A Lattice-Valued Linguistic-Based Decision-Making Method

Jun Liu, Yang Xu, Da Ruan, and Luis Martinez

Abstract— **The subject of this work is to establish a mathematical framework that provides the basis and tool for synthesis and evaluation analysis in decision making, especially from the logic point of view. This paper focuses on a flexible and realistic approach, i.e., the use of linguistic assessment in decision making, specially, the symbolic approach acts by direct computation on linguistic values. A lattice-valued linguistic algebra model, which is based on a logical algebraic structure, i.e., lattice implication algebra, is applied to represent imprecise information and deal with both comparable and incomparable linguistic values (i.e., non-ordered linguistic values). Within this framework, some known weighted aggregation functions are analyzed and extended to deal with these kinds of lattice-value linguistic information.**

***Index Terms*— decision-making, fuzzy logic, lattice, multivalued logic**

I. Introduction

In real decision making problem, most of information can be very qualitative in nature, e.g., with vague or imprecise knowledge. A more realistic approach in qualitative setting may be to use linguistic assessments instead of numerical values. The linguistic approach is an approximate technique appropriate for dealing with qualitative aspects of problems.

Considering the proposed linguistic approaches, two main different approaches can be found in order to aggregate linguistic values in decision making: the first, i.e., the approximation approach, uses the associated membership functions of the linguistic values [1], [2]. The second is the symbolic approach, acts by direct computation on linguistic values [3]. The latter kind of methods assumes that the linguistic value set is an ordered structure uniformly distributed on a scale. These methods seem natural when the linguistic approach is used, because the linguistic assessments are just approximations which are given and handled when it is impossible or unnecessary to obtain more accurate values. Thus, in this case, the use of membership functions in the former approach is unnecessary. Furthermore, they are computationally simple and quick [3].

A nice feature of linguistic variables is that their values are structured, which makes it possible to compute the representations of composed linguistic values from those of their composing parts. These linguistic values, in different natural language, seem difficult to distinguish their boundary sometime, but their meaning of common usage can be understood. Moreover, there are some "vague overlap districts" among some words which cannot be strictly linearly ordered on the universe. Accordingly, establishing certain suitable algebras to characterize and represent the values of linguistic variables is therefore desired.

It was shown that lattice has been a very useful and a well-developed branch of abstract algebra for modeling the ordering relations in real-world. Lattice-valued algebra for modeling linguistic values would be a good choice.

Based on the symbolic approaches and the above ideas, a lattice-valued linguistic algebra model, which is based on a logical algebraic structure, i.e., lattice implication algebra, is applied in decision making to represent imprecise information and deal with both comparable and incomparable linguistic values (i.e., non-ordered linguistic values). Within this framework, some known weighted aggregation functions are analysed and extended to deal with these kinds of lattice-value linguistic information.

The paper is organized as follows: Section II is an overview of the symbolic approach focusing on the issue of aggregation. Based on it, a lattice-valued linguistic approach for aggregation and decision-making is proposed in Section III with an example illustration. The paper is concluded in Section IV.

II. Weight Aggregation of Linguistic Assessments

The issue of aggregation has been studied extensively in many applications of fuzzy sets [4]. To manipulate the linguistic information in decision making, we shall work with operators for combining the linguistic un-weighted and weighted values by direct computation on labels. Specifically, we focus on the analysis of the operators like *Min*-type and *Max*-type weighted aggregation operators which are all introduced by Yager [5]-[7].

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The aggregation of weighted information involves the transformation of the weighted information under the importance degrees. The transformation form depends upon the type of aggregation of weighted information being performed. Yager [5]-[7] discussed the effect of the importance degrees in the types of aggregation *Max* and *Min* and suggested a class of functions for importance transformation in both types of aggregations, i.e.,

- ***Min*-type aggregation:**

$$D=Min(g(w_1, a_1), g(w_2, a_2),..., g(w_n, a_n)) \quad (1)$$

- ***Max*-type aggregation:**

$$D(x)=Max(g(w_1, a_1), g(w_2, a_2), ..., g(w_n, a_n)) \quad (2)$$

where g: $P\times P\rightarrow P$ is a weight transformation function, and P is a finite ordered set. For *Min* type aggregation he suggested a family of t-conorms acting on the weighted information and the negation of the weights, which presents the non-increasing monotonic property in the weights. For *Max* type aggregation he suggested a family of t-norms acting on the weighted information and the weight, which presents the non-decreasing monotonic property in the weights. Yager proposed a general specification of the requirements that any importance transformation function g: $P\times P\rightarrow P$, must have the following properties:

I. if $a>b$ then $g(w, a)\geq g(w, b)$;
II. $g(w, a)$ is monotone in w;
III. $g(0, a)$ = ID;
IV. IV. $g(1, a) = a$:

with a, $b\in P$ expressing the satisfaction with regard to a criterion, $w\in P$ the weight associated to the criterion, and ID an identity element, which is such that if we add it to our aggregations it doesn't change the aggregated value. Some justifications of conditions I-IV have been given in [5]-[7].

Note that the conditions I-IV are in fact a subset of general axioms required by a fuzzy implication operators [8]. As analyzed in [8] for the axioms hold by different fuzzy implication operators (about 18 implication operators), some of the implication operators satisfy conditions I-IV. These implication operators can be suggested as the manifestation of the transformation function g, which are used for *Min*-type aggregation operator. Because t-norms generally satisfy the conditions I-IV, some T-norms can also be given for *Max*-type aggregation operator.

Considering the aforementioned ideas and assuming a linguistic framework, i.e., a label set L_0, to express the information and the weights. Let $L_0=\{0=s_0<...<s_n=1\}$ be a finite set of linguistic terms, $n\in\{0\}\cup$N. Two general forms of the overall aggregation functions are given by

- ***Min*-type aggregation:**

$$D=\mathrm{T}(g(w_1, a_1), g(w_2, a_2),..., g(w_n, a_n)) \quad (4)$$

where w_i, $a_i\in L_0$ (i=1,...,n), g is an implication-type transform function satisfying the conditions I-IV. T is a t-norm. It is the aggregation rule used in the pessimistic strategy. For a linear scale, T can be taken as $\wedge$ or T_a (Bounded Difference [9]).

- ***Max*-type aggregation:**

$$D(x)=S(g(w_1, a_1), g(w_2, a_2), ..., g(w_n, a_n)) \quad (5)$$

Here w_i, $a_i\in L_0$ (i=1,...,n), g is a t-norm type transform function, and S is the corresponding t-conorm. It is the aggregation rule used in the optimistic strategy. For the linear scale here, accordingly, S can be taken as $\vee$=Max or S_a [9].

In the following section, we will consider the more general linguistic cases, i.e., lattice-valued linguistic terms by using the logical algebraic structure.

III. Lattice-Valued Linguistic Approach for Aggregation and Decision-Making

A. Lattice structure and lattice implication algebras

Considering the extension of two-valued logic to multi-valued logic, two important cases of L are of interest and often being used: when L is a finite simple ordered set; and when L is the unit interval [0, 1]. More general, L should be a lattice with suitable operations like $\wedge$, $\vee$, $\rightarrow$, $'$. The question of the appropriate operation and lattice structure has generated much literature [9-14]. Goguen [11] established L-fuzzy logic of which truth value set is a complete lattice-ordered monoid, also called a complete residuated lattice in Pavelka and Novak's L-fuzzy logic [9], [12]. Since this algebraic structure is quite general, it is relevant to ask whether one can specify the structure. In this note, we specify the algebraic structure to lattice implication algebras (LIA) introduced by Xu [13], [14], which was established by combining lattice and implication algebra with the attempt to model and deal with the comparable and incomparable information. There has been lots of work about LIAs, as well as the corresponding lattice valued logic system, reasoning theory and methods [13], [14]. The LIA [13] is defined axiomatically as:

Definition 3.1 (LIA) Let $(L, \vee, \wedge, ')$ be a bounded lattice with an order-reversing involution " $'$ " and the universal bounds O, I, $\rightarrow$: $L\times L\rightarrow L$ be a mapping. $(L, \vee, \wedge, ', \rightarrow)$ is called a *lattice implication algebra* (LIA) if the following axioms hold for all $x, y, z\in L$:

(A_1) $x\rightarrow(y\rightarrow z)=y\rightarrow(x\rightarrow z)$, (exchange property)
(A_2) $x\rightarrow x$=I, (identity)
(A_3) $x\rightarrow y=y'\rightarrow x'$, (contrapositive symmetry)
(A_4) $x\rightarrow y=y\rightarrow x$=I implies $x=y$, (equivalency)
(A_5) $(x\rightarrow y)\rightarrow y=(y\rightarrow x)\rightarrow x$,
(A_6) $(x\vee y)\rightarrow z=(x\rightarrow z)\wedge(y\rightarrow z)$,
(A_7) $(x\wedge y)\rightarrow z=(x\rightarrow z)\vee(y\rightarrow z)$.

Remarks:

(1) The implication operation in LIA satisfies the conditions I-IV of the importance transformation function. Hence the implication operation in LIA can be used to capture the transformation between the weight and the individual ratings in *Min*-type aggregation.

(2) The corresponding t-norm $\otimes$ [7], [12] in LIA can be taken as the transformation function in *Max*-type aggregation. And it also can be taken as the *Min*-type aggregation operator.

(3) The s-norm ⊕ [7], [12] in LIA can be taken as the *Max*-type aggregation operator.

B. Lattice-valued linguistic terms

In a natural language, there are some "vague overlap districts" among some words which cannot be strictly linearly ordered, as given in Fig. 1 of Example 3.1.

Example 3.1 The ordering relations in the linguistic terms:

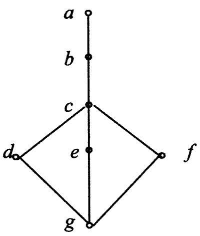

Fig. 1 Linguistic terms in an ordering. *a=very true, b=more true, c=true, d=approximately true, e=possibly true, f=more or less true, g=little true*

Note that *d=approximately true, e=possibly true, f=more or less true* are incomparable. You can not collapse that structure into a linearly ordered structure, because you would impose an ordering on *d*, *e*, and *f* which was originally not present. It means the set of linguistic values may not be strictly linearly ordered. Naturally, it should be suitable to represent these values by a partially ordered set or lattice.

Here we characterize the set of linguistic values by a LIA structure. In general, the value of a linguistic variable can be a linguistic expression involving a set of linguistic values such as "*high*," "*middle*," and "*low*," modifiers such as "*very*," "*more or less*" (called hedges [15]) and connectives (e.g., "*and*," "*or*"). Let us consider the domain of the linguistic variable "*truth*": domain (*truth*)={*true, false, very true, more or less true, possibly true, very false, ...*}, which can be regarded as a partially ordered set whose elements are ordered by their meanings and also regarded as an algebraically generated set from the generators *G*={*true, false*} by means of a set of linguistic modifiers M={*very, more or less, possibly, ...*}. The linguistic modifiers are strictly related to the notion of vague concept. The generators *G* can be regarded as the prime term, different prime terms correspond to the different linguistic variables.

Consider a set of linguistic hedges, e.g. H^+={*very, more or plus*}, H={*approximately, possibly, more or less, little*}, where H^+ consists of hedges which strengthen the meanings of "*true*" and the hedges in H weaken it. Put $H=H^+ \cup H$. H^+, H can be ordered by the degree of strengthening or weakening, e.g., one may assume that *very>more, little>approximately, possibly,* and *more or less*, and "*approximately*," "*possibly*," "*more or less*" are incomparable. We say that $a \le b$ iff $a(True) \le b(True)$ in the natural language, where *a* and *b* are linguistic hedges.

Applying the hedges of H to the primary term "*true*" or "*false*" we obtain a partially ordered set or lattice. For example, as represented in Fig. 2, we can obtain a lattice generated from "*true*" or "*false*" by means of operations in H. Moreover, one can define ∧, ∨, implication (→) and complement operation ′ on this lattice based on the LIA structure (Tables I and II).

Example 3.2 LIA of linguistic terms with 6 elements.

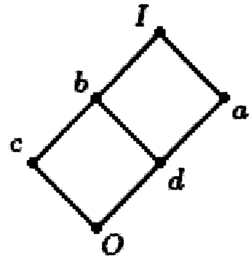

Fig. 2 The lattice-ordering structure of linguistic terms with 6 elements. Here *I=more true, a=less false, b=true, c=less true, d=false, O=more false*

TABLE I
AN ′ TABLE

x	*x′*
O	*I*
a	*c*
b	*d*
c	*a*
d	*b*
I	*O*

TABLE II
AN → TABLE

→	***O***	***a***	***b***	***c***	***d***	***I***
O	*I*	*I*	*I*	*I*	*I*	*I*
a	*c*	*I*	*b*	*c*	*b*	*I*
b	*d*	*a*	*I*	*b*	*a*	*I*
c	*a*	*a*	*I*	*I*	*a*	*I*
d	*b*	*I*	*I*	*b*	*I*	*I*
I	*O*	*a*	*b*	*c*	*d*	*I*

Note that *c=approximately true, d=possibly true* are incomparable. We set *L*={*O, a, b, c, d, M, e, f, g, h, I*}. }. The operations "∧" and "∨"can be shown in the Hasse diagram Fig. 2. The complement operation " ′ " is given in Table I, the implication operation is given in Table II, the corresponding t-norm ⊗ and the t-conorm ⊕ [7], [12] can be also given as according to the LIA structure and properties respectively: As we can show that (*L*, ∨, ∧, ′, →) is a LIA, all the properties of LIA will hold in *L*. So the finite set of linguistic values can be characterized by a finite LIA.

Accordingly, we can deal with the more general linguistic information, i.e., from a linear ordered linguistic label set to lattice-valued linguistic label set. According to Eqs. (4) and (5), we have the following extended aggregation function for lattice-valued linguistic terms: where w_i, $a_i \in L$, *L is* the finite set of linguistic values characterized by a finite LIA. While in Eq. (4), *g* is an implication operation in LIA, T=∧ or ⊗ in LIA. For Eq. (5), *g*=∧ or ⊗ in LIA. *S* can be taken as ∨ or ⊕.

For the implication operator *I*: $H \times H \to H$, here *H*=[0,1], L_0 (the set of the linearly order linguistic terms) or *L* (the set of the lattice-order linguistic terms based on LIA), we consider the following properties of *I*:

I. $I(w, a)$ is non-decreasing in *a*;
II. $I(w, a)$ is non-increasing in *w*;
III. $I(0, a) = 1$; IV. $I(1, a) = a$;
IV. $w \le a$ iff $I(w, a)=1$;
V. $w \le a$ iff $I(w, a)=1$;
VI. $I(w, a) \ge a$ for any *w*, *a*.

The following Table III gives a comparison among different implication operators I_* given in [8] as well as the implications in LIA on the properties for characterizing the transformation function *g*.

TABLE III
FULFILMENT OF SOME CONDITIONS FOR THE SELECTED IMPLICATION OPERATORS FOR THE FUNCTION G (Y=YES, N=NO)

	I_a	I_s	I_g	I_b	I_{b*}	I_Δ	I_*	I_{a*}	I_E	I_{LIA}
I	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y
II	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y
III	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y
IV	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y
V	Y	Y	Y	N	Y	Y	N	Y	N	Y
VI	Y	N	Y	Y	Y	N	Y	Y	Y	Y
Feasible for g on [0,1]	Y	Y	N	Y	Y	Y	Y	Y	Y	Y
Feasible for g on L_0	N	N	N	Y	Y	N	N	Y	N	Y
Feasible for g on L	N	N	N	Y	Y	N	N	N	N	Y
The order relation	$I_a \geq I_{b*} \geq I_b \geq I_g$, $I_{LIA} \geq I_{b*} \geq I_b$									

C. An example

To illustrate how the proposed lattice-valued linguistic method works, we consider a simple example to evaluate the set of cars $\{C_1$=Chevrolet, C_2= Toyota, C_3= Buick, C_4=Fiat$\}$.

Let $X=\{X_1, X_2, X_3,\}$ the set of objectives, where X_1=comfort, X_2= price and X_3=repair frequency.

Assume the evaluation set is the lattice of linguistic terms L as defined in Example 3.2 of 6-element LIA, where "*true*" is changed as "*high*", and "*false*" is changed as "*low*" respectively.

For the objective X_1, we have the following set of satisfaction degrees:

$A_1=\{I/C_1, c/C_2, b/C_3, d/C_4\}$;

Similarly, for the objective X_2 and X_3, we have

$A_2=\{O/C_1, b/C_2, c/C_3, a/C_4\}$; $A_3=\{c/C_1, d/C_2, I/C_3, b/C_4\}$.

Next, we evaluate the importance of each objective. In this case, we may assume weights of importance for X as $b_1=c$, $b_2=b$, $b_3=d$, respectively. Taking these weights into account, we have:

$M_1= b_1 \rightarrow A_1 =\{I/C_1, I/C_2, c/C_3, a/C_4\}$;

$M_2=b_2 \rightarrow A_2=\{d/C_1, I/C_2, b/C_3, a/C_4\}$;

$M_3= b_3 \rightarrow A_3 =\{b/C_1, I/C_2, I/C_3, I/C_4\}$.

Here $\rightarrow$ is an implication in LIA. By using Eq. (4), the final decision set

$D = M_1 \cap M_2 \cap M_3=\{d/C_1, I/C_2, c/C_3, a/C_4\}$.

The optimal alternative is the C_2={Toyota}$\in X$ that maximizes D. Here we take T as "min" operation in Eq. (4). The overall evaluation also can be seen from D.

IV. CONCLUSIONS

A lattice-valued linguistic decision approach was proposed in the paper. This method offers the advantage that it does not require the definition of the membership functions associated with the linguistic terms. Especially, we do use a finite set of linguistic terms with a rich lattice ordering algebraic structure to represent the weights of the criteria. So this procedure has another advantage of handling incomparable linguistic terms and the implication operation to combine the importance of criteria with the performance scores of alternatives in decision models. Computing with words is applied. Note that lattice is a more universal structure than the set of linguistic terms, and the implication operation in LIA is much general richer, it would be reasonable and realistic to design decision making models based on these methodologies. The further investigations on linguistic-valued decision making, linguistic-valued reasoning approach will be carried out based on the established lattice-valued reasoning approaches in [14], [16]-[18].

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