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A Linguistic Metric for Consensus Reaching Processes based on ELICIT Comprehensive Minimum Cost Consensus Models

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***Abstract*—Linguistic Group Decision-Making (LiGDM) aims at solving decision situations involving human decision-makers (DMs) whose opinions are modeled by using linguistic information. To achieve agreed solutions that increase DMs' satisfaction towards the collective solution, Linguistic Consensus Reaching Processes (LiCRPs) have been developed. These LiCRPs aim at suggesting DMs to change their original opinions to increase the group consensus degree, computed by a certain consensus measure. In recent years, these LiCRPs have been a prolific research line, and consequently numerous proposals have been introduced in the specialized literature. However, it has been pointed out the non-existence of objective metrics to compare these models and decide which one presents the best performance for each LiGDM problem. Therefore, this paper aims at introducing a metric to evaluate the performance of LiCRPs that takes into account the resulting consensus degree and the cost of modifying DMs' initial opinions. Such a metric is based on a linguistic Comprehensive Minimum Cost Consensus (CMCC) model based on ELICIT (Extended Comparative Linguistic Expressions with Symbolic Translation) information that models DMs' hesitancy and provides accurate Computing with Words processes. Additionally, the linguistic CMCC optimization model is linearized to speed up the computational model and improve its accuracy.**

***Index Terms*—Computing with words, ELICIT information, Fuzzy linguistic approach, Linguistic cost metric, Minimum cost consensus**

I. Introduction

In Group Decision-Making (GDM), a group of DMs faces a decision situation in which they provide their preferences to select the best alternative as a solution to the decision problem. Even though the participation of several DMs allows the consideration of several points of view in the decision process, it often implies the emergence of disagreements among them, which should be properly managed to avoid unsatisfactory results. Consensus Reaching Processes (CRPs) were designed to soften such discrepancies and drive the group toward an agreed solution [1]–[3]. Classically, a desired consensus threshold is fixed a priori, then a discussion process is carried out in which a moderator suggests the DMs to modify their preferences in order to increase the group consensus degree. A CRP is usually an iterative process which is repeated for several rounds until either the consensus degree surpasses the consensus threshold or the number of rounds exceeds a maximum limit [2].

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Real-world GDM problems and their CRPs are generally presented in uncertain contexts characterized by the absence of objective information, which increases the complexity of the decision situation. Under these circumstances, the DMs may have difficulties in providing their opinions by using numerical assessments. To offer more realistic and suitable frameworks for DMs to express their preferences according to their natural way of thinking, the use of the fuzzy linguistic approach and linguistic variables [4] has increased its popularity in recent years. When DMs provide their opinions through linguistic assessments, we talk about LiGDM [5] and LiCRPs [6]–[8].

Since achieving linguistic agreed solutions is essential in many real-world decision situations [9], [10], the interest of researchers has been aroused, leading to many LiCRP proposals in the specialized literature [1]. Although a priori having many proposals could make easier the resolution of LiGDM problems, the bibliographic analysis developed by García-Zamora et al. [1] pointed out that there is an evident lack of objective metrics to compare the performance of different LiCRPs and discern which one presents a better performance to deal with a certain LiGDM problem. The main consequence of this situation is that the authors justify the alleged well-performance of their proposals through the resolution of simple illustrative examples, which could easily be biased to obtain good results [1]. In this regard, authors have used different measures to compare consensus proposals such as the number of rounds necessary to reach the consensus threshold [11], [12], the trust among experts [11] or the consensus degree [2], [13]. However, these aspects could not be representative of the quality of the models because they do not provide enough information about their global performance, and, consequently, authors could show them in the most convenient way. For instance, a fast consensus model in terms of number of rounds may present several drawbacks related to the achieved consensus degree or the changes performed in the original preferences, which could have been excessively modified. In addition, these measures may allow comparing models in a specific case study, but they do not offer a global vision of the performance of the model when different DMs' opinions are used.

Therefore, the main goal of this paper is proposing the first linguistic metric to objectively compare linguistic consensus models and show which one presents the best performance

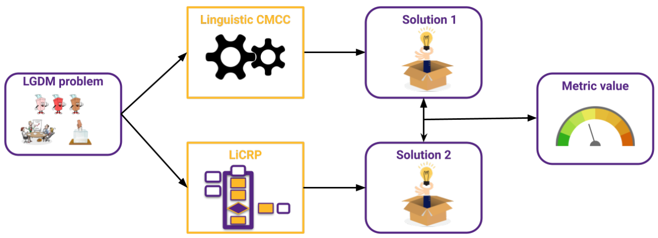

Fig. 1. A scheme of the proposed metric

in the resolution of a LiGDM problem. The proposed metric compares the results of the LiCRP with an ideal scenario in which the consensus threshold is achieved by making as few changes as possible to DMs' original opinions (see Fig. 1). This paper uses the CMCC [14], [15] models, which are automatic CRPs, to determine such ideal results but extending them to deal with linguistic information. Consequently, we raise the following research questions.

- RQ1: How to define CMCC models in a linguistic environment?
- RQ2: How to evaluate objectively the performance of LiCRPs?

To answer these questions, we first propose a linguistic CMCC model for ELICIT information [5], a recently proposed linguistic modelling approach that guarantees precise computations with linguistic information [4]. ELICIT information hybridizes the 2-tuple linguistic approach [16] and Hesitant Fuzzy Linguistic Term Sets (HFLTS) [17] by introducing a Computing with Words (CW) [18], [19] framework that guarantees precise computations with hesitant expressions without losing interpretability during the operational process [5]. These ELICIT-CMCC models inherit the properties of classic CMCC models [14] for numeric assessments, thus they provide modified DMs' preferences which preserve as much as possible the initial opinions and, in turn, guarantee the predefined consensus threshold. In addition, ELICIT-CMCC models follow the CW methodology [18], [19], i.e., linguistic results are obtained from linguistic inputs. Since such optimization models do not only require the use of many variables, but also the use of nonlinear constraints involving the absolute value, this proposal also includes a linearized version of the proposed ELICIT-CMCC models to speed up the computational model and improve the accuracy of the solution for the decision situation. Finally, these novel linguistic CMCC models are used as the basis to define a linguistic cost metric to evaluate LiCRPs that is based on two indicators to determine the quality of a consensus model: (i) the consensus degree achieved and (ii) the minimum changes necessary to obtain an agreed solution. The former is essential to ensure that the consensus process has been carried out successfully, i.e., it would be nonsense to score a consensus model that does not achieve the desired level of consensus with a high score [12], [13]. The latter guarantees that the original opinions of the DMs are not modified beyond the strictly necessary to reach the consensus threshold [14]. Therefore, a LiCRP that performs unnecessary changes on DMs' opinions to reach the consensus will receive a low mark.

To summarize, the main novelties of this proposal are:

- CMCC models for linguistic information are proposed following a CW approach.
- Such models are then linearized to accelerate computational cost, even with dealing with hundreds or thousands of DMs, and improve the precision of the results.
- From the linearized ELICIT-CMCC model, it is proposed a linguistic cost metric to objectively evaluate the performance of LiCRPs.

The reminder of this proposal is as follows. Section II includes some preliminary notions required to better understand this proposal related to LiGDM, 2-tuple and ELICIT linguistic representation schemes and MCC models. In Section III CMCC models for ELICIT information are proposed and then linearized. Here, it is also provided a brief analysis regarding the feasibility of such linear models when dealing with decision situations in which hundreds or thousands of DMs take part. Afterwards, Section IV introduces a linguistic cost metric based on the previous CMCC models and a couple of CRPs are evaluated to illustrate its working. Section V shows the CW nature of the ELICIT-CMCC models through the resolution of a LiGDM problem, and Section V-C includes a comparative analysis between the novel linguistic CMCC model for ELICIT information and other proposals. Finally, Section VI concludes the paper.

II. Background

This section introduces a revision of the basic notions related to the proposal. First, the basic concepts of LiGDM are revised. Afterwards, the linguistic 2-tuple model and ELICIT linguistic representation model are reviewed, and some notations are fixed to simplify their understanding. Finally, LiCRPs and MCC models are revised.

A. Linguistic Group Decision-Making

Decision processes are inherent in human beings' daily life. These decision situations consist of making the best possible choice among several possible solutions to a certain problem. Some decision problems are simple to solve and may involve just one individual. However, other decision problems are more complex and require several DMs, who may contribute with different points of view and knowledge. Formally, a GDM problem is modeled as a decision situation in which several individuals or DMs $E = \{e_1, e_2, ..., e_m\}$, $m \in \mathbb{N}$, have to decide which alternative from a set $X = \{x_1, x_2, ..., x_n\}$, $n \in N$, is the best solution to a problem [7], [20].

In addition, the complexity of GDM problems increases when the available information is not objective, but vague and imprecise. In such contexts, the stakeholders must address the decision situation from a subjective point of view by using qualitative assessments. In this regard, modeling DMs' opinions properly becomes crucial to managing the uncertainty inherent in these situations. Although some proposals translate qualitative information to a numerical scale, the goal of LiGDM is to model the uncertainty using linguistic

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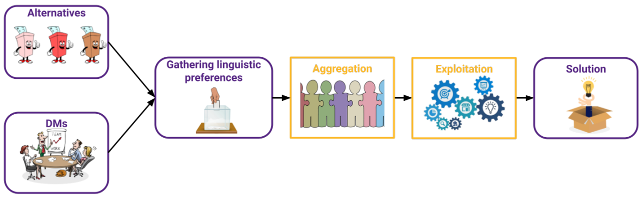

Fig. 2. LiGDM resolution scheme.

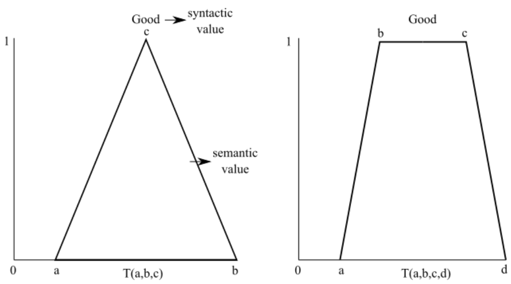

Fig. 3. Linguistic label.

expressions close to the natural way of human thinking (see Fig. 2).

This proposal uses the fuzzy linguistic approach [4] based on fuzzy sets theory [21], [22] to model uncertainty in LiGDM. This approach represents the linguistic information using linguistic variables [4], which usually model the information through parametric membership functions with triangular or trapezoidal graphical representation, among others (see Fig. 3).

The resolution of LiGDM problems implies to carry out computations with linguistic information. In this sense, the CW approach aims to provide linguistic solutions to problems formulated with linguistic expressions that emulate human thinking. There are several CW proposals in the literature [1] such as the linguistic model based on the fuzzy relation proposed by Tang and Zheng [23], linguistic distribution assessment proposed by Dong et al. [24] or the fuzzy set approach to treat determinacy and consistency of linguistic terms introduced by Ma et al. [25]. In particular, this proposal considers that the linguistic information is modelled by the 2-tuple linguistic model [16] and the ELICIT information [5], which highlight because they allow modelling uncertainty according to the fuzzy linguistic approach [22] without losing information or interpretability.

B. 2-tuple linguistic model

The 2-tuple linguistic model [16] aimed to overcome the lack of precision in classical linguistic computational approaches through a continuous fuzzy representation of the linguistic information and a computational model capable of carrying out simple symbolic precise computations without approximations, obtaining accurate linguistic results according to the CW scheme.

A 2-tuple linguistic value is a tuple $(s_i, \alpha) \in \overline{S} := S \times [-0.5, 0.5[$, where s_i is a linguistic term that belongs to a certain linguistic term set $S = \{s_0, s_1, \dots s_g\}$ (for a fixed even number $g \in \mathbb{N}$) and α is the so-called symbolic translation, i.e., a numerical value that represents the shifting of s_i fuzzy membership function (see Fig. 4). Note that for a linguistic 2-tuple value $(s_i, \alpha) \in \overline{S}$, the possible values for the symbolic translation α are:

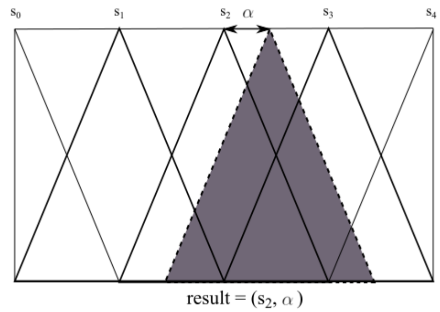

Fig. 4. Symbolic translation

$$\alpha \in \begin{cases} [-0.5, 0.5) & if \ s_i \in \{s_1, s_2, \dots, s_{g-1}\} \\ [0, 0.5) & if \ s_i = s_0 \\ [-0.5, 0] & if \ s_i = s_g \end{cases}$$

The key characteristic of 2-tuple linguistic expressions is the fact that they can be translated into a numerical quantity $x \in [0, g]$, which simplifies the computations:

Proposition 1. *[16] Let $S = \{s_0, \dots s_g\}$ be a linguistic term set. Then, the function $\Delta_S^{-1} : \overline{S} \to [0, g]$ defined by*

$$\Delta_S^{-1}(s_i, \alpha) = i + \alpha, \ \forall \ (s_i, \alpha) \in \overline{S}$$

is a bijection whose inverse $\Delta_S : [0, g] \to \overline{S}$ is given by

$$\Delta_S(x) = (s_{round(x)}, x - round(x)) \ \forall \ x \in [0, g],$$

where $round(\cdot)$ is the function that assigns the closest integer number $i \in \{0, \dots, g\}$.

Remark 1. *Note that any linguistic term $s_i \in S$ can be represented as a 2-tuple linguistic value by considering $(s_i, 0) \in \overline{S}$.*

C. ELICIT information

The 2-tuple linguistic framework follows a CW scheme to carry out computations, obtaining precise results that are easy to understand. However, it presents an important drawback regarding the lack of expressiveness, because the linguistic 2-tuple values are not able to model the DMs' hesitancy between several linguistic terms like HFLTS [17] do. Labella et al. [5] proposed the use of ELICIT information to address this limitation by introducing a linguistic approach that preserves the accuracy and understandability of the 2-tuple linguistic model and improves the expressiveness by hybridizing it with HFLTS.

Formally, ELICIT information is denoted here by an expression $[\overline{s}_i, \overline{s}_j]_{\gamma_1, \gamma_2}$, where $\overline{s}_i, \overline{s}_j \in \overline{S}, i \leq j$ are two 2-tuple linguistic values. In addition, ELICIT values also consider two parameters γ_1, γ_2 which guarantee that no information is lost during the computations with these expressions. It should be noted that any Trapezoidal Fuzzy Number (TrFN) [21], [22]

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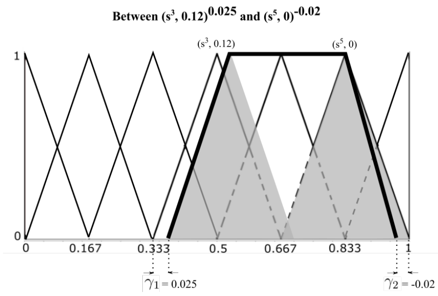

Fig. 5. Example of ELICIT linguistic expressions.

can be unequivocally represented as an ELICIT value (see Fig. 5).

Remark 2. *A TrFN is a function $T \equiv T(a,b,c,d) : [0,1] \to [0,1]$ of the form*

$$T(x) = \begin{cases} 0 & \text{if } 0 \le x \le a \\ \frac{x-a}{b-a} & \text{if } a < x < b \\ 1 & \text{if } b \le x \le c \\ \frac{d-x}{d-c} & \text{if } c < x < d \\ 0 & \text{if } d \le x \le 1 \end{cases} \quad \forall\, x \in [0,1]$$

for certain $0 \le a \le b \le c \le d \le 1$. For the sake of clarity, the set of all TrFNs on the interval $[0,1]$ will be denoted by

$$\mathcal{T} = \{T : [0,1] \to [0,1] \ : \ T \text{ is a TrFN}\}.$$

Proposition 2. *Let $\overline{\overline{S}}$ be the set of all possible ELICIT values. Then the mapping ζ given by:*

$$\zeta : \mathcal{T} \to \overline{\overline{S}}$$
$$T(a,b,c,d) \to [\overline{s}_1, \overline{s}_2]_{\gamma_1,\gamma_2}$$

where

$$\overline{s}_1 = \Delta_S(gb) \quad \gamma_1 = a - \max\left\{b - \frac{1}{g^2}, 0\right\}$$
$$\overline{s}_2 = \Delta_S(gc) \quad \gamma_2 = d - \min\left\{c + \frac{1}{g^2}, 1\right\}$$

is a bijection whose inverse ζ^{-1} is defined by:

$$\zeta^{-1} : \overline{\overline{S}} \to \mathcal{T}$$
$$[\overline{s}_1, \overline{s}_2]_{\gamma_1,\gamma_2} \to T(a,b,c,d)$$

and allows computing the fuzzy representation of an ELICIT expression as follows:

$$a = \gamma_1 + \max\left\{\frac{\Delta_S^{-1}(\overline{s}_1) - \frac{1}{g}}{g}, 0\right\}, b = \frac{\Delta_S^{-1}(\overline{s}_1)}{g},$$
$$c = \frac{\Delta_S^{-1}(\overline{s}_m)}{g}, d = \gamma_2 + \min\left\{\frac{\Delta_S^{-1}(\overline{s}_m) + \frac{1}{g}}{g}, 1\right\}.$$

Remark 3. *It must be highlighted that the notation $[\overline{s}_1, \overline{s}_2]_{\gamma_1,\gamma_2}$ is used for the sake of clarity, but the reader should keep in mind that, in spite of its formal nature, this notation resembles a linguistic expression. In other words, ELICIT information can be used to represent the hesitancy between several linguistic terms and perform precise computations on them by providing a linguistic result.*

The ELICIT computational model follows a CW approach that computes the fuzzy representation of the respective linguistic expressions, whose results are lately retranslated to ELICIT information. From a theoretical point of view, ELICIT expressions are generated by a context-free grammar which models comparative linguistic structures close to human language such as *at least bad*, *at most fast* or *between expensive and rather expensive* [5]. Thus, this context-free grammar together with a linguistic term set, for instance,

$$S = \{\text{Much Worse (MW)}, \text{Worse (W)}, \text{Slightly Worse (SW)}$$
$$\text{Equal (E)}, \text{Slightly Better (SB)}, \text{Better (B)}, \text{Much Better (MB)}\}$$

can model linguistic expressions such as, *at least* $(\text{W}, 0.2)^{0.2}$, *at most* $(\text{W}, 0.1)^{0.1}$ or *between* $(E, 0)^{-0.11}$ *and* $(SB, 0.32)^{0}$.

Remark 4. *Note that any linguistic term $s_i \in S$ can be represented as the ELICIT expression $(s_i, 0)_0 \equiv [(s_i, 0), (s_i, 0)]_{00}$. In the same way, an HFLTS $\{s_i, s_{i+1}, ..., s_j\}$, $i < j$, can be translated to the ELICIT value $[(s_i, 0), (s_j, 0)]_{00}$.*

To aggregate ELICIT values, Labella et al. [5], proposed the use of the fuzzy weighted average operator $A : \mathcal{T}^m \to \mathcal{T}$ defined by

$$A(T_1, T_2, ..., T_m) = (\sum_{k=1}^{m} \omega_k T_k^a, \sum_{k=1}^{m} \omega_k T_k^b, \sum_{k=1}^{m} \omega_k T_k^c, \sum_{k=1}^{m} \omega_k T_k^d),$$

where T_k^t denotes the t-th $t \in \{a,b,c,d\}$ coordinate of the TrFN $T_k, k = 1, 2, ..., m$ and $\omega_1, \omega_2, ..., \omega_m \ge 0$, $\sum_{k=1}^{m} \omega_k = 1$ are the weights for the DMs.

A comparison measure to order ELICIT values based on the method presented by Abbasbandy and Hajjari in [26] was also proposed. This method translates the fuzzy representation of the ELICIT values, given by a TrFN, into a numerical value called magnitude, which is defined by:

$$Mag([s_i, s_j]_{\gamma_1,\gamma_2}) = Mag(T(a,b,c,d)) = \frac{a + 5b + 5c + d}{12}.$$

To compare two ELICIT values, it suffices to compute the respective magnitudes. According to Labella et al. [5], the higher the magnitude, the larger the ELICIT value.

Furthermore, to measure the distance between two ELICIT values, Labella et al. [20] proposed using the geometric distance [27] between their respective associated TrFNs, $\delta : \mathcal{T} \times \mathcal{T} \to [0,1]$ defined by

$$\delta(T_1, T_2) = \frac{1}{4}(|a_1 - a_2| + |b_1 - b_2| + |c_1 - c_2| + |d_1 - d_2|)$$

where $T_1 \equiv (a_1, b_1, c_1, d_1)$, and $T_2 \equiv (a_2, b_2, c_2, d_2)$. Note that, even though the geometric distances were originally proposed as a parametric family [27], here we consider just the distance δ because it is defined in terms of absolute values rather than powers and this facilitates the linearization of the optimization models we aim at proposing in the following section.

The use of ELICIT information can be adapted in classical linguistic preference structures. In the following, we consider that DMs' opinions are modelled by using ELICIT Preference

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Relations (EPRs), i.e. matrices of ELICIT values whose associated TrFNs are additive reciprocal matrices of TrFNs.

Remark 5. *Let us define the set of matrices whose items are TrFN:*

$$\mathcal{M}_{n\times n}(\mathcal{T}) := \left\{(T^{ij})_{n\times n} \ : \ T^{ij} \in \mathcal{T} \ \forall\ 1 \leq i \leq n, 1 \leq j \leq n\right\}.$$

We will say that $T \in \mathcal{M}_{n\times n}(\mathcal{T})$ is additive reciprocal [28] if

$$\begin{aligned} T^{ij}[1] + T^{ji}[4] = 1 \\ T^{ij}[2] + T^{ji}[3] = 1 \\ T^{ij}[3] + T^{ji}[1] = 1 \\ T^{ij}[4] + T^{ji}[1] = 1 \end{aligned}$$

for any $i, j \in \{1, 2, ..., n\}$, where $T^{ij}[t]$, $t = 1, 2, 3, 4$ represents the t-th coordinate of the TrFN T^{ij}. Furthermore, we will use the notation $\mathcal{M}_{n\times n}(\mathcal{T})^*$ to denote the set of TrFN matrices that are additive reciprocal.

Therefore, EPRs allow the generalization of other commonly used preference structures based on linguistic pairwise comparison matrices that rely on triangular or TrFNs such as Linguistic Preference Relations (LPRs) [29] or Hesitant Fuzzy Linguistic Preference Relations (HFLPRs) [30]. For example, the HFLPR on the linguistic term set S given by:

$$\begin{pmatrix} E & W & \textit{Bt SW and E} \\ B & E & SB \\ \textit{Bt E and SB} & SW & E \end{pmatrix}$$

may be expressed as the EPR

$$\begin{pmatrix} (E,0)_0 & (W,0)_0 & [(SW,0),(E,0)]_{00} \\ (B,0)_0 & (E,0)_0 & (SB,0)_0 \\ [(E,0),(SB,0)]_{00} & (SW,0)_0 & (E,0)_0 \end{pmatrix}.$$

D. Linguistic Consensus Reaching Processes

In order to address GDM making problems, several rules have been proposed in the classical literature, such as the majority rule, the minority rule, unanimity, or the Borda count [31], [32]. However, even using these rules, some DMs may feel unsatisfied with the solution chosen by the group because their opinions have not been considered as much as they expected. This situation may especially be undesired in certain real-world problems which require a concrete level of agreement among the DMs.

To soften these disagreements, CRPs have been developed to guide DMs towards an agreed solution [7], [14], [20]. Usually in a CRP, a moderator or automatic moderator process suggests the DMs how to modify their opinions to lead the group to a greater agreement through different discussion rounds. Due to the increasing necessity of LiGDM, CRPs have also been adapted to manage linguistic information, emerging LiCRPs. The general scheme of a LiCRP follows the scheme of CRPs but includes the management of linguistic information and presents the following phases [2]

1) *Gathering preferences*. DMs' opinions are elicited by using linguistic information.
2) *Determining consensus degree*. In each round of discussion, the current consensus degree $\mu \in [0, 1]$ in the group is derived to evaluate the evolution of the consensus process.
3) *Consensus control.* After the discussion, the moderator computes if the group has reached a certain consensus threshold ($\mu_0 \in [0, 1]$). If so, the CRP stops and the exploitation process starts. If not, the discussion process continues for another round. In any case, if a predefined maximum number of rounds $MaxRounds \in \mathbb{N}$ is exceeded, the CRP stops.
4) *Recommendation process.* In case of the desired consensus threshold μ_0 is not achieved, those DMs whose opinions are furthest from the rest of the group are identified and modified if necessary.
5) *Exploitation*. After the desired consensus threshold is reached, the consensual modified opinions are aggregated in order to derive the group collective opinion.

Over the years, researchers have proposed many consensus models to support CRPs [15], [33]. For this reason, Palomares et al. [2] proposed a taxonomy to categorize them based on two characteristics related to consensus models:

- Type of recommendation process to modify DMs' opinions.
  - *Feedback mechanism*. The moderator asks the DMs if they want to change or not their preferences [7], [20].
  - *Automatic changes*. DMs' opinions are automatically modified according to a certain algorithm without asking the DMs [15], [33].
- Type of consensus measure to derive the consensus degree.
  - *Consensus measure of class 1*. The consensus degree among the DMs is computed by comparing the DMs' preferences with the collective opinion [15], [34], [35].
  - *Consensus measure of class 2*. The consensus degree among the DMs is computed by comparing the DMs' preferences with each other [15], [20], [36].

E. Comprehensive Minimum Cost Consensus

Ben-Arieh and Easton [33] proposed MCC models to study the cost of changing DMs' preferences in a consensus process. These models are automatic CRPs (without feedback mechanism) which minimize the cost of changing DMs' original preferences by assuring that a maximum absolute deviation ($\varepsilon \in]0, 1]$) between the individual assessments and the collective opinion is not surpassed. Formally, for the initial values of the preferences $(o_1, o_2, ..., o_m) \in \mathbb{R}$ and a cost vector $(c_1, c_2, ..., c_m) \in \mathbb{R}^+$ the proposed CRPs was defined by:

$$\begin{aligned} &\min \sum_{k=1}^{m} c_k |\overline{o}_k - o_k| \\ &s.t. \ |\overline{o}_k - \overline{o}| \leq \varepsilon, k = 1, 2, \ldots, m. \end{aligned} \tag{MCC}$$

where $(\overline{o}_1, \ldots, \overline{o}_m)$ are the adjusted opinions of the DMs, $\overline{o}$ represents the group collective opinion computed by using a weighted average operator and ε is the maximum acceptable distance of each DM to the collective opinion.

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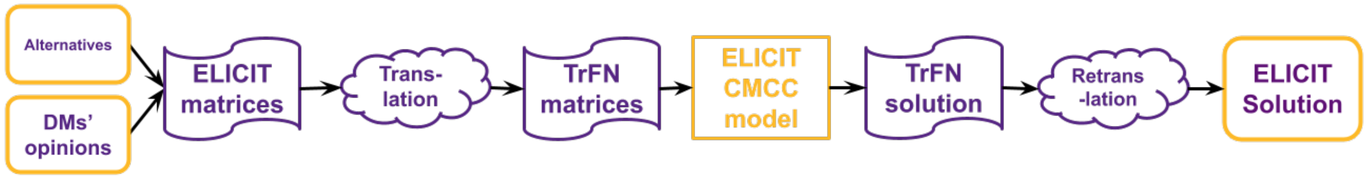

Fig. 6. ELICIT-CMCC scheme

Lately, Zhang et al. [37] studied the influence of the aggregation operator used to derive the collective opinion on the solution of the optimization problem. Consequently, they proposed a generalized version of MCC as follows

$$\begin{aligned} &\min \sum_{k=1}^{m} c_k|\overline{o}_k - o_k| \\ &s.t. \begin{cases} \overline{o} = F(\overline{o}_1, \ldots, \overline{o}_m) \\ |\overline{o}_k - \overline{o}| \leq \varepsilon, k = 1, 2, \ldots, m. \end{cases} \end{aligned} \quad \text{(MCC:AO)}$$

where $\overline{o}$ is now calculated using a different aggregation operator $F : \mathbb{R}^m \to \mathbb{R}$.

Even though these proposals allow translating a CRP situation into a mathematical programming problem, the constraint defined by ε is quite simple and does not guarantee that a certain consensus threshold $\mu_0 \in [0, 1[$ is achieved by the group. This drawback is solved by the CMCC models introduced by Labella et al. [14]. These models include the use of another constraint to control such a consensus threshold.

$$\begin{aligned} &\min \sum_{i=1}^{m} c_i|\overline{o}_i - o_i| \\ &s.t. \begin{cases} \overline{o} = F(\overline{o}_1, \ldots, \overline{o}_m) \\ |\overline{o}_i - \overline{o}| \leq \varepsilon, i = 1, 2, \ldots, m \\ consensus(\overline{o}_1, \ldots, \overline{o}_n) \geq \mu_0. \end{cases} \end{aligned} \quad \text{(CMCC)}$$

where $consensus(\cdot)$ represents the desired consensus measure.

III. ELICIT-CMCC MODELS FOR LIGDM

Keeping in mind that our main goal is to define an objective metric for measuring the performance of different LiCRPs, it is essential to compute some ideal values for the DMs' modified preferences. To obtain such optimal values, we follow the CMCC philosophy [14], which assumes that the best possible values for such modified opinions are those that, by satisfying the consensus threshold, are closest to their original preferences.

Even though MCC and CMCC models are focused on numerical assessments [14], [15], [33], [37], some proposals introduce extensions of the MCC models to a fuzzy environment. Nevertheless, the extended models either neglect the CW approach [38] or are not able to model hesitancy [16], [39]. Due to ELICIT information allows carrying out computations with linguistic expressions which model hesitancy without loss of information, this section extends the numeric CMCC models [14] to deal with ELICIT information and obtain an optimal adjustment consensus model for the CW approach.

The general scheme of this proposal is as follows: let us consider a LiGDM problem in which $E = \{e_1, e_2, ..., e_m\}$ DMs have to decide in a consensual way which alternative $X = \{x_1, x_2, ..., x_n\}$ is the best solution for a concrete problem. To do so, each DM provides a HFLPR [30], which is expressed in terms of ELICIT information as an EPR. The ELICIT information contained in these matrices is then expressed as the corresponding TrFNs by using the mapping ζ^{-1} (see Prop. 2). Such TrFNs are used as inputs for the ELICIT-CMCC model, whose output provides the agreed preferences which are closest to the original opinions given by the DMs. Finally, the modified preferences obtained of solving the optimization problem, represented by TrFNs, are retranslated into ELICIT information by using the mapping ζ. (see Fig. 6).

Let $O_1, O_2, ..., O_m \in \mathcal{M}_{n\times n}(\mathcal{T})^*$ be the additive reciprocal matrices of TrFNs corresponding to the translation via the mapping ζ^{-1} of DMs' original preferences expressed in form of EPRs and let $T_1, T_2, ..., T_m \in \mathcal{M}_{n\times n}(\mathcal{T})^*$ be the respective modified DMs' opinions. The cost function and the consensus measures for these values are modeled by using the distance δ revised in Section II-C. Consequently, the classical distance measure between DMs' opinions and the collective opinion $(0 < \varepsilon \leq 1)$ and the consensus threshold used in CMCC models $(0 \leq \mu_0 < 1)$ are adapted to the ELICIT-CMCC models as follows:

- *ELICIT-CMCC model considering a consensus measure of class 1:*

$$\begin{aligned} &\min_{T_1^{i,j},\ldots,T_m^{i,j}\in\mathcal{T}} \sum_{k=1}^{m}\sum_{i<j} c_k^{ij}\delta(T_k^{i,j}, O_k^{i,j}) \\ &s.t. \begin{cases} \overline{T}^{i,j} = A(T_1^{i,j}, T_2^{i,j}, \ldots, T_m^{i,j}), 1 \leq i < j \leq n, \\ \delta(T_k^{i,j}, \overline{T}^{i,j}) \leq \varepsilon, 1 \leq i < j \leq n, k = 1, 2, ..., m, \\ 1 - \frac{1}{N}\sum_{k=1}^{m}\sum_{i<j} w_k\delta(T_k^{i,j}, \overline{T}^{i,j}) \geq \mu_0, \end{cases} \end{aligned} \quad \text{(ELICIT-CMCC:1)}$$

- *ELICIT-CMCC model considering a consensus measure of class 2:*

$$\begin{aligned} &\min_{T_1^{i,j},\ldots,T_m^{i,j}\in\mathcal{T}} \sum_{k=1}^{m}\sum_{i<j} c_k^{ij}\delta(T_k^{i,j}, O_k^{i,j}) \\ &s.t. \begin{cases} \overline{T}^{i,j} = A(T_1^{i,j}, T_2^{i,j}, \ldots, T_m^{i,j}), 1 \leq i < j \leq n, \\ \delta(T_k^{i,j}, \overline{T}^{i,j}) \leq \varepsilon, 1 \leq i < j \leq n, k = 1, 2, ..., m, \\ 1 - \frac{1}{N}\sum_{k<l}\sum_{i<j} \frac{w_k+w_l}{m-1}\delta(T_k^{i,j}, T_l^{i,j}) \geq \mu_0, \end{cases} \end{aligned} \quad \text{(ELICIT-CMCC:2)}$$

where $c_k^{ij} \in [0, 1](\sum_{k=1}^{m}\sum_{i<j} c_k^{ij} = 1)$ models the cost of moving the DM e_k's preference of the alternative x_i over x_j, $w_1, w_2, ..., w_m \in [0, 1]$ $(\sum_{k=1}^{m} w_k = 1)$ are the weights for

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the DMs, $N = \frac{n(n-1)}{2}$, and $A : \mathcal{T}^m \to \mathcal{T}$ is a fuzzy weighted average operator.

Remark 6. *To adapt these linguistic models to return triangular fuzzy numbers, the condition $T_k^{ij}[1] \leq T_k^{ij}[2] \leq T_k^{ij}[3] \leq T_k^{ij}[4]$ should be replaced by $T_k^{ij}[1] \leq T_k^{ij}[2] = T_k^{ij}[3] \leq T_k^{ij}[4]$.*

It should be highlighted that both the inputs and the outputs of these models are represented by using linguistic information (EPRs), following a CW scheme which facilitates the understandability of the results by the involved DMs (RQ1).

Note that the resolution of the previous consensus models requires numerous variables and constraints of a nonlinear optimization problem, which may lead to a high time-consuming [7]. To overcome this drawback, we introduce below linearized versions of both ELICIT-CMCC:1 and ELICIT-CMCC:2. For the sake of clarity, the domains of the constraints in the models below use the notation $\mathcal{I}_a^b := [a, b] \cap \mathbb{N}$ for any pair $a < b \in \mathbb{N}$.

Theorem 1 (Linear ELICIT-CMCC:1). *Let $O_k^{ij}[t]$ be the t-th coordinate ($t = 1, 2, 3, 4$) of the TrFN O_k^{ij} which represents the initial rating about the alternative x_i over x_j provided by the DM e_k. In the same way, $T_k^{ij}[t]$ $t = 1, 2, 3, 4$ denotes the corresponding modified opinions. Then, the model ELICIT-CMCC:1 is linearized as follows:*

$$
\min_{T_k^{i,j}[t]\in[0,1]} \frac{1}{4}\sum_{k=1}^{m}\sum_{i<j} c_k^{ij} \sum_{t=1}^{4} v_k^{ij}[t]
$$
$$
s.t. \begin{cases}
0 \leq v_k^{ij}[t] \leq 1, k \in \mathcal{I}_1^m, i \in \mathcal{I}_1^{n-1}, j \in \mathcal{I}_{i+1}^n, t \in \mathcal{I}_1^4 \\
-1 \leq u_k^{ij}[t] \leq 1, k \in \mathcal{I}_1^m, i \in \mathcal{I}_1^{n-1}, j \in \mathcal{I}_{i+1}^n, t \in \mathcal{I}_1^4 \\
u_k^{ij}[t] = T_k^{ij}[t] - O_k^{ij}[t], k \in \mathcal{I}_1^m, i \in \mathcal{I}_1^{n-1}, j \in \mathcal{I}_{i+1}^n, t \in \mathcal{I}_1^4 \\
v_k^{ij}[t] \geq u_k^{ij}[t], k \in \mathcal{I}_1^m, i \in \mathcal{I}_1^{n-1}, j \in \mathcal{I}_{i+1}^n, t \in \mathcal{I}_1^4 \\
v_k^{ij}[t] \geq -u_k^{ij}[t], k \in \mathcal{I}_1^m, i \in \mathcal{I}_1^{n-1}, j \in \mathcal{I}_{i+1}^n, t \in \mathcal{I}_1^4 \\
\overline{T}^{ij}[t] = \sum_{k=1}^m \omega_k T_k^{ij}[t], i \in \mathcal{I}_1^{n-1}, j \in \mathcal{I}_{i+1}^n, t \in \mathcal{I}_1^4 \\
0 \leq z_k^{ij}[t] \leq 1, k \in \mathcal{I}_1^m, i \in \mathcal{I}_1^{n-1}, j \in \mathcal{I}_{i+1}^n, t \in \mathcal{I}_1^4 \\
-1 \leq y_k^{ij}[t] \leq 1, k \in \mathcal{I}_1^m, i \in \mathcal{I}_1^{n-1}, j \in \mathcal{I}_{i+1}^n, t \in \mathcal{I}_1^4 \\
y_k^{ij}[t] = T_k^{ij}[t] - \overline{T}_k^{ij}[t], k \in \mathcal{I}_1^m, i \in \mathcal{I}_1^{n-1}, j \in \mathcal{I}_{i+1}^n, t \in \mathcal{I}_1^4 \\
z_k^{ij}[t] \geq y_k^{ij}[t], k \in \mathcal{I}_1^m, i \in \mathcal{I}_1^{n-1}, j \in \mathcal{I}_{i+1}^n, t \in \mathcal{I}_1^4 \\
z_k^{ij}[t] \geq -y_k^{ij}[t], k \in \mathcal{I}_1^m, i \in \mathcal{I}_1^{n-1}, j \in \mathcal{I}_{i+1}^n, t \in \mathcal{I}_1^4 \\
T_k^{ij}[1] \leq T_k^{ij}[2] \leq T_k^{ij}[3] \leq T_k^{ij}[4], k \in \mathcal{I}_1^m, i \in \mathcal{I}_1^{n-1}, j \in \mathcal{I}_{i+1}^n \\
\frac{1}{4}\sum_{t=1}^4 z_k^{ij}[t] \leq \varepsilon, k \in \mathcal{I}_1^m, i \in \mathcal{I}_1^{n-1}, j \in \mathcal{I}_{i+1}^n \\
1 - \frac{1}{4N}\sum_{k=1}^m \sum_{i<j} w_k \sum_{t=1}^4 z_k^{ij}[t] \geq \mu_0
\end{cases} .
\tag{L-ELICIT-CMCC:1}
$$

where $c_k^{ij} \in [0,1] (\sum_{k=1}^m \sum_{i<j} c_k^{ij} = 1)$ model the cost of moving the DM e_k's preference of the alternative x_i over x_j, $w_1, w_2, ..., w_m \in [0,1]$ ($\sum_{k=1}^m w_k = 1$) are the weights for the DMs, $N = \frac{n(n-1)}{2}$ and $\omega_1, \omega_2, ..., \omega_m \in [0,1]$ ($\sum_{k=1}^m \omega_k = 1$) are the weights for a fuzzy weighted average operator.

Theorem 2 (Linear ELICIT-CMCC:2). *Let $O_k^{ij}[t]$ be the t-th coordinate ($t = 1, 2, 3, 4$) of the TrFN O_k^{ij} which represents the initial rating about the alternative x_i over x_j provided by the DM e_k. In the same way, $T_k^{ij}[t]$ $t = 1, 2, 3, 4$ denotes the for the corresponding modified opinions. Then, the linearized version of the model ELICIT-CMCC:2 is given by:*

$$
\min_{T_k^{i,j}[t]\in[0,1]} \frac{1}{4}\sum_{k=1}^{m}\sum_{i<j} c_k^{ij} \sum_{t=1}^{4} v_k^{ij}[t]
$$
$$
s.t. \begin{cases}
0 \leq v_k^{ij}[t] \leq 1, k \in \mathcal{I}_1^m, i \in \mathcal{I}_1^{n-1}, j \in \mathcal{I}_{i+1}^n, t \in \mathcal{I}_1^4 \\
-1 \leq u_k^{ij}[t] \leq 1, k \in \mathcal{I}_1^m, i \in \mathcal{I}_1^{n-1}, j \in \mathcal{I}_{i+1}^n, t \in \mathcal{I}_1^4 \\
u_k^{ij}[t] = T_k^{ij}[t] - O_k^{ij}[t], k \in \mathcal{I}_1^m, i \in \mathcal{I}_1^{n-1}, j \in \mathcal{I}_{i+1}^n, t \in \mathcal{I}_1^4 \\
v_k^{ij}[t] \geq u_k^{ij}[t], k \in \mathcal{I}_1^m, i \in \mathcal{I}_1^{n-1}, j \in \mathcal{I}_{i+1}^n, t \in \mathcal{I}_1^4 \\
v_k^{ij}[t] \geq -u_k^{ij}[t], k \in \mathcal{I}_1^m, i \in \mathcal{I}_1^{n-1}, j \in \mathcal{I}_{i+1}^n, t \in \mathcal{I}_1^4 \\
\overline{T}^{ij}[t] = \sum_{k=1}^m \omega_k T_k^{ij}[t] \\
0 \leq z_k^{ij}[t] \leq 1, i \in \mathcal{I}_1^{n-1}, j \in \mathcal{I}_{i+1}^n, t \in \mathcal{I}_1^4 \\
-1 \leq y_k^{ij}[t] \leq 1, k \in \mathcal{I}_1^m, i \in \mathcal{I}_1^{n-1}, j \in \mathcal{I}_{i+1}^n, t \in \mathcal{I}_1^4 \\
y_k^{ij}[t] = T_k^{ij}[t] - \overline{T}_k^{ij}[t], k \in \mathcal{I}_1^m, i \in \mathcal{I}_1^{n-1}, j \in \mathcal{I}_{i+1}^n, t \in \mathcal{I}_1^4 \\
z_k^{ij}[t] \geq y_k^{ij}[t], k \in \mathcal{I}_1^m, i \in \mathcal{I}_1^{n-1}, j \in \mathcal{I}_{i+1}^n, t \in \mathcal{I}_1^4 \\
z_k^{ij}[t] \geq -y_k^{ij}[t], k \in \mathcal{I}_1^m, i \in \mathcal{I}_1^{n-1}, j \in \mathcal{I}_{i+1}^n, t \in \mathcal{I}_1^4 \\
T_k^{ij}[1] \leq T_k^{ij}[2] \leq T_k^{ij}[3] \leq T_k^{ij}[4], k \in \mathcal{I}_1^m, i \in \mathcal{I}_1^{n-1}, j \in \mathcal{I}_{i+1}^n \\
\frac{1}{4}\sum_{t=1}^4 z_k^{ij}[t] \leq \varepsilon, k \in \mathcal{I}_1^m, i \in \mathcal{I}_1^{n-1}, j \in \mathcal{I}_{i+1}^n \\
0 \leq q_{kl}^{ij}[t] \leq 1, k \in \mathcal{I}_1^{m-1}, l \in \mathcal{I}_{k+1}^m i \in \mathcal{I}_1^{n-1}, j \in \mathcal{I}_{i+1}^n, t \in \mathcal{I}_1^4 \\
-1 \leq p_{kl}^{ij}[t] \leq 1, k \in \mathcal{I}_1^{m-1}, l \in \mathcal{I}_{k+1}^m i \in \mathcal{I}_1^{n-1}, j \in \mathcal{I}_{i+1}^n, t \in \mathcal{I}_1^4 \\
p_{kl}^{ij}[t] = T_k^{ij}[t] - T_l^{ij}[t], k \in \mathcal{I}_1^{m-1}, l \in \mathcal{I}_{k+1}^m i \in \mathcal{I}_1^{n-1}, j \in \mathcal{I}_{i+1}^n, t \in \mathcal{I}_1^4 \\
q_{kl}^{ij}[t] \geq p_{kl}^{ij}[t], k \in \mathcal{I}_1^{m-1}, l \in \mathcal{I}_{k+1}^m i \in \mathcal{I}_1^{n-1}, j \in \mathcal{I}_{i+1}^n, t \in \mathcal{I}_1^4 \\
q_{kl}^{ij}[t] \geq -p_{kl}^{ij}[t], k \in \mathcal{I}_1^{m-1}, l \in \mathcal{I}_{k+1}^m i \in \mathcal{I}_1^{n-1}, j \in \mathcal{I}_{i+1}^n, t \in \mathcal{I}_1^4 \\
1 - \frac{1}{4N}\sum_{k=1}^m \sum_{i<j} \frac{w_k + w_l}{m-1} \sum_{t=1}^4 q_{kl}^{ij}[t] \geq \mu_0,
\end{cases}
\tag{L-ELICIT-CMCC:2}
$$

where $c_k^{ij} \in [0,1] (\sum_{k=1}^m \sum_{i<j} c_k^{ij} = 1)$ model the cost of moving the DM e_k's preference of the alternative x_i over x_j, $w_1, w_2, ..., w_m \in [0,1]$ ($\sum_{k=1}^m w_k = 1$) are the weights for the DMs, $N = \frac{n(n-1)}{2}$ and $\omega_1, \omega_2, ..., \omega_m \in [0,1]$ ($\sum_{k=1}^m \omega_k = 1$) are the weights for a fuzzy weighted average operator.

Proof. The proof of these results are provided in Appendix A. □

This linear formulation of the ELICIT-CMCC models allows to considerably accelerate the resolution of the optimization problem and improve the accuracy of the results provided by computational solvers. Indeed, the linear formulation also allows applying these models in large-scale GDM problems [1], [40], namely, decision situations in which hundreds or thousands of DMs may take part. In this regard, we have tested the performance of the proposal in such contexts under randomly-generated initial preferences. The simulations have considered $n = 4$, $\mu_0 = 0.8$ and $\varepsilon = 0.2$ and have been carried out by using the solver *Clp* for the programming language Julia 1.6 [41] on the cloud service Google Colaboratory [42] (2.20GHz Intel(R) Xeon(R) CPU and 13 GB RAM). These simulations have shown that the model ELICIT-CMCC:1 is able to deal with problems involving hundreds of DMs in a few seconds and just needs around four minutes to solve problems with 2000 DMs. However, since the volume of constraints and variables required to linearize ELICIT-CMCC:2 is much higher, the latter requires around 26 minutes to solve problems in which 200 DMs are considered.

Remark 7. *Note that, according to the literature review carried out by García-Zamora et al. in [1], most of the existing large-scale CRPs are evaluated by using GDM problems involving just 20 or 50 DMs.*

IV. A Linguistic Cost Metric based on ELICIT-CMCC

The high prevalence of LiGDM problems in society has attracted the attention of researchers, who have proposed many LiCRPs based on the fuzzy linguistic approach [7], [20]. However, this large number of proposals implies a considerable problematic related to choose the most suitable consensus model for solving a certain LiGDM problem. Even though several authors carry out a comparative analysis with other proposals in order to show their advantages, the lack of objective metrics prevents from categorically claiming that one model is better than another. In addition, this absence of metrics harms the research in the area, since there is no filter to evaluate the novel CRPs from a performance point of view [1].

Hereafter, it is introduced a linguistic metric based on the ELICIT-CMCC models presented in the previous section. This linguistic metric aims at measuring the performance of those LiCRPs which model the linguistic information by means of linguistic variables with a triangular or trapezoidal membership function representation because they can be easily written in terms of ELICIT information. As in the previous section, here it is considered a LiGDM problem in which m DMs want to reach a consensus about which alternative, from a set of n, is the most suitable one with a consensus threshold $\mu_0 \in [0, 1[$.

To do so, their judgements, which are elicited by using linguistic expressions and pairwise comparisons, are first translated into TrFNs. If two TrFN matrices T, T' which are additive reciprocal are given, the distance between them is computed by using the function $\nu : \mathcal{M}_{n\times n}(\mathcal{T})^* \times \mathcal{M}_{n\times n}(\mathcal{T})^* \to [0, 1]$ defined by:

$$\nu(T, T') = \frac{2}{n(n-1)} \sum_{i<j} \delta(T^{ij}, T'^{ij}) =$$

$$\frac{2}{n(n-1)} \sum_{i<j} \frac{1}{4} \sum_{t=1}^{4} |T^{ij}[t] - T'^{ij}[t]| \ \forall\ (T, T') \in \mathcal{M}_{n\times n}(\mathcal{T})^* \times \mathcal{M}_{n\times n}(\mathcal{T})^*$$

where δ is the geometric distance between TrFNs defined in Section II-C and $T^{ij}[t], T'^{ij}[t]$ $t = 1, 2, 3, 4$ denote the t-th coordinates of the TrFNs T^{ij} and T'^{ij} respectively.

Let $O = \{O_1, O_2, ..., O_m\} \subset \mathcal{M}_{n\times n}(\mathcal{T})^*$ be the TrFN matrices corresponding to the initial values of DMs' preferences for the aforementioned LiGDM problem, and let $T = \left\{\hat{T}_1, \hat{T}_2, \dots, \hat{T}_m\right\} \subset \mathcal{M}_{n\times n}(\mathcal{T})^*$ be the set of modified agreed preferences obtained as output from a certain LiCRP. In the same way, the set $T^0 = \left\{T_1^0, \dots, T_m^0\right\} \in \mathcal{M}_{n\times n}(\mathcal{T})^*$ denotes the optimal solution obtained for the consensus threshold μ_0 by using either the model ELICIT-CMCC:1 if the LiCRP uses a consensus measure of class 1 or ELICIT-CMCC:2 if the LiCRP uses a consensus measure of class 2. From these TrFN matrices, the mean distance between the outputs of the corresponding consensus models and the original preferences are computed:

$$d := \frac{1}{m} \sum_{k=1}^{m} \nu(T_k, O_k) \in [0, 1]$$

$$d_0 := \frac{1}{m} \sum_{k=1}^{m} \nu(T_k^0, O_k) \in [0, 1]$$

Note that these values strongly depend on the original values of the DMs' preferences, but a such dependency is not reflected in the notation for the sake of simplicity.

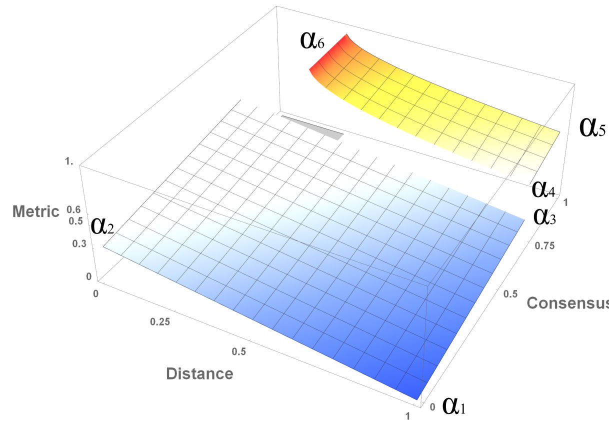

Fig. 7. Sketch of the graph of $\Phi_{0.25,0.75}$.

To analyze the performance of the LiCRP, the distance d computed from the corresponding modified preferences is compared to the distance d_0 computed by using the ELICIT-CMCC model, which provides the preferences that require the lowest changes to reach the consensus threshold μ_0 (when $\varepsilon = 1$).

To compare these values, we use the metric $\Phi_{d_0,\mu_0} : [0, 1] \times [0, 1] \to [-1, \alpha_6]$ given by:

$$\Phi_{d_0,\mu_0}(x, y) = \begin{cases} (\alpha_1 - \alpha_2)x + \dfrac{\alpha_3 - \alpha_2}{\mu_0} y + \alpha_2 & 0 \le y < \mu_0 \\ -1 & \begin{cases} \mu_0 \le y \le 1 \\ 0 \le x < d_0 \end{cases} \\ \begin{array}{l} \left(\frac{x-d_0}{1-d_0}\right)^{\frac{1}{3}}(\alpha_4 - \alpha_6 + \\ \left(\frac{y-\mu_0}{1-\mu_0}\right)(\alpha_5 - \alpha_4)) + \alpha_6 \end{array} & \begin{cases} \mu_0 \le y \le 1 \\ d_0 \le x \le 1 \end{cases} \end{cases}$$

$\forall\ x, y \in [0, 1]$, where $0 \le \alpha_1 < \alpha_2 < \alpha_3 \le \alpha_4 < \alpha_5 < \alpha_6$ are some parameters to configure the scale. In this regard, we propose the use of the default values $\alpha_1 = 0.0, \alpha_2 = 0.3, \alpha_3 = 0.5, \alpha_4 = 0.5, \alpha_5 = 0.6, \alpha_6 = 1.0$, which guarantee that the function Φ_{d_0,μ_0} is valuated in the interval $[0, 1]$. For such values, the graph shown in Fig. 7 is obtained when the distance between the minimal solution to the ELICIT-CMCC optimization problem and the original preferences is $d_0 = 0.25$ and the consensus threshold is $\mu_0 = 0.75$.

Note that this metric provides a numeric rating in a $[0, 1]$ scale which is higher when the performance of the analyzed LiCRP is better. Consequently, to objectively evaluate the performance of a LiCRP in a certain LiGDM problem, it suffices to compute the value of $\Phi_{d_0,\mu_0}(d, \mu)$, where d is the distance between the original preferences and the modified opinions provided as output of the evaluated LiCRP and μ is the consensus degree of such modified preferences.

Remark 8. *It should be highlighted that changing the values of the parameters $\alpha_1, \alpha_2, ..., \alpha_6$ imply a change of the scale in which the marks of the CRPs are given, but the better CRPs will still receive the higher marks.*

Let us analyze the geometrical interpretation of the value $\Phi_{d_0,\mu_0}(d, \mu)$:

- $0 \le \mu < \mu_0$. In this case, the consensus degree μ obtained by the LiCRP is worse than the consensus threshold μ_0.

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TABLE I
LABELLA ET AL. [20] SIMULATIONS RESULTS FOR $\mu_0 = 0.8$.

Simulations	S1	S2	S3	S4	S5	S6	S7	S8	S9	S10	Average
d_0	0.07	0.09	0.08	0.04	0.09	0.09	0.1	0.09	0.08	0.06	0.08
d	0.09	0.13	0.09	0.06	0.13	0.12	0.16	0.15	0.1	0.07	0.11
μ	0.81	0.83	0.8	0.81	0.83	0.8	0.83	0.82	0.81	0.8	0.81
$\Phi_{d_0,\mu_0}(d,\mu)$	0.855	0.831	0.876	0.875	0.841	0.852	0.811	0.803	0.863	0.886	0.849

TABLE II
RODRÍGUEZ ET AL. [7] SIMULATIONS RESULTS FOR $\mu_0 = 0.8$.

Simulations	S1	S2	S3	S4	S5	S6	S7	S8	S9	S10	Average
d_0	0.06	0.12	0.04	0.1	0.06	0.09	0.04	0.06	0.11	0.05	0.07
d	0.14	0.16	0.07	0.19	0.11	0.15	0.07	0.15	0.15	0.13	0.13
μ	0.85	0.81	0.82	0.82	0.82	0.82	0.82	0.88	0.8	0.86	0.82
$\Phi_{d_0,\mu_0}(d,\mu)$	0.788	0.815	0.846	0.77	0.813	0.802	0.848	0.79	0.816	0.795	0.808

In this case, the worst scenario is $\Phi_{d_0,\mu_0}(1,0) = \alpha_1$ and the best ones are those close to the pair $(0,\mu_0)$, which receives a value close to α_3. α_2 is the value assigned to the pairs close to $(0,0)$.

- $\mu_0 \leq \mu \leq 1$. In the case in which the LiCRP reaches the consensus threshold, it is necessary to differentiate two scenarios:
  - $0 \leq d < d_0$. This case is unfeasible in practice because to achieve the consensus threshold μ_0 the minimum distance required is d_0. Therefore, the metric assigns -1 to the values in this region.
  - $d_0 \leq d \leq 1$. In this case, the LiCRP achieves the consensus threshold μ_0, but the distance d between the modified preferences and the original ones may not be close to the optimal distance d_0. The best pairs are those in which the distance d is equal to the optimal, and therefore the metric receives the value α_6. If the LiCRP reaches the consensus threshold but makes unnecessary changes (d close to 1), the metric returns values close to α_4. The value α_5 is obtained when the distance is maximal, but the consensus level is close to 1.

The metric Φ_{d_0,μ_0} allows testing the performance of a model by comparing it with the optimal modified preferences obtained from the ELICIT-CMCC models (RQ2). However, the value of $\Phi^{\alpha}_{d_0,\mu_0}(d,\mu)$ highly depends on the original values of the preferences given by the DMs $O = \{O_1, O_2, ..., O_m\}$. To provide fair comparisons, the value of this metric should be computed for different LiGDM problems. To do that, the consensus model should be tested under several contexts $O^1, O^2, ..., O^r$ in order to better evaluate its performance, thus obtaining an average value $\Phi_{\mu_0} := \frac{1}{r}\sum_{s=1}^{r} \Phi_{d_0^r,\mu_0}(d^s,\mu^s)$, where d_0^r is the minimum value of the cost function for the initial preferences O^s, d_0^r is the value of the cost function for the preferences modified by the LiCRP and μ^s the corresponding consensus degree. Therefore, we propose solving the same LiGDM problem for several randomized preferences and computing the average value of the metric.

For instance, this metric has been used to evaluate the performance of two LiCRPs: the consensus model for ELICIT information introduced by Labella et al. [20] and the model proposed by Rodriguez et al. [7] for large scale dealing with Comparative Linguistic Expressions (CLEs). To do so, 10 simulations with random preferences have been carried out in both models. In each simulation, five DMs have to decide which alternative within a collection of four possible choices is the best one from a consensual point of view. The consensus threshold has been established in $\mu_0 = 0.8$ and the maximum number of allowed rounds is $MaxRounds = 5$.

The results of both models are respectively shown in Tables I and II. Whereas the average value of our metric for the Labella et al. model is 0.849, the Rodríguez et al. model obtained an average mark of 0.808. Although both models usually reach the consensus threshold $\mu_0 = 0.8$, the Rodríguez et al. model has shown a slightly worse performance because it changes DMs' initial opinions more than Labella et al. model, i.e. the average value $d - d_0$ is larger for the Rodríguez et al. model.

Finally, in order to perform a comparative analysis of this metric with other proposals, a search in Web of Science of the topics "metric" and "Consensus Reaching Process" reveals that there is only one proper related paper proposed by Labella et al. [14]. Even though, such work also considers as input the cost of modifying experts' opinions, the metric here proposed includes the following novelties regarding the one in [14]:

- The proposed metric in this paper is capable to deal with flexible comparative linguistic information, which allows applying the metric in LiCRPs that require the modeling of decision makers' hesitancy with expressions closer to their way of thinking.
- It can be used to rate consensus models for large-scale LiGDM problems due to the linearization of the ELICIT-CMCC model.
- Whereas the Labella et al. metric [14] assigns the same value to models with similar cost, the proposed metric assigns the metric value according to not only the cost but also the consensus degree reached by the consensus model. Consequently, the mathematical definition of the proposed metric is completely different to the one given in [14] (see Fig. 7) to ensure that the models are evaluated according to different scenarios that are determined by the consensus threshold and the minimum feasible cost.
- The metric proposed in [14] is valuated in [-1,1], where 0 is the best scenario in terms of cost and 1 and -1 are

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bad scenarios with different meanings. On the contrary, the metric here introduced returns a value in a 0-1 scale that increases according to the quality of the evaluated model. This new metric, even though it is formally more complex, simplifies the comparison process because the higher the value of the metric, the better the quality of the model.

V. Applying the LiCRP metric to LiGDM problems

Here, the performance of both the ELICIT-CMCC models and the proposed linguistic cost metric are shown. First, in Section V-A, an illustrative LiGDM problem is introduced. Afterwards, Section V-B solves such a LiGDM problem by using the CW ELICIT-CMCC:2 model. Finally, in Section V-C, two LiCRPs proposed in the literature [7], [20] are used to solve the same LiGDM problem in order to compare their performances through the linguistic cost metric. Since the purpose of this section is not solving a real-world problem, but showing how to use our proposals, we consider a toy problem with 5 DMs to simplify the process.

A. Illustrative LiGDM problem description

The LiGDM problem we aim at solving consists of a group of five friends $m = 5$ who want to decide in a consensual way (to avoid none of them feel unsatisfied with the chosen alternative) which movie franchise is the most preferred by the group to do a marathon. The possible alternatives are x_1 : Avengers, x_2 : Harry Potter, x_3 : Star Wars and x_4 : The Lord of the Rings. In order to facilitate the decision process, they are asked to provide linguistic assessments by comparing the alternatives to each other. Since they may doubt in their preferences, we use HFLPRs to model their opinions. The linguistic expression domain is as follows.

$$S = \{\text{Much Worse (MW)}, \text{Worse (W)}, \text{Slightly Worse (SW)} \\ \text{Equal (E)}, \text{Slightly Better (SB)}, \text{Better (B)}, \text{Much Better (MB)}\}.$$

The initial values provided by the three DMs are compiled in Appendix B.A.

B. Solving the LiGDM problem with ELICIT-CMCC models

Here, the resolution of the illustrative LiGDM problem using the ELICIT-CMCC:2 model is carried out. First, the HFLPRs provided by the DMs (see Appendix B.A) are rewritten as EPRs (Appendix B.B) and then expressed as TrFNs by using the mapping ζ^{-1} (Appendix B.C).

To obtain the results of the linearized optimization problem, we have used the programming language Julia [41], concretely the package *Clp* which allows solving linear optimization problems. For a consensus threshold established as $\mu_0 = 0.8$ and a maximal distance between DMs and the collective opinion $\varepsilon = 0.2$, the optimal agreed preferences obtained for the ELICIT-CMCC:2 model are shown in Appendix B.D, and their translation into ELICIT values are in B.E.

From the collective values, the ELICIT expressions corresponding to the dominance degree [43], [44] of each alternative over the others are computed by using the fuzzy weighted average. For each one of such dominances, the respective value of its magnitude [5] (see Section II-C) is computed in order to determine the ranking of the alternatives. Both the dominances and their magnitudes are summarized in Table III.

TABLE III
Dominances and magnitudes for determining the ranking of each alternative

Alternative	ELICIT expression	Magnitude
x_1	$Bt\ (SW, -0.33)^{0.002}\ and\ (SW, 0.01)^{-0.002}$	0.30620
x_2	$Bt\ (E, 0.28)^{0.019}\ and\ (SB, 0.17)^{-0.032}$	0.61935
x_3	$Bt\ (SB, -0.34)^{0.003}\ and\ (B, -0.13)^{-0.053}$	0.70694
x_4	$Bt\ (SW, -0.15)^{0.025}\ and\ (E, -0.5)^{0.035}$	0.36750

Therefore, the ranking of the alternatives is: $x_3 \succ x_2 \succ x_4 \succ x_1$. In other words, choosing the alternative x_3 : Star Wars is the best option from a consensual point of view, which requires the lowest cost.

C. Comparative Analysis

This section is devoted to compare the performance of two different LiCRPs to the ELICIT-CMCC approach when facing the problem described in the previous section. To do so, several aspects of these models are analyzed, such as the value of the metric Φ_{μ_0} or the number of rounds required to reach the desired consensus under different scenarios.

The selected consensus models for this comparative analysis are the consensus model for ELICIT information introduced by Labella et al. [20] and the consensus model that deals with CLEs proposed by Rodríguez et al. [7]. Both proposals have solved the problem previously introduced under two different scenarios:

1) Scenario 1: $\mu_0 = 0.8$ and $MaxRounds = 5$ (Table IV).
2) Scenario 2: $\mu_0 = 0.9$ and $MaxRounds = 5$ (Table V).

In addition, the value for the parameter ε used in the ELICIT-CMCC:2 is set as $\varepsilon = 0.2$. This model is also evaluated under the two aforementioned consensus situations.

In the first scenario, the Labella et al. model [20] achieves a consensus degree $\mu = 0.81$ in 1 discussion round, and the Rodríguez et al. [7] model achieves a consensus degree of $\mu = 0.85$ in 2 discussion rounds. Regarding the maximal distance between DMs and collective opinion, note that the condition $\varepsilon \leq 0.2$ guarantees such a maximal distance in ELICIT-CMCC:2 (see Table IV). However, such distance is much higher in both Labella et al. and Rodríguez et al. models, which can be appreciated in Fig. 8.

In the second scenario, the consensus degree obtained by Labella et al. model is $\mu = 0.91$ in 5 rounds and the obtained by Rodríguez et al. model is $\mu = 0.92$ in 3 rounds. In this scenario, the distance between modified preferences and the collective opinion is lower than before for the Labella et al. model (0.12), but still higher than $\varepsilon = 0.2$ for the Rodríguez et al. (see Table V and Fig. 8).

As expected, the costs obtained in ELICIT-CMCC:2 (0.06 and 0.12) are lower than the costs of both Labella et al. [20] (0.08 and 0.15) and Rodríguez et al. [7] (0.14 and 0.17) models. In this regard, the ELICIT:CMCC:2 stands out because of its efficiency.

TABLE IV
COMPARATIVE RESULTS OF LABELLA ET AL. [20], RODRÍGUEZ ET AL. [7] AND ELICIT-CMCC:2 FOR $\mu_0 = 0.8$ AND $d_0 = 0.06$

Consensus model	Consensus degree (μ)	Distance to collective (ε)	Cost	Rounds Required	Metric (Φ_{d_0,μ_0})
ELICIT-CMCC:2	0.8	0.2	0.06	-	0.939
Labella et al. [20]	0.81	0.44	0.08	1	0.875
Rodríguez et al. [7]	0.85	0.4	0.14	2	0.801

TABLE V
COMPARATIVE RESULTS OF LABELLA ET AL. [20], RODRÍGUEZ ET AL. [7] AND ELICIT-CMCC:2 FOR $\mu_0 = 0.9$ AND $d_0 = 0.12$

Consensus model	Consensus degree (μ)	Distance to collective (ε)	Cost	Rounds Required	Metric (Φ_{d_0,μ_0})
ELICIT-CMCC:2	0.9	0.2	0.12	-	1.0
Labella et al. [20]	0.91	0.12	0.15	5	0.843
Rodríguez et al. [7]	0.92	0.37	0.17	3	0.815

Regarding the marks provided by our metric for these three approaches, in the $\mu_0 = 0.8$ scenario, Labella et al. CRP gets a score of 0.875 whereas Rodríguez et al. proposal obtains a score equal to 0.801. The performance of both models to solve this specific LiGDM problem in terms of "extra cost" could be considered "good" but far from the optimal modified preferences provided by the ELICIT-CMCC model, whose mark is 0.939.

In the $\mu_0 = 0.9$ case, the Labella et al. model is still better than the Rodríguez et al. approach, but their marks are closer than in the previous scenario (0.843 and 0.815, respectively). Meanwhile, the ELICIT-CMCC:2 proposal gets an approximate mark of 1, which means that, for these values of the initial preferences, the solutions for the optimization problems corresponding to $\varepsilon = 1$, which provides the ideal modified preferences used in the metric, and $\varepsilon = 0.2$, which is the value used to derive the agreed solution in this illustrative example, are very close.

To sum up, the marks provided by the cost metric are quite simple and intuitive and allow evaluating properly the performance of LiCRPs, because it compares the output provided by the LiCRPs with the one provided by the ELICIT-CMCC model in terms of cost and consensus degree achieved.

VI. CONCLUSIONS

This paper proposes a cost metric for LiCRPs, which takes into account both the cost of modifying the original DMs' preferences and the final consensus degree obtained by the group.

The definition of such a metric relies on ELICIT-CMCC models, a novel extension of CMCC models to manage linguistic information. The use of ELICIT information guarantees the manipulation of linguistic values without losing information in the process and assuring the interpretability of the results. Concretely, the output obtained from ELICIT-CMCC models present the following properties:

- It is expressed in a linguistic domain.
- It minimizes the cost of moving DMs' preferences.
- It guarantees a maximal absolute deviation ε between the modified opinions and the collective one.
- The obtained consensus degree is equal or greater than a predefined consensus threshold μ_0.

In order to improve the computational performance of these ELICIT-CMCC models, we have also proposed the corresponding linearized version, which additionally grants more precise solutions when it is implemented in a computer solver. Furthermore, the performance of these linear ELICIT-CMCC in GDM problems involving hundreds or thousands of DMs have been briefly discussed.

The inherent features of the previous models have also allowed us to address one of the most recurrent limitations in the LiCRPs literature, the lack of metrics capable to evaluate the performance of these processes. In this sense, the proposed linguistic cost metric compares the optimal cost necessary to reach the desired consensus threshold, which is obtained from solving an ELICIT-CMCC model (ELICIT-CMCC:1 or ELICIT-CMCC:2), with the changes made by the LiCRP. In addition, if the resulting consensus degree after the LiCRP is lower than the desired consensus threshold, the metric will rate such LiCRP with a low mark. This metric has also been used to evaluate the performance of two linguistic consensus models already defined in the specialized literature [7], [20] to show its implementation in practice.

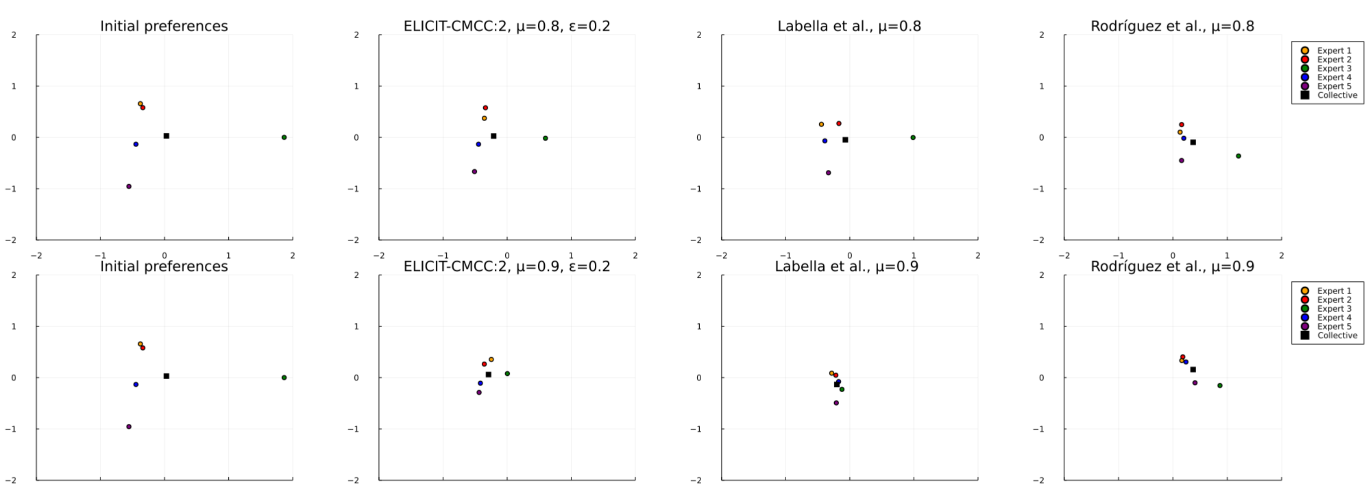

Fig. 8. Graphical visualization regarding the DMs' preferences in the different simulations and consensus models.

Finally, we have developed a comparative analysis that reveals that ELICIT-CMCC models are much better in terms of efficiency (lower cost and better values for μ and ε) than two LiCRPs [7], [20].

To summarize, the main contributions of this paper are:

- Linguistic CMCC models for LiGDM based on ELICIT information which follow a CW approach.
- Linearization of the ELICIT-CMCC models to improve their performance and expand their use to LiCRP with many DMs.
- A linguistic cost metric to evaluate LiCRPs.

As future works, we will analyze some formal aspects such as the use of other linguistic preference structures to propose ELICIT-CMCC, instead of pairwise comparison matrices, such as utility linguistic vectors. Furthermore, it will be studied the impact of using different aggregation operators to compute the collective opinion to improve the scope of ELICIT-CMCC, as well as the use of different weighting mechanisms to determine experts' importance [45]. In addition, the influence of the parameters μ_0 and ε in the resolution of the GDM problem should be discussed. From the application point of view, ELICIT-CMCC will be used to solve real-world decision problems with hundreds or thousands of DMs. Last but not least, the proposed metric must be applied to the evaluation of novel proposed LiCRPs to draw conclusions about their capability.

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