# OWG Operators that Maintain Reciprocity Property in the Aggregation of Multiplicative Preference Relations

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## Resumen

In multicriteria decision making (MCDM) problems, multiplicative preference relations are widely used to represent assessments on the set of alternatives with respect to all criteria. Multiplicative preference relations are usually assumed to be reciprocal. In [6] we present a decision model to solve a MCDM problem under multiplicative preference relations based on two steps: i) aggregation of the individual relations in a collective one and ii) explotation of the collective relation. However, it is well known that reciprocity is not generally preserved after aggregation is carried out.

In this paper, we give conditions for maintaining reciprocity in the aggregation of reciprocal multiplicative preference relations using an OWG operator guided by a relative linguistic quantifier.

**Key words:** Multiplicative preference relations, aggregation, reciprocity.

## 1 INTRODUCTION

In any multicriteria decision making (MCDM) process the final solution must be obtained from the synthesis (aggregation) of performance degrees of the *majority* of criteria. The majority is traditionally defined as a threshold number of elements. This concept is not always included in the MCDM process. The *fuzzy logic* provides one way to model it.

*Fuzzy majority* is a soft majority concept expressed by a *fuzzy quantifier* [13], which is manipulated via a fuzzy logic calculus of linguistically quantified propositions. Therefore, using fuzzy majority guided aggregation operators we can incorporate the concept of majority in the computation of the solution. These operators reflect the fuzzy majority calculating their weights by means of a fuzzy quantifier, as for example, the *Ordered Weighted Averaging (OWA)* operator [12].

The MCDM problem when assessments on the alternatives are expressed using multiplicative preference relations has been solved by Saaty using the decision AHP, which obtains the set of solution alternatives by means of the eigenvector method [11]. However, this decision process is not guided by the concept of majority [7, 8, 9]. As shown in [1, 2], the proper aggregation operator of ratio-scale measurements is not the *arithmetic mean* but the *geometric mean*. However, this operator does not allow to incorporate the concept of fuzzy majority in the decision processes. We could use the OWA operator, but as it presents a similar behavior to the arithmetic mean this is not possible.

In [3] we define the *Ordered Weighted Geometric (OWG)* operator, which is a fuzzy majority guided aggregation operator to aggregate information given on a ratio scale. It is based on the geometric mean and the OWA operator and it allows us to incorporate the concept of fuzzy majority in decision processes when the information is provided using a ratio scale. Then, using the OWG operator in [6] we present a decision model alternative to the AHP for MCDM problems under multiplicative preference relations. This model obtains the solution in two steps: *aggregation* of the individual multiplicative preference relations in a collective one using the OWG operator and *explotation* of that collective multiplicative preference relation using choice functions based on the OWG operator.

Multiplicative preference relations are usually assumed to be reciprocal [4, 6, 11]. However, it is well known that reciprocity is not generally preserved after aggregation is carried out. In this contribution, we determine the set of OWG operators guided by a relative linguistic quantifier that maintain the reciprocity property when aggregating multiplicative reciprocal preference relations.

In order to do this, the paper is set out as follows. In Section 2 we present the problem. In Section 3, we study conditions under which reciprocity property is maintained when aggregating reciprocal multiplicative preference relations using an OWG operator guided by a relative linguistic quantifier. Finally, in Section 4 some concluding remarks are pointed out.

## 2 PRESENTATION OF THE PROBLEM

We assume MCDM problems where the alternatives, $X = \{x_1, \ldots, x_n\}$, are assessed according to different criteria, $E = \{e_1, \ldots, e_m\}$, by means of multiplicative preference relations, $\{A^1, \ldots, A^m\}$, where $A^k = \left(a_{ij}^k\right)$, indicating $a_{ij}^k$ a ratio of the preference intensity of alternative $x_i$ to that of $x_j$, i.e., it is interpreted as $x_i$ is $a_{ij}^k$ times as good as $x_j$. According to Miller's study [10], Saaty suggests measuring $a_{ij}^k$ using a ratio scale, and in particular the 1 to 9 scale [11]: $a_{ij}^k = 1$ indicates indifference between $x_i$ and $x_j$, $a_{ij}^k = 9$ indicates that $x_i$ is unanimously preferred to $x_j$, and $a_{ij}^k \in \{2, 3, \ldots, 8\}$ indicates intermediate evaluations. It is usual to assume the multiplicative reciprocity property $a_{ij}^k \cdot a_{ji}^k = 1 \; \forall i, j$.

As we have said, using an OWG operator $\phi_Q^G$ guided by a linguistic quantifier $Q$, we obtain a collective multiplicative preference relation $A^c = \left(a_{ij}^c\right)$. Each value $a_{ij}^c \in [1/9, 9]$, represents the preference of alternative $x_i$ over alternative $x_j$ according to the majority of criteria, represented by $Q$. In this case,

$$a_{ij}^c = \phi_Q^G(a_{ij}^1, \ldots, a_{ij}^m) = \prod_{k=1}^{m} \left(b_{ij}^k\right)^{w_k}$$

where $b_{ij}^k$ is the k-th largest value in the set $\{a_{ij}^1, \ldots, a_{ij}^m\}$, and

$$w_k = Q\left(\frac{k}{m}\right) - Q\left(\frac{k-1}{m}\right), \forall k.$$

Our objective in this paper is to study conditions under which reciprocity property is maintained when aggregating reciprocal multiplicative preference relations using an OWG operator guided by a relative linguistic quantifier, i.e.,

$$a_{ij}^c \cdot a_{ji}^c = 1 \; \forall i, j.$$

## 3 RECIPROCITY OF THE COLLECTIVE MULTIPLICATIVE PREFERENCE RELATION

As we are assuming $A^k$ reciprocal then $a_{ji}^k = 1/a_{ij}^k$, and therefore if $\{b_{ij}^1, \ldots, b_{ij}^m\}$ are ordered from largest to lowest, $\{b_{ji}^1, \ldots, b_{ji}^m\}$, being $b_{ji}^k = 1/b_{ij}^k$, are ordered form lowest to largest, and in consequence we have:

$$\begin{aligned} a_{ij}^c \cdot a_{ji}^c &= \prod_{k=1}^{m} \left(b_{ij}^k\right)^{w_k} \cdot \prod_{k=1}^{m} \left(b_{ji}^k\right)^{w_{m-k+1}} \\ &= \prod_{k=1}^{m} \left(b_{ij}^k\right)^{w_k} \cdot \prod_{k=1}^{m} \left(\frac{1}{b_{ij}^k}\right)^{w_{m-k+1}} \\ &= \prod_{k=1}^{m} \left(b_{ij}^k\right)^{w_k - w_{m-k+1}} = \prod_{k=1}^{m} \left(b_{ij}^k\right)^{\overline{w}_k}, \end{aligned}$$

where

$$\overline{w}_k = \left[Q\left(\frac{k}{m}\right) - Q\left(\frac{k-1}{m}\right)\right] - \left[Q\left(\frac{m-k+1}{m}\right) - Q\left(\frac{m-k}{m}\right)\right].$$

If we denote

$$A(k) = Q\left(\frac{k}{m}\right) + Q\left(1 - \frac{k}{m}\right)$$

then

$$\overline{w}_k = A(k) - A(k-1).$$

The following result is obvious:

**Proposition 1** *If $Q$ is a linguistic quantifier with membership function verifying*

$$Q(1-x) = 1 - Q(x)$$

*then the collective multiplicative preference relation, $A^c$, obtained by aggregating the set of multiplicative preference relations, $\{A^1, \ldots, A^m\}$, using the OWG operator, $\phi_Q^G$, guided by $Q$, is reciprocal.*

*Proof:* If $Q(1-x) = 1 - Q(x)$ then $A(k) = 1, \forall k$ and in consequence $\overline{w}_k = A(k) - A(k-1) = 0, \forall k$. This implies that

$$a_{ij}^c \cdot a_{ji}^c = \prod_{k=1}^{m} \left(b_{ij}^k\right)^{\overline{w}_k} = \prod_{k=1}^{m} \left(b_{ij}^k\right)^0 = \prod_{k=1}^{m} 1 = 1, \forall i, j.$$

In the case that $Q$ is a relative non-decreasing linguistic quantifier with membership function:

$$Q(x) = \begin{cases} 0 & 0 \le x < a \\ \frac{x-a}{b-a} & a \le x \le b \\ 1 & b < x \le 1 \end{cases} \qquad a, \; b \in [0, 1],$$

the election of a suitable relative quantifier, for representing the concept of fuzzy majority that we desire to implement in our MCDM problem, reduces to selecting adequate values for parameters $a$ and $b$. Therefore, the problem to solve is:

*What condition do parameters $a$ and $b$ have to verify so that $a^c_{ij} \cdot a^c_{ji} = 1, \forall i, j$?.*

We distinguish three possible cases, according to the value of $a+b$: $(A)$ $a+b=1$, $(B)$ $a+b<1$, $(C)$ $a+b>1$.

### 3.1 CASE A: $a+b=1$

In this case $1-a=b$ $(1-b=a)$ and therefore:

$$Q(1-x) = \left\{ \begin{array}{cc} 0 & 0 \le 1-x < a \\ \frac{1-x-a}{b-a} & a \le 1-x \le b \\ 1 & b < 1-x \le 1 \end{array} \right\} =$$

$$\left\{ \begin{array}{cc} 1-0 & 0 \le x < a \\ 1-\frac{x-a}{b-a} & a \le x \le b \\ 1-1 & b < x \le 1 \end{array} \right\} = 1 - Q(x).$$

Applying proposition 1 we have that in this case reciprocity property is maintained after the aggregation process is carried out.

The geometric mean operator is a particular case of OWG operator guided by a relative non-decreasing linguistic quantifier verifying this case and it is obtained using $(a,b)=(0,1)$.

The above is summarized in the following proposition:

**Proposition 2** *If $Q$ is a relative non-decreasing linguistic quantifier with parameters $a$ and $b$ verifying $a+b=1$, then the OWG operator guided by $Q$ maintains multiplicative reciprocity.*

**Example 1** *Assume that we have a set of four criteria, $E=\{e_1,e_2,e_3,e_4\}$, and a set of three alternatives, $X=\{x_1,x_2,x_3\}$. Suppose that assessements are given by means of the following multiplicative preference relations:*

$$A^1 = \begin{bmatrix} 1 & 3 & 5 \\ \frac{1}{3} & 1 & 2 \\ \frac{1}{5} & \frac{1}{2} & 1 \end{bmatrix}, A^2 = \begin{bmatrix} 1 & 2 & 7 \\ \frac{1}{2} & 1 & 5 \\ \frac{1}{7} & \frac{1}{5} & 1 \end{bmatrix},$$

$$A^3 = \begin{bmatrix} 1 & 2 & 3 \\ \frac{1}{2} & 1 & 2 \\ \frac{1}{3} & \frac{1}{2} & 1 \end{bmatrix}, A^4 = \begin{bmatrix} 1 & 5 & 3 \\ \frac{1}{5} & 1 & 9 \\ \frac{1}{3} & \frac{1}{9} & 1 \end{bmatrix}.$$

*Using the linguistic quantifier with the pair of values (0.25,0.75) and the corresponding OWG operator with weight vector $(0, 0.5, 0.5, 0)$, the reciprocal collective multiplicative preference relation obtained is*

$$A^c = \begin{bmatrix} 1 & \sqrt{6} & \sqrt{15} \\ \frac{\sqrt{6}}{6} & 1 & \sqrt{10} \\ \frac{\sqrt{15}}{15} & \frac{\sqrt{10}}{10} & 1 \end{bmatrix}.$$

### 3.2 CASE B: $a+b<1$

In this case, we have that $1-a>b$ $(1-b>a)$ and as a consequence of being $a \le b$ we have $a<1/2$. We will start by assuming that $b \ge 1/2$, what implies that $1-b \le b$, and consequently:

$$Q(x) = \begin{cases} 0 & 0 \le x < a \\ \frac{x-a}{b-a} & a \le x < 1-b \\ \frac{x-a}{b-a} & 1-b \le x < b \\ 1 & b \le x < 1-a \\ 1 & 1-a \le x < 1, \end{cases}$$

$$Q(1-x) = \begin{cases} 1 & 0 \le x < a \\ 1 & a \le x < 1-b \\ \frac{1-x-a}{b-a} & 1-b \le x < b \\ \frac{1-x-a}{b-a} & b \le x < 1-a \\ 0 & 1-a \le x < 1, \end{cases}$$

and

$$A(x) = \begin{cases} 1 & 0 \le x < ma \\ \frac{x+m(b-2a)}{m(b-a)} & ma \le x < m(1-b) \\ \frac{1-2a}{b-a} & m(1-b) \le x < mb \\ \frac{m-x-m(b-2a)}{m(b-a)} & mb \le x < m(1-a) \\ 1 & m(1-a) \le x < m \end{cases}$$

It is clear that there exist $h_1, h_2, h_3, h_4 \in \{1, \ldots, m\}$ such that

$$h_1 - 1 < ma \le h_1,$$

$$h_2 - 1 < m(1-b) \le h_2,$$

$$h_3 - 1 < mb \le h_3,$$

$$h_4 - 1 < m(1-a) \le h_4,$$

and in consequence:

$$A(0) = \ldots = A(h_1 - 1) = 1$$

$$A(k) = \tfrac{k+m(b-2a)}{m(b-a)}, k = h_1, \ldots, h_2 - 1$$

$$A(j) = \tfrac{1-2a}{b-a}, j = h_2, \ldots, h_3 - 1$$

$$A(l) = \tfrac{m-l-m(b-2a)}{m(b-a)}, l = h_3, \ldots, h_4 - 1$$

$$A(h_4) = \ldots = A(m) = 1$$

Moreover, it is clear that $m - h_4 = h_1 - 1$, $m - h_3 = h_2 - 1$, so:

$$\overline{w}_1 = \ldots = \overline{w}_{h_1-1} = 0 = \overline{w}_{h_4+1} = \ldots = \overline{w}_m$$

$$\overline{w}_{h_1} = \tfrac{h_1 - ma}{m(b-a)} = -\overline{w}_{h_4} \geq 0$$

$$\overline{w}_{h_1+1} = \ldots = \overline{w}_{h_2-1} = \tfrac{1}{m(b-a)} =$$

$$-\overline{w}_{h_3+1} = \ldots = -\overline{w}_{h_4-1} \geq 0$$

$$\overline{w}_{h_2} = \tfrac{h_3 - mb}{m(b-a)} = -\overline{w}_{h_3} \geq 0$$

$$\overline{w}_{h_2+1} = \ldots = \overline{w}_{h_3-1} = 0.$$

The expression of $a_{ij}^c \cdot a_{ji}^c$ reduces to:

$$a_{ij}^c \cdot a_{ji}^c = \left(\frac{b_{ij}^{h_1}}{b_{ij}^{h_4}}\right)^{\overline{w}_{h_1}} \cdot \prod_{k=h_1+1}^{h_2-1} \left(\frac{b_{ij}^k}{b_{ij}^{m-k+1}}\right)^{\frac{1}{m(b-a)}} \cdot \left(\frac{b_{ij}^{h_2}}{b_{ij}^{h_3}}\right)^{\overline{w}_{h_2}}$$

For being $\{b_{ij}^1, \ldots, b_{ij}^m\}$ ordered from largest to lowest, we have that every fraction in the above expression is greater or equal to 1, and so the whole product is, i.e., $a_{ij}^c \cdot a_{ji}^c \geq 1$. In the case of being $b < 1/2$, following a similar reasoning, we get the same conclusion. Summarizing, we have obtained the following result:

**Proposition 3** *Let $\{A^1, \ldots, A^m\}$ be a finite set of individual reciprocal multiplicative preference relations, and $Q$ a non-decreasing relative quantifier with membership function*

$$Q(x) = \begin{cases} 0 & 0 \leq x < a \\ \frac{x-a}{b-a} & a \leq x \leq b \\ 1 & b < x \leq 1 \end{cases}$$

*and $a + b < 1$. Then, the collective multiplicative preference relation obtained using the OWG operator $\phi_Q^G$, $A^c = \left(a_{ij}^c\right)$, $a_{ij}^c = \phi_Q^G\left(a_{ij}^1, \ldots, a_{ij}^m\right)$, verifies $a_{ij}^c \cdot a_{ji}^c \geq 1$.*

**Example 2** *Suppose again the same set of reciprocal multiplicative preference relations given in example 1. Using the linguistic quantifier "at least half" with the pair of values $(0, 0.5)$ and the corresponding OWG operator with weight vector $(0.5, 0.5, 0, 0)$, the collective multiplicative preference relation is:*

$$A^c = \begin{bmatrix} 1 & \sqrt{15} & \sqrt{35} \\ \frac{1}{2} & 1 & 3\sqrt{5} \\ \frac{1}{3} & \frac{1}{2} & 1 \end{bmatrix}.$$

### 3.3 CASE C: $a + b > 1$

As in the previous subsection, we have to distinguish two subcases: (i) $a < 1/2$ and (ii) $a \geq 1/2$. We will study just the first one because, following a similar reasoning, the same result is obtained in both subcases.

The expressions of $A(x)$ when $a < 1/2$ is the following:

$$A(x) = \begin{cases} 1 & 0 \leq x < m(1-b) \\ \frac{m-x-ma}{m(b-a)} & m(1-b) \leq x < ma \\ \frac{1-2a}{b-a} & ma \leq x < m(1-a) \\ \frac{x-ma}{m(b-a)} & m(1-a) \leq x < mb \\ 1 & mb \leq x < m. \end{cases}$$

As in the previous case, there exist $r_1, r_2, r_3, r_4 \in \{1, \ldots, m\}$ such that

$$r_1 - 1 < m(1-b) \leq r_1,$$

$$r_2 - 1 < ma \leq r_2,$$

$$r_3 - 1 < m(1-a) \leq r_3,$$

$$r_4 - 1 < mb \leq r_4,$$

$$m - r_4 = r_1 - 1,$$

$$m - r_3 = r_2 - 1,$$

and therefore:

$$\overline{w}_1 = \ldots = \overline{w}_{r_1-1} = 0 = \overline{w}_{r_4+1} = \ldots = \overline{w}_m$$

$$\overline{w}_{r_1} = \tfrac{m - r_1 - mb}{m(b-a)} = -\overline{w}_{r_4} \leq 0$$

$$\overline{w}_{r_1+1} = \ldots = \overline{w}_{r_2-1} = \tfrac{-1}{m(b-a)} =$$

$$-\overline{w}_{r_3+1} = \ldots = -\overline{w}_{r_4-1} \leq 0$$

$$\overline{w}_{r_2} = \tfrac{r_2 - 1 - ma}{m(b-a)} = -\overline{w}_{r_3} \leq 0$$

$$\overline{w}_{r_2+1} = \ldots = \overline{w}_{r_3-1} = 0.$$

The expression of $a_{ij}^c \cdot a_{ji}^c$ reduces to:

$$a_{ij}^c \cdot a_{ji}^c = \left(\frac{b_{ij}^{r_4}}{b_{ij}^{r_1}}\right)^{\overline{w}_{r_4}} \cdot \prod_{k=r_1+1}^{r_2-1} \left(\frac{b_{ij}^{m-k+1}}{b_{ij}^k}\right)^{\frac{1}{m(b-a)}} \cdot \left(\frac{b_{ij}^{r_3}}{b_{ij}^{r_2}}\right)^{\overline{w}_{r_3}}$$

For being $\{b_{ij}^1, \ldots, b_{ij}^m\}$ ordered from largest to lowest, we have that every fraction in the above expression is lower or equal to 1 and so the whole product is, i.e., $a_{ij}^c \cdot a_{ji}^c \leq 1$. Summarizing, we have obtained the following result:

**Proposition 4** *Let $\{A^1, \ldots, A^m\}$ be a finite set of individual reciprocal multiplicative preference relations,* *and $Q$ a non-decreasing relative quantifier with membership function*

$$Q(x) = \begin{cases} 0 & 0 \leq x < a \\ \frac{x-a}{b-a} & a \leq x \leq b \\ 1 & b < x \leq 1 \end{cases}$$

*and $a + b > 1$. Then, the collective multiplicative preference relation obtained using the OWG operator $\phi_Q^G$, $A^c = \left(a_{ij}^c\right)$, $a_{ij}^c = \phi_Q^G\left(a_{ij}^1, \ldots, a_{ij}^m\right)$, verifies $a_{ij}^c \cdot a_{ji}^c \leq 1$.*

**Example 3** *Using, in this case, the linguistic quantifier "as many as possible" with the pair of values* $(0.5, 1)$ *and the corresponding OWG operator with weight vector* $(0, 0, 0.5, 0.5)$*, then the collective multiplicative preference relation is:*

$$A^c = \begin{bmatrix} 1 & 2 & 3 \\ \\ \frac{\sqrt{15}}{15} & 1 & 2 \\ \\ \frac{\sqrt{35}}{35} & \frac{\sqrt{5}}{15} & 1 \end{bmatrix}.$$

## 4 Conclusions

We have studied conditions to maintain reciprocity when aggregating a finite set of multiplicative reciprocal relations using OWG operators guided by linguistic quantifiers. In the case of a relative non-decreasing quantifier with parameters $(a, b)$, reciprocity is maintained when $a+b = 1$ and not otherwise. Furthermore, the greater the value of $|a + b - 1|$ the more distant the collective multiplicative preference relation is from being reciprocal as we show in the case of fuzzy preference relations [5].

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