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MODELING ATTITUDES TOWARDS CONSENSUS IN GROUP DECISION MAKING BY MEANS OF OWA OPERATORS

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Nowadays important decisions that have a significant impact either in societies or in organizations are commonly made by a group rather than a single decision maker, which might require more than a majority rule to obtain a real acceptance. Consensus Reaching Processes provide a way to drive group decisions which are more accepted and appreciated by people affected by such a decision. These processes care about different consensus measures to determine the agreement in the group. The adequate choice of a consensus measure is a key issue for improving and optimizing consensus reaching processes, which still requires further research. This paper focuses on the attitude of the group to achieve the consensus and introduces this attitude in the consensus measure through a novel aggregation operator based on OWA operators, so-called Attitude-OWA (AOWA).

Keywords: Group Decision Making; Consensus; OWA; RIM Quantifiers.

1. Introduction

Group Decision Making (GDM) problems are required throughout most companies and organizations nowadays, in order to guarantee a right development in them. GDM problems can be defined as decision situations where two or more decision makers or *experts* try to achieve a common solution to a decision problem consisting of two or more possible solutions or *alternatives*.[1]

In real world GDM problems, a range of situations including collaboration and competitiveness among individuals, compatible approaches or incompatible proposals could occur. Some guiding rules have been proposed to support decision making in such situations, for example the majority rule, minority rule, unique person-based decision and unanimity.[2] In democratic political systems, for instance, the majority rule is the most usual rule for dealing with GDM problems.[3] However, in many real world GDM problems that can affect many people or the whole society (civil rights, political or religious issues) agreed solutions are highly appreciated. Therefore, the necessity of making decisions under consensus is becoming increasingly common in a variety of social situations.

Consensus Reaching Processes (CRP)[2,4] pursue an experts' agreement about the problem before making the decision, thus yielding a more accepted solution by the whole group. Different authors have proposed distinct approaches to handle CRPs, where Kacprzyk's *soft consensus* approach stands out.[1] In this approach, later considered by many authors, the concept of fuzzy linguistic majority is used to measure consensus between individuals in a flexible way. Later on, major achievements have been reached with the development different consensus models, including the use of different preference structures,[5] management of incomplete preferences[6,7] or even the introduction of adaptive consensus models based on the process' performance.[8]

However, some crucial aspects in CRPs still require a further study, for example the study of *consensus measures*, i.e. measurements to evaluate the level of agreement in the group. Consensus measures normally require *similarity measures* to calculate closeness among opinions based on distance metrics, as well as *aggregation operators* to obtain a collective consensus degree.[9] In this aggregation process, it would be very important reflecting the decision group's policies or *attitudes* regarding how to measure consensus as faithfully as possible, since different attitudes might be considered by them depending on the context of the decision problem.

Additionally, it is important to consider the application of GDM problems with a large number of experts, because although real-world consensus processes usually involve many experts, most developed models provide examples of performance with a small number of experts only.

In this paper, we aim to integrate the attitude towards consensus of groups participating in GDM problems with many experts. Our goal consists in introducing this attitude in the aggregation of agreement degrees between experts. To do this, we present the Attitude-OWA Operator (AOWA), that extends OWA operator[10] so that it lets us determine those agreement positions among experts which will be considered in the measurement of consensus in the group, and therefore reflect the group's attitude towards consensus. The approach will be tested in an automatic consensus support system to solve problems with a large number of experts.

The rest of this paper is organized as follows: in Section 2 we present some preliminaries related to consensus processes in GDM, and review OWA operators and linguistic quantifiers, used to compute OWA weights. In Section 3, we present our proposal to reflect group's attitudes by means of AOWA operators. A case study with the use of different AOWA operators reflecting distinct group's attitudes is presented in Section 4. Finally, in Section 5, we draw the main conclusions and consider some future works.

2. Preliminaries

In this section, we present an overview of GDM problems and CRPs. We then briefly review OWA operators and linguistic quantifiers.

2.1. *Group Decision Making (GDM)*

GDM problems are characterized by the participation of two or more experts in a decision problem, where a set of alternatives or possible solutions to the problem are presented.[1,2] Formally, the main elements found in any GDM problem are:

- A set of possible *alternatives* to choose as possible solutions to the problem.

$$X = \{x_1, \ldots, x_n\} (n \geq 2) \tag{2.1}$$

- A set of individuals or *experts* who express their judgements or opinions on the alternatives in X.

$$E = \{e_1, \ldots, e_m\} (m \geq 2) \tag{2.2}$$

Each expert e_i provides his/her opinions over alternatives in X by means of a preference structure. Different preference structures have been widely used in GDM approaches, including preference orderings, preference relations and utility vectors.[5,11] One of the most usual preference structures, which also has shown to be especially effective when dealing with uncertainty, is the so-called fuzzy preference relation.

Definition 2.1. A *fuzzy preference relation* P_i associated to an expert e_i on a set of alternatives X is a fuzzy set on $X \times X$, which is characterized by the membership function $\mu_{P_i} : X \times X \longrightarrow [0, 1]$. When the number of alternatives n is finite, P_i may be represented by a $n \times n$ matrix as follows:

$$P_i = \begin{pmatrix} p_i^{11} & \ldots & p_i^{1n} \\ \vdots & \ddots & \vdots \\ p_i^{n1} & \ldots & p_i^{nn} \end{pmatrix}$$

where the numerical opinion or *assessment* $p_i^{lk} = \mu_{P_i}(x_l, x_k)\ \forall l, k \in \{1, \ldots, n\}$ is the degree of preference of alternative x_l over x_k assessed by expert e_i, and:

- $p_i^{lk} > 1/2$ indicates that x_l is preferred to x_k, and $p_i^{lk} = 1$ indicates absolute preference of x_l over x_k.
- $p_i^{lk} < 1/2$ indicates that x_k is preferred to x_l, and $p_i^{lk} = 0$ indicates absolute preference of x_k over x_l.
- $p_i^{lk} = 1/2$ indicates indifference between x_l and x_k.

The solution for a GDM problem may be obtained either by a *direct approach* or an *indirect approach*.[12] In the former, the solution is immediately obtained from the experts' individual opinions or preferences. In the latter, a social or collective opinion is computed to determine the chosen alternative/s. Regardless of the approach considered, two main phases are required to solve a GDM problem (Fig. 1):

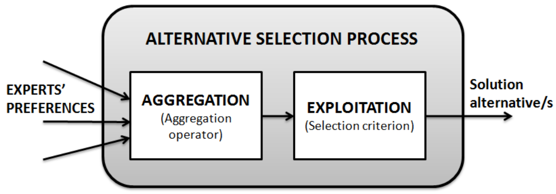

Fig. 1. Selection process in GDM problems.

(i) an *Aggregation phase*, where experts' preferences are combined, and (ii) an *Exploitation phase*, which consists in obtaining an alternative or subset of alternatives as the solution to the problem.

2.2. *Consensus Reaching Processes (CRPs)*

One of the main shortcomings found in classic GDM rules, such as the majority rule or minority rule, is the possible disagreement shown by one or more experts with the achieved solution, because they might consider that their opinions have not been taken into account sufficiently. Given the importance of obtaining an accepted solution by the whole group, CRPs as part of the decision process have attained a great attention. *Consensus* can be understood as a state of mutual agreement among members of a group,[2,4] where the decision made satisfies all of them. Reaching a consensus usually requires that experts modify their initial opinions in a discussion process, making them closer to each other and towards a collective opinion which must be satisfactory for all of them.

The notion of consensus can be interpreted in different ways, ranging from consensus as total agreement (*unanimity*), to a more flexible approach.[13] The traditional or *strict* notion of consensus assumes that consensus exists only if all experts have achieved a mutual agreement in all their opinions, which is commonly known as *unanimity*.[3,14] Consensus as unanimity may be quite difficult or even impossible to achieve in practice, and in the cases it could be achieved, the cost derived from the CRP would be unacceptable, and it may sometimes have been achieved through a *normative* point of view, through intimidation and other social strategies.[15] Subsequently, more flexible notions of consensus have been proposed to soften the strict view of consensus as unanimity. These flexible approaches, more feasible in practice, consider different degrees of partial agreement among experts to decide about the existence of consensus, thus indicating how far a group of experts is from ideal consensus or unanimity.

One of the most widely accepted approaches for a flexible measurement of consensus is the so-called notion of *soft consensus*, proposed by Kacprzyk in Ref. 1. This approach introduces the concept of fuzzy linguistic majority, which establishes that there exists consensus if *most experts participating in a problem agree with the most important alternatives*. *Soft consensus* based approaches have been used in different

GDM problems, providing satisfactory results.[16,17,18,19,20] Consensus measures based on *soft consensus* are more human-consistent and ideal for reflecting human perceptions of the meaning of consensus in practice [21]. The aforementioned concept of fuzzy linguistic majority has been captured by using linguistic quantifiers [22].

Next, some related works on consensus models and a general scheme for conducting CRPs are described.

2.2.1. *Related Work on CRPs*

Here we review a selection of some representative consensus models to conduct CRPs, together with their main features. Kacprzyk et al. proposed in the 80's and early 90's some consensus models based on soft computing techniques.[1,21,23,24] Their work introduced the use of fuzzy preference relations combined with the notion of *soft consensus* and fuzzy majority, where linguistic quantifiers are used to reflect a 'humanly-consistent' degree of consensus.[25] Later models were proposed by them in Refs. 18,26.

Saint et al. proposed in Ref. 4 a theoretical consensus model to describe CRPs as they usually occur in real situations in companies and organizations. This model addresses social aspects such as the initial proposal's presentation and acceptation, disagreement solution and some alternative actions in view of fails to reach a consensus, and introduces some roles to support moderator in the process supervision.

The model proposed by Herrera-Viedma et al. in Ref. 5 is characterized by the use of different preference structures by experts. A transformation function is applied on them so that they are expressed under a common uniform preference structure, i.e. fuzzy preference relations. Later on, Herrera-Viedma et al. presented in Ref. 27 a new model incorporating the use of multi-granular linguistic preference relations, making possible to manage CRPs where experts may have different levels of knowledge on the problem considered. In Refs. 6,7, they introduced the problem of dealing with incomplete preference relations. More recently, the work of Mata et al.[8] presents an adaptive consensus model, which adapts its behavior to the level of agreement achieved in each discussion round.

In Ref. 28, Cabrerizo et al. presented a consistent consensus model characterized by the use of incomplete and unbalanced linguistic preferences. The approach includes a methodology to manage unbalanced fuzzy sets effectively.[20,29] Cabrerizo et al. also conducted in Ref. 13 a study on different flexible and adaptive consensus approaches in fuzzy GDM problems, highlighting the strong and weak points in all of them.

Current efforts on the improvement of CRPs regard not only the increase in automation degree, but also an effective resolution of consensus processes with uncertain information and considerably large groups of experts. Although many consensus models have been proposed and put in practice, it is still necessary a higher scalability to manage large groups. The consensus support system, so-called COMAS, proposed by Mata et al. in Refs. 30,31 is based on a MAS (multi-agent sys-

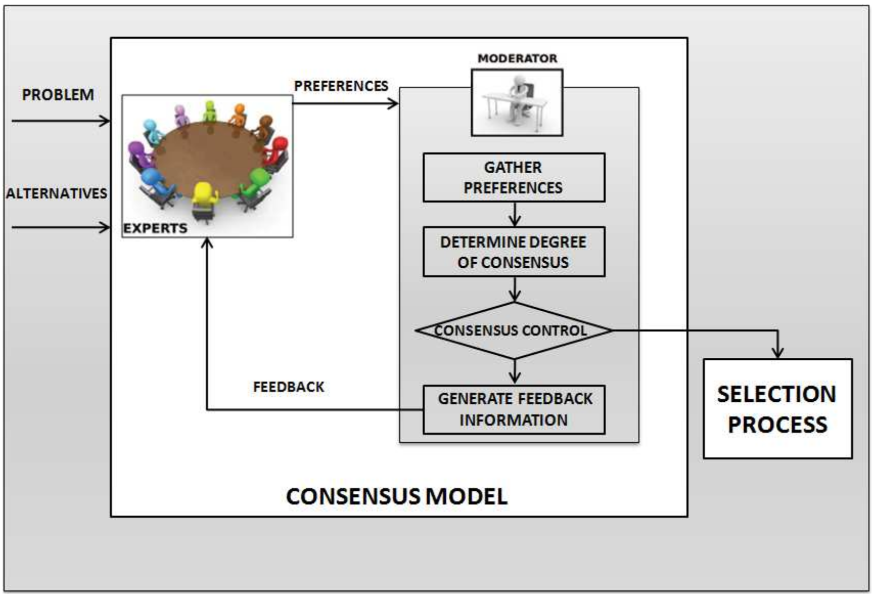

Fig. 2. General consensus process scheme in GDM problems.

tem) architecture, thus allowing higher computational capabilities and distributed processing.

2.2.2. *General CRP Scheme and Notation*

The process to reach a consensus in GDM problems is a dynamic and iterative discussion process,[4] frequently coordinated by a human figure known as *moderator*, who plays a key role in consensus processes.[32] The moderator's main responsibilities are:

- Evaluate the degree of agreement achieved in each round of discussion, and decide whether it is enough or not to accept the existence of consensus.
- Identify those alternatives that hamper reaching a consensus.
- Give feedback to experts, regarding changes they should perform in their opinions on the previously identified alternatives, in order to increase the level of agreement in the next rounds.

A general scheme for conducting CRPs is shown in Fig. 2. A brief description and some necessary notation on the steps shown in this process are given below.

- *Gather preferences*: Each expert e_i provides moderator a fuzzy preference relation on X, P_i.
- *Determine Degree of Consensus*: For each different pair of experts e_i, e_j $(i < j)$, a similarity value $sm_{ij}^{lk} \in [0, 1]$ is computed on each pair of alternatives (x_l, x_k),

$l, k \in \{1, \dots, n\}$ as a function of the distance between their assessments p_i^{lk}, p_j^{lk} on that pair,

$$sm_{ij}^{lk} = 1 - |p_i^{lk} - p_j^{lk}| \tag{2.3}$$

Pairwise similarities are aggregated to obtain the consensus degree on each pair of alternatives, $cm^{lk} \in [0,1]$, as

$$cm^{lk} = \phi(sm_{12}^{lk}, sm_{13}^{lk}, \dots, sm_{1m}^{lk}, sm_{23}^{lk}, \dots, sm_{2m}^{lk}, \dots, sm_{(m-1)m}^{lk}) \tag{2.4}$$

where ϕ is the aggregation operator used. Our proposal in this paper shall use a novel AOWA operator to reflect a group's attitude in this aggregation procedure, in order to improve the CRP adapting it to such attitude.

Afterwards, an average operator is used to obtain an overall consensus degree $cr \in [0,1]$ based on consensus degrees on pairs of alternatives cm^{lk}.

- *Consensus Control*: cr is compared with a consensus threshold μ, i.e. the minimum level of agreement desired by the group. If $cr \geq \mu$, the group moves on to the selection process, otherwise, moderator must move to the feedback generation phase.
- *Generate Feedback Information*: A collective preference P_c is obtained by aggregating experts' preferences. The moderator determines those furthest opinions p_i^{lk} from the collective opinion p_c^{lk} for each expert e_i and pair of alternatives (x_l, x_k), and suggest experts increasing or decreasing them in order to increase consensus degree in next rounds.

2.3. *OWA (Ordered Weighted Averaging) Operators*

Frequently, aggregation processes in multiple criteria decision making and GDM problems require assigning weights on values to be aggregated, thus using the so-called *weighted operators*.[33] One of the most widely applied families of weighted operators in different GDM approaches in the literature are the so-called OWA (Ordered Weighted Averaging) operators, introduced by Yager in Ref. 10. OWA operators are defined for values $a_i \in [0,1]$ $(i = 1, \dots, n)$ as follows:

Definition 2.2. An OWA operator of dimension n is a mapping $OWA_W : [0,1]^n \rightarrow [0,1]$, with an associated weighting vector $W = [w_1 w_2 \dots w_n]^\top$, where $w_i \in [0,1]$, $\sum_i w_i = 1$ and,

$$OWA_W(a_1, \dots, a_n) = \sum_{j=1}^{n} w_j b_j \tag{2.5}$$

where b_j is the jth largest of the a_i values.

Note that a weight w_i is associated with a particular ordered position instead of a particular element, i.e. w_i is associated with the ith largest element in $a_1, \dots, a_n$.

OWA operators are idempotent, continuous, monotone, neutral and compensative.[34,35] In addition, they fulfill some behavioral properties, including the reflection on attitudes in the aggregation, as well as an easy adjusting of the degree of optimism by an appropriate choice of the weights w_i employed. In order to classify an OWA operator basing on its optimism degree accordingly, a measure of *orness* associated with W was introduced, and defined as

$$orness(W) = \frac{1}{n-1}\sum_{i=1}^{n}(n-i)w_i \tag{2.6}$$

While optimistic or OR-LIKE OWA operators are those whose $orness(W) > 0.5$, in pessimistic or AND-LIKE operators we have $orness(W) < 0.5$.[10,37]

Another measure, the *dispersion*,[36] was introduced to let a further distinction amongst different OWA operators with equal optimism degree. This measure is also defined by the vector W as

$$Disp(W) = -\sum_{i=1}^{n} w_i \ln w_i \tag{2.7}$$

This measure can be used as an indicator of the degree to which information contained in values $a_1, \ldots, a_n$ is really used in aggregation.

Yager provides in Ref. 37 a detailed review on different families of OWA operators, including some special cases of them. Next, we point out three widely known special cases of OWA operators:

- Let W^* be the weighting vector defined as $w_1 = 1$ and $w_i = 0$, $i \neq 1$. Then $OWA_{W^*}(a_1, \ldots, a_n) = \max(a_1, \ldots, a_n)$ and $orness(W^*) = 1$.
- Let W_* be the weighting vector defined as $w_n = 1$ and $w_i = 0$, $i \neq n$. Then $OWA_{W_*}(a_1, \ldots, a_n) = \min(a_1, \ldots, a_n)$ and $orness(W_*) = 0$.
- For W_A defined as $w_i = 1/n\ \forall i$, we have the *arithmetic mean*, with and $orness(W_A) = 0.5$.

2.4. *OWA Weights Computation*

Several approaches have been proposed to perform OWA weights identification,[34] including methods based on maximum entropy,[38] through a learning mechanism based on previous observations of decision makers performance,[39] and by semantic considerations by using linguistic quantifiers,[40] as will be considered in this paper, more specifically, with *Regular Increasing Monotone* (RIM) quantifiers.

Linguistic quantifiers, were introduced by Zadeh in Ref. 22. They can be used to semantically express aggregation policies and actually capture Kacprzyk's notion of *soft consensus* in consensus models.[1,25] Examples of linguistic quantifiers are *most, almost all, few, many, all* and *at least one*.[22,40]

OWA weights identification based on linguistic quantifiers is possible thanks to fuzzy set theory.[39,41] Zadeh distinguished in Ref. 22 two types of linguistic quantifiers: absolute and relative. A *relative* linguistic quantifier can be represented as a fuzzy subset Q of the unit interval, where for a given proportion $r \in [0, 1]$, $Q(r)$ indicates the extent to which this proportion satisfies the semantics defined in Q.[40]

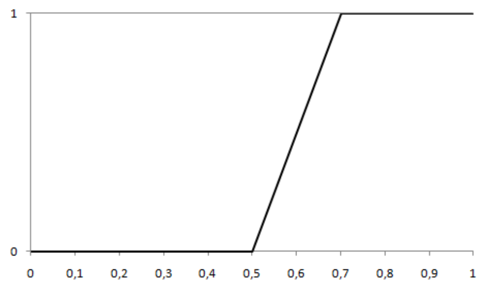

Fig. 3. Example of RIM quantifier $Q =$"*most*".

For example, given $Q =$"*most*" (see Fig. 3), if $Q(0.7) = 1$ then we say that a proportion of 70% totally satisfies the idea conveyed by the quantifier *most*, whereas $Q(0.55) = 0.25$ indicates that the proportion 55% is barely compatible (only 0.25) with this concept.

Yager provided in Refs. 40,42 a further classification of relative linguistic quantifiers into three categories: Regular Increasing Monotone (RIM) quantifier, Regular Decreasing Monotone (RDM) quantifier and Regular UniModal (RUM) quantifier. We are interested in the use of linear RIM quantifiers,[40,43] which are appropriate to apply the notion of *soft consensus* and the concept of fuzzy linguistic majority. RIM quantifiers present the following properties: (i) $Q(0) = 0$, (ii) $Q(1) = 1$ and (iii) if $r_1 > r_2$ then $Q(r_1) \geq Q(r_2)$.

In Ref. 10, Yager proposed the following method to compute OWA weights with the use of RIM quantifiers:

$$w_i = Q\left(\frac{i}{n}\right) - Q\left(\frac{i-1}{n}\right), i = 1, \dots, n \tag{2.8}$$

where the linear membership function of a RIM quantifier $Q(r)$ is defined by the use of two parameters $\alpha, \beta \in [0, 1]$ as

$$Q(r) = \begin{cases} 0 & \text{if } r < \alpha, \\ \dfrac{r-\alpha}{\beta-\alpha} & \text{if } \alpha \leq r \leq \beta, \\ 1 & \text{if } r > \beta. \end{cases} \tag{2.9}$$

Although these parameters are usually denoted in the literature as a and b, here we use an alternative notation, α, β to avoid confusion with previously denoted aggregation elements a_i and b_i in OWA operators.

3. Attitude-OWA Operator in Consensus Reaching Processes

As it was previously pointed out, the aim of this paper is to introduce and manage the concept of group's attitude towards consensus in CRPs, by means of a new aggregation operator based on OWA that allows reflecting and managing this concept regarding the measurement of consensus (see Eq. (2.4)). Therefore, in this section we introduce the idea of attitude towards the achievement of consensus and the so-called Attitude-OWA operator (AOWA) used to reflect it in aggregation. Attitude towards the measurement of consensus must be understood as follows:

- *Optimistic attitude*: those positions (pairs of experts) in the group whose level of agreement is higher will be given more importance in the aggregation process, so that when experts modify their opinions to make them closer each other, the desired consensus degree will be achieved more quickly.
- *Pessimistic attitude*: those positions in the group whose agreement is lower will be given more importance in the aggregation process, so that when experts modify their opinions to make them closer to each other, the desired consensus degree will require more discussion rounds to be achieved.

The choice of an appropriate attitude depends on the prospects considered by experts in the group and the nature of the decision problem to address. Our proposal begins introducing the attitudinal parameters which can be used by the group to reflect their attitude towards consensus, then AOWA operator is defined to capture the reflected attitude in the aggregation of similarity values, and finally the relations between parameters are described. Figure 4 shows with more detail the phase of determining consensus degree previously presented in the general consensus scheme in Fig. 2, including the aggregation step where we are integrating attitudes reflection with the use of AOWA.

3.1. *Attitudinal parameters and AOWA Operator*

In Section 2.4, we reviewed RIM quantifiers and stated the membership function for a linear RIM quantifier upon two parameters α, β. Note that $[\alpha, \beta] \subseteq [0, 1]$ $(\alpha \leq \beta)$, defines the range of proportions r where the membership function $Q(r)$ increases, i.e. the *slope* of the RIM quantifier. Therefore, we have either $Q(r) = 0$ or $Q(r) = 1$ for any r situated to the left or to the right side of the slope, respectively. For a slope $[a, b]$, its amplitude d is defined as $d = \beta - \alpha$. We will denote a RIM quantifier Q, regarding values of α and d, as $Q_{(\alpha,d)}$.

When computing OWA weights from $Q_{(\alpha,d)}$ using Eq. (2.8), non null weights w_i are assigned to elements b_i whose $r = i/n$ is situated inside the quantifier's slope, i.e. $r \in [\alpha, \alpha + d]$. As we can see, α and d can be used to indicate which and how many similarity values are considered in the aggregation. In addition, $orness(W)$ indicates how optimistic this aggregation is. These three elements will be considered the *attitudinal parameters* which can be used by the decision group to reflect an attitude towards consensus.

- $orness(W) \in [0, 1]$ represents the group's attitude in aggregation of pairwise similarities. This attitude can be either optimistic if $orness(W) > 0.5$, pessimistic if $orness(W) < 0.5$ or neutral if $orness(W) = 0.5$.

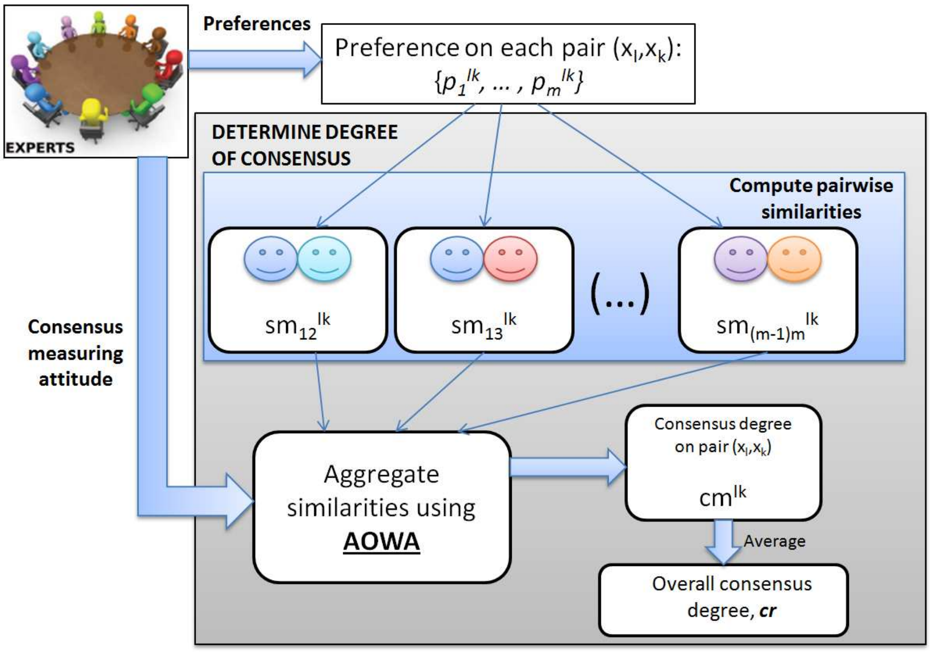

Fig. 4. Procedure to measure consensus based on group's attitude.

- $\alpha \in [0,1]$ indicates whether higher or lower similarity values are assigned a non-null weight when aggregating (assuming these similarities are ranked in decreasing order). The lower α, the higher ranked values are considered.
- $d \in [0,1]$ indicates the amount of similarity values which are given non-null weight and therefore are considered in the aggregation. This parameter has a relation with the dispersion of the corresponding operator. The higher d, the wider range of ranked similarity values are given non-null weight and the higher dispersion in the OWA operator defined by $Q_{(\alpha,d)}$.

These parameters are related to each other, so it is not necessary that the decision group uses all of them to express an attitude towards consensus. We can now define a class of OWA operator so-called AOWA for reflecting specific aggregation attitudes as follows:

Definition 3.1. An AOWA operator of dimension n on $A = \{a_1, \ldots, a_n\}$ is an OWA operator based on two attitudinal parameters ϑ, φ given by a decision group to indicate how to measure consensus between their members,

$$AOWA_W(A, \vartheta, \varphi) = \sum_{j=1}^{n} w_j b_j \tag{3.1}$$

where b_j is the jth largest of the a_i values, $\vartheta, \varphi \in [0,1]$ are two attitudinal parameters and weights W are computed using RIM quantifier $Q_{(\alpha,d)}$. Our proposal focuses on considering $\vartheta = orness(W)$ and $\varphi = d$ as the input attitudinal parameters to define an AOWA operator.

The attitude or optimism degree ϑ of an AOWA operator can be determined given the associated RIM quantifier $Q_{(\alpha,\varphi)}$, when the number of elements to aggregate n is infinitely large, as follows[40,43]

Theorem 3.1. *Let ϑ be the* attitude *of an AOWA operator based on a RIM quantifier denoted as $Q_{(\alpha,\varphi)}$. Then for $n \to \infty$, $\vartheta \in [0,1]$ is determined as follows*

$$\vartheta = 1 - \alpha - \frac{\varphi}{2} \tag{3.2}$$

The detailed analytical proof to obtain this expression is given as follows:

Proof. Based on Eq. (2.6) and Eq. (2.8), we have

$$\vartheta = \frac{1}{n-1}\sum_{i=1}^{n}(n-i)\left[Q\left(\frac{i}{n}\right) - Q\left(\frac{i-1}{n}\right)\right] \tag{3.3}$$

To calculate ϑ when n is infinitely large, $n \to \infty$,

$$\begin{aligned}\vartheta &= \lim_{n\to\infty}\frac{1}{n-1}\sum_{i=1}^{n}(n-i)\left[Q\left(\frac{i}{n}\right) - Q\left(\frac{i-1}{n}\right)\right]\\ &= \lim_{n\to\infty}\frac{1}{n-1}\sum_{i=1}^{n-1}(n-i)\left[Q\left(\frac{i}{n}\right) - Q\left(\frac{i-1}{n}\right)\right]\end{aligned} \tag{3.4}$$

If we consider $P = \sum_{i=1}^{n-1}(n-i)\left[Q\left(\frac{i}{n}\right) - Q\left(\frac{i-1}{n}\right)\right]$ then we have

$$\begin{aligned}P &= \sum_{i=1}^{n-1}\left[n\left[Q\left(\frac{i}{n}\right) - Q\left(\frac{i-1}{n}\right)\right] - i\left[Q\left(\frac{i}{n}\right) - Q\left(\frac{i-1}{n}\right)\right]\right]\\ &= \sum_{i=1}^{n-1}\left[n\left[Q\left(\frac{i}{n}\right) - Q\left(\frac{i-1}{n}\right)\right]\right] - \sum_{i=1}^{n-1}\left[i\left[Q\left(\frac{i}{n}\right) - Q\left(\frac{i-1}{n}\right)\right]\right]\\ &= n\sum_{i=1}^{n-1}\left[Q\left(\frac{i}{n}\right) - Q\left(\frac{i-1}{n}\right)\right] - \sum_{i=1}^{n-1}\left[i\left[Q\left(\frac{i}{n}\right) - Q\left(\frac{i-1}{n}\right)\right]\right]\end{aligned} \tag{3.5}$$

where, expanding it into the sum form, some terms are mutually deleted, having finally

$$\begin{aligned}P &= nQ\left(\frac{n-1}{n}\right) - \left[-\left[\sum_{i=1}^{n-2}Q\left(\frac{i}{n}\right)\right] + (n-1)Q\left(\frac{n-1}{n}\right)\right]\\ &= nQ\left(\frac{n-1}{n}\right) - (n-1)Q\left(\frac{n-1}{n}\right) + \sum_{i=1}^{n-2}Q\left(\frac{i}{n}\right)\\ &= Q\left(\frac{n-1}{n}\right) + \sum_{i=1}^{n-2}Q\left(\frac{i}{n}\right) = \sum_{i=1}^{n-1}Q\left(\frac{i}{n}\right)\end{aligned} \tag{3.6}$$

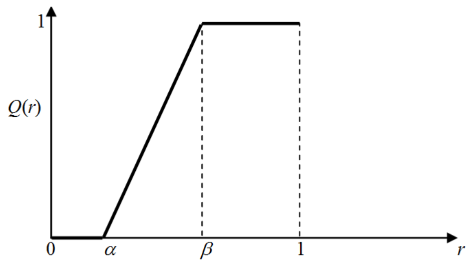

Fig. 5. Membership function in RIM quantifiers considered.

Therefore,

$$\vartheta = \lim_{n\to\infty} \frac{1}{n-1} \sum_{i=1}^{n} (n-i) \left[Q\left(\frac{i}{n}\right) - Q\left(\frac{i-1}{n}\right) \right] = \lim_{n\to\infty} \frac{1}{n-1} \sum_{i=1}^{n-1} Q\left(\frac{i}{n}\right) \tag{3.7}$$

When $n \to \infty$, it follows from the limit definition of definite integral that[40]

$$\vartheta = \lim_{n\to\infty} \frac{1}{n-1} \sum_{i=1}^{n-1} Q\left(\frac{i}{n}\right) = \int_0^1 Q(r)dr = \int_\alpha^1 Q(r)dr \tag{3.8}$$

where $r = i/n$. Notice that the interval $[\alpha, 1]$ defines the *support* of quantifier $Q_{(\alpha,\varphi)}$. The obtained integral states that ϑ is equal to the area under the membership function $Q(r)$. Considering Eq. (2.9) where function $Q(r)$ was defined (see Figure 5), the area under $Q(r)$ is actually as follows:

$$\vartheta = \int_\alpha^1 Q(r)dr = Area(Q) = \left[\frac{1}{2}(\beta-\alpha)\right] + [1-\beta] = \frac{1}{2}\varphi + 1 - (\alpha+\varphi) = 1 - \alpha - \frac{\varphi}{2} \tag{3.9}$$

Notice here that $\beta - \alpha = d = \varphi$. This completes the proof of Theorem 3.1. □

Therefore, ϑ may closely approximate to the result of Eq. (3.2) when measuring consensus in large groups, where a high number of pairs of experts' similarities must be aggregated to measure consensus.

As a result, since we are interested in integrating a group's attitude towards consensus by means of ϑ and φ, we use Eq. (3.2) to determine the value of α, necessary to define the RIM quantifier, as follows:

$$\alpha = 1 - \vartheta - \frac{\varphi}{2} \tag{3.10}$$

3.2. *Relations and restrictions between attitudinal parameters*

Attitudinal parameters' values are related to each other, so it is convenient to clarify some existing relations and restrictions between them. As stated earlier, α and φ are used to univocally define a RIM quantifier $Q_{(\alpha,\varphi)}$, but the following condition must be fulfilled to define a valid RIM quantifier and therefore integrate a valid attitude in the process:

Theorem 3.2. *Given $\alpha, \varphi \in [0, 1]$, a valid attitude given by ϑ can be guaranteed only if $\alpha + \varphi \leq 1$.*

Proof. Let us suppose $\alpha + \varphi > 1$. Considering that $\varphi = \beta - \alpha$, Eq. (3.2) leads to

$$\vartheta = 1 - \alpha - \frac{\varphi}{2} = 1 - \frac{\alpha + \beta}{2} \tag{3.11}$$

where $\frac{\alpha + \beta}{2}$ is the central value of the quantifier's slope, so that

$$\begin{aligned} \alpha \leq \frac{\alpha + \beta}{2} \leq \beta \\ 1 - \alpha \geq 1 - \frac{\alpha + \beta}{2} \geq 1 - \beta \\ 1 - \alpha \geq \vartheta \geq 1 - (\alpha + \varphi) \end{aligned} \tag{3.12}$$

where $\beta = \alpha + \varphi$. Notice here that if $\alpha + \varphi > 1$ as we supposed, then ϑ can be negative, therefore $\alpha + \varphi$ must be equal or less than one to ensure a valid attitude is defined. This completes the proof of Theorem 3.2. □

In order to avoid expressing invalid attitudinal parameters, we present the restrictions to be considered by the decision group when providing them.

Corollary 3.1. *The following condition must be fulfilled when the group provides a value of ϑ:*

$$\frac{\varphi}{2} \leq \vartheta \leq 1 - \frac{\varphi}{2} \tag{3.13}$$

Proof. According to Eq. (3.10), α is negative if $(\vartheta + \varphi/2) > 1$. We need $\alpha \geq 0$, i.e.

$$\begin{aligned} 1 - \vartheta - \frac{\varphi}{2} \geq 0 \\ \vartheta + \frac{\varphi}{2} \leq 1 \\ \vartheta \leq 1 - \frac{\varphi}{2} \end{aligned} \tag{3.14}$$

However, according to Theorem 3.2, it is also necessary to guarantee $\alpha + \varphi \leq 1$. Based on Eq. (3.10) we have,

$$\begin{aligned} \alpha + \varphi = 1 - \vartheta - \frac{\varphi}{2} + \varphi \leq 1 \\ 1 - \vartheta + \frac{\varphi}{2} \leq 1 \\ \vartheta \geq \frac{\varphi}{2} \end{aligned} \tag{3.15}$$

The fulfillment of both inequalities leads to the aforementioned restriction and completes the proof of Corollary 3.1. □

As a result, the higher the proportion of values to consider in aggregation (given by φ), the narrower range of possible attitudes or optimism degrees (given by ϑ) can be considered.

Corollary 3.2. *The following condition must be fulfilled when the group provides a value of* φ*:*

$$\varphi \leq 1 - |2\vartheta - 1| \tag{3.16}$$

Proof. Based on the previous proof in Corollary 3.1, $\alpha \geq 0$ requires

$$\begin{aligned} \vartheta + \frac{\varphi}{2} \leq 1 \quad \textit{i.e.}, \\ \varphi \leq 2(1 - \vartheta) \end{aligned} \tag{3.17}$$

which is valid for $\vartheta \in [0.5, 1]$, but may give rise to $\varphi > 1$ and fail to fulfill Theorem 3.2 when $\vartheta < 0.5$. Let us consider Theorem 3.2 and Eq. (3.10). We then have

$$\begin{aligned} \alpha + \varphi = 1 - \vartheta - \frac{\varphi}{2} + \varphi \leq 1 \\ 1 - \vartheta + \frac{\varphi}{2} \leq 1 \\ \varphi \leq 2\vartheta \end{aligned} \tag{3.18}$$

which is valid for $\vartheta \in [0, 0.5]$ but $\varphi > 1$ may still be possible when $\vartheta > 0.5$, hence a valid quantifier can be defined only if these restrictions are satisfied,

$$\begin{cases} \varphi \leq 2\vartheta & \text{if } \vartheta \in [0, 0.5] \\ \varphi \leq 2(1 - \vartheta) & \text{if } \vartheta \in [0.5, 1] \end{cases} \tag{3.19}$$

We finally proceed to find a single expression which considers both restrictions. On the one hand, we have

$$2\vartheta = 1 - (-2\vartheta + 1) \tag{3.20}$$

where, when $\vartheta \in [0, 0.5]$, the term $(-2\vartheta + 1) \leq 0$. On the other hand,

$$2(1 - \vartheta) = 1 - (2\vartheta - 1) \tag{3.21}$$

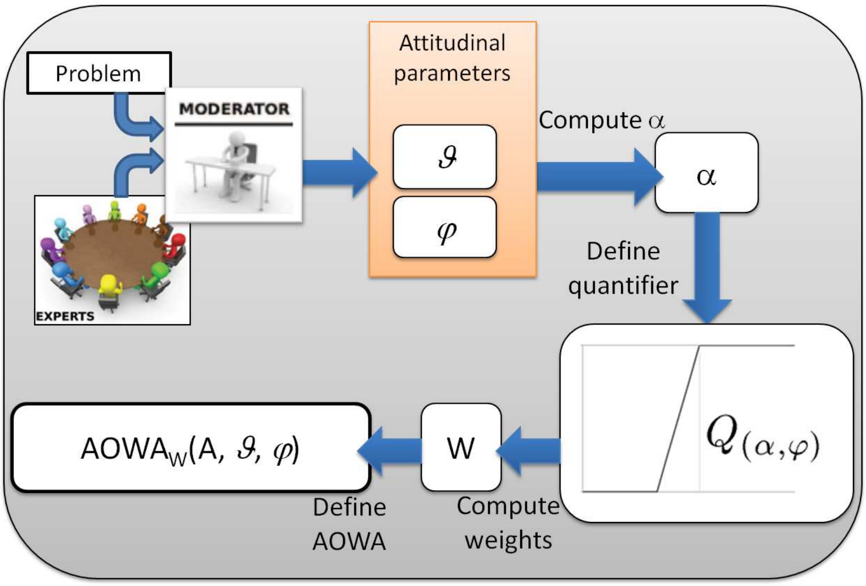

Fig. 6. Process to determine the AOWA operator used to measure consensus based on the group's attitudinal parameters ϑ and φ.

where, when $\vartheta \in [0.5, 0.1]$, the term $(2\vartheta - 1) \geq 0$. This means we can consider the absolute value of the term $(2\vartheta - 1)$ to integrate both restrictions as

$$\varphi \leq 1 - |2\vartheta - 1| \tag{3.22}$$

This completes the proof of Corollary 3.2. □

This restriction can be interpreted as the fact that the closer ϑ is to a neutral attitude (0.5), the wider range of possible degrees for φ can be considered.

If restrictions pointed out in Eqs. (3.13) and (3.16) are taken into account when expressing any two values for input attitudinal parameters (ϑ, φ), then a valid RIM quantifier is always defined, thus guaranteing a valid AOWA operator.

3.3. *Attitude Integration in Consensus Processes*

Once presented the concept of attitude towards consensus and the main features of the AOWA operator used to reflect it, next we describe how this attitude can be integrated in CRPs. We propose conducting the task of expressing group's attitude as part of the *pre-consensus process*,[4] where the moderator is responsible for reflecting a group's attitude towards measuring consensus before the discussion process begins. In order to express the appropriate attitudinal parameters (ϑ, φ), the moderator may consider both the context and characteristics of the decision problem to solve, and the experts' individual concerns and level of knowledge about the domain where the problem is defined. He might also not consider all experts in the group to determine an attitude, but rather a subset of them (due to their status, high level of knowledge, etc.) or even a single person's concerns only, because of his/her core position in the group.

Figure 6 shows the procedure to determine a group's attitude towards the achievement of consensus and integrate it in the CRP, defining the corresponding AOWA operator to measure consensus.

4. Experimental simulation

In this section, we use a multi-agent based consensus support system[30,31] to simulate the resolution of a consensus problem under uncertainty, with different AOWA operators based on different group attitudes to achieve consensus, having a considerable number of experts in the group.

Our main hypothesis mainly focuses on the effect of using different attitudes towards consensus in the process, and states that optimism, given by OR-LIKE operators, may favor a greater convergence towards consensus with a lower number of rounds; whereas pessimism, given by AND-LIKE operators, may favor a lower convergence towards consensus and, therefore, more rounds of discussion are required.

Table 1. Attitudinal parameters and RIM quantifiers used.

Attitude	ϑ	φ	α	$Q_{(\alpha,\varphi)}$
Pessimistic	0.25	0.1	0.7	$Q_{(0.7,0.1)}$
	0.25	0.3	0.6	$Q_{(0.6,0.3)}$
	0.25	0.5	0.5	$Q_{(0.5,0.5)}$
Indifferent	0.5	0.2	0.4	$Q_{(0.4,0.2)}$
	0.5	0.6	0.2	$Q_{(0.2,0.6)}$
	0.5	1	0	$Q_{(0,1)}$
Optimistic	0.75	0.1	0.2	$Q_{(0.2,0.1)}$
	0.75	0.3	0.1	$Q_{(0.1,0.3)}$
	0.75	0.5	0	$Q_{(0,0.5)}$

The GDM problem we have simulated consists of four alternatives $X = \{x_1, x_2, x_3, x_4\}$ and 50 experts, $E = \{e_1, \ldots, e_{50}\}$. In addition, a consensus threshold $\mu = 0.85$ was defined. The experiments consisted in defining a total of nine different attitudes towards consensus, where both optimistic, indifferent and pessimistic attitudes are reflected, and applying the CRP introduced in Section 2.2.2. For each one, three different degrees of the amount of information used, given by φ, have been considered. Table 1 shows the different group attitudes used in simulations, the obtained value of α (see Eq. (3.10)) and the subsequent definition of

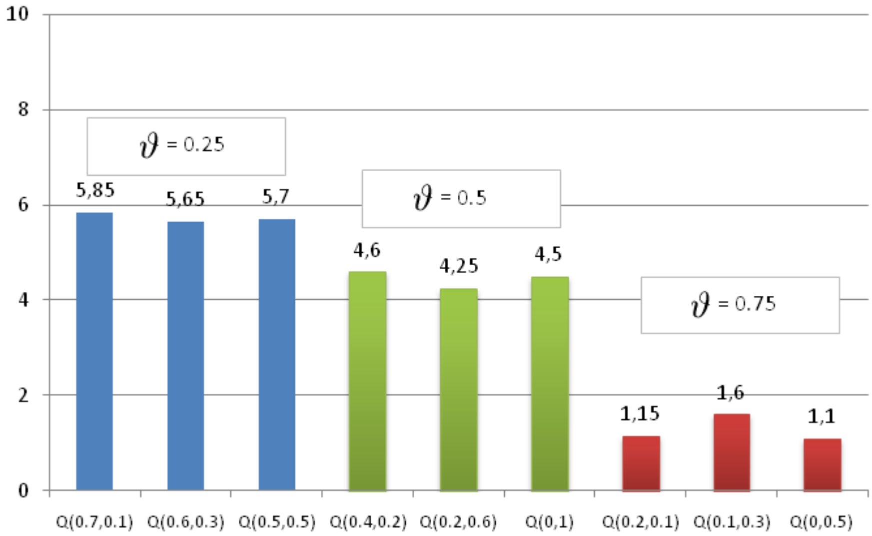

Fig. 7. Average number of required rounds of discussion for RIM quantifier-based AOWA operators with different attitudinal parameters given by ϑ and φ.

nine different RIM quantifiers used in experiments. For each instance of AOWA, 20 experiments have been run.

Results from experiments include the convergence to consensus achieved, i.e. the average number of rounds of discussion required to reach a consensus for each AOWA operator defined upon a RIM quantifier $Q_{(\alpha,\varphi)}$. These results are shown in Figure 7. Results allow us to confirm our hypothesis that the use of AOWA operator based on an optimistic attitude favors a greater convergence towards consensus, whereas the use of AOWA operator based on a pessimistic attitude favors a lower convergence and a further discussion process. Note that attitude is faithfully reflected in the process performance regardless of the proportion of values considered, φ, which is not as influential as ϑ in the process performance.

Therefore, if decision makers' priority is achieving a consensus in a fast discussion process and they don't care about considering rather the highest agreement positions, they would adopt an optimistic attitude. On the other hand, if they consider that the problem requires further discussion and they want to ensure that the even the most disagreing experts finally reach an agreement, they would rather consider a pessimistic attitude.

5. Conclusions and Future works

In this paper, we have presented AOWA operator for expressing group's attitudes in consensus reaching processes. Basing on the definition of attitudinal parameters and their relations, we have proposed an approach where a decision group can easily reflect the attitude they consider towards the measurement of consensus in the group. A simulation has been carried out in an automatic consensus support system in order to prove the importance of integrating these attitudes in the process, having shown the effect of using optimistic/pessimistic attitudes in the number of discussion rounds necessary to achieve an agreement.

Our future works are currently focused on a further analysis of defined attitudinal parameters, as well as introducing the possibility that experts can express their desired attitudes in a linguistic background, thus giving them an even more natural way to provide attitudinal information. We also aim to develop an adaptive consensus model with the integration of dynamic attitudes based on the process performance.

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