A Fuzzy Representation for the Semantics of Hesitant Fuzzy Linguistic Term Sets

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Abstract Recently, a concept of hesitant fuzzy linguistic term sets (HFLTS) to facilitate elicitation of linguistic information when experts hesitate among several linguistic terms to express their preferences in linguistic decision problems was presented. To carry out computations with such type of information, it was introduced the concept of envelope of an HFLTS. This envelope is represented by a symbolic linguistic interval, hence the results do not keep the fuzzy representation. Therefore, this contribution proposes a new fuzzy envelope to represent HFLTS by means of a fuzzy membership function that keeps the fuzzy representation in computational processes with HFLTS.

1 Introduction

Decision-making problems are usually defined under uncertain and imprecise situations. There are different approaches to deal with this type of uncertainty, such as, evidential reasoning approach [11], type-2 fuzzy sets [2], fuzzy linguistic approach [12], etc. The use of linguistic information implies to carry out computing with words (CWW), processes [6, 13] that can be accomplished by different computational models [6]. Usually, in decision-making problems defined in qualitative settings with vague and imprecise information, experts hesitate among

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different linguistic terms and need more flexible and richer expressions than single linguistic terms to express their preferences.

Recently, Rodríguez et al. proposed the concept of hesitant fuzzy linguistic term sets (HFLTS) [7] to improve the elicitation of linguistic information in decision making when experts hesitate among several linguistic terms to express their preferences. This approach allows experts to express their preferences by means of comparative linguistic expressions close to human beings' cognitive model by using context-free grammars that formalize the generation of linguistic expressions. The use of comparative linguistic expressions and HFLTS has been applied in different decision-making problems [7, 8] where computational processes are carried out by the envelope of the HFLTS which is a linguistic interval. This envelope manages in a symbolic way the HFLTS, thus the results obtained lose the initial fuzzy representation.

In this contribution, we propose a new fuzzy envelope for HFLTS that represents the comparative linguistic expressions by means of a fuzzy membership function obtained by aggregating the multiple linguistic terms of the HFLTS with the OWA operator [10]. This fuzzy envelope facilitates the CWW processes in fuzzy decision models [1] that manage HFLTS.

The remainder of the paper is structured as follows: Sect. 2 revises some basic concepts of the HFLTS and OWA operator. Section 3 proposes a fuzzy representation for HFLTS based on a fuzzy membership function. Finally, Sect. 4 draws some conclusions.

2 Preliminaries

This section reviews the elicitation of comparative linguistic expressions represented by HFLTS and the OWA operator, which is used to aggregate the multiple linguistic terms of the HFLTS and obtain the new fuzzy envelope.

2.1 Comparative Linguistic Expressions Represented by Hesitant Fuzzy Linguistic Term Sets

In some hesitant decision situations, experts might hesitate among several linguistic terms to express their preferences and need more flexible expressions than single linguistic terms to provide their assessments in a more precise way. In order to deal with such hesitant situations, Rodríguez et al. introduced the concept of HFLTS [7], which is based on the fuzzy linguistic approach [12].

Definition 1 [7] Let S be a linguistic term set, $S = \{s_0, \ldots, s_g\}$, H_S be an HFLTS, which is an ordered finite subset of consecutive linguistic terms of S,$H_S = \{s_i, s_{i+1}, \ldots, s_j\}$, such that $s_k \in S, k \in \{i, \ldots, j\}$.

However, experts usually do not express their preferences by using multiple linguistic terms, but rather more elaborated expressions similar to those used by humans in decision-making problems. The literature has presented different proposals in [5, 9] that improve the flexibility of the elicitation of linguistic information in hesitant decision situations, but none of them generate expressions close to the human cognitive model or provide a formalization to build the linguistic expressions.

Rodríguez et al. [7] proposed the use of context-free grammars to generate comparative linguistic expressions in a formal way, and defined the following context-free grammar G_H:

Definition 2 [8] Let G_H be a context-free grammar and $S = \{s_0, \ldots, s_g\}$ be a linguistic term set. The elements of $G_H = (V_N, V_T, I, P)$ are defined as follows:

$$
\begin{aligned}
&V_N = \{\langle \text{primary term} \rangle, \langle \text{composite term} \rangle, \langle \text{unary relation} \rangle,\\
&\langle \text{binary relation} \rangle, \langle \text{conjunction} \rangle\}\\
&V_T = \{\text{at most, at least, between, and}, s_0, \ldots, s_g\},\\
&I \in V_N\\
&P = \{I ::= \langle \text{primary term} \rangle | \langle \text{composite term} \rangle\\
&\langle \text{composite term} \rangle ::= \langle \text{unary relation} \rangle \langle \text{primary term} \rangle | \langle \text{binary relation} \rangle\\
&\langle \text{primary term} \rangle \langle \text{conjunction} \rangle \langle \text{primary term} \rangle\\
&\langle \text{primary term} \rangle ::= s_0 | s_1 | \ldots | s_g\\
&\langle \text{unary relation} \rangle ::= \text{at most} | \text{at least}\\
&\langle \text{binary relation} \rangle ::= \text{between}\\
&\langle \text{conjunction} \rangle ::= \text{and}\}.
\end{aligned}
$$

Such expressions cannot be directly used to carry out the computational processes, hence in [8] was introduced a transformation function E_{G_H}, to convert the comparative linguistic expressions into HFLTS.

Definition 3 Let E_{G_H} be a function that transforms linguistic expressions ll, obtained by G_H, into HFLTS $\mathrm{H_S}$, where S is the linguistic term set used by G_H and S_{ll} the expression domain generated by G_H,

$$E_{G_H} : S_{ll} \to H_S. \tag{1}$$

To facilitate the computations with HFLTS, it was proposed the concept of envelope of an HFLTS defined as follows:

Definition 4 [7] The envelope of an HFLTS $env(H_S)$, is a linguistic interval whose limits are obtained by means of its upper and lower bounds:

$$\text{env}(H_S) = [H_{S^-}, H_{S^+}], H_{S^-} \leq H_{S^+} \tag{2}$$

where the upper bound is defined as $H_{S^+} = max\{s_k\}$ and the lower bound $H_{S^-} = \min\{s_k\}$, $\forall s_k \in H_S$, $k \in \{i, \ldots, j\}$

Different decision models introduced in the literature [7, 8] carry out the computations with such linguistic intervals in a symbolic way losing the initial fuzzy representation. Therefore, this contribution proposes a new fuzzy envelope that improves the fuzzy representation for HFLTS.

2.2 OWA Operator

According to the definition of a linguistic variable [12], a linguistic term set has defined a syntax and a semantics given by fuzzy numbers. Therefore, it seems reasonable that the comparative linguistic expressions generated by a context-free grammar are represented by means of fuzzy membership functions. To do so, the fuzzy membership functions of the linguistic terms that compound the HFLTS are aggregated by the OWA operator [10]. The result will be a fuzzy membership function that represents the HFLTS.

Definition 5 [10] Let $A = \{a_1, \ldots, a_n\}$ be a set of n values to aggregate. An OWA operator is a mapping $OWA : R^n \rightarrow R$, with an associated weighting vector $W = (w_1, \ldots, w_n)^T$ where $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$, such that

$$OWA(a_1, \ldots, a_n) = \sum_{j=1}^{n} w_j b_j \tag{3}$$

being b_j the jth largest of the a_i values.

OWA operators can be classified according to their optimism degree by means of the orness measure associated to the weighting vector W.

Taking into account that an HFLTS consists of multiple linguistic terms and the hesitation among different linguistic terms might imply different importance of such terms, the orness measure is used to compute the weights of the linguistic terms of the HFLTS.

Definition 6 [10] The orness measure associated with a weighting vector $W = (w_1, \ldots, w_n)^T$ of an OWA operator is defined as

$$\text{orness}(W) = \sum_{i=1}^{n} w_i \left(\frac{n-i}{n-1} \right). \tag{4}$$

While optimistic or OR-LIKE OWA operators are those whose $\text{orness}(W) > 0.5$, in pessimistic or AND-LIKE operators we have $\text{orness}(W) < 0.5$.

3 Fuzzy Representation for Hesitant Fuzzy Linguistic Term Sets

The use of HFLTS provides a flexible way to manage comparative linguistic expressions in decision-making problems. To carry out CWW processes with HFLTS, we propose a new fuzzy envelope, which is a trapezoidal fuzzy membership function obtained by aggregating the membership functions of the linguistic terms that compound the HFLTS.

This section presents a general process to compute the fuzzy envelope for HFLTS which will be applied for the comparative linguistic expressions generated by the context-free grammar G_H (see Def. 2).

3.1 General Process to Obtain a Fuzzy Envelope

The general process to compute the new fuzzy envelope is divided into four steps as shown in Fig. 1.

1. *Obtaining the parameters to aggregate*: Let $H_S = \{s_i, s_{i+1}, \ldots, s_j\}$ be an HFLTS, such that, $s_k \in S = \{s_0, \ldots, s_g\}$, $k \in \{i, \ldots, j\}$, to obtain the trapezoidal fuzzy membership function, all the linguistic terms that compound the HFLTS are considered. We assume that the linguistic terms $s_k \in S$ are defined by trapezoidal membership functions $A^k = T(a_L^k, a_M^k, a_M^k, a_R^k)$, $k = \{0, \ldots, g\}$ Therefore, the elements to aggregate are the following ones:

$$T = \{a_L^i, a_M^i, a_M^{i+1}, \ldots, a_M^j, a_R^j\}.$$

2. *Computing the fuzzy membership function*: Taking into account that a trapezoidal fuzzy membership function $A = T(a, b, c, d)$ is used to represent a comparative linguistic expression, the definition domain of A should be the same as the linguistic terms $H_S = \{s_i, \ldots, s_j\}$. Therefore, the left and right limits of A are obtained as follows:

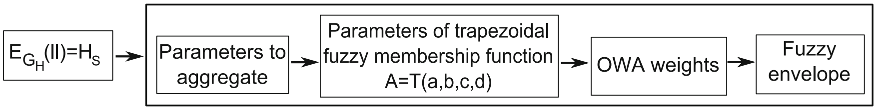

Fig. 1 Scheme of the general process to obtain the fuzzy envelope

$$\begin{aligned} a &= \min\{a_L^i, a_M^i, a_M^{i+1}, \ldots, a_M^j, a_R^j\} = a_L^i, \\ d &= \max\{a_L^i, a_M^i, a_M^{i+1}, \ldots, a_M^j, a_R^j\} = a_R^j. \end{aligned}$$

The parameters b and c are computed by aggregating the remaining elements a_M^i, a_M^{i+1}, …, $a_M^j \in T$ by using the OWA operator,

$$b = OWA_{W^s}(a_M^i, a_M^{i+1}, \ldots, a_M^j), c = OWA_{W^t}(a_M^i, a_M^{i+1}, \ldots, a_M^j).$$

Remark 1 The OWA weights used to compute b and c are in the form W^s and W^t respectively, being $s, t = 1, 2, s \neq t$ or $s = t$. The latter case implies the same form to compute the weights, but the values of the parameters in the two weighting vectors are different, thus the associated weights are different.

3. *Obtaining the importance of the linguistic terms*: The hesitation among the linguistic terms that compound a HFLTS might imply different importance of such terms, which will be reflected by means of the OWA weights. There are different approaches to compute the OWA weights. In this proposal, we have used the method presented in [3].

Definition 7 [3] Let α be a parameter, $\alpha \in [0, 1]$, the OWA weights W^1 are defined as

$$w_1^1 = \alpha, w_2^1 = \alpha(1-\alpha), \ldots, w_{n-1}^1 = \alpha(1-\alpha)^{n-2}, w_n^1 = (1-\alpha)^{n-1}. \tag{5}$$

And the OWA weights W^2 are

$$w_1^2 = \alpha^{n-1}, w_2^2 = (1-\alpha)\alpha^{n-2}, \ldots, w_{n-1}^2 = (1-\alpha)\alpha, w_n^2 = 1-\alpha. \tag{6}$$

We have chosen the OWA weights W^1 and W^2 because they facilitate the computations of the OWA weights with respect to different numbers n, if the parameter α is known for each n. Thus, it is necessary to obtain the parameter α to compute the OWA weights (for further details see [4]).

4. *Obtaining the fuzzy envelope*: The fuzzy envelope $env_F(H_S)$, of an *HFLTS* H_S, is defined as a trapezoidal fuzzy membership function:

$$\mathrm{env}_F(H_S) = T(a,b,c,d)$$

being the parameters (a, b, c, d) computed by the previous steps.

3.2 Using the General Process to Represent the Semantics of Hesitant Fuzzy Linguistic Term Sets

The general process presented previously can be applied for any context-free grammar G, that generates expressions based on HFLTS. We will use the context-free grammar G_H, introduced in Def. 2 and apply the general process for the comparative linguistic expressions obtained with such a grammar.

- *Fuzzy envelope for the expression at least* s_i

This comparative linguistic expression is transformed into HFLTS using the following transformation function:

$$E_{G_H}(\text{at least } s_i) = \{s_i, s_{i+1}, \ldots, s_g\}.$$

To compute the fuzzy envelope, we follow the steps of the general process.

1. *Obtaining the parameters to aggregate*: Taking into account that $a_R^{k-1} = a_M^k = a_L^{k+1}, k = \{1, \ldots, g-1\}$, the elements to aggregate are:

$$T = \{a_L^i, a_M^i, a_M^{i+1}, \ldots, a_M^g, a_R^g\}.$$

2. *Computing the fuzzy membership function*: The parameters of the trapezoidal fuzzy membership function are computed as follows:

$$\begin{aligned} a = &\ \min\{a_L^i, a_M^i, a_M^{i+1}, \ldots, a_M^g, a_R^g\} = a_L^i, \\ d = &\ \max\{a_L^i, a_M^i, a_M^{i+1}, \ldots, a_M^g, a_R^g\} = a_R^g, \end{aligned}$$

$$b = OWA_{W^2}(a_M^i, a_M^{i+1}, \ldots, a_M^g), \tag{7}$$

$$c = OWA_{W^2}(a_M^i, a_M^{i+1}, \ldots, a_M^g). \tag{8}$$

3. *Obtaining the importance of the linguistic terms*: The importance of the linguistic terms of the HFLTS is reflected by means of the computation of the

OWA weights. The weights to compute b are obtained using W^2 with $n = g - i + 1$, where

$$w_1^2 = \alpha^{g-i}, w_2^2 = (1-\alpha)\alpha^{g-i-1}, w_3^2 = (1-\alpha)\alpha^{g-i-2}, \ldots, w_{g-i}^2 = (1-\alpha)\alpha,$$
$$w_{g-i+1}^2 = 1-\alpha. \tag{9}$$

The orness measure $\text{orness}(W^2) > 0.5$ implies the closeness of b to the maximum value, hence the greater importance of the maximum linguistic term s_g in the HFLTS. Therefore, the orness measure $\text{orness}(W^2) < 0.5$ implies the opposite.

The weights to compute c are also in the form of W^2 defined by Eq. (9) with $\alpha = 1$. Therefore $c = a_M^g$.

4. *Obtaining the fuzzy envelope*: Finally, the fuzzy envelope of the comparative linguistic expression at leasts_i, is the trapezoidal fuzzy membership function $env_F(E_{G_H}) = T(a_L^i, b, a_M^g, a_R^g)$.

Remark 2 For a fixed s_i in the linguistic expression at least s_i, if $\alpha \to 0$, then $b \to a_M^i$; if $\alpha \gg 0$, then $b \gg a_M^i$; if $\alpha \to 1$, then $b \to a_M^i$. Therefore, the value α increases from 0 to 1 as s_i increases from s_0 to s_g.

The value α is computed by the following function,

$$f_1 : [0, g] \to [0, 1], \text{such that}\, \alpha = f_1(i) = \frac{i}{g}. \tag{10}$$

The reason for using the associated weighting vector W^2 is explained in [4].

- *Fuzzy envelope for the expression at most s_i*

The HFLTS obtained from this comparative linguistic expression is

$$E_{G_H}(\text{at most}\, s_i) = \{s_0, s_1, \ldots, s_i\}.$$

Following the general process, the fuzzy envelope is computed as follows.

1. *Obtaining the parameters to aggregate*: The arguments to combine are

$$T = \{a_L^0, a_M^0, a_M^1 \ldots, a_M^i, a_R^i\}.$$

2. Computing the fuzzy membership function: The trapezoidal fuzzy membership function $A = T(a, b, c, d)$ is obtained as follows:

$$
\begin{aligned}
a &= \min\{a_L^0, a_M^0, a_M^1, \ldots, a_M^i, a_R^i\} = a_L^0,\\
d &= \max\{a_L^0, a_M^0, a_M^1, \ldots, a_M^i, a_R^i\} = a_R^i,
\end{aligned}
$$

$$b = OWA_{W^1}(a_M^0, a_M^1, \ldots, a_M^i), \tag{11}$$

$$c = OWA_{W^1}(a_M^0, a_M^1, \ldots, a_M^i). \tag{12}$$

3. *Obtaining the importance of the linguistic terms*: To compute the parameter b, first the weights are obtained using W^1 with $n = i + 1$ and $\alpha = 0$, where

$$
\begin{aligned}
&w_1^1 = \alpha, w_2^1 = \alpha(1-\alpha), w_3^1 = \alpha(1-\alpha)^2, \ldots, w_i^1 = \alpha(1-\alpha)^{i-1},\\
&w_{i+1}^1 = (1-\alpha)^i.
\end{aligned}
\tag{13}
$$

Thus, $b = a_M^0$. The parameter c is also computed using Eq. (13).

The orness measure $\text{orness}(W^1) > 0.5$ implies the closeness of c to the maximum value, hence the importance of the maximum linguistic term s_i in the HFLTS. If the orness measure *orness* $(W^2) < 0.5$, the parameter c is closer to the minimum value, hence the linguistic term s_0 has greater importance in the HFLTS.

4. *Obtaining the fuzzy envelope*: The fuzzy envelope of the comparative linguistic expression at most s_i is $\text{env}_F(E_{G_H}) = T(a_L^0, a_M^0, c, a_R^i)$.

Remark 3 For a fixed s_i, if $\alpha \to 0$, then $c \to a_M^0$, if $\alpha \gg 0$, then $c \gg a_M^0$, if $\alpha \to 1$, then $c \to a_M^i$.

The value α is computed by using Eq. (10), and the reason for using the weighting vector W^1 is detailed in [4].

- *Fuzzy envelope for the expression between s_i and s_j*

The transformation function to convert the expression *between s_i and s_j* into HFLTS is the following one:

$$E_{G_H}(\text{between}\, s_i\, \text{and}\, s_j) = \{s_i, s_{i+1}, \ldots, s_j\}.$$

1. *Obtaining the parameters to aggregate*: The elements to combine are

$$T = \{a_L^i, a_M^i, a_M^{i+1}, \ldots, a_M^j, a_R^j\}$$

2. *Computing the fuzzy membership function*: The points a and d of the fuzzy envelope are computed as follows:

$$a = \min\{a_L^i, a_M^i, a_M^{i+1}, \ldots, a_M^j, a_R^j\} = a_L^i,$$
$$d = \max\{a_L^i, a_M^i, a_M^{i+1}, \ldots, a_M^j, a_R^j\} = a_R^j.$$

To compute the points b and c, it is considered the number of the linguistic terms that compound the HFLTS. And we use the OWA operator.

a. If $i + j$ is odd, then

i. If $i + 1 = j$, then the linguistic terms s_i and s_j are equally important, therefore, $b = a_M^i$ and $c = a_M^{i+1}$.

ii. If $i + 1 < j$, then

$$b = OWA_{W^2}\left(a_M^i, a_M^{i+1}, \ldots, a_M^{\frac{i+j-1}{2}}\right), \quad (14)$$

$$c = OWA_{W^1}\left(a_M^j, a_M^{j-1}, \ldots, a_M^{\frac{i+j+1}{2}}\right). \quad (15)$$

b. If $i + j$ is even, then

$$b = OWA_{W^2}\left(a_M^i, a_M^{i+1}, \ldots, a_M^{\frac{i+j}{2}}\right), \quad (16)$$

$$c = OWA_{W^1}\left(a_M^j, a_M^{j-1}, \ldots, a_M^{\frac{i+j}{2}}\right). \quad (17)$$

3. *Obtaining the importance of the linguistic terms*: In this comparative linguistic expression, the weights are obtained by using W^1 and W^2 according to the following situations:

a. If $i + j$ is odd, then the weights $W^2 = (w_1^2, w_2^2, \ldots, w_{(j-i+1)/2}^2)^T$, are computed as follows

$$w_1^2 = \alpha_1^{\frac{j-i-1}{2}}, w_2^2 = (1-\alpha_1)\alpha_1^{\frac{j-i-3}{2}}, \ldots, w_{\frac{j-i-1}{2}}^2 = (1-\alpha_1)\alpha_1, w_{\frac{j-i+1}{2}}^2 = 1-\alpha_1. \quad (18)$$

And the weights $W^1 = (w_1^1, w_2^1, \ldots, w_{(j-i+1)/2}^1)^T$ are

$$w_1^1 = \alpha_2, w_2^1 = \alpha_2(1-\alpha_2), \ldots, w_{\frac{j-i-1}{2}}^1 = \alpha_2(1-\alpha_2)^{\frac{j-i-3}{2}}, w_{\frac{j-i+1}{2}}^1 = (1-\alpha_2)^{\frac{j-i-1}{2}}. \quad (19)$$

b. If $i+j$ is even, then the weights $W^2 = \left(w_1^2, w_2^2, \ldots, w_{(j-i+2)/2}^2\right)^T$, are obtained as

$$w_1^2 = \alpha_1^{\frac{j-i}{2}}, w_2^2 = (1-\alpha_1)\alpha_1^{\frac{j-i-2}{2}}, \ldots, w_{\frac{j-i}{2}}^2 = (1-\alpha_1)\alpha_1, w_{\frac{j-i+2}{2}}^2 = 1-\alpha_1. \quad (20)$$

And the weights $W^1 = \left(w_1, w_2^1, \ldots, w_{(j-i+2)/2}^1\right)^T$ are

$$w_1^1 = \alpha_2, w_2^1 = \alpha_2(1-\alpha_2), \ldots, w_{\frac{j-i}{2}}^1 = \alpha_2(1-\alpha_2)^{\frac{j-i-2}{2}}, w_{\frac{j-i+2}{2}}^1 = (1-\alpha_2)^{\frac{j-i}{2}}. \quad (21)$$

4. *Obtaining the fuzzy envelope*: Finally, it is obtained the fuzzy envelope $\mathrm{env}_F(E_{G_H}) = T\left(a_L^i, b, c, a_R^i\right)$.

   In [4] can be found some properties regarding parameters b and c.

3.3 Examples of Fuzzy Envelopes

Some examples are shown to understand the proposal presented in this contribution. Let $S = \{s_0 : \text{nothing}, s_1 : \text{very low}, s_2 : \text{low}, s_3 : \text{medium}, s_4 : \text{high}, s_5 : \text{very high}, s_6 : \text{perfect}\}$ be the linguistic term set used by the context-free grammar G_H, introduced in Definition 2.

- The fuzzy envelope for the HFLTS, $E_{G_H}(\text{at least} s_4) = \{s_4, s_5, s_6\}$
  - Obtaining the parameters to aggregate:

$$T = \{a_L^4, a_M^4, a_L^5, a_R^4, a_M^5, a_L^6, a_R^5, a_M^6, a_R^6\}$$

where $a_M^4 = a_L^5, a_R^4 = a_M^5 = a_L^6, \text{and} a_R^5 = a_M^6$. Therefore, the elements to aggregate are $T = \{a_L^4, a_M^4, a_M^5, a_M^6, a_R^6\}$.

  - Computing the fuzzy membership function: $\mathrm{env}_F(H_S) = T(a_1, b_1, c_1, d_1)$

$$\begin{aligned} a_1 &= \min\{a_L^4, a_M^4, a_M^5, a_M^6, a_R^6\} = a_L^4 = 0.5, \\ d_1 &= \max\{a_L^4, a_M^4, a_M^5, a_M^6, a_R^6\} = a_R^6 = 1, \\ c_1 &= a_M^6 = 1, \\ b_1 &= w_1 \cdot a_M^6 + w_2 \cdot a_M^5 + w_3 \cdot a_M^4. \end{aligned}$$

- Obtaining the importance of the linguistic terms: To obtain the OWA weights, we use the Eq. 9 with $\alpha = i/g = 4/6$,

$$W^2 = \left(\left(\frac{4}{6}\right)^2, \left(1 - \frac{4}{6}\right) \cdot \frac{4}{6}, \left(1 - \frac{4}{6}\right)\right)^T,$$
$$b_1 = \left(\frac{4}{6}\right)^2 \cdot 1 + \left(1 - \frac{4}{6}\right) \cdot \frac{4}{6} \cdot 0.83 + \left(1 - \frac{4}{6}\right) \cdot 0.67 \approx 0.85.$$

– Obtaining the fuzzy envelope: $\mathrm{env}_F(H_S) = T(0.5, 0.85, 1, 1)$

- The fuzzy envelope for the HFLTS, $E_{G_H}(\text{between}\, s_3 \text{ and}\, s_5) = \{s_3, s_4, s_5\}$
  – Obtaining the parameters to aggregate:

$$T = \{a_L^3, a_M^3, a_M^4, a_M^5, a_R^5\}.$$

– Computing the fuzzy membership function: $\mathrm{env}_F(H_S) = T(a_2, b_2, c_2, d_2)$

$$\begin{aligned} a_2 &= \min\{a_L^3, a_M^3, a_M^4, a_M^5, a_R^5\} = a_L^3 = 0.33, \\ d_2 &= \max\{a_L^3, a_M^3, a_M^4, a_M^5, a_R^5\} = a_R^5 = 1, \\ b_2 &= w_1 \cdot a_M^4 + w_2 \cdot a_M^3, \\ c_2 &= 2a_M^4 - b_3. \end{aligned}$$

– Obtaining the importance of the linguistic terms: Given that $3 + 5$ is even, the OWA weights are computed by using the Eq. (20) with $\alpha_1 = 4/5$.

$$b_3 = \frac{4}{5} \cdot 0.67 + \left(1 - \frac{4}{5}\right) \cdot 0.5 \approx 0.64, c_3 = 0.7$$

– Obtaining the fuzzy envelope: $\mathrm{env}_F(H_S) = T(0.33, 0.64, 0.7, 1)$.

4 Conclusions

In this contribution a fuzzy representation for the semantics of HFLTS is presented by aggregating the membership functions of the linguistic terms of the HFLTS and considering the importance of each term. The result is a fuzzy membership function that facilitates the computational processes with HFLTS.

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