}^{A,\beta}]\,\}$, where $r(A,\beta)\in\mathbb{N}$. We denote by $\mathcal{A}_\beta^A$ to the aggregated value $\mathcal{A}(c_1^{A,\beta},\dots,c_{s(A,\beta)}^{A,\beta})$ where $\{c_1^{A,\beta},\dots,c_{s(A,\beta)}^{A,\beta}\}$ is the set of all extremes $\{\,a_1^{A,\beta},b_1^{A,\beta},\dots,a_{r(A,\beta)}^{A,\beta},b_{r(A,\beta)}^{A,\beta}\,\}$ of the connected components of A_β under two assumptions: they are strictly and increasingly ordered and there is no repeated numbers (if some extremes are equal, we remove some of them in order to avoid repetitions). If $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_n)$ is a membership degree vector, we denote $\mathcal{A}_\alpha^A=(\mathcal{A}_{\alpha_1}^A,\mathcal{A}_{\alpha_2}^A,\dots,\mathcal{A}_{\alpha_n}^A)\in[0,1]^n$.

The α-order to compare two fuzzy sets of C is based on the following real numbers associated to each fuzzy set $A\in C$:

$$\upsilon_{\alpha,\omega,\vartheta,\mathcal{A}}^k(A)=\langle\,\vartheta^k(\omega),\mathcal{A}_\alpha^A\,\rangle\qquad\text{and}\qquad\upsilon_{\alpha,\omega,\varphi_\alpha,\mathcal{A}}^k(A)=\langle\,\varphi_\alpha^k(\omega),\mathcal{A}_\alpha^A\,\rangle, \tag{5}$$

where $k\in\{0,1,2,\dots\}$, $\omega=(\omega_1,\omega_2,\dots,\omega_n)$ is a weight vector, $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_n)$ is a membership vector, ϑ and φ_α are the weight distribution functions and $\mathcal{A}$ is an aggregation function of the real line. In order to simplify the notation, when the weight vector ω, the membership vector α and the aggregation function $\mathcal{A}$ are given by the context, we will denote $\upsilon_{\alpha,\omega,\vartheta,\mathcal{A}}^k$ by υ_ϑ^k and $\upsilon_{\alpha,\omega,\varphi_\alpha,\mathcal{A}}^k$ by $\upsilon_{\varphi_\alpha}^k$. By employing this notation and such algebraic tools, the α-order was introduced in [18, Definition 34] by defining some previous binary relations ($\underset{m}{\overset{\vartheta}{=}}$ and $\underset{m}{\overset{\varphi_\alpha}{=}}$). In the case of triangular fuzzy numbers of TFN, where a traditional aggregation is involved, we synthetize the notions of $\overset{\alpha}{=}$, $\overset{\alpha}{<}$ and $\overset{\alpha}{\leq}$ in the following result.

Proposition 2.2. *Let $\omega=(\omega_1,\omega_2,\dots,\omega_n)\in W_n$ be a weight vector, let $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_n)$ be a membership vector and let $\mathcal{A}:[0,1]^2\to[0,1]$ be an aggregation function on $[0,1]$. Then the binary relations $\overset{\alpha}{=}$, $\overset{\alpha}{<}$ and $\overset{\alpha}{\leq}$ are given on* TFN *by:*

- $A\overset{\alpha}{=}B\ \Leftrightarrow\ \left[\ \upsilon_\vartheta^k(A)=\upsilon_\vartheta^k(B)\quad\textit{and}\quad\upsilon_{\varphi_\alpha}^k(A)=\upsilon_{\varphi_\alpha}^k(B)\quad\textit{for all } k\in\{0,1,\dots,n-1\}\ \right];$

- $A \overset{\alpha}{<} B \;\Leftrightarrow\;$ *there is* $k_0 \in \{0,1,\dots,n-1\}$ *such that*

$$\begin{cases} \textit{either} \begin{cases} \upsilon_{\vartheta}^{j}(A) = \upsilon_{\vartheta}^{j}(B) \quad \textit{for all } j \in \{0,1,\dots,k_0-1\}, \quad \textit{and} \\ \upsilon_{\vartheta}^{k_0}(A) < \upsilon_{\vartheta}^{k_0}(B), \end{cases} \\[2ex] \textit{or} \quad \begin{cases} \upsilon_{\vartheta}^{j}(A) = \upsilon_{\vartheta}^{j}(B) \quad \textit{for all } j \in \{0,1,\dots,n-1\}, \quad \textit{and} \\ \upsilon_{\varphi_\alpha}^{j}(A) = \upsilon_{\varphi_\alpha}^{j}(B) \quad \textit{for all } j \in \{0,1,\dots,k_0-1\}, \quad \textit{and} \\ \upsilon_{\varphi_\alpha}^{k_0}(A) < \upsilon_{\varphi_\alpha}^{k_0}(B); \end{cases} \end{cases}$$

- $A \overset{\alpha}{\leq} B \;\Leftrightarrow\; \left[\, \textit{either } A \overset{\alpha}{=} B \textit{ or } A \overset{\alpha}{<} B \,\right].$

We highlight three facts: (1) the α-order is not an order because it is reflexive and transitive, but it is not antisymmetric (it would only be an order on the quotient set whose classes are given by α-equal fuzzy numbers); however, it is total, so, as we will explain later, it is a perfect example of the type of binary relations that we will use to define the fuzzy Choquet integral; (2) the α-order was introduced by employing an aggregation function of the real line $\mathcal{A} : \cup_{m\in\mathbb{N}}\mathbb{R}^m \to \mathbb{R}$; however, for our purposes (which are only introductory), it will be sufficient to consider a classic aggregation function $\mathcal{A} : [0,1]^2 \to [0,1]$ of two real variables; (3) although we will use the notation $\overset{\alpha}{\leq}$, the α-order actually depends on a weight vector $\omega = (\omega_1, \omega_2, \dots, \omega_n) \in W_n$, on a membership vector $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ and on an aggregation function $\mathcal{A} : [0,1]^2 \to [0,1]$.

2.5. Restricted dissimilarity functions

Definition 2.3. ([6]) A function $\delta : [0,1]^2 \to [0,1]$ is a *restricted dissimilarity function on* $[0,1]$ if it satisfies, for all $t, s, r \in [0,1]$, the following properties:

1. $\delta(t,s) = \delta(s,t)$;
2. $\delta(t,s) = 1$ if and only if $\{t,s\} = \{0,1\}$;
3. $\delta(t,s) = 0$ if and only if $t = s$;
4. if $t \leq s \leq r$, then $\delta(t,s) \leq \delta(t,r)$ and $\delta(s,r) \leq \delta(t,r)$.

The basic example of a restricted dissimilarity function on $[0,1]$ is $\delta(t,s) = |\, t - s \,|$ for all $t, s \in [0,1]$. However, this family is very extensive. Some applications of restricted dissimilarity functions can be found on [23,13].

2.6. The discrete Choquet integral

Given $m \in \mathbb{N}$, let $\mathcal{P}_m$ denote the family of all subsets (or parts) of $\{1,2,\dots,m\}$ (sometimes, it is also denoted by $2^{[m]}$).

Definition 2.4. A *fuzzy measure on* $\mathcal{P}_m$ is a function $\mu : \mathcal{P}_m \to [0,1]$ such that: (1) (*non-decreasingness*) if $F_1 \subseteq F_2 \subseteq \{1,2,\dots,m\}$, then $\mu(F_1) \leq \mu(F_2)$; (2) (*boundary conditions*) $\mu(\emptyset) = 0$ and $\mu(\{1,2,\dots,m\}) = 1$.

As a result, $0 \leq \mu(F) \leq 1$ for each $F \in \mathcal{P}_m$.

Example 2.5. The minimum and the maximum fuzzy measures are $\mu_\perp, \mu_\top : \mathcal{P}_m \to [0,1]$ defined by:

$$\mu_\perp(F) = \begin{cases} 1, & \text{if } F = \{1,2,\dots,m\}, \\ 0, & \text{if } F \neq \{1,2,\dots,m\}; \end{cases} \qquad \mu_\top(F) = \begin{cases} 1, & \text{if } F \neq \emptyset, \\ 0, & \text{if } F = \emptyset. \end{cases}$$

If μ is a fuzzy measure on $\mathcal{P}_m$, then $\mu_\perp \leq \mu \leq \mu_\top$.

Definition 2.6. ([6]) Given $n \in \mathbb{N}$, $n \geq 2$, the *discrete Choquet integral on* $[0,1]$ *with respect to a fuzzy measure* $\mu : \mathcal{P}_n \to [0,1]$ *and a restricted dissimilarity function* $\delta : [0,1]^2 \to [0,1]$ is defined as the mapping $C_{\mu,\delta} : [0,1]^n \to [0,1]$ such that

$$C_{\mu,\delta}(t_1, t_2, \dots, t_n) = \sum_{i=1}^{n} \delta(t_{\sigma(i)}, t_{\sigma(i-1)}) \cdot \mu(F_{\sigma(i)}),$$

where $\sigma : \{1,2,\dots,n\} \to \{1,2,\dots,n\}$ is a permutation satisfying $t_{\sigma(1)} \leq t_{\sigma(2)} \leq \ldots \leq t_{\sigma(n)}$ with the convention $t_{\sigma(0)} = 0$ and $F_{\sigma(i)} = \{\sigma(i), \sigma(i+1), \dots, \sigma(n)\}$.

When $\delta(t,s) = |\,t-s\,|$ for all $t,s \in [0,1]$, we recover the definition of the classical discrete Choquet integral because:

$$C_{\mu,\delta}(t_1,t_2,\ldots,t_n) = \sum_{i=1}^{n} |\,t_{\sigma(i)} - t_{\sigma(i-1)}\,| \cdot \mu(F_{\sigma(i)}) = \sum_{i=1}^{n} (t_{\sigma(i)} - t_{\sigma(i-1)}) \cdot \mu(F_{\sigma(i)}).$$

3. Some properties of the weight distribution functions

In this section we describe some basic properties of the mappings ϑ and φ_α that we will use in the rest of the paper. Unless otherwise stated, from now on, $n \in \mathbb{N}$ will represent the number of selected α-cuts $\alpha_1,\alpha_2,\ldots,\alpha_n \in (0,1]$, $\alpha = (\alpha_1,\alpha_2,\ldots,\alpha_n)$ will denote a membership vector and $\omega = (\omega_1,\omega_2,\ldots,\omega_n) \in W_n$ will stand for a weight vector ($\omega_1+\omega_2+\ldots+\omega_n = 1$). In order not to excessively repeat these involved tools, we will suppose that such three notions are implicitly considered on each of the next statements without explicitly mention them (we will only highlight them on notable theorems).

Proposition 3.1. *Let $\alpha = (\alpha_1,\alpha_2,\ldots,\alpha_n)$ be a membership vector and let $\omega = (\omega_1,\omega_2,\ldots,\omega_n) \in W_n$ be a weight vector.*

1. *Either $\vartheta(\omega) = \mathbf{0}$ or $\vartheta(\omega)$ is another weight vector.*
2. *Either $\varphi_\alpha(\omega) = \mathbf{0}$ or $\varphi_\alpha(\omega)$ is another weight vector.*
3. *The vectors $\vartheta(\omega)$ and $\varphi_\alpha(\omega)$ have strictly more coordinates with value zero than the vector ω.*
4. *If $k \in \{1,2,\ldots,n\}$, at least k coordinates of $\vartheta^k(\omega)$ and $\varphi_\alpha^k(\omega)$ are zero.*
5. *If $k \geq n$, then $\vartheta^k(\omega) = \varphi_\alpha^k(\omega) = \mathbf{0}$ for all $\omega \in W_n \cup \{\mathbf{0}\}$.*

In the next result we show that the successive vectors $\vartheta^0(\omega), \vartheta^1(\omega), \ldots, \vartheta^k(\omega), \ldots$ are linearly independent up to come to the vector zero, which is the moment in which the rest of the sequence only contains the null vector.

Corollary 3.2. *The following properties hold.*

1. *Given a weight vector $\omega = (\omega_1,\omega_2,\ldots,\omega_n) \in W_n$, there is $p_\omega \in \{1,2,\ldots,n\}$ such that the set $\{\, \vartheta^0(\omega), \vartheta^1(\omega), \ldots, \vartheta^{p_\omega - 1}(\omega) \,\}$ is linearly independent in $\mathbb{R}^n$ and $\vartheta^k(\omega) = \mathbf{0}_n \in \mathbb{R}^n$ for all $k \geq p_\omega$.*
2. *Given a membership vector $\alpha = (\alpha_1,\alpha_2,\ldots,\alpha_n)$ and a weight vector $\omega = (\omega_1,\omega_2,\ldots,\omega_n) \in W_n$, there is $q_{\alpha,\omega} \in \{1,2,\ldots,n\}$ such that the set $\{\, \varphi_\alpha^0(\omega), \varphi_\alpha^1(\omega), \ldots, \varphi_\alpha^{q_{\alpha,\omega}-1}(\omega) \,\}$ is linearly independent in $\mathbb{R}^n$ and $\varphi_\alpha^k(\omega) = \mathbf{0}_n \in \mathbb{R}^n$ for all $k \geq q_{\alpha,\omega}$.*

4. The α-order for triangular fuzzy numbers

In this section we describe some basic properties of the α-order $\overset{\alpha}{\leq}$ on the family TFN of all triangular fuzzy numbers on $[0,1]$. Furthermore, we characterize the binary relation $\overset{\alpha}{\leq}$ in terms of a valuations vector in $\mathbb{R}^{2n-1}$.

4.1. Some properties of the α-order on triangular fuzzy numbers

In this subsection we particularize the α-order to the family of all triangular fuzzy numbers on $[0,1]$, detailing the simple tools that are needed to consider it, and we introduce some of its first properties. Given $\beta \in (0,1]$, the β-cut of a triangular fuzzy number on $[0,1]$ $A = (a_1/a_2/a_3) \in \text{TFN}$ is the closed interval $A_\beta = [\,\underline{a}_\beta, \overline{a}_\beta\,] = \big[a_1 + (a_2-a_1)\beta,\ a_3 - (a_3-a_2)\beta\big] \subseteq [0,1]$. To aggregate its extremes, we only need a bivariate aggregation function $\mathcal{A} : [0,1]^2 \to [0,1]$ (we agree $\mathcal{A}(t) = t$ for all $t \in [0,1]$ when a unique argument is involved). Therefore:

$$\mathcal{A}_\beta^A = \mathcal{A}(\,\underline{a}_\beta, \overline{a}_\beta\,) = \mathcal{A}(a_1 + (a_2-a_1)\beta,\ a_3 - (a_3-a_2)\beta), \tag{6}$$

where $0 \leq a_1 \leq \underline{a}_\beta \leq a_2 \leq \overline{a}_\beta \leq a_3 \leq 1$ for all $\beta \in (0,1]$. Taking into account (5), $v_\vartheta^k(A) = v_{\varphi_\alpha}^k(A) = 0$ for each $k \geq n$ because $\vartheta^k(\omega) = \varphi_\alpha^k(\omega) = 0$ if $k \geq n$, so we only involve the real numbers $v_\vartheta^k(A)$ and $v_{\varphi_\alpha}^k(A)$ for $k \in \{0,1,\ldots,n-1\}$.

Proposition 4.1. *If $\mathcal{A}$ is an aggregation function on $[0,1]$ and $A \in \text{TFN}$, then $v_\vartheta^k(A) \in [0,1]$ and $v_{\varphi_\alpha}^k(A) \in [0,1]$ for each $k \in \{0,1,\ldots,n-1\}$.*

As we will explain later, a critical property of the α-order $\overset{\alpha}{\leq}$ is the following one.

Proposition 4.2. *If $A = (a_1/a_2/a_3), B = (b_1/b_2/b_3) \in \text{TFN}$ are two triangular fuzzy numbers such that $a_i \leq b_i$ for each $i \in \{1,2,3\}$, then $A \overset{\alpha}{\leq} B$.*

It is interesting to know how the functions v_ϑ^k and $v_{\varphi_\alpha}^k$ act on crisp fuzzy numbers.

Proposition 4.3. *Let $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_n)$ be a membership vector, let $\omega=(\omega_1,\omega_2,\dots,\omega_n)\in W_n$ be a weight vector and let $\mathcal{A}$ be an extended aggregation function on $\mathbb{R}$. Therefore, for each $r\in\mathbb{R}$ and all $k\in\{0,1,\dots,n-1\}$,*

$$\upsilon_{\vartheta}^{k}(\widetilde{r})=\begin{cases} r, & \text{if } \vartheta^k(\omega)\neq\mathbf{0},\\ 0, & \text{if } \vartheta^k(\omega)=\mathbf{0},\end{cases}\qquad \text{and}\qquad \upsilon_{\varphi_\alpha}^{k}(\widetilde{r})=\begin{cases} r, & \text{if } \varphi_\alpha^k(\omega)\neq\mathbf{0},\\ 0, & \text{if } \varphi_\alpha^k(\omega)=\mathbf{0}.\end{cases}$$

Remark 4.4. Since $\vartheta^0(\omega)=\omega\neq\mathbf{0}$, the order $\overset{\alpha}{\leq}$ is coherent with the usual order on real numbers, that is, given $r,s\in\mathbb{R}$, we conclude that $\widetilde{r}\overset{\alpha}{<}\widetilde{s}$ if and only if $r<s$, and $\widetilde{r}\overset{\alpha}{=}\widetilde{s}$ if and only if $r=s$.

4.2. The valuations vector $V_{\alpha,\omega,\mathcal{A}}$

In this subsection we consider the vector of numbers that determines the binary relation $\overset{\alpha}{\leq}$, which can be useful to α-order two distinct fuzzy numbers by employing the binary relation $\sqsubseteq$ that we will introduce in Definition 4.6.

Definition 4.5. Let $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_n)$ be a membership vector, let $\omega=(\omega_1,\omega_2,\dots,\omega_n)\in W_n$ be a weight vector and let $\mathcal{A}$ be an aggregation function. Given $A\in\mathrm{TFN}$, we denote by $V_{\alpha,\omega,\mathcal{A}}(A)\in\mathbb{R}^{2n-1}$ the vector

$$V_{\alpha,\omega,\mathcal{A}}(A)=\left(V^1_{\alpha,\omega,\mathcal{A}}(A),\,V^2_{\alpha,\omega,\mathcal{A}}(A),\dots,\,V^{2n-1}_{\alpha,\omega,\mathcal{A}}(A)\right)$$

whose components are:

$$V^k_{\alpha,\omega,\mathcal{A}}(A)=\begin{cases}\upsilon_{\vartheta}^{k-1}(A), & \text{if } k\in\{1,2,\dots,n\},\\[2ex] \upsilon_{\varphi_\alpha}^{k-n}(A), & \text{if } k\in\{n+1,n+2,\dots,2n-1\}.\end{cases}$$

In other words,

$$V_{\alpha,\omega,\mathcal{A}}(A)=\Big(\underbrace{\upsilon_{\vartheta}^{0}(A),\,\upsilon_{\vartheta}^{1}(A),\dots,\,\upsilon_{\vartheta}^{n-1}(A)}_{n\text{ components computed by using }\vartheta},\ \underbrace{\upsilon_{\varphi_\alpha}^{1}(A),\,\upsilon_{\varphi_\alpha}^{2}(A),\dots,\,\upsilon_{\varphi_\alpha}^{n-1}(A)}_{n-1\text{ components computed by using }\varphi_\alpha}\Big).$$

As before, in order to simplify the notation, when the weight vector ω, the membership vector α and the aggregation function $\mathcal{A}$ are given by the context, we will denote the real value $V^k_{\alpha,\omega,\mathcal{A}}(A)$ by $V^k(A)$, and the valuations vector $V_{\alpha,\omega,\mathcal{A}}(A)$ by $V(A)$. The first n components of $V(A)$ are worked out by using the mapping ϑ, and the last $n-1$, by using φ_α. Notice that we have avoided to include the term $\upsilon^0_{\varphi_\alpha}(A)$ because $\upsilon^0_{\vartheta}(A)=\upsilon^0_{\varphi_\alpha}(A)$ since $\vartheta^0(\omega)=\varphi^0_\alpha(\omega)=\omega$.

In order to compare the valuations vectors of distinct triangular fuzzy numbers, we introduce a binary relation in $\mathbb{R}^m$ (denoted by $\sqsubseteq$) as follows.

Definition 4.6. Given $m\in\mathbb{N}$ and $\mathbf{u}=(u_1,u_2,\dots,u_m),\mathbf{v}=(v_1,v_2,\dots,v_m)\in\mathbb{R}^m$, we denote:

- $\mathbf{u}\sqsubset\mathbf{v}$ if either $u_1<v_1$ or there is $k_0\in\{2,3,\dots m\}$ such that $u_i=v_i$ for all $i\in\{1,2,\dots,k_0-1\}$ and $u_{k_0}<v_{k_0}$;
- $\mathbf{u}\sqsubseteq\mathbf{v}$ if either $\mathbf{u}=\mathbf{v}$ or $\mathbf{u}\sqsubset\mathbf{v}$.

Notice that if $\mathbf{u}\sqsubset\mathbf{v}$ then $\mathbf{u}\neq\mathbf{v}$. The binary relation $\sqsubseteq$ on $\mathbb{R}^m$ is just the *lexicographic order*, so it is a total order on any non-empty subset of $\mathbb{R}^m$. In the next result, we use $\sqsubseteq$ to characterize the ranking $A\overset{\alpha}{\leq}B$ in terms of their valuations vectors.

Proposition 4.7. *Let $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_n)$ be a membership vector, let $\omega=(\omega_1,\omega_2,\dots,\omega_n)\in W_n$ be a weight vector and let $\mathcal{A}$ be an aggregation function. Given $A,B\in\mathrm{TFN}$, then $A\overset{\alpha}{\leq}B$ (respectively, $A\overset{\alpha}{=}B$, $A\overset{\alpha}{<}B$) if and only if $V(A)\sqsubseteq V(B)$ (respectively, $V(A)=V(B)$, $V(A)\sqsubset V(B)$).*

In the following corollary we use the notation $\mathbf{0}_m=(0,0,\dots,0)\in[0,1]^m$ and $\mathbf{1}_m=(1,1,\dots,1)\in[0,1]^m$.

Corollary 4.8. *Given a membership vector $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_n)\in(0,1]^n$ and a weight vector $\omega=(\omega_1,\omega_2,\dots,\omega_n)\in W_n$, let $p_\omega,q_{\alpha,\omega}\in\{0,1,\dots,n-1\}$ be the integer numbers given in Corollary 3.2. Then, for each aggregation function $\mathcal{A}$ and each $A\in\mathrm{TFN}$:*

$$\mathbf{0}_{2n-1}=V(\widetilde{0})\sqsubseteq V(A)\sqsubseteq V(\widetilde{1})=(\mathbf{1}_{p_\omega},\mathbf{0}_{n-p_\omega},\mathbf{1}_{q_{\alpha,\omega}-1},\mathbf{0}_{n-q_{\alpha,\omega}}).$$

Notice that we obtain $\mathbf{1}_{q_{\alpha,\omega}-1}$ because we avoid the value $\upsilon^0_{\varphi_\alpha}(\widetilde{1})=1$ in the vector $V(\widetilde{1})$.

5. Dissimilarities of fuzzy numbers

Having in mind the notion of *restricted dissimilarity function* and its increasingly interesting applications, in this section we introduce a very general idea to face the problem of defining the concept of *dissimilarity of fuzzy numbers*. Two drawbacks naturally arise: on the one hand, the interval $[0,1]$ is canonically ordered by the total order $\leq$, but the set TFN of all fuzzy numbers in $[0,1]$ (or even the set $\mathrm{TFN}(\mathbb{R})$) is not ordered by a globally accepted binary relation that extends the real order; on the other hand, the interval $[0,1]$ is bounded, but $\mathrm{TFN}(\mathbb{R})$ is not. We introduce the notion of *FRE-set* to overcome these obstacles, which let us to present a fuzzy version of the notion of dissimilarity and to show some of its first properties.

5.1. FRE-sets

The following definition comes from the idea to handle a set of fuzzy numbers endowed with a binary relation for which there exist absolute extremes. Obviously, we are mainly interested in the set TFN endowed with the binary relation $\overset{\alpha}{\leq}$.

Definition 5.1. A *fuzzy binary related set with absolute extremes* (for short, a *FRE-set*[1]) is a quadruple $(\mathcal{S},\precsim,A_{\min},A_{\max})$ where $\mathcal{S}$ is a subset of fuzzy numbers endowed with a binary relation $\precsim$, and $A_{\min},A_{\max}\in\mathcal{S}$ are elements such that $A_{\min}\precsim A\precsim A_{\max}$ for each $A\in\mathcal{S}$.

Notice that in the previous definition we are not assuming any condition on the binary relation $\precsim$ because the binary relations we can consider on a non-empty set of fuzzy numbers can be extraordinarily complex and intricate. We prefer only to highlight that the points of the non-empty set $\mathcal{S}$ are, in some cases, "*related*" by $\precsim$ (it could exist pairs of points that are not $\precsim$-related as well) and there exist "*absolute*"extremes, denoted by $A_{\min}$ and $A_{\max}$. Obviously, on each context, the binary relation $\precsim$ will satisfy the concrete properties in which the researcher could be interested. For instance, it has sense to assume that $\precsim$ is reflexive, transitive or total on $\mathcal{S}$, which are some of the main properties of the binary relation $\overset{\alpha}{\leq}$ when it is considered on the set TFN. Observe that such binary relation is not bounded when it is considered on $\mathrm{TFN}(\mathbb{R})$, which is a problem when we try to extend the notion of "*restricted dissimilarity function*"from the real setting to the fuzzy framework: in the real case, the extremes 0 and 1 of the interval $[0,1]$ plays a crucial role in Definition 2.3.

Obviously, the set $\{\,\widetilde{r}:r\in[0,1]\,\}$ is a FRE-set with minimum $\widetilde{0}$ and maximum $\widetilde{1}$ provided it is endowed with a binary relation $\precsim$ satisfying (3). Anyway, we are mainly interested in the following example.

Example 5.2. $(\mathrm{TFN},\overset{\alpha}{\leq},\widetilde{0},\widetilde{1})$ is a FRE-set in which the binary relation $\overset{\alpha}{\leq}$ is reflexive, transitive, total and, by Proposition 4.2, it satisfies the condition:

$$A,B\in\mathrm{TFN},\quad\left[\ \underline{a}_\beta\leq\underline{b}_\beta,\ \overline{a}_\beta\leq\overline{b}_\beta\ \text{ for all }\beta\in(0,1]\ \right]\ \Rightarrow\ A\overset{\alpha}{\leq}B. \tag{7}$$

5.2. Dissimilarities on FRE-sets

In this section we introduce the notion of *dissimilarity* on a general FRE-set, and we comment some of the difficulties that appear when working with such algebraic structure.

In the following definition, in the setting of a FRE-set $(\mathcal{S},\precsim,A_{\min},A_{\max})$, we use the notation "$\{A,B\}\sim\{C,D\}$" to indicate that $A\sim C$ and $B\sim D$, or vice versa.

Definition 5.3. Let $(\mathcal{S},\precsim,A_{\min},A_{\max})$ and $(\mathcal{S}',\precsim',A'_{\min},A'_{\max})$ be two FRE-sets. A mapping $\Delta:\mathcal{S}^2\to\mathcal{S}'$ is a *(fuzzy) dissimilarity* if it satisfies, for all $A,B,C\in\mathcal{S}$, the following properties:

- (D_1) $\Delta(A,B)=\Delta(B,A)$;
- (D_2) $\Delta(A,B)=A'_{\max}$ if and only if $\{A,B\}\sim\{A_{\min},A_{\max}\}$;
- (D_3) $\Delta(A,B)=A'_{\min}$ if and only if $A\sim B$;
- (D_4) if $A\precsim B\precsim C$, then $\Delta(A,B)\precsim'\Delta(A,C)$ and $\Delta(B,C)\precsim'\Delta(A,C)$.

Although the previous definition has been written in a very general context, in this paper we are mainly interested in the case in which both FRE-sets are equal (especially in the case $\mathcal{S}=\mathcal{S}'=\mathrm{TFN}$), that is, $(\mathcal{S}',\precsim',A'_{\min},A'_{\max})=(\mathcal{S},\precsim,A_{\min},A_{\max})$, so we will write the dissimilarity as $\Delta:\mathcal{S}^2\to\mathcal{S}$. The case $\mathcal{S}'\subseteq\mathcal{S}$ with the same binary relation and absolute extrema is also covered under this point of view. A particularly interesting case that we will consider on Section 6 occurs when $\mathcal{S}'=\{\,\widetilde{r}:r\in[0,1]\,\}\subseteq\mathcal{S}$.

In the case of the FRE-set $(\mathrm{TFN},\overset{\alpha}{\leq},\widetilde{0},\widetilde{1})$, the previous constraints can be stated as follows:

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[1] F = fuzzy, R = (binary) related, E = (absolute) extremes.

(D'_2) $\quad \Delta(A,B)=\widetilde{1}$ if and only if $\{A,B\} \overset{\alpha}{=} \{\widetilde{0},\widetilde{1}\}$;

(D'_3) $\quad \Delta(A,B)=\widetilde{0}$ if and only if $A \overset{\alpha}{=} B$;

(D'_4) $\quad$ if $A \overset{\alpha}{\leq} B \overset{\alpha}{\leq} C$, then $\Delta(A,B) \overset{\alpha}{\leq} \Delta(A,C)$ and $\Delta(B,C) \overset{\alpha}{\leq} \Delta(A,C)$.

It is not easy to show examples of dissimilarities on certain families of fuzzy numbers because the binary relation $\precsim$ has a decisive role in the property (D_4). We will face this task in the following section, involving a simplified version of those dissimilarities. Anyway, the particular case $\alpha = \mathbf{1}_n$ let us introduce a first family of dissimilarities showing that any restricted dissimilarity function can be interpreted as a fuzzy dissimilarity.

Proposition 5.4. *Let us consider the membership vector $\alpha = \mathbf{1}_n = (1,1,\ldots,1)$, let $\omega = (\omega_1,\omega_2,\ldots,\omega_n) \in W_n$ be a weight vector and let $\mathcal{A} : [0,1]^2 \to [0,1]$ be an aggregation function on $[0,1]$. Given a restricted dissimilarity function $\delta : [0,1]^2 \to [0,1]$, let us define $\Delta_\delta : \mathrm{TFN}^2 \to \mathrm{TFN}$, for each $A=(a_1/a_2/a_3), B=(b_1/b_2/b_3) \in \mathrm{TFN}$, as:*

$$\Delta_\delta(A,B) = \mathrm{cr}\left[\delta\left(a_2,b_2\right)\right].$$

Then, Δ_δ is a dissimilarity on $(\mathrm{TFN}, \overset{\mathbf{1}}{\leq}, \widetilde{0}, \widetilde{1})$ satisfying

$$\Delta_\delta(\widetilde{t},\widetilde{s}) = \mathrm{cr}\left[\delta(t,s)\right] \qquad \text{for each } t,s \in [0,1].$$

5.3. Some properties of dissimilarities

Lemma 5.5. *Let $\Delta : \mathcal{S}^2 \to \mathcal{S}$ be a dissimilarity on a FRE-set $(\mathcal{S},\precsim,A_{\min},A_{\max})$ and suppose that the binary relation $\precsim$ is transitive on $\mathcal{S}$. Then the following properties hold.*

1. *If $A,B \in \mathcal{S}$ satisfy $A \sim B$, then $\Delta(A,B) = A_{\min}$.*
2. *If $A,B,C \in \mathcal{S}$ satisfy $A \sim B$ and A is $\precsim$-related to C, then $\Delta(A,C) \sim \Delta(B,C)$.*
3. *If $A,B,C,D \in \mathcal{S}$ satisfy $A \sim B$, $C \sim D$ and one element in $\{A,B\}$ is $\precsim$-related to one element in $\{C,D\}$, then $\Delta(A,C) \sim \Delta(B,D)$.*

Proof. **(1)** It is an implication of condition (D_3).

(2) As A is $\precsim$-related to C, then $A \precsim C$ or $C \precsim A$. Suppose, for instance, that $A \precsim C$ (the other case is similar). Since $B \precsim A$ and $A \precsim C$, then $B \precsim C$. Therefore, by axiom (D_4),

$$B \precsim A \precsim C \quad \Rightarrow \quad \Delta(A,C) \precsim \Delta(B,C),$$
$$A \precsim B \precsim C \quad \Rightarrow \quad \Delta(B,C) \precsim \Delta(A,C).$$

Therefore, $\Delta(A,C) \sim \Delta(B,C)$.

(3) Since one element in $\{A,B\}$ is $\precsim$-related to one element in $\{C,D\}$, we can assume, without loss of generality, that $A \precsim C$ (other cases are similar). Taking into account that $\precsim$ is transitive, then all elements in $\{A,B\}$ are $\precsim$-related to all elements in $\{C,D\}$ (for instance, $B \precsim A \precsim C \precsim D$ implies that $B \precsim D$). By item (2), $\Delta(A,C) \sim \Delta(B,C)$. Similarly, as $C \sim D$ and B is $\precsim$-related to C, then $\Delta(C,B) \sim \Delta(D,B)$. As Δ is symmetric and $\precsim$ is transitive, the relations $\Delta(A,C) \sim \Delta(B,C) = \Delta(C,B) \sim \Delta(D,B) = \Delta(B,D)$ guarantee that $\Delta(A,C) \sim \Delta(B,D)$. ■

Corollary 5.6. *If $\Delta : \mathrm{TFN}^2 \to \mathrm{TFN}$ is a dissimilarity on $(\mathrm{TFN}, \overset{\alpha}{\leq}, \widetilde{0}, \widetilde{1})$, then the following properties hold.*

1. *If $A,B \in \mathrm{TFN}$ satisfy $A \overset{\alpha}{=} B$, then $\Delta(A,A) = \Delta(B,B) = \Delta(A,B) = \widetilde{0}$.*
2. *If $A,B,C \in \mathrm{TFN}$ satisfy $A \overset{\alpha}{=} B$, then $\Delta(A,C) \overset{\alpha}{=} \Delta(B,C)$.*
3. *If $A,B,C,D \in \mathrm{TFN}$ satisfy $A \overset{\alpha}{=} B$ and $C \overset{\alpha}{=} D$, then $\Delta(A,C) \overset{\alpha}{=} \Delta(B,D)$.*

6. Crisp dissimilarities on FRE-sets

Proposition 5.4 guarantees that the family of dissimilarities on TFN is at least as large as the family of restricted dissimilarity functions on $[0,1]$. However, it is a complex task to show great families of dissimilarities of fuzzy numbers. In this section we introduce the notion of *crisp dissimilarity* of fuzzy numbers and, after showing some examples of such notion, we use it to illustrate some families of *dissimilarities of fuzzy numbers*.

6.1. Crisp dissimilarities

Definition 6.1. A mapping $\varrho : \mathcal{S}^2 \to [0,1]$ is a *crisp dissimilarity on a FRE-set* $(\mathcal{S},\precsim,A_{\min},A_{\max})$ if it satisfies, for all $A,B,C \in \mathcal{S}$, the following properties:

(SD_1) $\varrho(A,B)=\varrho(B,A)$;
(SD_2) $\varrho(A,B)=1$ if and only if $\{A,B\}\sim\{A_{\min},A_{\max}\}$;
(SD_3) $\varrho(A,B)=0$ if and only if $A\sim B$;
(SD_4) if $A\precsim B\precsim C$, then $\varrho(A,B)\le\varrho(A,C)$ and $\varrho(B,C)\le\varrho(A,C)$.

In the setting of $(\mathrm{TFN},\overset{\alpha}{\le},\widetilde{0},\widetilde{1})$, the last three axioms can be stated as:

(SD_2') $\varrho(A,B)=1$ if and only if $\{A,B\}\overset{\alpha}{=}\{\widetilde{0},\widetilde{1}\}$;
(SD_3') $\varrho(A,B)=0$ if and only if $A\overset{\alpha}{=}B$;
(SD_4') if $A\overset{\alpha}{\le}B\overset{\alpha}{\le}C$, then $\varrho(A,B)\le\varrho(A,C)$ and $\varrho(B,C)\le\varrho(A,C)$.

The previous notion extends the concept of "*restricted dissimilarity function*" if we additionally assume that $\{\widetilde{r}: r\in[0,1]\}\subseteq\mathcal{S}$. In this case, a crisp dissimilarity is a fuzzy dissimilarity taking $\mathcal{S}'=\{\widetilde{r}: r\in[0,1]\}$ in Definition 5.3. First we show that each FRE-set $(\mathcal{S},\precsim,A_{\min},A_{\max})$ where $\precsim$ is transitive can be endowed with a crisp dissimilarity.

Proposition 6.2. *Given a FRE-set $(\mathcal{S},\precsim,A_{\min},A_{\max})$ where $\precsim$ is transitive and $\lambda\in(0,1)$, let us define $\varrho_\lambda:\mathcal{S}^2\to[0,1]$, for each $A,B\in\mathcal{S}$, by:*

$$\varrho_\lambda(A,B)=\begin{cases}0, & \text{if } A\sim B,\\ 1, & \text{if } \{A,B\}\sim\{A_{\min},A_{\max}\},\\ \lambda, & \text{otherwise.}\end{cases}$$

Then ϱ_λ is a crisp dissimilarity on $(\mathcal{S},\precsim,A_{\min},A_{\max})$.

As a second family of examples, in the next result we check that each restricted dissimilarity function on $[0,1]$ can be extended to a crisp dissimilarity on TFN under a particular binary relation. The proof directly follows from the definitions, taking into account that $A\sim\widetilde{0}$ if and only if $A=\widetilde{0}$, and also $A\sim\widetilde{1}$ if and only if $A=\widetilde{1}$.

Lemma 6.3. *Given a restricted dissimilarity function $\delta:[0,1]^2\to[0,1]$, let us define $\varrho_\delta:\mathrm{TFN}^2\to[0,1]$ for each $A=(a_1/a_2/a_3), B=(b_1/b_2/b_3)\in\mathrm{TFN}$, by:*

$$\varrho_\delta(A,B)=\delta\left(\frac{a_1+a_2+a_3}{3},\frac{b_1+b_2+b_3}{3}\right).$$

Then ϱ_δ is a crisp dissimilarity on TFN with respect to the reflexive, transitive and total binary relation $\precsim_\Sigma$ on TFN defined by:

$$A=(a_1/a_2/a_3), B=(b_1/b_2/b_3)\in\mathrm{TFN},\qquad A\precsim_\Sigma B\quad\text{if}\quad \frac{a_1+a_2+a_3}{3}\le\frac{b_1+b_2+b_3}{3}.$$

Furthermore, ϱ_δ extends δ to TFN in the sense that:

$$\delta(r,s)=\varrho_\delta(\widetilde{r},\widetilde{s})\quad\text{for all } r,s\in[0,1].$$

6.2. ***The mapping*** *m and non trivial examples of crisp dissimilarities*

In order to show non trivial examples of crisp dissimilarities, it is necessary to know the action of the binary relation $\precsim$. Hence, from now on, we center our efforts on the FRE-set $(\mathrm{TFN},\overset{\alpha}{\le},\widetilde{0},\widetilde{1})$, where the α-order $\overset{\alpha}{\le}$ is complex enough to greatly complicate this task. To face this problem, in the following results, given $A,B\in\mathrm{TFN}$, we will use the notation $m(A,B)$ to indicate the first component in which the vectors $V(A)$ and $V(B)$ (recall Definition 4.5) are distinct (which can only occur when $A\overset{\alpha}{<}B$ or $B\overset{\alpha}{<}A$).

Definition 6.4. Let $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_n)$ be a membership vector, let $\omega=(\omega_1,\omega_2,\dots,\omega_n)\in W_n$ be a weight vector and let $\mathcal{A}:[0,1]^2\to[0,1]$ be an aggregation function. Given $A,B\in\mathrm{TFN}$ such that $V(A)\ne V(B)$, let denote by $m(A,B)$ the minimum value $k_0\in\{1,2,\dots,2n-1\}$ such that

$$V^j(A)=V^j(B)\quad\text{for all } j\in\{1,2,\dots,k_0-1\},\qquad\text{and}\qquad V^{k_0}(A)\ne V^{k_0}(B).$$

If $V(A)=V(B)$, that is, if $A\overset{\alpha}{=}B$, we define $m(A,B)=2n$.

A first property of the discrete function $m:\mathrm{TFN}^2\to\{1,2,\dots,2n\}$ is a consequence of the transitivity of the binary relation $\overset{\alpha}{\le}$.

Lemma 6.5. *If $A, B, C \in \mathrm{TFN}$ are three triangular fuzzy numbers such that $A \overset{\alpha}{\leq} B \overset{\alpha}{\leq} C$, then*

$$m(A,C) = \min\{m(A,B), m(B,C)\}.$$

Proof. We consider the following cases.

Case 1) Suppose that $A \overset{\alpha}{=} B$. In this case, $V(A) = V(B)$ and $m(A,B) = 2n$, which is the greatest possible value for $m(A,B)$. Then $\min\{m(A,B), m(B,C)\} = m(B,C)$, and we have to prove that $m(A,C) = m(B,C)$.

- If $m(B,C) = 2n$, then $V(B) = V(C)$. In this case, $V(A) = V(B) = V(C)$, so $m(A,B) = m(B,C) = m(A,C)$.
- If $m(B,C) \leq 2n-1$,

$$V^j(B) = V^j(C) \quad \text{for all } j \in \{1,2,\ldots,m(B,C)-1\}, \quad \text{and} \quad V^{m(B,C)}(B) < V^{m(B,C)}(C).$$

Since $V^j(A) = V^j(B)$ for all $j \in \{1,2,\ldots,2n-1\}$, then

$$V^j(A) = V^j(C) \quad \text{for all } j \in \{1,2,\ldots,m(B,C)-1\}, \quad \text{and} \quad V^{m(B,C)}(A) < V^{m(B,C)}(C).$$

Hence $m(A,C) = m(B,C)$.

Case 2) Suppose that $B \overset{\alpha}{=} C$. We can repeat the arguments of the first case.

Case 3) Suppose that $A \overset{\alpha}{<} B \overset{\alpha}{<} C$. Since $\overset{\alpha}{<}$ is transitive, then $A \overset{\alpha}{<} C$. Therefore $m(A,B), m(B,C), m(A,C) \in \{1,2,\ldots,2n-1\}$. In this case,

$$V^j(A) = V^j(B) \quad \text{for all } j \in \{1,2,\ldots,m(A,B)-1\}, \quad \text{and} \quad V^{m(A,B)}(A) < V^{m(A,B)}(B);$$

$$V^j(B) = V^j(C) \quad \text{for all } j \in \{1,2,\ldots,m(B,C)-1\}, \quad \text{and} \quad V^{m(B,C)}(B) < V^{m(B,C)}(C).$$

We consider the following two new subcases.

Subcase 3.1) Suppose that $m(A,B) \leq m(B,C)$. In this case,

$$V^j(A) = V^j(B) = V^j(C) \quad \text{for all } j \in \{1,2,\ldots,m(A,B)-1\}, \quad \text{and}$$
$$V^{m(A,B)}(A) < V^{m(A,B)}(B) \leq V^{m(A,B)}(C).$$

Therefore, $m(A,C) = m(A,B) = \min\{m(A,B), m(B,C)\}$.

Subcase 3.2) Suppose that $m(A,B) > m(B,C)$. Similarly,

$$V^j(A) = V^j(B) = V^j(C) \quad \text{for all } j \in \{1,2,\ldots,m(B,C)-1\}, \quad \text{and}$$
$$V^{m(B,C)}(A) \leq V^{m(B,C)}(B) < V^{m(B,C)}(C).$$

Therefore, $m(A,C) = m(B,C) = \min\{m(A,B), m(B,C)\}$. ■

In the next result we show a parametric family of crisp dissimilarities on TFN with respect to the binary relation $\overset{\alpha}{\leq}$.

Theorem 6.6. *Let α be a membership vector, let ω be a weight vector and let $\mathcal{A} : [0,1]^2 \to [0,1]$ be an aggregation function on $[0,1]$. Given $k \in (0,+\infty)$, let us define $\varrho_k : \mathrm{TFN}^2 \to [0,1]$, for each $A, B \in \mathrm{TFN}$, as:*

$$\varrho_k(A,B) = \begin{cases} 0, & \text{if } A \overset{\alpha}{=} B, \\ 1, & \text{if } \{A,B\} \overset{\alpha}{=} \{\widetilde{0}, \widetilde{1}\}; \\ 1 - \dfrac{m(A,B) + k - |V^{m(A,B)}(A) - V^{m(A,B)}(B)|}{2n+k}, & \text{otherwise.} \end{cases} \tag{8}$$

Then, ϱ_k is a crisp dissimilarity on $(\mathrm{TFN}, \overset{\alpha}{\leq}, \widetilde{0}, \widetilde{1})$.

Proof. First of all, we check that ϱ is well defined, that is, $\varrho(A,B) \in [0,1]$ for all $A, B \in \mathrm{TFN}$. Indeed, given $A, B \in \mathrm{TFN}$, it is known that $V^{m(A,B)}(A), V^{m(A,B)}(B) \in [0,1]$. Therefore, $|V^{m(A,B)}(A) - V^{m(A,B)}(B)| \in [0,1]$. We have two cases.

- If $V(A) = V(B)$, then $A \overset{\alpha}{=} B$, so $\varrho(A,B) = 0 \in [0,1]$.
- If $V(A) \neq V(B)$, then $m(A,B) \in \{1,2,\ldots,2n-1\}$, so

$$0 \leq m(A,B) + k - |V^{m(A,B)}(A) - V^{m(A,B)}(B)| \leq (2n-1) + k + 1 = 2n + k.$$

Hence

$$\frac{m(A,B)+k-|\,V^{m(A,B)}(A)-V^{m(A,B)}(B)\,|}{2n+k}\in[0,1],$$

so $\varrho(A,B)\in[0,1]$.

Next we prove that ϱ satisfies the properties (SD_1)-(SD_4). Property (SD_1) is obvious.

(SD_3') If $A\overset{\alpha}{=}B$, then $\varrho(A,B)=0$ by definition. Conversely, suppose that $\varrho(A,B)=0$. In order to prove that $A\overset{\alpha}{=}B$, suppose that A and B are not α-equal and we will get a contradiction. In such a case, $m(A,B)\le 2n-1$ and $V^{m(A,B)}(A)\neq V^{m(A,B)}(B)$. Then

$$\begin{aligned}\varrho(A,B)=0\quad&\Leftrightarrow\quad\frac{m(A,B)+k-|\,V^{m(A,B)}(A)-V^{m(A,B)}(B)\,|}{2n+k}=1\\&\Leftrightarrow\quad m(A,B)+k-|\,V^{m(A,B)}(A)-V^{m(A,B)}(B)\,|=2n+k\\&\Leftrightarrow\quad m(A,B)-|\,V^{m(A,B)}(A)-V^{m(A,B)}(B)\,|=2n-1.\end{aligned}$$

Since $m(A,B)\le 2n-1$, then

$$2n-1=m(A,B)-|\,V^{m(A,B)}(A)-V^{m(A,B)}(B)\,|\le(2n-1)-|\,V^{m(A,B)}(A)-V^{m(A,B)}(B)\,|.$$

Hence $m(A,B)=2n-1$ and $|\,V^{m(A,B)}(A)-V^{m(A,B)}(B)\,|=0$, which contradicts the fact that $V^{m(A,B)}(A)\neq V^{m(A,B)}(B)$.

(SD_2') If $\{A,B\}\overset{\alpha}{=}\{\widetilde{0},\widetilde{1}\}$, then $\varrho(A,B)=1$ by definition. Conversely, suppose that $\varrho(A,B)=1$. In this case, the condition "$A\overset{\alpha}{=}B$" is false (in such a case, $\varrho(A,B)=0$, which is false). Therefore $m(A,B)\in\{1,2,\dots,2n-1\}$ and $V^{m(A,B)}(A)\neq V^{m(A,B)}(B)$. Thus,

$$\begin{aligned}\varrho(A,B)=1\quad&\Leftrightarrow\quad\frac{m(A,B)+k-|\,V^{m(A,B)}(A)-V^{m(A,B)}(B)\,|}{2n+k}=0\\&\Leftrightarrow\quad k+(m(A,B)-|\,V^{m(A,B)}(A)-V^{m(A,B)}(B)\,|)=0.\end{aligned}$$

However this equality is false because $k>0$ and $m(A,B)-|\,V^{m(A,B)}(A)-V^{m(A,B)}(B)\,|\ge 0$. This contradiction proves that, necessarily, we must be in the case $\{A,B\}\overset{\alpha}{=}\{\widetilde{0},\widetilde{1}\}$.

(SD_4') Let $A,B,C\in\mathrm{TFN}$ be such that $A\overset{\alpha}{\le}B\overset{\alpha}{\le}C$. We study the following cases.

- *Suppose that $A\overset{\alpha}{=}B\overset{\alpha}{=}C$.* Then $\varrho(A,B)=\varrho(B,C)=0$ by definition, so $\varrho(A,B)=0\le\varrho(A,C)$ and $\varrho(B,C)=0\le\varrho(A,C)$.
- *Suppose that $A\overset{\alpha}{=}B$ and $B\overset{\alpha}{<}C$.* Then $\varrho(A,B)=0\le\varrho(A,C)$. To prove that $\varrho(B,C)\le\varrho(A,C)$, observe that

$$V^j(B)=V^j(C)\quad\text{for all } j\in\{1,2,\dots,m(B,C)-1\},\quad\text{and}\quad V^{m(B,C)}(B)<V^{m(B,C)}(C).$$

Let $k_1=m(B,C)\in\{1,2,\dots,2n-1\}$. Since $A\overset{\alpha}{=}B$,

$$V^{k_1}(A)=V^{k_1}(B)<V^{k_1}(C).\tag{9}$$

As $m(A,B)=2n$, then $m(A,C)=\min\{m(A,B),m(B,C)\}=m(B,C)=k_1$. Then:

$$\begin{aligned}&\varrho(B,C)\le\varrho(A,C)\\&\quad\Leftrightarrow\quad 1-\frac{m(B,C)+k-|\,V^{m(B,C)}(B)-V^{m(B,C)}(C)\,|}{2n+k}\\&\qquad\qquad\le 1-\frac{m(A,C)+k-|\,V^{m(A,C)}(A)-V^{m(A,C)}(C)\,|}{2n+k}\\&\quad\Leftrightarrow\quad\frac{k_1+k-|\,V^{k_1}(A)-V^{k_1}(C)\,|}{2n+k}\le\frac{k_1+k-|\,V^{k_1}(B)-V^{k_1}(C)\,|}{2n+k}\\&\quad\Leftrightarrow\quad k_1+k-|\,V^{k_1}(A)-V^{k_1}(C)\,|\le k_1+k-|\,V^{k_1}(B)-V^{k_1}(C)\,|\\&\quad\Leftrightarrow\quad|\,V^{k_1}(B)-V^{k_1}(C)\,|\le|\,V^{k_1}(A)-V^{k_1}(C)\,|,\end{aligned}$$

which holds because of (9).

- *Suppose that $A\overset{\alpha}{<}B$ and $B\overset{\alpha}{=}C$.* This case can be analyzed in a similar way than the previous one.
- *Suppose that $A\overset{\alpha}{<}B$ and $B\overset{\alpha}{<}C$.* Therefore $m(A,B),m(B,C)\in\{1,2,\dots,2n-1\}$, and

$$V^j(A)=V^j(B)\quad\text{for all } j\in\{1,2,\dots,m(A,B)-1\},\quad\text{and}\quad V^{m(A,B)}(A)<V^{m(A,B)}(B),\tag{10}$$

$$V^j(B)=V^j(C)\quad\text{for all } j\in\{1,2,\dots,m(B,C)-1\},\quad\text{and}\quad V^{m(B,C)}(B)<V^{m(B,C)}(C).\tag{11}$$

Lemma 6.5 guarantees that

$$m(A, C) = \min\{m(A, B), m(B, C)\}. \tag{12}$$

- *Suppose that* $m(A, B) \leq m(B, C)$. By (12), let us denote

  $$k_1 = m(A, C) = m(A, B) \quad \text{and} \quad k_2 = m(B, C).$$

  By hypothesis, $k_1 \leq k_2$. In this case,

  $$0 \leq V^{m(A,B)}(A) < V^{m(A,B)}(B) \leq V^{m(A,B)}(C) \leq 1,$$

  which can be rewritten as

  $$0 \leq V^{k_1}(A) < V^{k_1}(B) \leq V^{k_1}(C) \leq 1. \tag{13}$$

  Then:

  $$\begin{aligned} &\varrho(A, B) \leq \varrho(A, C) \\ &\quad \Leftrightarrow \quad 1 - \frac{m(A, B) + k - |V^{m(A,B)}(A) - V^{m(A,B)}(B)|}{2n + k} \\ &\qquad\qquad \leq 1 - \frac{m(A, C) + k - |V^{m(A,C)}(A) - V^{m(A,C)}(C)|}{2n + k} \\ &\quad \Leftrightarrow \quad \frac{k_1 + k - |V^{k_1}(A) - V^{k_1}(C)|}{2n + k} \leq \frac{k_1 + k - |V^{k_1}(A) - V^{k_1}(B)|}{2n + k} \\ &\quad \Leftrightarrow \quad k_1 + k - |V^{k_1}(A) - V^{k_1}(C)| \leq k_1 + k - |V^{k_1}(A) - V^{k_1}(B)| \\ &\quad \Leftrightarrow \quad |V^{k_1}(A) - V^{k_1}(B)| \leq |V^{k_1}(A) - V^{k_1}(C)|, \end{aligned}$$

  and this last condition holds because of (13). Similarly:

  $$\begin{aligned} &\varrho(B, C) \leq \varrho(A, C) \\ &\quad \Leftrightarrow \quad 1 - \frac{m(B, C) + k - |V^{m(B,C)}(B) - V^{m(B,C)}(C)|}{2n + k} \\ &\qquad\qquad \leq 1 - \frac{m(A, C) + k - |V^{m(A,C)}(A) - V^{m(A,C)}(C)|}{2n + k} \\ &\quad \Leftrightarrow \quad \frac{k_1 + k - |V^{k_1}(A) - V^{k_1}(C)|}{2n + k} \leq \frac{k_2 + k - |V^{k_2}(B) - V^{k_2}(C)|}{2n + k} \\ &\quad \Leftrightarrow \quad k_1 + k - |V^{k_1}(A) - V^{k_1}(C)| \leq k_2 + k - |V^{k_2}(B) - V^{k_2}(C)| \\ &\quad \Leftrightarrow \quad |V^{k_2}(B) - V^{k_2}(C)| \leq (k_2 - k_1) + |V^{k_1}(A) - V^{k_1}(C)|. \end{aligned} \tag{14}$$

  If $k_1 = k_2$, then (14) holds because of (13). On the contrary, if $k_1 < k_2$, then (14) also holds because:

  $$|V^{k_2}(B) - V^{k_2}(C)| \leq 1 \leq k_2 - k_1 \leq (k_2 - k_1) + |V^{k_1}(A) - V^{k_1}(C)|.$$

  In any case, $\varrho(B, C) \leq \varrho(A, C)$ also holds.
- *Suppose that* $m(B, C) < m(A, B)$. By (12), let denote

  $$k_1 = m(A, C) = m(B, C) \quad \text{and} \quad k_2 = m(A, B).$$

  By hypothesis, $k_1 < k_2$. In this case, by (10)-(11),

  $$0 \leq V^{m(B,C)}(A) = V^{m(B,C)}(B) < V^{m(B,C)}(C) \leq 1,$$

  which can be rewritten as

  $$0 \leq V^{k_1}(A) = V^{k_1}(B) < V^{k_1}(C) \leq 1. \tag{15}$$

  Then, as before,

  $$\begin{aligned} &\varrho(A, B) \leq \varrho(A, C) \\ &\quad \Leftrightarrow \quad 1 - \frac{m(A, B) + k - |V^{m(A,B)}(A) - V^{m(A,B)}(B)|}{2n + k} \\ &\qquad\qquad \leq 1 - \frac{m(A, C) + k - |V^{m(A,C)}(A) - V^{m(A,C)}(C)|}{2n + k} \end{aligned}$$

$$\Leftrightarrow \quad \frac{k_1 + k - |V^{k_1}(A) - V^{k_1}(C)|}{2n+k} \le \frac{k_2 + k - |V^{k_2}(A) - V^{k_2}(B)|}{2n+k}$$
$$\Leftrightarrow \quad k_1 + k - |V^{k_1}(A) - V^{k_1}(C)| \le k_2 + k - |V^{k_2}(A) - V^{k_2}(B)|$$
$$\Leftrightarrow \quad |V^{k_1}(A) - V^{k_1}(B)| \le (k_2 - k_1) + |V^{k_1}(A) - V^{k_1}(C)|,$$

which holds because $k_2 - k_1 \ge 1$. Similarly,

$$\varrho(B,C) \le \varrho(A,C)$$
$$\Leftrightarrow \quad 1 - \frac{m(B,C) + k - |V^{m(B,C)}(B) - V^{m(B,C)}(C)|}{2n+k}$$
$$\le 1 - \frac{m(A,C) + k - |V^{m(A,C)}(A) - V^{m(A,C)}(C)|}{2n+k}$$
$$\Leftrightarrow \quad \frac{k_1 + k - |V^{k_1}(A) - V^{k_1}(C)|}{2n+k} \le \frac{k_1 + k - |V^{k_1}(B) - V^{k_1}(C)|}{2n+k}$$
$$\Leftrightarrow \quad k_1 + k - |V^{k_1}(A) - V^{k_1}(C)| \le k_1 + k - |V^{k_1}(B) - V^{k_1}(C)|$$
$$\Leftrightarrow \quad |V^{k_1}(B) - V^{k_1}(C)| \le |V^{k_1}(A) - V^{k_1}(C)|,$$

which holds because of (15).

The previous arguments prove that $\varrho(A,B) \le \varrho(A,C)$ and $\varrho(B,C) \le \varrho(A,C)$, which definitively demonstrates the axiom (SD'_4).

■

The following results help us to guarantee that the family of all crisp dissimilarities is very extensive.

Lemma 6.7. *Let $F : [0,1]^m \to [0,1]$ be a 0-dependent and 1-dependent aggregation function and let $\{\varrho_i : \mathcal{S}^2 \to [0,1]\}_{i=1}^m$ be a family of m crisp dissimilarities on a FRE-set $(\mathcal{S}, \precsim, A_{\min}, A_{\max})$. Let us define $F(\varrho_1, \varrho_2, \ldots, \varrho_m) : \mathcal{S}^2 \to [0,1]$, for each $A, B \in \mathcal{S}$, by*

$$[\,F(\varrho_1, \varrho_2, \ldots, \varrho_m)\,](A,B) = F(\varrho_1(A,B), \varrho_2(A,B), \ldots, \varrho_m(A,B)).$$

Then, $F(\varrho_1, \varrho_2, \ldots, \varrho_m)$ is a crisp dissimilarity on $(\mathcal{S}, \precsim, A_{\min}, A_{\max})$.

The previous result can be particularized in multiple ways. Among all of them, we highlight the case of the convex combinations.

Corollary 6.8. *Let $\{\lambda_i\}_{i=1}^m \subset (0,1)$ be a set of m real numbers such that $\lambda_1 + \lambda_2 + \ldots + \lambda_m = 1$ and let $\{\varrho_i : \mathcal{S}^2 \to [0,1]\}_{i=1}^m$ be a family of m crisp dissimilarity on a FRE-set $(\mathcal{S}, \precsim, A_{\min}, A_{\max})$. Let us define $\Sigma\lambda_i\varrho_i : \mathcal{S}^2 \to [0,1]$, for each $A, B \in \mathcal{S}$, by*

$$[\,\Sigma\lambda_i\varrho_i\,](A,B) = \sum_{i=1}^m \lambda_i \varrho_i(A,B).$$

Then, $\Sigma\lambda_i\varrho_i$ is a crisp dissimilarity on $(\mathcal{S}, \precsim, A_{\min}, A_{\max})$.

6.3. From crisp dissimilarities to dissimilarities of fuzzy numbers

In the following results we show some families of dissimilarities on TFN by using crisp dissimilarities on TFN.

Proposition 6.9. *Let $\precsim$ be a binary relation on TFN such that, given $r, s \in [0,1]$, $\widetilde{r} \precsim \widetilde{s} \Leftrightarrow r \le s$. Given a function $\varrho : \mathrm{TFN}^2 \to [0,1]$, let us define $\Delta_\varrho : \mathrm{TFN}^2 \to \mathrm{TFN}$, for each $A, B \in \mathrm{TFN}$, as*

$$\Delta_\varrho(A,B) = \Big(\ \varrho(A,B) \,/\, \varrho(A,B) \,/\, \varrho(A,B)\ \Big). \tag{16}$$

Then, ϱ is a crisp dissimilarity on $(\mathrm{TFN}, \precsim, \widetilde{0}, \widetilde{1})$ if and only if Δ_ϱ is a dissimilarity on $(\mathrm{TFN}, \precsim, \widetilde{0}, \widetilde{1})$.

Proof. Notice that $\Delta_\varrho(A,B) = \widetilde{\varrho(A,B)}$ for each $A, B \in \mathrm{TFN}$. Clearly Δ_ϱ is symmetric if and only if ϱ is too.

- $\Delta_\varrho(A,B) = \widetilde{1} \Leftrightarrow \varrho(A,B) = 1$, so any of the previous conditions can be equivalent to $\{A,B\} \sim \{\widetilde{0}, \widetilde{1}\}$ at the same time.
- $\Delta_\varrho(A,B) = \widetilde{0} \Leftrightarrow \varrho(A,B) = 0$, so any of the previous conditions can be equivalent to $A \sim B$ at the same time.

- Notice that

$$\varrho(A,B) \le \varrho(A,C) \quad \Leftrightarrow \quad \widetilde{\varrho(A,B)} \precsim \widetilde{\varrho(A,C)} \quad \Leftrightarrow \quad \Delta_\varrho(A,B) \precsim \Delta_\varrho(A,C), \qquad \text{and}$$
$$\varrho(B,C) \le \varrho(A,C) \quad \Leftrightarrow \quad \widetilde{\varrho(B,C)} \precsim \widetilde{\varrho(A,C)} \quad \Leftrightarrow \quad \Delta_\varrho(B,C) \precsim \Delta_\varrho(A,C),$$

  so the conditions (SD_4) and (D_4) are equivalent.

■

Corollary 6.10. *Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ be a membership vector, let $\omega = (\omega_1, \omega_2, \dots, \omega_n) \in W_n$ be a weight vector and let $\mathcal{A} : [0,1]^2 \to [0,1]$ be an aggregation function on $[0,1]$. Given a function $\varrho : \mathrm{TFN}^2 \to [0,1]$, let us define $\Delta_\varrho : \mathrm{TFN}^2 \to \mathrm{TFN}$ as in (16). Then $\varrho : \mathrm{TFN}^2 \to [0,1]$ is a crisp dissimilarity on $(\mathrm{TFN}, \overset{\alpha}{\preceq}, \widetilde{0}, \widetilde{1})$ if and only if Δ_ϱ is a fuzzy dissimilarity on $(\mathrm{TFN}, \overset{\alpha}{\preceq}, \widetilde{0}, \widetilde{1})$.*

Proof. It directly follows from Proposition 6.9 taking into account that $\overset{\alpha}{\preceq}$ is a binary relation on TFN such that $\widetilde{r} \overset{\alpha}{\preceq} \widetilde{s} \Leftrightarrow r \le s$ for any $r, s \in [0,1]$. ■

In order to show another great family of dissimilarities on $(\mathrm{TFN}, \overset{\alpha}{\preceq})$ we need the following property.

Proposition 6.11. *Let $a \in [0,\infty)$ and $b, \kappa, \kappa' \in (0,\infty)$ be such that $a \le b$ and $\kappa \le \kappa'$. Then*

$$\frac{a+\kappa}{b+\kappa} \le \frac{a+\kappa'}{b+\kappa'}, \qquad (\le 1)$$

and the equality holds if and only if $\kappa = \kappa'$ or $a = b$.

Theorem 6.12. *Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ be a membership vector, let $\omega = (\omega_1, \omega_2, \dots, \omega_n) \in W_n$ be a weight vector and let $\mathcal{A} : [0,1]^2 \to [0,1]$ be an aggregation function on $[0,1]$. Given $\kappa, \kappa', \kappa'' \in (0,+\infty)$ such that $\kappa \le \kappa' \le \kappa''$, let define $\Delta_{\kappa,\kappa',\kappa''} : \mathrm{TFN}^2 \to \mathrm{TFN}$, for each $A, B \in$ TFN, as:*

$$\Delta_{\kappa,\kappa',\kappa''}(A,B) = \Big(\varrho_{\kappa''}(A,B) \,/\, \varrho_{\kappa'}(A,B) \,/\, \varrho_{\kappa}(A,B) \Big),$$

where $\varrho_\kappa, \varrho_{\kappa'}, \varrho_{\kappa''} : \mathrm{TFN}^2 \to [0,1]$ are the functions defined by (8). Then, $\Delta_{\kappa,\kappa',\kappa''}$ is a dissimilarity on $(\mathrm{TFN}, \overset{\alpha}{\preceq}, \widetilde{0}, \widetilde{1})$.

Proof. Since κ, κ', κ'' are fixed, for simplicity, through this proof, we denote $\Delta_{\kappa,\kappa',\kappa''}$ by Δ. Applying Theorem 6.6, the functions ϱ_κ, $\varrho_{\kappa'}$ and $\varrho_{\kappa''}$ are crisp dissimilarities on $(\mathrm{TFN}, \overset{\alpha}{\preceq}, \widetilde{0}, \widetilde{1})$. We check that Δ is well defined. Let $A, B \in$ TFN. On the one hand,

- $\Delta(A,B) = \widetilde{1} \quad \Leftrightarrow \quad \varrho_\kappa(A,B) = \varrho_{\kappa'}(A,B) = \varrho_{\kappa''}(A,B) = 1 \quad \Leftrightarrow \quad \{A,B\} \overset{\alpha}{=} \{\widetilde{0}, \widetilde{1}\};$ (17)
- $\Delta(A,B) = \widetilde{0} \quad \Leftrightarrow \quad \varrho_\kappa(A,B) = \varrho_{\kappa'}(A,B) = \varrho_{\kappa''}(A,B) = 0 \quad \Leftrightarrow \quad A \overset{\alpha}{=} B.$ (18)

Therefore, Δ is well defined when "$\{A,B\} \overset{\alpha}{=} \{\widetilde{0}, \widetilde{1}\}$" or "$A \overset{\alpha}{=} B$". On the other hand, if A and B are otherwise, using $a = m(A,B) - |V^{m(A,B)}(A) - V^{m(A,B)}(B)| \ge 0$ and $b = 2n > 0$ in Proposition 6.11, we deduce that

$$\frac{m(A,B) - |V^{m(A,B)}(A) - V^{m(A,B)}(B)| + \kappa}{2n+\kappa} \le \frac{m(A,B) - |V^{m(A,B)}(A) - V^{m(A,B)}(B)| + \kappa'}{2n+\kappa'},$$

so

$$\begin{aligned} 0 \le \varrho_{\kappa'}(A,B) &= 1 - \frac{m(A,B) + \kappa' - |V^{m(A,B)}(A) - V^{m(A,B)}(B)|}{2n+\kappa'} \\ &\le 1 - \frac{m(A,B) + \kappa - |V^{m(A,B)}(A) - V^{m(A,B)}(B)|}{2n+\kappa} = \varrho_\kappa(A,B) \le 1. \end{aligned}$$

Hence $0 \le \varrho_{\kappa'}(A,B) \le \varrho_\kappa(A,B) \le 1$. Similarly, it can be proved that $0 \le \varrho_{\kappa''}(A,B) \le \varrho_{\kappa'}(A,B) \le 1$, so Δ is well defined. Notice that Δ is symmetric because ϱ_κ, $\varrho_{\kappa'}$ and $\varrho_{\kappa''}$ are so. Properties (D_2) and (D_3) follow from (17) and (18), respectively. To prove the condition (D_4), let $A, B, C \in$ TFN be such that $A \overset{\alpha}{\preceq} B \overset{\alpha}{\preceq} C$. Since ϱ_κ, $\varrho_{\kappa'}$ and $\varrho_{\kappa''}$ are crisp dissimilarity functions on $(\mathrm{TFN}, \overset{\alpha}{\preceq}, \widetilde{0}, \widetilde{1})$, then

$$\begin{aligned} &\varrho_\kappa(A,B) \le \varrho_\kappa(A,C) \quad \text{and} \quad \varrho_\kappa(B,C) \le \varrho_\kappa(A,C), \\ &\varrho_{\kappa'}(A,B) \le \varrho_{\kappa'}(A,C) \quad \text{and} \quad \varrho_{\kappa'}(B,C) \le \varrho_{\kappa'}(A,C), \\ &\varrho_{\kappa''}(A,B) \le \varrho_{\kappa''}(A,C) \quad \text{and} \quad \varrho_{\kappa''}(B,C) \le \varrho_{\kappa''}(A,C). \end{aligned}$$

Using Proposition 4.2, we deduce that

$$\Delta(A,B)=\left(\,\varrho_{\kappa''}(A,B)\,/\,\varrho_{\kappa'}(A,B)\,/\,\varrho_{\kappa}(A,B)\,\right)\overset{\alpha}{\leq}\left(\,\varrho_{\kappa''}(A,C)\,/\,\varrho_{\kappa'}(A,C)\,/\,\varrho_{\kappa}(A,C)\,\right)=\Delta(A,C)$$

and

$$\Delta(B,C)=\left(\,\varrho_{\kappa''}(B,C)\,/\,\varrho_{\kappa'}(B,C)\,/\,\varrho_{\kappa}(B,C)\,\right)\overset{\alpha}{\leq}\left(\,\varrho_{\kappa''}(A,C)\,/\,\varrho_{\kappa'}(A,C)\,/\,\varrho_{\kappa}(A,C)\,\right)=\Delta(A,C).$$

Hence Δ is a dissimilarity on $(\mathrm{TFN},\overset{\alpha}{\leq},\widetilde{0},\widetilde{1})$. ■

The same arguments of the previous proof let us to demonstrate the following result.

Theorem 6.13. *Let $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_n)$ be a membership vector, let $\omega=(\omega_1,\omega_2,\dots,\omega_n)\in W_n$ be a weight vector and let $\mathcal{A}:[0,1]^2\to[0,1]$ be an aggregation function on $[0,1]$. Given three crisp dissimilarities $\varrho_1,\varrho_2,\varrho_3:\mathrm{TFN}^2\to[0,1]$ on $(\mathrm{TFN},\overset{\alpha}{\leq},\widetilde{0},\widetilde{1})$ such that*

$$\varrho_1(A,B)\leq\varrho_2(A,B)\leq\varrho_3(A,B)\quad\textit{for each } A,B\in\mathrm{TFN},$$

let define $\Delta_{\varrho_1,\varrho_2,\varrho_3}:\mathrm{TFN}^2\to\mathrm{TFN}$, for each $A,B\in\mathrm{TFN}$, as:

$$\Delta_{\varrho_1,\varrho_2,\varrho_3}(A,B)=\Big(\,\varrho_1(A,B)\,/\,\varrho_2(A,B)\,/\,\varrho_3(A,B)\,\Big).$$

Then $\Delta_{\varrho_1,\varrho_2,\varrho_3}$ is a dissimilarity on $(\mathrm{TFN},\overset{\alpha}{\leq},\widetilde{0},\widetilde{1})$.

7. The (discrete) fuzzy (μ,Δ)-Choquet integral of triangular fuzzy numbers on [0,1]

In this section we introduce the notion of (discrete) fuzzy (μ,Δ)-Choquet integral with triangular fuzzy numbers. Although the definition we are going to introduce is very general, we are mainly interested on the family TFN of all triangular fuzzy numbers whose supports are included in $[0,1]$.

The most tentative definition for the notion of "*Choquet integral of fuzzy numbers*" (which present some shortcomings) is the following one. Given $n\in\mathbb{N}$, let $\mu:\mathcal{P}_n\to[0,1]$ be a measure on $\mathcal{P}_n$ and let $\Delta:\mathcal{S}^2\to\mathcal{S}$ be a dissimilarity on $(\mathcal{S},\precsim)$. As a first approach, the (μ,Δ)-*Choquet integral of the fuzzy numbers* $A_1,A_2,\dots,A_n\in\mathcal{S}$ could be the fuzzy number:

$$C_{\mu,\Delta}(A_1,A_2,\dots,A_n)=\sum_{i=1}^{n}\Delta(A_{\sigma(i)},A_{\sigma(i-1)})\cdot\mu(F_{\sigma(i)}).$$

where $A_{\sigma(0)}=\widetilde{0}$, $\sigma:\{1,2,\dots,n\}\to\{1,2,\dots,n\}$ is a permutation such that $A_{\sigma(i-1)}\precsim A_{\sigma(i)}$ for all $i\in\{1,2,\dots,n\}$ and $F_{\sigma(i)}$ is the set $\{\sigma(i),\sigma(i+1),\dots,\sigma(n)\}\in\mathcal{P}_n$. This definition inevitably entails several drawbacks.

- Firstly, we need to order any family of fuzzy numbers $A_1,A_2,\dots,A_n$, so we can assume that the binary relation $\precsim$ is total on $\mathcal{S}$.
- Secondly, since the relation $\precsim$ is not necessarily antisymmetric, the fuzzy value $C_{\mu,\Delta}(A_1,A_2,\dots,A_n)$ could depend on the permutation σ, especially when there are some $\precsim$-equivalent fuzzy numbers on $\{A_1,A_2,\dots,A_n\}$. In other words, if $A_i\sim A_j$ for each $i,j\in\{r,r+1,\dots,s\}\subseteq\{1,2,\dots,n\}$, how must we reorder the fuzzy numbers $\{A_r,A_{r+1},\dots,A_s\}$? Such ordering does not only affect to $\Delta(A_{\sigma(i)},A_{\sigma(i-1)})$, but also to the fuzzy measures of the type $\mu(F_{\sigma(i)})$.
- Thirdly, $C_{\mu,\Delta}(A_1,A_2,\dots,A_n)$ is a fuzzy number but, does it belong to $\mathcal{S}$?

Next we discuss about how to overcome such difficulties in order to introduce an appropriate notion of Choquet integral among fuzzy numbers.

7.1. Algebraic background to formalize the fuzzy Choquet integral

In this section we formalize some necessary background in order to establish a consistent notion of fuzzy Choquet integral. The proof of the following properties can be found on elementary textbooks (see [12,19]).

Proposition 7.1. *If $\precsim$ is a total and transitive binary relation on X and $Y\subseteq X$ is a finite and non-empty subset of X, then there is $y_0\in Y$ such that $y_0\precsim y$ for each $y\in Y$.*

In the following result, we use the notation $\prec$ for the binary relation introduced in (2).

Lemma 7.2. *If $\precsim$ is a total and transitive binary relation on $\widetilde{X}$ and $X\subseteq\widetilde{X}$ is a finite and non-empty subset of $\widetilde{X}$, then there is a partition*

$$X=X_1\cup X_2\cup\ldots\cup X_r\tag{19}$$

such that

$$\begin{cases} x_1 \prec x_2 \prec \ldots \prec x_r & \text{for each } x_1 \in X_1,\ x_2 \in X_2,\ \ldots,\ x_r \in X_r, \\ x_i \prec x_j & \text{for each } x_i \in X_i \text{ and } x_j \in X_j \text{ provided that } i < j, \\ z \sim z' & \text{for each } z, z' \in X_k \text{ for any } k \in \{1, 2, \ldots, r\}. \end{cases} \tag{20}$$

Furthermore, the decomposition (19) is unique in the following sense: if $X = X'_1 \cup X'_2 \cup \ldots \cup X'_r$ is another decomposition of X such that

$$\begin{cases} x'_1 \prec x'_2 \prec \ldots \prec x'_r & \text{for each } x'_1 \in X'_1,\ x'_2 \in X'_2,\ \ldots,\ x'_{r'} \in X'_{r'}, \\ z \sim z' & \text{for each } z, z' \in X'_k, \text{ for any } k \in \{1, 2, \ldots, r'\}, \end{cases}$$

then $r' = r$ and $X'_k = X_k$ for all $k \in \{1, 2, \ldots, r\}$.

Notice that (19) is the partition of X by equivalence classes of the binary relation $\sim$.

Definition 7.3. According to Lemma 7.2, if $X_1, X_2, \ldots, X_r$ are non-empty subsets of $(X, \precsim)$, we will use the notation $X_1 \prec X_2 \prec \ldots \prec X_r$ to indicate that $x_1 \prec x_2 \prec \ldots \prec x_r$ for each $x_1 \in X_1$, $x_2 \in X_2$, $\ldots$, $x_r \in X_r$.

7.2. Compatible aggregation functions

A concrete class of aggregation functions will be of interest in this paper.

Definition 7.4. An aggregation function $\mathcal{A} : [0,1]^2 \to [0,1]$ is *mean-compatible with the cuts of triangular fuzzy numbers* if

$$\mathcal{A}\left(\frac{\underline{a_1}_\beta + \underline{a_2}_\beta + \ldots + \underline{a_m}_\beta}{m}, \frac{\overline{a_1}_\beta + \overline{a_2}_\beta + \ldots + \overline{a_m}_\beta}{m} \right) = \frac{\mathcal{A}(\underline{a_1}_\beta, \overline{a_1}_\beta) + \mathcal{A}(\underline{a_2}_\beta, \overline{a_2}_\beta) + \ldots + \mathcal{A}(\underline{a_m}_\beta, \overline{a_m}_\beta)}{m}$$

for each $A_1, A_2, \ldots, A_m \in \mathrm{TFN}$ and all $\beta \in (0,1]$.

Proposition 7.5. *The aggregation functions mean M, minimum* min*, maximum* max *and weighted mean $W_\gamma M$, $\gamma \in (0,1)$, are mean-compatible with the cuts of triangular fuzzy numbers.*

Proof. It follows from the fact that $\underline{a}_\beta \le \overline{a}_\beta$ for each $A \in \mathrm{TFN}$ and each $\beta \in (0,1]$. For instance, we check this property for the weighted mean $W_\gamma M$, $\gamma \in (0,1)$:

$$\begin{aligned} & W_\gamma M \left(\frac{\underline{a_1}_\beta + \underline{a_2}_\beta + \ldots + \underline{a_m}_\beta}{m}, \frac{\overline{a_1}_\beta + \overline{a_2}_\beta + \ldots + \overline{a_m}_\beta}{m} \right) \\ & = (1-\gamma) \frac{\underline{a_1}_\beta + \underline{a_2}_\beta + \ldots + \underline{a_m}_\beta}{m} + \gamma \frac{\overline{a_1}_\beta + \overline{a_2}_\beta + \ldots + \overline{a_m}_\beta}{m} \\ & = \frac{\left[(1-\gamma)\underline{a_1}_\beta + \gamma \overline{a_1}_\beta\right] + \left[(1-\gamma)\underline{a_2}_\beta + \gamma \overline{a_2}_\beta\right] + \ldots + \left[(1-\gamma)\underline{a_m}_\beta + \gamma \overline{a_m}_\beta\right]}{m} \\ & = \frac{W_\gamma M(\underline{a_1}_\beta, \overline{a_1}_\beta) + W_\gamma M(\underline{a_2}_\beta, \overline{a_2}_\beta) + \ldots + W_\gamma M(\underline{a_m}_\beta, \overline{a_m}_\beta)}{m}. \end{aligned}$$

■

The mean-compatible aggregation functions enjoy good properties when applied to α-order.

Lemma 7.6. *Let α be a membership vector, let ω be a weight vector and let $\mathcal{A} : [0,1]^2 \to [0,1]$ be an aggregation function on $[0,1]$ which is mean-compatible with the cuts of triangular fuzzy numbers. If $A_1, A_2, \ldots, A_m, B, C \in \mathrm{TFN}$ are triangular fuzzy numbers such that $B \overset{\alpha}{<} A_1 \overset{\alpha}{=} A_2 \overset{\alpha}{=} \ldots \overset{\alpha}{=} A_m \overset{\alpha}{<} C$, then*

$$B \overset{\alpha}{<} \frac{A_1 + A_2 + \ldots + A_m}{m} \overset{\alpha}{<} C.$$

Proof. Let $D = (A_1 + A_2 + \ldots + A_m)/m \in \mathrm{TFN}$ be the mean of $A_1, A_2, \ldots, A_m$. By definition, for each $\beta \in (0,1]$, the β-cut of D is:

$$D_\beta = \left[\frac{\underline{a_1}_\beta + \underline{a_2}_\beta + \ldots + \underline{a_m}_\beta}{m}, \frac{\overline{a_1}_\beta + \overline{a_2}_\beta + \ldots + \overline{a_m}_\beta}{m} \right].$$

Therefore, as $\mathcal{A}$ is M-compatible with the cuts of triangular fuzzy numbers,

$$\mathcal{A}^D_\beta = \mathcal{A}\left(\frac{\underline{a_1}_\beta + \underline{a_2}_\beta + \ldots + \underline{a_m}_\beta}{m}, \frac{\overline{a_1}_\beta + \overline{a_2}_\beta + \ldots + \overline{a_m}_\beta}{m}\right) =$$

$$= \frac{\mathcal{A}(\underline{a_1}_\beta, \overline{a_1}_\beta) + \mathcal{A}(\underline{a_2}_\beta, \overline{a_2}_\beta) + \ldots + \mathcal{A}(\underline{a_m}_\beta, \overline{a_m}_\beta)}{m} = \frac{\mathcal{A}^{A_1}_\beta + \mathcal{A}^{A_2}_\beta + \ldots + \mathcal{A}^{A_m}_\beta}{m}.$$

Hence, for $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$,

$$\mathcal{A}^D_\alpha = \frac{1}{m}\mathcal{A}^{A_1}_\alpha + \frac{1}{m}\mathcal{A}^{A_2}_\alpha + \ldots + \frac{1}{m}\mathcal{A}^{A_m}_\alpha.$$

Using Proposition 4.7, since $A_1 \overset{\alpha}{=} A_2 \overset{\alpha}{=} \ldots \overset{\alpha}{=} A_m$, for each $i, j \in \{1, 2, \ldots, m\}$ and each $k \in \{0, 1, 2n-2\}$,

$$V^k(A_i) = V^k(A_j).$$

By Proposition 4.1, for each $i, j \in \{1, 2, \ldots, m\}$ and each $k \in \{0, 1, \ldots, n-1\}$,

$$\langle \vartheta^k(\omega), \mathcal{A}^{A_i}_\alpha \rangle = v^k_\vartheta(A_i) = v^k_\vartheta(A_j) = \langle \vartheta^k(\omega), \mathcal{A}^{A_j}_\alpha \rangle \qquad \text{and}$$

$$\langle \varphi^k_\alpha(\omega), \mathcal{A}^{A_i}_\alpha \rangle = v^k_{\varphi_\alpha}(A_i) = v^k_{\varphi_\alpha}(A_j) = \langle \varphi^k_\alpha(\omega), \mathcal{A}^{A_j}_\alpha \rangle.$$

For each $k \in \{0, 1, \ldots, n-1\}$, let denote:

$$v^k_1 = \langle \vartheta^k(\omega), \mathcal{A}^{A_1}_\alpha \rangle = \langle \vartheta^k(\omega), \mathcal{A}^{A_2}_\alpha \rangle = \ldots = \langle \vartheta^k(\omega), \mathcal{A}^{A_m}_\alpha \rangle \qquad \text{and}$$

$$v^k_2 = \langle \varphi^k_\alpha(\omega), \mathcal{A}^{A_1}_\alpha \rangle = \langle \varphi^k_\alpha(\omega), \mathcal{A}^{A_2}_\alpha \rangle = \ldots = \langle \varphi^k_\alpha(\omega), \mathcal{A}^{A_m}_\alpha \rangle.$$

Therefore, for each $k \in \{0, 1, \ldots, n-1\}$,

$$\begin{aligned} V^k(D) = v^k_\vartheta(D) = \langle \vartheta^k(\omega), \mathcal{A}^D_\alpha \rangle &= \left\langle \vartheta^k(\omega), \frac{1}{m}\mathcal{A}^{A_1}_\beta + \frac{1}{m}\mathcal{A}^{A_2}_\beta + \ldots + \frac{1}{m}\mathcal{A}^{A_m}_\beta \right\rangle \\ &= \frac{1}{m}\langle \vartheta^k(\omega), \mathcal{A}^{A_1}_\alpha \rangle + \frac{1}{m}\langle \vartheta^k(\omega), \mathcal{A}^{A_2}_\alpha \rangle + \ldots + \frac{1}{m}\langle \vartheta^k(\omega), \mathcal{A}^{A_m}_\alpha \rangle \\ &= \frac{1}{m}v^k_1 + \frac{1}{m}v^k_1 + \ldots + \frac{1}{m}v^k_1 = v^k_1, \end{aligned}$$

and, similarly, if $k \in \{n, n+1, \ldots, 2n-2\}$, then

$$V^k(D) = v^{k-(n-1)}_{\varphi_\alpha}(D) = \langle \vartheta^{k-(n-1)}(\omega), \mathcal{A}^D_\alpha \rangle = \ldots = v^{k-(n-1)}_2.$$

Since $B \overset{\alpha}{<} A_1 \overset{\alpha}{=} A_2 \overset{\alpha}{=} \ldots \overset{\alpha}{=} A_m$, then there is $k_0 \in \{n, n+1, \ldots, 2n-2\}$ such that

$$V^j(B) = V^j(A_1) \quad \text{for each } j \in \{0, 1, \ldots, k_0 - 1\}, \quad \text{and} \quad V^{k_0}(B) < V^{k_0}(A_1).$$

Therefore, for such k_0,

$$V^j(B) = V^j(D) \quad \text{for each } j \in \{0, 1, \ldots, k_0 - 1\}, \quad \text{and} \quad V^{k_0}(B) < V^{k_0}(D),$$

which means that $B \overset{\alpha}{<} D$. In the same way, it can be proved that as $A_m \overset{\alpha}{<} C$, then $D \overset{\alpha}{<} C$, which demonstrates that $B \overset{\alpha}{<} D \overset{\alpha}{<} C$. ■

7.3. Closed under means binary relations

The notions we have introduced up to this moment (for instance, FRE-sets and fuzzy dissimilarities) do not make use of any kind of algebraic operations. They can be introduced in a more general setting, by involving abstract elements, from a purely axiomatic point of view. It is not necessary that the underlying set is endowed with a richer structure in which sums, differences and products can be considered. We are now going to involve the algebraic operations of fuzzy numbers, but not to their full extent: we will only use arithmetic means, that is, additions and divisions between crisp fuzzy numbers.

Definition 7.7. A binary relation $\precsim$ on a non-empty set $\mathcal{S}$ of fuzzy numbers is *closed under the arithmetic mean of $\precsim$-equivalent fuzzy numbers* if

$$A_1, A_2, \ldots, A_m \in \mathcal{S} \text{ and } A_1 \sim A_2 \sim \ldots \sim A_m \implies \frac{A_1 + A_2 + \ldots + A_m}{m} \in \mathcal{S}.$$

For instance, the mean of triangular fuzzy numbers is also a triangular fuzzy number. In the following result we refine this property.

Corollary 7.8. *If α is a membership vector, ω is a weight vector and $\mathcal{A} : [0,1]^2 \to [0,1]$ is an aggregation function on $[0,1]$ which is mean-compatible with the cuts of triangular fuzzy numbers, then the binary relation $\overset{\alpha}{\leq}$ is closed under means of α-equal fuzzy numbers in* TFN *(and also in* TFN(ℝ)*). Furthermore, if $A_1, A_2, \ldots, A_m \in \mathrm{TFN}(\mathbb{R})$ satisfy $A_1 \overset{\alpha}{=} A_2 \overset{\alpha}{=} \ldots \overset{\alpha}{=} A_m$, then*

$$\frac{A_1 + A_2 + \ldots + A_m}{m} \overset{\alpha}{=} A_1 \overset{\alpha}{=} A_2 \overset{\alpha}{=} \ldots \overset{\alpha}{=} A_m. \tag{21}$$

Proof. If $A_1, A_2, \ldots, A_m \in \mathrm{TFN}(\mathbb{R})$, then $D = (A_1 + A_2 + \ldots + A_m)/m$ is another triangular fuzzy number on $\mathbb{R}$. If $A_1 \overset{\alpha}{=} A_2 \overset{\alpha}{=} \ldots \overset{\alpha}{=} A_m$, following the previous proof, we have proved that, for all $k \in \{0, 1, \ldots, n-1\}$,

$$V^k(D) = v_1^k = V^k(A_1) = V^k(A_2) = \ldots = V^k(A_m),$$

and for all $k \in \{n, n+1, \ldots, 2n-2\}$,

$$V^k(D) = v_2^{k-(n-1)} = V^{k-(n-1)}(A_1) = V^{k-(n-1)}(A_2) = \ldots = V^{k-(n-1)}(A_m).$$

Therefore, $V(D) = V(A_j)$ for all $j \in \{1, 2, \ldots, m\}$, so (21) holds. ■

7.4. The (discrete) fuzzy (μ, Δ)-Choquet integral of triangular fuzzy numbers on $[0,1]$

In order to define an appropriate notion of "Choquet integral for fuzzy numbers", let $\precsim$ be a total and transitive binary relation on $\mathcal{S}$. Given n fuzzy numbers $A_1, A_2, \ldots, A_n \in \mathcal{S}$, let us apply Lemma 7.2 to the finite set $\{A_1, A_2, \ldots, A_n\} \subseteq \mathcal{S}$. Such result let us to distribute the fuzzy numbers into r groups according to the binary relation $\precsim$, by using a permutation $\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ satisfying:

$$\begin{aligned}\{ A_{\sigma(1)} \sim A_{\sigma(2)} \sim \ldots \sim A_{\sigma(i_1)} \} &\prec \{ A_{\sigma(i_1+1)} \sim A_{\sigma(i_1+2)} \sim \ldots \sim A_{\sigma(i_2)} \} \\ &\prec \ldots \prec \{ A_{\sigma(i_{r-1}+1)} \sim A_{\sigma(i_{r-1}+2)} \sim \ldots \sim A_{\sigma(i_r)} \},\end{aligned} \tag{22}$$

where $0 = i_0 < i_1 < i_2 < \ldots < i_{r-1} < i_r = n$ (for convenience, we agree that $i_0 = 0$). Notice that the permutation σ is not unique: in fact, any other permutation preserving the groups $F_k^\sigma = \{\sigma(i_k+1), \sigma(i_k+2), \ldots, \sigma(i_{k+1})\}_{k=0}^{r-1}$ could lead to the same ordering. For each group, let consider its arithmetic mean as follows:

$$D_k^\sigma = \begin{cases} \widetilde{0}, & \text{if } k = 0, \\ \dfrac{A_{\sigma(i_{k-1}+1)} + A_{\sigma(i_{k-1}+2)} + \ldots + A_{\sigma(i_k)}}{i_k - i_{k-1}}, & \text{if } k \in \{1, 2, \ldots, r\}. \end{cases} \tag{23}$$

Associated to the permutation σ, for each $k \in \{1, 2, \ldots, r\}$, let us denote:

$$F_k^\sigma = \{ \sigma(i_{k-1}+1), \sigma(i_{k-1}+2), \ldots, \sigma(i_k) \}$$

and

$$F_{k,\sigma} = F_k^\sigma \cup F_{k+1}^\sigma \cup \ldots \cup F_r^\sigma. \tag{24}$$

Notice that

$$\begin{aligned} F_{1,\sigma} &= \{ \sigma(1), \sigma(2), \ldots, \sigma(i_1) \} \cup \{ \sigma(i_1+1), \sigma(i_1+2), \ldots, \sigma(i_2) \} \\ &\qquad \cup \ldots \cup \{ \sigma(i_{r-1}+1), \sigma(i_{r-1}+2), \ldots, \sigma(i_r) \} \\ &= \{ \sigma(1), \sigma(2), \ldots, \sigma(n) \} = \{1, 2, \ldots, n\}. \end{aligned}$$

First of all, let us check that the previous definitions do not depend on the permutation σ.

Lemma 7.9. *Let $\precsim$ be a total and transitive binary relation on $\mathcal{S}$. Given $A_1, A_2, \ldots, A_n \in \mathcal{S}$, let $\sigma, \sigma' : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ be two permutations such that*

$$\begin{aligned}\{ A_{\sigma(1)} \sim A_{\sigma(2)} \sim \ldots \sim A_{\sigma(i_1)} \} &\prec \{ A_{\sigma(i_1+1)} \sim A_{\sigma(i_1+2)} \sim \ldots \sim A_{\sigma(i_2)} \} \\ &\prec \ldots \prec \{ A_{\sigma(i_{r-1}+1)} \sim A_{\sigma(i_{r-1}+2)} \sim \ldots \sim A_{\sigma(i_r)} \}\end{aligned} \tag{25}$$

and

$$\begin{aligned}\{ A_{\sigma'(1)} \sim A_{\sigma'(2)} \sim \ldots \sim A_{\sigma'(j_1)} \} &\prec \{ A_{\sigma'(i_1+1)} \sim A_{\sigma'(i_1+2)} \sim \ldots \sim A_{\sigma'(j_2)} \} \\ &\prec \ldots \prec \{ A_{\sigma'(j_{r'-1}+1)} \sim A_{\sigma'(j_{r'-1}+2)} \sim \ldots \sim A_{\sigma'(j_{r'})} \}.\end{aligned} \tag{26}$$

Then $r = r'$ (the number of groups is equal) and $i_k = j_k$, $D_k^\sigma = D_k^{\sigma'}$, $\digamma_k^\sigma = \digamma_k^{\sigma'}$ and $\digamma_{k,\sigma} = \digamma_{k,\sigma'}$ for all $k \in \{0,1,\dots,r\}$.

Proof. The first part follows from Lemma 7.2 because the partition is unique. Then, associated to both permutations, we deduce that $r = r'$, and $i_k = j_k$ and $\digamma_k^\sigma = \digamma_k^{\sigma'}$ for all $k \in \{0,1,\dots,r\}$. Since $\digamma_k^\sigma = \digamma_k^{\sigma'}$, then $D_k^\sigma = D_k^{\sigma'}$ because the fuzzy number D_k^σ is defined as the arithmetic mean of the fuzzy numbers $\{ A_j : j \in \digamma_k^\sigma \}$. ■

Since $D_k^\sigma = D_k^{\sigma'}$ and $\digamma_{k,\sigma} = \digamma_{k,\sigma'}$ for all $k \in \{0,1,\dots,r\}$, where σ and σ' are two permutations of $\{1,2,\dots,n\}$ satisfying the properties (25) and (26), then we can define

$$\begin{cases} D_k = D_k^\sigma & \text{for all } k \in \{0,1,\dots,r\}, \\ \digamma_k = \digamma_{k,\sigma} & \text{for all } k \in \{1,2,\dots,r\}, \end{cases} \tag{27}$$

whatever the permutation $\sigma : \{1,2,\dots,n\} \to \{1,2,\dots,n\}$ satisfying (25). Using the previous notation, we introduce the following notion.

Definition 7.10. Given $n \in \mathbb{N}$, let $\mu : \mathcal{P}_n \to [0,1]$ be a fuzzy measure on $\mathcal{P}_n$ and let $\Delta : \mathcal{S}^2 \to \mathcal{S}$ be a dissimilarity on a FRE-set $(\mathcal{S}, \precsim, \widetilde{0}, \widetilde{1})$, where $\precsim$ is a total, transitive and closed under means of $\precsim$-equivalent fuzzy numbers binary relation on $\mathcal{S}$. The (*discrete*) *fuzzy* (μ,Δ)*-Choquet integral of the fuzzy numbers* $A_1, A_2, \dots, A_n \in \mathcal{S}$ is the fuzzy number:

$$C_{\mu,\Delta}(A_1, A_2, \dots, A_n) = \sum_{k=1}^{r} \Delta(D_k, D_{k-1}) \cdot \mu(\digamma_k), \tag{28}$$

where r is the number of groups in (22), and $\{D_k\}_{k=0}^r$ and $\{\digamma_k\}_{k=1}^r$ are defined by (27). We will refer to $\{D_k\}_{k=1}^r$ (respectively, to $\{\digamma_k\}_{k=1}^r$) as the *means* (respectively, *measurable sets*) associated to the partition (22).

Remark 7.11. One of the key ideas on the previous definition is to replace each set of $\precsim$-equivalent fuzzy numbers $\{ A_{\sigma(i_{k-1}+1)} \sim A_{\sigma(i_{k-1}+2)} \sim \dots \sim A_{\sigma(i_k)} \}$ by its arithmetic mean:

$$D_k = \frac{A_{\sigma(i_{k-1}+1)} + A_{\sigma(i_{k-1}+2)} + \dots + A_{\sigma(i_k)}}{i_k - i_{k-1}} \in \mathcal{S}.$$

This replacement was inspired by Lemma 7.6. However, other possible replacement could be also considered in prospect works.

The first property we must check is that this definition generalizes the usual notion of *discrete Choquet integral* for real numbers on $[0,1]$ when a restricted dissimilarity function δ is involved. To do it, we show that it is only necessary to consider the case $\alpha = \mathbf{1}_n$. For convenience, in the following result we denote $\mathrm{cr}(t) = \widetilde{t} = t \cdot \widetilde{1}$ for each $t \in [0,1]$, and, in order to recover the discrete Choquet integral associated to μ and δ, we use the defuzzification $\mathcal{D}_c : \mathrm{FN}(\mathbb{R}) \to \mathbb{R}$ that associates, to each fuzzy number $A \in \mathrm{FN}(\mathbb{R})$, the midpoint of its kernel, that is, $\mathcal{D}_c(A) = (\underline{a}_1 + \overline{a}_1)/2$.

Theorem 7.12. *Let us consider the membership vector $\alpha = \mathbf{1}_n = (1,1,\dots,1)$, let $\omega = (\omega_1, \omega_2, \dots, \omega_n) \in W_n$ be a weight vector and let $\mathcal{A} : [0,1]^2 \to [0,1]$ be an aggregation function on $[0,1]$. If $\delta : [0,1]^2 \to [0,1]$ is a restricted dissimilarity function on $[0,1]$ and $\mu : \mathcal{P}_n \to [0,1]$ is a fuzzy measure on $\mathcal{P}_n$, then*

$$C_{\mu,\Delta_\delta}(\mathrm{cr}(t_1), \mathrm{cr}(t_2), \dots, \mathrm{cr}(t_n)) = \mathrm{cr}\left[C_{\mu,\delta}(t_1, t_2, \dots, t_n) \right] \quad \text{for each } t_1, t_2, \dots, t_n \in [0,1],$$

where $\Delta_\delta : \mathrm{TFN}^2 \to \mathrm{TFN}$ is the dissimilarity function on $(\mathrm{TFN}, \overset{\alpha}{\leq}, \widetilde{0}, \widetilde{1})$ defined, for each $A = (a_1/a_2/a_3)$, $B = (b_1/b_2/b_3) \in \mathrm{TFN}$, as

$$\Delta_\delta(A, B) = \mathrm{cr}\left[\delta\left(a_2, b_2\right) \right].$$

As a consequence, the discrete Choquet integral $C_{\mu,\delta}$ associated to μ and δ is, for each $t_1, t_2, \dots, t_n \in [0,1]$,

$$C_{\mu,\delta}(t_1, t_2, \dots, t_n) = \mathcal{D}_c\left(C_{\mu,\Delta_\delta}(\mathrm{cr}(t_1), \mathrm{cr}(t_2), \dots, \mathrm{cr}(t_n)) \right).$$

The previous result is a particular case of a more general statement whose proof shows the consistency of the involved algebraic tools.

Theorem 7.13. *Let $\mathcal{S}$ be a subset of fuzzy numbers containing the crisp fuzzy numbers $\{ \widetilde{r} : r \in [0,1] \}$ and let $\precsim$ be a total and transitive binary relation on $\mathcal{S}$ such that:*

$$\text{if } t, s \in [0,1], \qquad \widetilde{t} \precsim \widetilde{s} \Leftrightarrow t \leq s. \tag{29}$$

Given a restricted dissimilarity function $\delta : [0,1]^2 \to [0,1]$, let $\Delta : \mathcal{S}^2 \to \mathcal{S}$ be a dissimilarity on $(\mathcal{S}, \precsim)$ such that

$$\Delta(\widetilde{t}, \widetilde{s}) = \mathrm{cr}\left[\delta(t,s)\right] \quad \text{for each } t, s \in [0,1]. \tag{30}$$

If $\mu : \mathcal{P}_n \to [0,1]$ is a fuzzy measure on $\mathcal{P}_n$, then

$$C_{\mu,\Delta_\delta}(\mathrm{cr}(t_1),\mathrm{cr}(t_2),\ldots,\mathrm{cr}(t_n)) = \mathrm{cr}\left[C_{\mu,\delta}(t_1,t_2,\ldots,t_n)\right] \quad \textit{for each } t_1,t_2,\ldots,t_n \in [0,1].$$

As a consequence, the discrete Choquet integral $C_{\mu,\delta}$ associated to μ and δ is, for each $t_1,t_2,\ldots,t_n \in [0,1]$,

$$C_{\mu,\delta}(t_1,t_2,\ldots,t_n) = \mathcal{D}_c\left(C_{\mu,\Delta_\delta}(\mathrm{cr}(t_1),\mathrm{cr}(t_2),\ldots,\mathrm{cr}(t_n))\right).$$

Proof. Notice that $\precsim$ verifies, for each $t,s \in [0,1]$,

$$\widetilde{t} \precsim \widetilde{s} \Leftrightarrow t \le s, \qquad \widetilde{t} \sim \widetilde{s} \Leftrightarrow t = s \qquad \text{and} \qquad \widetilde{t} \prec \widetilde{s} \Leftrightarrow t < s.$$

Let $t_1,t_2,\ldots,t_n \in [0,1]$ and let $\sigma : \{1,2,\ldots,n\} \to \{1,2,\ldots,n\}$ be a permutation such that $0 \le t_{\sigma(1)} \le t_{\sigma(2)} \le \ldots \le t_{\sigma(n)} \le 1$. We denote $F_{\sigma(k)} = \{\sigma(k),\sigma(k+1),\ldots,\sigma(n)\}$ for all $k \in \{1,2,\ldots,n\}$. Following Definition 2.6, the discrete Choquet integral for real numbers on $[0,1]$ associated to μ and δ is:

$$C_{\mu,\delta}(t_1,t_2,\ldots,t_n) = \sum_{k=1}^{n} \delta(t_{\sigma(k)},t_{\sigma(k-1)}) \cdot \mu(F_{\sigma(k)}).$$

We need to determine what numbers are repeated in the ordering $t_{\sigma(1)} \le t_{\sigma(2)} \le \ldots \le t_{\sigma(n)}$. Hence, we can find r groups such that

$$\begin{aligned}\{\, t_{\sigma(1)} = t_{\sigma(2)} = \ldots = t_{\sigma(i_1)} \,\} &< \{\, t_{\sigma(i_1+1)} = t_{\sigma(i_1+2)} = \ldots = t_{\sigma(i_2)} \,\} \\ &< \ldots < \{\, t_{\sigma(i_{r-1}+1)} = t_{\sigma(i_{r-1}+2)} = \ldots = t_{\sigma(i_r)} \,\}\end{aligned}$$

(we use the braces to indicate the subsets of numbers that are equal). Using (29), the ordering of the fuzzy numbers $\{\mathrm{cr}(t_{\sigma(k)})\}_{k=1}^{n}$ must be the same, that is:

$$\begin{aligned}\{\, \mathrm{cr}(t_{\sigma(1)}) \sim \mathrm{cr}(t_{\sigma(2)}) \sim \ldots \sim \mathrm{cr}(t_{\sigma(i_1)}) \,\} &\prec \{\, \mathrm{cr}(t_{\sigma(i_1+1)}) \sim \mathrm{cr}(t_{\sigma(i_1+2)}) \sim \ldots \sim \mathrm{cr}(t_{\sigma(i_2)}) \,\} \\ &\prec \ldots \prec \{\, \mathrm{cr}(t_{\sigma(i_{r-1}+1)}) \sim \mathrm{cr}(t_{\sigma(i_{r-1}+2)}) \sim \ldots \sim \mathrm{cr}(t_{\sigma(i_r)}) \,\}.\end{aligned}$$

Let us denote

$$\begin{cases} s_1 = t_{\sigma(1)} = t_{\sigma(2)} = \ldots = t_{\sigma(i_1)}, \\ s_2 = t_{\sigma(i_1+1)} = t_{\sigma(i_1+2)} = \ldots = t_{\sigma(i_2)}, \\ \quad \vdots \\ s_r = t_{\sigma(i_{r-1}+1)} = t_{\sigma(i_{r-1}+2)} = \ldots = t_{\sigma(i_r)} \end{cases}$$

to the distinct numbers we can find in the set $\{t_{\sigma(1)},t_{\sigma(2)},\ldots,t_{\sigma(n)}\}$. Clearly $s_1 < s_2 < \ldots < s_r$. Let define $i_0 = t_{\sigma(0)} = s_0 = 0$. Notice that:

$$t_{\sigma(k)} = t_{\sigma(k-1)} \qquad \text{for all } k \in \{1,2,\ldots,n\} \setminus \{\, i_0+1,\, i_1+1,\, i_2+1,\ldots,\, i_{r-1}+1 \,\}.$$

As a consequence,

$$\delta(t_{\sigma(k)},t_{\sigma(k-1)}) = 0 \qquad \text{for all } k \in \{1,2,\ldots,n\} \setminus \{\, i_0+1,\, i_1+1,\, i_2+1,\ldots,\, i_{r-1}+1 \,\}.$$

Therefore:

$$\begin{aligned}C_{\mu,\delta}(t_1,t_2,\ldots,t_n) &= \sum_{k=1}^{n} \delta(t_{\sigma(k)},t_{\sigma(k-1)}) \cdot \mu(F_{\sigma(k)}) = \sum_{j=0}^{r-1} \delta(t_{\sigma(i_j+1)},t_{\sigma(i_j)}) \cdot \mu(F_{\sigma(i_j+1)}) \\ &= \sum_{j=0}^{r-1} \delta(s_{j+1},s_j) \cdot \mu(F_{\sigma(i_j+1)}).\end{aligned}$$

The sets $\{F_{\sigma(i_j+1)}\}_{j=0}^{r-1}$ are:

$$\begin{aligned}F_{\sigma(i_0+1)} &= F_{\sigma(1)} = \{\sigma(1),\sigma(2),\ldots,\sigma(n)\} = \{1,2,\ldots,n\} = F_1, \\ F_{\sigma(i_1+1)} &= \{\sigma(i_1+1),\sigma(i_1+2),\ldots,\sigma(n)\} = F_2, \\ F_{\sigma(i_2+1)} &= \{\sigma(i_2+1),\sigma(i_2+2),\ldots,\sigma(n)\} = F_3, \\ &\vdots \\ F_{\sigma(i_{r-1}+1)} &= \{\sigma(i_{r-1}+1),\sigma(i_{r-1}+2),\ldots,\sigma(n)\} = F_r,\end{aligned}$$

so

Table 1
Computations to determine the α-ordering of the TFNs A_1, A_2, A_3 and A_4 of Subsection 7.5.

A_i	j	α_j	$\mathbb{I}_{\alpha_j}^{A_i}$	$\overline{M}_{\alpha_j}^{A_i}$	ω_j	$v_{\vartheta}^0(A_i)$	$\vartheta^1(\omega)(j)$	$v_{\vartheta}^1(A_i)$
A_1	1	1	$\{[0.7, 0.7]\}$	0.7	0.7	0.6925	1	0.7
	2	0.5	$\{[0.55, 0.8]\}$	0.675	0.3		0	
A_2	1	1	$\{[0.5, 0.5]\}$	0.5	0.7	0.5	1	0.5
	2	0.5	$\{[0.45, 0.55]\}$	0.5	0.3		0	
A_3	1	1	$\{[0.4, 0.4]\}$	0.4	0.7	0.4375	1	0.4
	2	0.5	$\{[0.35, 0.7]\}$	0.525	0.3		0	
A_4	1	1	$\{[0.5, 0.5]\}$	0.5	0.7	0.5	1	0.5
	2	0.5	$\{[0.4, 0.6]\}$	0.5	0.3		0	

$$C_{\mu,\delta}(t_1, t_2, \ldots, t_n) = \sum_{j=0}^{r-1} \delta(s_{j+1}, s_j) \cdot \mu(F_{\sigma(i_j+1)}) = \sum_{j=0}^{r-1} \delta(s_{j+1}, s_j) \cdot \mu(F_{j+1})$$
$$= \sum_{j=1}^{r} \delta(s_j, s_{j-1}) \cdot \mu(F_j).$$

On the other hand, by (23), $D_0 = \widetilde{0} = \mathrm{cr}(0) = \mathrm{cr}(s_0)$ and, for all $k \in \{1, 2, \ldots, r\}$,

$$D_k = \frac{A_{\sigma(i_{k-1}+1)} + A_{\sigma(i_{k-1}+2)} + \ldots + A_{\sigma(i_k)}}{i_k - i_{k-1}} = \frac{\mathrm{cr}(s_k) + \mathrm{cr}(s_k) + \ldots + \mathrm{cr}(s_k)}{i_k - i_{k-1}}$$
$$= \frac{(i_k - i_{k-1})\, \mathrm{cr}(s_k)}{i_k - i_{k-1}} = \mathrm{cr}(s_k).$$

As the classical real operations on $[0, 1]$ are similar to the extended operations with crisp fuzzy numbers, using (30), we deduce that:

$$\mathrm{cr}\left[C_{\mu,\delta}(t_1, t_2, \ldots, t_n)\right] = \mathrm{cr}\left[\sum_{j=1}^{r} \delta(s_j, s_{j-1}) \cdot \mu(F_j)\right] = \sum_{j=1}^{r} \mathrm{cr}\left[\delta(s_j, s_{j-1})\right] \cdot \mu(F_j)$$
$$= \sum_{j=1}^{r} \Delta(\mathrm{cr}(s_j), \mathrm{cr}(s_{j-1})) \cdot \mu(F_j) = \sum_{j=1}^{r} \Delta(D_j, D_{j-1}) \cdot \mu(F_j),$$

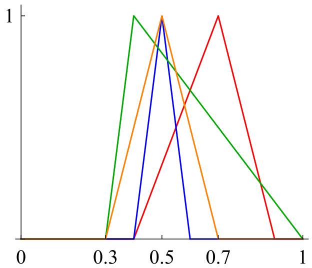
and, by (28), this linear combination is, exactly, $C_{\mu,\Delta_\delta}(\mathrm{cr}(t_1), \mathrm{cr}(t_2), \ldots, \mathrm{cr}(t_n))$. ■

Proof (of Theorem 7.12). On the one hand, when $\alpha = \mathbf{1}_n$, the binary relation $\overset{\mathbf{1}}{\preceq}$ satisfies $A \overset{\mathbf{1}}{\preceq} B \Leftrightarrow a_2 \leq b_2$, where $A = (a_1/a_2/a_3)$, $B = (b_1/b_2/b_3) \in \mathrm{TFN}$. Therefore it satisfies (29). On the other hand, Proposition 5.4 guarantees that Δ_δ is a dissimilarity on $(\mathrm{TFN}, \overset{\mathbf{1}}{\preceq}, \widetilde{0}, \widetilde{1})$ satisfying (30). Then Theorem 7.13 is applicable. ■

7.5. An example of the computation of the fuzzy (Δ, μ)-Choquet integral

As a toy example in the FRE-set $(\mathrm{TFN}, \overset{\alpha}{\preceq}, \widetilde{0}, \widetilde{1})$, choose $\alpha = (1, 0.5)$, $\omega = (0.7, 0.3)$ and the mean aggregation function $\mathcal{A} = \overline{M}$. Then $n = 2$ (the number of components of α). We consider the following set of four fuzzy numbers of TFN:

$$A_1 = (0.4/0.7/0.9),$$
$$A_2 = (0.4/0.5/0.6),$$
$$A_3 = (0.3/0.4/1),$$
$$A_4 = (0.3/0.5/0.7).$$



We first compute the $\overset{\alpha}{\preceq}$-ordering of these fuzzy numbers. To do it, notice that $\vartheta^0(\omega) = \varphi_\alpha^0(\omega) = \omega$, $\vartheta^1(\omega) = \varphi_\alpha^1(\omega) = (1, 0)$ and $\vartheta^k(\omega) = \varphi_\alpha^k(\omega) = (0, 0)$ for all $k \geq 2$. We carry out the computations to complete Table 1.

Since $\vartheta^k(\omega) = \varphi_\alpha^k(\omega)$ for all $k \geq 0$, then $v_\vartheta^k(A_i) = v_{\varphi_\alpha}^k(A_i)$ for all $k \geq 0$. Therefore, their valuation vectors are $V(A_1) = (0.6925, 0.7, 0.7)$, $V(A_2) = V(A_4) = (0.5, 0.5, 0.5)$ and $V(A_3) = (0.4375, 0.4, 0.4)$. Thus, $A_3 \overset{\alpha}{<} A_2 \overset{\alpha}{=} A_4 \overset{\alpha}{<} A_1$, so we have the partition $\{A_3\} \overset{\alpha}{<} \{A_2, A_4\} \overset{\alpha}{<} \{A_1\}$. To each class, the following fuzzy number is associated:

$$D_0=\widetilde{0} \text{ (by definition)}, \quad D_1=A_3, \quad D_2=\frac{A_2+A_4}{2}=(0.35/0.5/0.65), \quad D_3=A_1.$$

Then:

$$V(D_0)=V(\widetilde{0})=(0,0,0), \quad V(D_1)=V(A_3)=(0.4375,0.4,0.4),$$
$$V(D_2)=V(A_2)=(0.5,0.5,0.5) \quad \text{and} \quad V(D_3)=V(A_1)=(0.6925,0.7,0.7).$$

Since the first terms of each valuation vector are distinct, taking into account that $2n=4$, we derive that, for all $i,j\in\{0,1,2,3\}$,

$$m(D_i,D_j)=\begin{cases} 1, & \text{if } i\neq j,\\ 4, & \text{if } i=j.\end{cases}$$

To compute the fuzzy (μ,Δ)-Choquet integral of A_1, A_2, A_3 and A_4, choose μ as the usual fuzzy measure (given as the quotient of the number of elements over the total number of elements) in $\mathcal{P}_4=\{1,2,3,4\}$ and the dissimilarity $\Delta=\Delta_{1,2,3}$ defined as in Theorem 6.12. On the one hand, using the previous partition, we deduce that $\digamma_1=\{3\}$, $\digamma_2=\{2,3,4\}$ and $\digamma_3=\{1,2,3,4\}$. Hence, $\mu(\digamma_1)=1/4$, $\mu(\digamma_2)=3/4$ and $\mu(\digamma_3)=1$. On the other hand, given $\kappa>0$ and $k\in\{1,2,3\}$, we compute the values:

$$\begin{aligned}\varrho_\kappa(D_k,D_{k-1})&=1-\frac{m(D_k,D_{k-1})+\kappa-\left|V^{m(D_k,D_{k-1})}(D_k)-V^{m(D_k,D_{k-1})}(D_{k-1})\right|}{2n+\kappa}=\\ &=1-\frac{1+\kappa-\left|V^1(D_k)-V^1(D_{k-1})\right|}{4+\kappa}=\frac{3+\left|V^1(D_k)-V^1(D_{k-1})\right|}{4+\kappa}.\end{aligned}$$

Then:

$$\begin{aligned}\varrho_\kappa(D_1,D_0)&=\frac{3+\left|V^1(D_1)-V^1(D_0)\right|}{4+\kappa}=\frac{3+|0.4375-0|}{4+\kappa}=\frac{3.4375}{4+\kappa},\\ \varrho_\kappa(D_2,D_1)&=\frac{3+\left|V^1(D_2)-V^1(D_1)\right|}{4+\kappa}=\frac{3+|0.5-0.4375|}{4+\kappa}=\frac{3.0625}{4+\kappa},\\ \varrho_\kappa(D_3,D_2)&=\frac{3+\left|V^1(D_3)-V^1(D_2)\right|}{4+\kappa}=\frac{3+|0.6925-0.5|}{4+\kappa}=\frac{3.1925}{4+\kappa}.\end{aligned}$$

As a result,

$$\begin{aligned}\Delta_{1,2,3}(D_1,D_0)&=\left(\varrho_3(D_1,D_0)\,/\,\varrho_2(D_1,D_0)\,/\,\varrho_1(D_1,D_0)\right)\\ &=\left(\frac{3.4375}{4+3}\Big/\frac{3.4375}{4+2}\Big/\frac{3.4375}{4+1}\right)=3.4375\left(\frac{1}{7}\Big/\frac{1}{6}\Big/\frac{1}{5}\right),\\ \Delta_{1,2,3}(D_2,D_1)&=\ldots=3.0625\left(\frac{1}{7}\Big/\frac{1}{6}\Big/\frac{1}{5}\right),\\ \Delta_{1,2,3}(D_3,D_2)&=\ldots=3.1925\left(\frac{1}{7}\Big/\frac{1}{6}\Big/\frac{1}{5}\right).\end{aligned}$$

Finally, the fuzzy $(\mu,\Delta_{1,2,3})$-Choquet integral of A_1, A_2, A_3 and A_4 is

$$\begin{aligned}C_{\mu,\Delta_{1,2,3}}(A_1,A_2,A_3,A_4)&=\sum_{k=1}^{3}\Delta_{1,2,3}(D_k,D_{k-1})\cdot\mu(\digamma_k)=3.4375\left(\frac{1}{7}\Big/\frac{1}{6}\Big/\frac{1}{5}\right)\cdot\frac{1}{4}\\ &\quad+3.0625\left(\frac{1}{7}\Big/\frac{1}{6}\Big/\frac{1}{5}\right)\cdot\frac{3}{4}+3.1925\left(\frac{1}{7}\Big/\frac{1}{6}\Big/\frac{1}{5}\right)\cdot 1\\ &=6.34875\left(\frac{1}{7}\Big/\frac{1}{6}\Big/\frac{1}{5}\right)\approx(0.90696\,/\,1.058125\,/\,1.26975).\end{aligned}$$

This example shows another important characteristic of the fuzzy Choquet integral. In the real case, when m real values of the interval $[0,1]$ are aggregated by the Choquet integral, the final result belongs to the interval $[0,m]$, that is, it can be out of the interval $[0,1]$. In the above example, we also notice that the support of $C_{\mu,\Delta}(A_1,A_2,\ldots,A_n)$ can be out of the interval $[0,1]$, i.e., the aggregate value of n fuzzy numbers of TFN may be a fuzzy number that is outside of TFN.

8. Conclusions and prospect work

In this paper we have introduced the notion of CI in the context of fuzzy numbers. For this purpose, we have used a total preorder on a suitable set of fuzzy numbers with absolute extrema, which is the natural underlying framework for the development of this concept. As a basic example, we have used the set of triangular fuzzy numbers with support on $[0,1]$ and the α-order. To achieve the main objective, we have also introduced the concept of dissimilarity between fuzzy numbers, and we have described some examples, starting with the simple case of crisp numbers and successively extending the family of examples. Finally, we have proved that the introduced notion of CI extends to the real notion.

This paper is basically introductory. For reasons of extension, we have not shown enough examples of the introduced concepts, which is a task left for prospect works. Hence, the main properties of the fuzzy CI and the potential applications of this notion open a great field of study. First, it is necessary to analyze whether this notion could satisfy similar/reasonable properties to CI in the real context. Second, it seems interesting to particularize the CI by employing several preorders, and to study what additional properties can be deduced when handling an order. Finally, the field of applications is at least as large as the field of CI applications in the real case, so it is worth developing experiments in which the input data are fuzzy numbers.

CRediT authorship contribution statement

A.F. Roldán López de Hierro: Writing – review & editing, Writing – original draft, Investigation. **A. Cruz:** Writing – review & editing, Writing – original draft, Investigation. **R.H.N. Santiago:** Writing – review & editing, Writing – original draft, Investigation. **C. Roldán:** Writing – review & editing, Writing – original draft, Investigation. **D. García-Zamora:** Writing – review & editing, Writing – original draft, Investigation. **F. Neres:** Writing – review & editing, Writing – original draft, Investigation. **H. Bustince:** Writing – review & editing, Writing – original draft, Investigation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A

Here we include the proofs of some mentioned results.

Lemma 8.1. *Let V be a vector space of dimension m and let ß be an ordered basis of V. Let $\mathcal{H} = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k\} \subset V$ be a set of non-null vectors of V and, for each $i \in \{1, 2, \dots, k\}$, let denote by $\mathbf{v}_i(a^i_1, a^i_2, \dots, a^i_m) \in \mathbb{R}^m$ the vector of coordinates of $\mathbf{u}_i$ w.r.t. the basis ß. Suppose that:*

- *for each $i \in \{1, 2, \dots, k-1\}$, the vector $\mathbf{v}_{i+1}$ has strictly more zeros than the vector $\mathbf{v}_i$;*
- *if the j-th coordinate of $\mathbf{v}_i$ is zero, then the j-th coordinate of $\mathbf{v}_{i+1}$ is also zero ($a^i_j = 0 \Rightarrow a^{i+1}_j = 0$).*

Then the set $\mathcal{H}$ is linearly independent.

Proof. By the second assumption, $\mathbf{v}_1$ is the vector with the least number of coordinates zero, $\mathbf{v}_2$ is the vector with the second least number of coordinates zero, and so on. Then $\mathbf{v}_k$ is the vector with the greatest number of coordinates zero, but it cannot be zero because the vectors of $\mathcal{H}$ are non-null. Suppose that $V = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_m\}$. Then, we can reorder the vectors of the basis ß using the following criteria:

- we consider first the vectors $ß_1 = \{\mathbf{w}_{\sigma(1)}, \mathbf{w}_{\sigma(2)}, \dots, \mathbf{w}_{\sigma(n_1)}\}$ of ß corresponding to coordinates zero of $\mathbf{v}_1$;
- secondly, we enlarge $ß_1$ with the vectors $ß_2 = \{\mathbf{w}_{\sigma(n_1+1)}, \mathbf{w}_{\sigma(n_1+2)}, \dots, \mathbf{w}_{\sigma(n_2)}\}$ corresponding to coordinates with value zero of $\mathbf{v}_2$ that are not null for $\mathbf{v}_1$;
- thirdly, we enlarge $ß_1 \cup ß_2$ with the vectors $ß_3 = \{\mathbf{w}_{\sigma(n_2+1)}, \mathbf{w}_{\sigma(n_2+2)}, \dots, \mathbf{w}_{\sigma(n_3)}\}$ corresponding to coordinates with value zero of $\mathbf{v}_3$ that are not null for $\mathbf{v}_2$;
- etc.

This process continues in successive steps. After k steps, we have the union $ß_1 \cup ß_2 \cup \ldots \cup ß_k$ of vectors of the basis ß that can be completed with the rest $ß'$ of vectors of ß. In this way, $\widetilde{ß} = ß_1 \cup ß_2 \cup \ldots \cup ß_k \cup ß'$ is an ordered basis of V that comes from a re-ordenation of the vectors of the basis ß. In this basis, if we write (by rows) the coordinates of the vectors of $\mathcal{H}$, we obtain a matrix with k rows and m columns as the following one:

$$\left(\begin{array}{cccc|cccccccccccccccc}
0 & 0 & \dots & 0 & * & * & \dots & * & * & * & \dots & * & \dots & * & * & \dots & * & * & * & \dots & * \\
0 & 0 & \dots & 0 & \overline{0} & \overline{0} & \overline{\dots} & \overline{0}\,| & * & * & \dots & * & \dots & * & * & \dots & * & * & * & \dots & * \\
0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \overline{0} & \overline{0} & \overline{\dots} & \overline{0}\,| & \dots & * & * & \dots & * & * & * & \dots & * \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \dots & \overline{0} & \overline{0} & \overline{\dots} & \overline{0}\,| & * & * & \dots & *
\end{array}\right),$$

where positions with $*$ correspond to non zero values. Notice that the last row contains non zero values. Hence, the range of this matrix is k (it is easy to determine a non zero minor of order k), so the vectors of the set $\mathcal{H}$ are linearly independent. ■

Proof (of Corollary 3.2, page 7). Clearly, $\omega \neq \mathbf{0}$, so $\vartheta^0(\omega) = \omega \neq \mathbf{0}$. By item 1 of Proposition 3.1, either $\vartheta^1(\omega) = \vartheta(\omega) = \mathbf{0}$ or $\vartheta^1(\omega)$ is another weight vector. In the first case, for each $k \geq 2$, $\vartheta^k(\omega) = \vartheta^{k-1}(\vartheta^1(\omega)) = \vartheta^{k-1}(\mathbf{0}) = \mathbf{0} \in \mathbb{R}^n$. In this case, $p_\omega = 1$. On the contrary case, suppose that $\vartheta^1(\omega) \neq \mathbf{0}$. Then $\omega^1 = \vartheta^1(\omega)$ is a weight vector. Again, if we call $\omega^2 = \vartheta^1(\omega^1) = \vartheta^2(\omega)$, item 1 of Proposition 3.1 guarantees that either $\vartheta^2(\omega) = \mathbf{0}$ or $\vartheta^2(\omega)$ is another weight vector. In the first case, for each $k \geq 3$, $\vartheta^k(\omega) = \vartheta^{k-2}(\vartheta^2(\omega)) = \vartheta^{k-2}(\mathbf{0}) = \mathbf{0} \in \mathbb{R}^n$, and we can take $p_\omega = 2$. In the second case, $\omega^2 = \vartheta^2(\omega)$ is a weight vector and the process continues. Since the item 5 of Proposition 3.1 states that $\vartheta^k(\omega) = \mathbf{0}$ for all $k \geq n$, the previous process has to finish in such way that $\vartheta^0(\omega), \vartheta^1(\omega), \ldots, \vartheta^{p_\omega - 1}(\omega)$ are non null vectors and $\vartheta^k(\omega) = \mathbf{0}$ for all $k \geq p_\omega$. In this case, Lemma 8.1 guarantees that the vectors of the set $\{\vartheta^0(\omega), \vartheta^1(\omega), \ldots, \vartheta^{p_\omega - 1}(\omega)\}$ are linearly independent because: (1) by item 3 of Proposition 3.1, for each $k \in \{0, 1, \ldots, p_\omega - 2\}$, the vector $\vartheta^{k+1}(\omega)$ has strictly more coordinates whose values are zero than the vector $\vartheta^k(\omega)$; and (2) if the j-th coordinate of $\vartheta^k(\omega)$ is zero, then the j-th coordinate of $\vartheta^{k+1}(\omega)$ is also zero.

The second part can be proved in a similar way (but it depends on α and on ω). ■

Proof (of Proposition 4.1, page 7). Since $A \in \text{TFN}$, then $\mathcal{A}_\beta^A = \mathcal{A}(\underline{a}_\beta, \overline{a}_\beta) \in [0,1]$. As a result, for each $k \in \{0, 1, \ldots, n-1\}$:

$$0 = \sum_{j=1}^{n} \left[\vartheta^k(\omega)(j) \cdot 0 \right] \leq \sum_{j=1}^{n} \left[\vartheta^k(\omega)(j) \cdot \mathcal{A}_{\alpha_j}^A \right] \leq \sum_{j=1}^{n} \left[\vartheta^k(\omega)(j) \cdot 1 \right] \leq 1,$$

so $v_\vartheta^k(A) \in [0,1]$. The rest is similar. ■

Proof (of Proposition 4.2, page 7). Since $a_i \leq b_i$ for each $i \in \{1,2,3\}$, then

$$\underline{a}_\beta = a_1(1-\beta) + a_2\beta \leq b_1(1-\beta) + b_2\beta = \underline{b}_\beta$$

and similarly $\overline{a}_\beta \leq \overline{b}_\beta$ for all $\beta \in (0,1]$. As $\mathcal{A}$ is non-decreasing on each argument, then $\mathcal{A}(\underline{a}_\beta, \overline{a}_\beta) \leq \mathcal{A}(\underline{b}_\beta, \overline{b}_\beta)$ for all $\beta \in (0,1]$. Thus $0 \leq \mathcal{A}_{\alpha_j}^A \leq \mathcal{A}_{\alpha_j}^B \leq 1$ for all $j \in \{0, 1, \ldots, n-1\}$. Hence

$$v_\vartheta^k(A) = \sum_{j=1}^{n} \left[\vartheta^k(\omega)(j) \cdot \mathcal{A}_{\alpha_j}^A \right] \leq \sum_{j=1}^{n} \left[\vartheta^k(\omega)(j) \cdot \mathcal{A}_{\alpha_j}^B \right] = v_\vartheta^k(B), \qquad \text{and}$$

$$v_{\varphi_\alpha}^k(A) = \sum_{j=1}^{n} \left[\varphi_\alpha^k(\omega)(j) \cdot \mathcal{A}_{\alpha_j}^A \right] \leq \sum_{j=1}^{n} \left[\varphi_\alpha^k(\omega)(j) \cdot \mathcal{A}_{\alpha_j}^B \right] = v_{\varphi_\alpha}^k(B).$$

Then necessarily $A \overset{\alpha}{\leq} B$. ■

Proof (of Proposition 4.3, page 8). Given $\beta \in (0,1]$, taking into account that the β-level set of the crisp fuzzy number $\widetilde{r}$ is $\widetilde{r}_\beta = \{r\}$, then $\mathbb{I}_\beta^{\widetilde{r}} = \{r\}$, so $\mathcal{A}_\beta^{\widetilde{r}} = \mathcal{A}(r) = r$ (in dimension 1, the aggregation function acts as the identity mapping). Hence

$$\begin{aligned} v_\vartheta^k(\widetilde{r}) &= \sum_{j=1}^{n} \left[\vartheta^k(\omega)(j) \cdot \mathcal{A}_{\alpha_j}^{\widetilde{r}} \right] = \sum_{j=1}^{n} \left[\vartheta^k(\omega)(j) \cdot r \right] = r \cdot \sum_{j=1}^{n} \vartheta^k(\omega)(j) = \\ &= \left\{ \begin{array}{ll} r \cdot 1, & \text{if } \vartheta^k(\omega) \neq \mathbf{0}, \\ r \cdot 0, & \text{if } \vartheta^k(\omega) = \mathbf{0} \end{array} \right\} = \left\{ \begin{array}{ll} r, & \text{if } \vartheta^k(\omega) \neq \mathbf{0}, \\ 0, & \text{if } \vartheta^k(\omega) = \mathbf{0}. \end{array} \right. \end{aligned}$$

The same argument can be done for $v_{\alpha,\omega,\varphi_\alpha,\mathcal{A}}^k$. ■

Proof (of Corollary 4.8, page 8). It is clear that $\widetilde{0} \overset{\alpha}{\leq} A \overset{\alpha}{\leq} \widetilde{1}$ for each $A \in \text{TFN}$, so $\mathbf{0}_{2n-1} = V(\widetilde{0}) \sqsubseteq V(A) \sqsubseteq V(\widetilde{1})$ from Proposition 4.7. ■

Proof (of Proposition 5.4, page 10). Suppose that $\alpha = \mathbf{1}_n = (1, 1, \ldots, 1)$. Let ω be a weight vector and let $\mathcal{A} : [0,1]^2 \to [0,1]$ be an aggregation function on $[0,1]$. Given $A = (a_1/a_2/a_3) \in \text{TFN}$ and $\beta = 1$, $\mathbb{I}_1^A = \{a_2\}$, so $\mathcal{A}_1^A = \mathcal{A}(a_2) = a_2$. Therefore, for each $k \in \{0, 1, \ldots, n-1\}$,

$$v_{\mathbf{1},\omega,\vartheta,\mathcal{A}}^k(A) = \langle \vartheta^k(\omega), \mathcal{A}_\mathbf{1}^A \rangle = \langle \vartheta^k(\omega), a_2\,\mathbf{1} \rangle = a_2 \langle \vartheta^k(\omega), \mathbf{1} \rangle = \left\{ \begin{array}{ll} a_2, & \text{if } \vartheta^k(\omega) \neq \mathbf{0}, \\ 0, & \text{if } \vartheta^k(\omega) = \mathbf{0}, \end{array} \right.$$

and, similarly,

$$v_{\mathbf{1},\omega,\varphi_\mathbf{1},\mathcal{A}}^k(A) = \left\{ \begin{array}{ll} a_2, & \text{if } \varphi_\mathbf{1}^k(\omega) \neq \mathbf{0}, \\ 0, & \text{if } \varphi_\mathbf{1}^k(\omega) = \mathbf{0}. \end{array} \right.$$

This implies that, when $\alpha = \mathbf{1}_n$,

$$A \overset{\mathbf{1}}{\leq} B \Leftrightarrow a_2 \leq b_2, \quad A \overset{\mathbf{1}}{=} B \Leftrightarrow a_2 = b_2, \quad A \overset{\mathbf{1}}{<} B \Leftrightarrow a_2 < b_2,$$
$$A \overset{\mathbf{1}}{=} \widetilde{0} \Leftrightarrow a_2 = 0 \quad \text{and} \quad A \overset{\mathbf{1}}{=} \widetilde{1} \Leftrightarrow a_2 = 1. \tag{31}$$

Notice that, in this case, $\langle \vartheta^k(\omega), \mathbf{1} \rangle = 1 = \langle \omega, \mathbf{1} \rangle$ for all $k \in \{0, 1, \ldots, p_\omega - 1\}$ and also $\langle \varphi_{\mathbf{1}}^j(\omega), \mathbf{1} \rangle = 1 = \langle \omega, \mathbf{1} \rangle$ for all $j \in \{0, 1, \ldots, q_{\alpha,\omega} - 1\}$.

Next, we prove that Δ_δ is a dissimilarity on (TFN, $\overset{\mathbf{1}}{\leq}, \widetilde{0}, \widetilde{1}$). Clearly Δ_δ is symmetric.

(D_2) $\Delta_\delta(A, B) = \widetilde{1} \Leftrightarrow \delta\left(a_2, b_2\right) = 1 \Leftrightarrow$ [$a_2 = 0$ and $b_2 = 1$, or vice versa] $\Leftrightarrow$ [$A \overset{\mathbf{1}}{=} \widetilde{0}$ and $B \overset{\mathbf{1}}{=} \widetilde{1}$, or vice versa] $\Leftrightarrow \{A, B\} \overset{\mathbf{1}}{=} \{\widetilde{0}, \widetilde{1}\}$;

(D_3) $\Delta_\delta(A, B) = \widetilde{0} \Leftrightarrow \delta\left(a_2, b_2\right) = 0 \Leftrightarrow a_2 = b_2 \Leftrightarrow A \overset{\mathbf{1}}{=} B$.

(D_4) Suppose that $A \overset{\mathbf{1}}{\leq} B \overset{\mathbf{1}}{\leq} C$. Then $a_2 \leq b_2 \leq c_2$, so $\delta(a_2, b_2) \leq \delta(a_2, c_2)$ and $\delta(b_2, c_2) \leq \delta(a_2, c_2)$. Therefore $\Delta_\delta(A, B) = \widetilde{\delta\left(a_2, b_2\right)} \overset{\mathbf{1}}{\leq} \widetilde{\delta\left(a_2, c_2\right)} = \Delta_\delta(A, C)$, and, similarly, $\Delta_\delta(A, B) \overset{\mathbf{1}}{\leq} \Delta_\delta(A, C)$.

Then Δ_δ is a dissimilarity on (TFN, $\overset{\mathbf{1}}{\leq}, \widetilde{0}, \widetilde{1}$). ■

Proof (of Corollary 5.6, page 10). It follows from Lemma 5.5 taking into account that the binary relation $\overset{\alpha}{\leq}$ is reflexive, transitive and total on TFN. ■

Proof (of Proposition 6.2, page 11). Clearly, the function ϱ_λ satisfies (SD_1)-(SD_3). To prove (SD_4), let $A, B, C \in \mathcal{S}$ be such that $A \precsim B \precsim C$. We demonstrate that $\varrho_\lambda(A, B) \leq \varrho_\lambda(A, C)$ (the other inequality is similar). If $\varrho_\lambda(A, B) = 0$ or $\varrho_\lambda(A, C) = 1$, the previous inequality is obvious. Then, suppose that $\varrho_\lambda(A, B) \geq \lambda$ and $\varrho_\lambda(A, C) \leq \lambda$. Let us show two impossible cases.

- *Case 1: suppose that* $\varrho_\lambda(A, B) = 1$. Then $\{A, B\} \sim \{A_{\min}, A_{\max}\}$. Since $A \precsim B$, assume that $A \sim A_{\min}$ and $B \sim A_{\max}$. Hence $A_{\max} \precsim B \precsim C \precsim A_{\max}$. Since $\precsim$ is transitive, $C \sim A_{\max}$, so $\varrho_\lambda(A, C) = 1$, which contradicts the fact that $\varrho_\lambda(A, C) \leq \lambda$.
- *Case 2: suppose that* $\varrho_\lambda(A, C) = 0$. Then $A \sim C$, so $A \precsim B \precsim C \precsim A$. Since $\precsim$ is transitive, then $A \sim B$, so $\varrho_\lambda(A, B) = 0$, which contradicts the fact that $\varrho_\lambda(A, B) \geq \lambda$.

As the previous two cases are impossible, then $\varrho_\lambda(A, B) \geq \lambda$ and $\varrho_\lambda(A, C) \leq \lambda$ lead to $\varrho_\lambda(A, B) = \lambda = \varrho_\lambda(A, C)$, which finally proves the inequality $\varrho_\lambda(A, B) \leq \varrho_\lambda(A, C)$ in any case. ■

Proof (of Lemma 6.3, page 11). We check all properties. The symmetry of ϱ_δ is apparent.

(SD_2) Notice that

$$\varrho_\delta(A, B) = 1 \;\Leftrightarrow\; \delta\left(\frac{a_1 + a_2 + a_3}{3}, \frac{b_1 + b_2 + b_3}{3}\right) = 1 \;\Leftrightarrow\; \left\{\frac{a_1 + a_2 + a_3}{3}, \frac{b_1 + b_2 + b_3}{3}\right\} = \{0, 1\}$$
$$\Leftrightarrow \quad \{A, B\} = \{\widetilde{0}, \widetilde{1}\} \quad \Leftrightarrow \quad \{A, B\} \sim_{\precsim_\Sigma} \{\widetilde{0}, \widetilde{1}\}.$$

(SD_3) We observe that:

$$\varrho_\delta(A, B) = 0 \quad \Leftrightarrow \quad \delta\left(\frac{a_1 + a_2 + a_3}{3}, \frac{b_1 + b_2 + b_3}{3}\right) = 0$$
$$\Leftrightarrow \quad \frac{a_1 + a_2 + a_3}{3} = \frac{b_1 + b_2 + b_3}{3} \quad \Leftrightarrow \quad A \sim_{\precsim_\Sigma} B.$$

(SD_4) Let $A, B, C \in$ TFN be such that $A \precsim_\Sigma B \precsim_\Sigma C$. Then

$$\frac{a_1 + a_2 + a_3}{3} \leq \frac{b_1 + b_2 + b_3}{3} \leq \frac{c_1 + c_2 + c_3}{3}.$$

Therefore

$$\delta\left(\frac{a_1 + a_2 + a_3}{3}, \frac{b_1 + b_2 + b_3}{3}\right) \leq \delta\left(\frac{a_1 + a_2 + a_3}{3}, \frac{c_1 + c_2 + c_3}{3}\right) \qquad \text{and}$$
$$\delta\left(\frac{b_1 + b_2 + b_3}{3}, \frac{c_1 + c_2 + c_3}{3}\right) \leq \delta\left(\frac{a_1 + a_2 + a_3}{3}, \frac{c_1 + c_2 + c_3}{3}\right),$$

which means that $\varrho_\delta(A, B) \leq \varrho_\delta(A, C)$ and $\varrho_\delta(B, C) \leq \varrho_\delta(A, C)$.

■

Proof (of Lemma 6.7, page 15). Clearly $F(\varrho_1, \varrho_2, \ldots, \varrho_m)$ is symmetric. Let check the other properties.

(SD_2) Since F is 1-dependent and each ϱ_i is a crisp dissimilarity, then

$$[\,F(\varrho_1, \varrho_2, \ldots, \varrho_m)\,](A,B) = 1 \quad \Leftrightarrow \quad F(\varrho_1(A,B), \varrho_2(A,B), \ldots, \varrho_m(A,B)) = 1$$
$$\Leftrightarrow \quad \left[\, \varrho_i(A,B) = 1 \text{ for each } i \in \{1,2,\ldots,m\} \,\right] \quad \Leftrightarrow \quad \{A,B\} \sim \{A_{\min}, A_{\max}\}.$$

(SD_3) Similarly, since F is 0-dependent,

$$[\,F(\varrho_1, \varrho_2, \ldots, \varrho_m)\,](A,B) = 0 \quad \Leftrightarrow \quad F(\varrho_1(A,B), \varrho_2(A,B), \ldots, \varrho_m(A,B)) = 0$$
$$\Leftrightarrow \quad \left[\, \varrho_i(A,B) = 0 \text{ for each } i \in \{1,2,\ldots,m\} \,\right] \quad \Leftrightarrow \quad A \sim B.$$

(SD_4) Let $A, B, C \in \mathcal{S}$ be such that $A \precsim B \precsim C$. Since each ϱ_i is a crisp dissimilarity on $(\mathcal{S}, \precsim, A_{\min}, A_{\max})$, then $\varrho_i(A,B) \le \varrho_i(A,C)$ and $\varrho_i(B,C) \le \varrho_i(A,C)$ for all $i \in \{1,2,\ldots,m\}$. Thus, as F is non-decreasing on each argument;

$$\begin{cases} F(\varrho_1(A,B), \varrho_2(A,B), \ldots, \varrho_m(A,B)) \le F(\varrho_1(A,C), \varrho_2(A,C), \ldots, \varrho_m(A,C)), \\ F(\varrho_1(B,C), \varrho_2(B,C), \ldots, \varrho_m(B,C)) \le F(\varrho_1(A,C), \varrho_2(A,C), \ldots, \varrho_m(A,C)), \end{cases}$$

which means that $[\,F(\varrho_1, \varrho_2, \ldots, \varrho_m)\,](A,B) \le [\,F(\varrho_1, \varrho_2, \ldots, \varrho_m)\,](A,C)$ and $[\,F(\varrho_1, \varrho_2, \ldots, \varrho_m)\,](B,C) \le [\,F(\varrho_1, \varrho_2, \ldots, \varrho_m)\,](A,C)$.

■

Proof (of Corollary 6.8, page 15). It follows from Lemma 6.7 using $F(t_1, t_2, \ldots, t_m) = \sum_{i=1}^{n} \lambda_i t_i$ for each $t_1, t_2, \ldots, t_m \in [0,1]$. ■

Proof (of Proposition 6.11, page 16). If $\kappa = \kappa'$, the result is obvious. Suppose that $\kappa < \kappa'$. Since $a \le b$ and $\kappa' - \kappa > 0$, we deduce that:

$$\begin{aligned} \frac{a+\kappa}{b+\kappa} \le \frac{a+\kappa'}{b+\kappa'} \quad &\Leftrightarrow \quad (a+\kappa)(b+\kappa') \le (a+\kappa')(b+\kappa) \\ &\Leftrightarrow \quad ab + a\kappa' + b\kappa + \kappa\kappa' \le ab + b\kappa' + a\kappa + \kappa\kappa' \\ &\Leftrightarrow \quad a\kappa' + b\kappa \le b\kappa' + a\kappa \quad \Leftrightarrow \quad a\kappa' - a\kappa \le b\kappa' - b\kappa \\ &\Leftrightarrow \quad a(\kappa' - \kappa) \le b(\kappa' - \kappa) \quad \Leftrightarrow \quad a \le b, \end{aligned}$$

and the equality holds in one of the previous inequalities if and only if it holds in all of them. ■

Proof (of Lemma 7.2, page 17). By Proposition 7.1, let $x_1^0 \in X$ be such that

$$x_1^0 \precsim x \qquad \text{for all } x \in X. \tag{32}$$

Let define $X_1 = \{\, x \in X \,:\, x \sim x_1^0 \,\}$. As $\precsim$ is total and transitive, it can be easily checked that:

$$\begin{cases} \bullet\ z \sim z' \quad \text{for each } z, z' \in X_1, \\ \bullet\ x_1 \precsim x \quad \text{for each } x_1 \in X_1 \text{ and each } x \in X, \\ \bullet\ \text{if } x_1 \in X_1 \text{ and } x \in X \diagdown X_1, \text{ then } x_1 \prec x. \end{cases} \tag{33}$$

Let prove the last property. Let $x_1 \in X_1$ and $x \in X \diagdown X_1$. Since $x_1 \in X_1$, then $x_1 \sim x_1^0$. Using (32), then $x_1 \precsim x_1^0 \precsim x$. As $\precsim$ is transitive, then $x_1 \precsim x$. If we suppose that $x \precsim x_1$, then $x \precsim x_1 \precsim x_1^0$, which leads to both $x \precsim x_1^0$ (because $\precsim$ is transitive), and $x_1^0 \precsim x$ by (32). Then $x \sim x_1^0$, so $x \in X_1$, which contradicts the fact that $x \in X \diagdown X_1$. This proves that "$x_1 \precsim x$" holds but "$x \precsim x_1$" is false. Hence $x_1 \prec x$.

If $X_1 = X$, the decomposition is finished. On the contrary case, suppose that $X \diagdown X_1 \neq \varnothing$. As this set is finite, Proposition 7.1 guarantees that there is $x_2^0 \in X \diagdown X_1$ such that:

$$x_2^0 \precsim x \qquad \text{for all } x \in X \diagdown X_1. \tag{34}$$

Let define $X_2 = \{\, x \in X \,:\, x \sim x_2^0 \,\}$. We claim that:

$$\begin{cases} \bullet\ X_1 \cap X_2 = \varnothing, \\ \bullet\ z \sim z' \quad \text{for each } z, z' \in X_2, \\ \bullet\ x_2 \precsim x \quad \text{for each } x_2 \in X_2 \text{ and each } x \in X \diagdown X_1, \\ \bullet\ \text{if } x_1 \in X_1,\ x_2 \in X_2 \text{ and } x \in X \diagdown (X_1 \cup X_2), \text{ then } x_1 \prec x_2 \prec x. \end{cases} \tag{35}$$

Suppose that there is $z_0 \in X_1 \cap X_2$. Since $z_0 \in X_1$, then $z_0 \sim x_1^0$, and as $z_0 \in X_2$, then $z_0 \sim x_2^0$. Then $x_1^0 \sim z_0 \sim x_2^0$. As $\precsim$ is transitive, then $x_1^0 \sim x_2^0$, so $x_2^0 \in X_1$, which contradicts the fact that $x_2^0 \in X \diagdown X_1$. Hence $X_1 \cap X_2 = \varnothing$. The transitivity of $\precsim$ implies the second property of (35), and its third property follows from (34). To prove the fourth property of (35), suppose that $x_1 \in X_1$, $x_2 \in X_2$ and $x \in X \setminus (X_1 \cup X_2)$. By the third property of (33), $x_1 \prec x_2$, and by the third property of (35), $x_2 \precsim x$. Suppose, by contradiction, that $x \precsim x_2$. Then $x_2^0 \precsim x_2 \precsim x \precsim x_2 \precsim x_2^0$. As $\precsim$ is transitive, then $x \sim x_2^0$, so $x \in X_2$, which contradicts that $x \in X \setminus (X_1 \cup X_2)$. Then "$x_2 \precsim x$" holds but "$x \precsim x_2$" is false, so $x_2 \prec x$.

Repeating this process, we can find the partition $X = X_1 \cup X_2 \cup \ldots \cup X_r$ satisfying the properties detailed in (20). In fact, (19) is the partition of X by equivalence classes of the binary relation $\sim$ by using the points $x_1^0, x_2^0, \ldots, x_r^0$. The process must be finite because X is finite. The uniqueness follows because $X = X_1 \cup X_2 \cup \ldots \cup X_r$ is the partition of X by equivalence classes of the binary relation $\sim$. ■

Data availability

No data was used for the research described in the article.

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