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A Fuzzy Envelope of Hesitant Fuzzy Linguistic Term Set and Its Application to Multicriteria Decision Making

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Abstract

Decision making is a common process for human beings. The uncertainty and fuzziness of problems demand the use of the fuzzy linguistic approach to model qualitative aspects of the decision problems. The recent proposal of hesitant fuzzy linguistic term sets supports the elicitation of comparative linguistic expressions in hesitant situations when experts hesitate among different linguistic terms to provide their assessments. To facilitate the computing with words processes with such expressions was introduced the use of linguistic intervals whose results lose the initial fuzzy representation. The aim of this paper is to present a new representation of the hesitant fuzzy linguistic term sets by means of a fuzzy envelope to carry out the computing with words processes. This new fuzzy envelope can be straightly applied in fuzzy multicriteria decision making models. An illustrative example of its application by using fuzzy TOPSIS in a supplier selection problem is presented.

Keywords: Hesitant fuzzy linguistic term set, fuzzy envelope, comparative linguistic expression, OWA operator, multicriteria decision making.

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1. Introduction

Decision making is a universal process in human beings' life, which can be the final outcome of some mental and reasoning processes that lead to the selection of the best alternative or set of alternatives. Decision making problems [18] are usually defined under uncertain and imprecise situations.

In such cases, it is suitable for experts to provide their preferences or assessments by using linguistic information rather than quantitative values. This fact has led to the use of different approaches, such as, fuzzy logic [33] and fuzzy linguistic approach [34], to model this type of uncertainty and vagueness in decision making problems. The use of linguistic information implies the need of computing with words (CWW) processes [15, 35] that can be accomplished by different linguistic computational models [15]. These models follow the computational scheme depicted in Figure 1 in which Yager [32] highlights the translation and retranslation phases in the CWW processes. The former involves taking linguistic information and translating it into a machine manipulative format, and the latter consists of taking the results from the machine manipulative format and transforming them into linguistic information to facilitate the understanding by human beings, that is one of the main objectives of CWW [16].

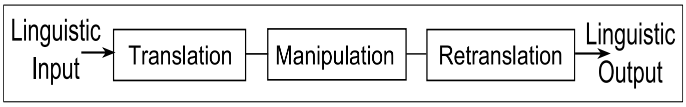

Figure 1: Computing with words scheme

The complexity of real world decision problems is often induced by the uncertainty of the alternatives, and for managing it, the use of linguistic information has provided successful results. However, sometimes it is limited because of the linguistic models use just one linguistic term and experts may not reflect exactly what they mean. Usually, in decision problems defined in a linguistic context with a high degree of uncertainty, experts might hesitate among different linguistic terms and need richer linguistic expressions to express their assessments. Different linguistic proposals have been introduced in the literature to provide richer linguistic expressions than single linguistic terms. Wang and Hao [25] proposed the use of proportional 2-tuple based on the proportion of two consecutive linguistic terms. Ma et al. [14] presented a linguistic model to increase the flexibility of the linguistic expressions merging different single linguistic terms into a new synthesized term. Tang and Zheng [23] introduced another linguistic model that manages linguistic expressions built by logical connectives. Nevertheless, these proposals generate expressions far away from the natural language used by experts in decision problems and do not have any defined formalization.

A recent proposal was introduced by Rodríguez et al. [21] to improve the elicitation of linguistic information in decision making by using hesitant fuzzy linguistic term sets (HFLTS) when experts hesitate among several linguistic terms to express their assessments. This approach provides experts a greater flexibility to elicit comparative linguistic expressions close to human beings' cognitive model by using context-free grammars that formalize the generation of flexible linguistic expressions.

The use of comparative linguistic expressions based on context-free grammars and HFLTS has been applied in different decision making problems [20, 21] in which the computational linguistic model deals with linguistic intervals obtained by the envelope of HFLTS [21] and operates on them by a symbolic model that finally obtains crisp values losing the initial fuzzy representation. Keeping in mind the fuzzy linguistic approach in which the linguistic terms are represented by a syntax and fuzzy semantics, it seems reasonable that the semantics of the comparative linguistic expressions based on a context-free grammar and HFLTS would be represented by fuzzy membership functions that model the uncertainty and vagueness expressed in such comparative linguistic expressions.

The aim of this paper is to introduce a fuzzy representation for comparative linguistic expressions that will be based on a new fuzzy envelope for HFLTS that will represent the expressions by a fuzzy membership function obtained from the multiple linguistic terms that compound the HFLTS, and aggregated by using the OWA operator [27]. Such a fuzzy representation will facilitate the CWW processes in fuzzy multicriteria decision making models [11, 17] that deal with HFLTS. To show the performance of the proposed fuzzy envelope, a supplier selection multicriteria decision making problem is presented and solved by a fuzzy TOPSIS model [2, 5, 26] dealing with comparative linguistic expressions.

The remaining of the paper is structured as follows: Section 2 reviews the fuzzy linguistic approach basis of the HFLTS, the elicitation of comparative linguistic expressions based on context-free grammars and HFLTS, and the OWA operator used to compute the novel fuzzy envelope. Section 3 proposes a fuzzy envelope for HFLTS based on fuzzy membership functions. Section 4 shows the application of the fuzzy envelope in a supplier selection multicriteria decision making problem. And finally, section 5 points out some concluding remarks.

2. Preliminaries

This section reviews the fuzzy linguistic approach basis of the HFLTS, the elicitation of comparative linguistic expressions and some basic concepts about the OWA operator, which is used to obtain the proposed fuzzy envelope for HFLTS.

2.1. Fuzzy linguistic approach

In many real decision situations is straightforward the use of linguistic information rather than numerical information due to the imprecise framework in which are defined such problems. In such situations, the fuzzy linguistic approach [34] represents the linguistic information by means of linguistic variables.

Zadeh introduced the concept of "linguistic variable" as *a variable whose values are not numbers but words or sentences in a natural or artificial language*. It is not so precise as a number but it is closer to human beings' cognitive processes. It is defined as follows:

Definition 1. *[34] A linguistic variable is characterized by a quintuple* $(H, T(H), U, G, M)$ *in which* H *is the name of the variable;* $T(H)$ *is the term set of* H*, i.e., the collection of its linguistic values;* U *is a universe of discourse;* G *is a syntactic rule which generates the terms in* $T(H)$*; and* M *is a semantic rule which associates with each linguistic value* X *its meaning,* $M(X)$ *denotes a fuzzy subset of* U*.*

To deal with linguistic variables, it is necessary to choose appropriate linguistic descriptors of the linguistic terms and their semantics. There are different approaches [19] for such selection. To choose the linguistic descriptors we will use an approach that consists of applying directly the term set by considering all the terms distributed on a scale that has a defined order [30]. In such cases, it is required that a linguistic term set $S = \{s_0, s_1, \ldots, s_g\}$ satisfies the following conditions:

1. An order of the terms of S: $s_i \leq s_j$ if $i \leq j$;

2. A negation operator $Neg(s_i) = s_j$ such that $j = g - i$ ($g + 1$ is the granularity of S);

3. A maximization operator and a minimization operator: $\max(s_i, s_j) = s_i, \min(s_i, s_j) = s_j$ if $i \geq j$.

The usual approach to define the semantics of the linguistic descriptors is based on membership functions [3, 6, 24]. This approach defines the semantics of the linguistic term set by using fuzzy numbers defined in the interval $[0, 1]$, described by membership functions [24].

An efficient method from a computational point of view to obtain a fuzzy number is to use a representation based on parameters of its membership function [1, 6]. Because of the linguistic values provided by experts are approximate assessments, several authors [7, 8] consider that the trapezoidal fuzzy membership functions are good enough to capture and represent the uncertainty and vagueness of such linguistic assessments.

Definition 2. *[34] A fuzzy number $A = T(a, b, c, d)$ is said to be a trapezoidal fuzzy number if its membership function is given by*

$$\mu_A(x) = \begin{cases} 0, & x < a \\ \frac{x-a}{b-a}, & a \leq x \leq b \\ 1, & b < x < c \\ \frac{d-x}{d-c}, & c \leq x \leq d \\ 0, & x > d \end{cases} \tag{1}$$

where the left middle point b and the right middle point c indicate between which the membership degree is 1, with a and d indicating the left and right limits of the definition domain of the trapezoidal membership function.

A special case of this type of membership function is the triangular membership function in which $b = c$.

2.2. Elicitation of comparative linguistic expressions in decision making

Most of linguistic models in decision making [4, 6] provide experts a vocabulary to express their preferences by using single linguistic terms. Nevertheless in the literature, different authors [14, 23, 25] point out the necessity of richer expressions, mainly in decision making problems with high degree of uncertainty in which experts might hesitate among different linguistic terms to express their preferences. Despite these proposals [14, 23, 25] provide higher flexibility to express linguistic expressions in hesitate decision situations, none of them is close to human beings' cognitive model and does not provide rules to generate the linguistic expressions.

Recently, Rodríguez et al. have introduced an approach [21] to improve the elicitation of linguistic information in decision making by using context-free grammars which provide a formal way to build comparative linguistic expressions. A context-free grammar G is a 4-tuple (V_N, V_T, I, P), where V_N is the set of non-terminal symbols, V_T is the set of terminals' symbols, I is the starting symbol, and P the production rules defined in an extended Backus Naur Form [3].

The definition of the context-free grammar G, depends on the decision making problem. Therefore, it is very important to define suitably each element.

In [21] is presented a context-free grammar G_H, that generates comparative linguistic expressions similar to common language used by experts in real world decision making problems. Such comparative linguistic expressions cannot be directly used to carry out the CWW processes, thus in [21] it was defined a transformation function to transform them into HFLTS.

Definition 3. *[21] A HFLTS H_S, is an ordered finite subset of consecutive linguistic terms of $S = \{s_0, \ldots, s_g\}$.*

Example 1. *Let S be a linguistic term set such as $S = \{s_0 : nothing,\ s_1 : very\ bad,\ s_2 : bad,\ s_3 : medium,\ s_4 : good,\ s_5 : very\ good,\ s_6 : perfect\}$ and ϑ be a linguistic variable, a HFLTS might be:*

$$H_S(\vartheta) = \{\textit{very bad, bad, medium}\}.$$

The transformation function is defined as follows,,

$$E_{G_H} : S_{ll} \to H_S \tag{2}$$

where S_{ll} is the expression domain generated by G_H.

This function depends on the comparative linguistic expressions generated by means of the context-free grammar G_H.

To facilitate the computations with HFLTS, it was introduced the concept of *envelope* of a HFLTS.

Definition 4. *[21] The envelope of a HFLTS, $env(H_S)$, is a linguistic interval whose limits are obtained by means of its upper bound and lower bound:*

$$env(H_S) = [H_{S^-}, H_{S^+}],\ H_{S^-} \le H_{S^+}, \tag{3}$$

where the upper bound and lower bound are defined as:

$$\begin{aligned} H_{S^+} &= max\{s_i\} = s_j,\ s_i \le s_j \textit{ and } s_i \in H_S, \forall i, \\ H_{S^-} &= min\{s_i\} = s_j,\ s_i \ge s_j \textit{ and } s_i \in H_S, \forall i. \end{aligned}$$

Following the previous example, the envelope of the HFLTS $H_S(\vartheta)$= $\{very\ bad, bad, medium\}$, is

$$env(H_S(\vartheta)) = [very\ bad, medium].$$

Different operators and models [20, 21] have been defined to compute on such linguistic intervals by a symbolic model that finally obtains crisp values losing the initial fuzzy representation. Therefore, in this paper, we propose a fuzzy representation for comparative linguistic expressions based on a new fuzzy envelope for HFLTS.

2.3. The OWA operator

Taking into account the basis on the fuzzy linguistic approach in which the linguistic terms have defined a syntax and fuzzy semantics, it seems suitable that the semantics of the comparative linguistic expressions are represented by fuzzy membership functions. Hence, to build the new fuzzy envelope for HFLTS, the fuzzy membership functions of the linguistic terms of the HFLTS are aggregated by using the OWA operator [28] to obtain a fuzzy membership function that represents the HFLTS. This operator has been chosen in our proposal because its fundamental aspect of re-ordering adapts to our aim.

Definition 5. *[28] An OWA operator of dimension n is a mapping $OWA: R^n \to R$, such that*

$$OWA(a_1, a_2, \ldots, a_n) = \sum_{j=1}^{n} w_j b_j \tag{4}$$

where b_j is the jth largest of the aggregated arguments $a_1, a_2, \ldots, a_n$, and $W = (w_1, w_2, \ldots, w_n)^T$ is the associated weighting vector satisfying $w_i \in [0, 1]$, $i = 1, 2, \ldots, n$ and $\sum_{i=1}^{n} w_i = 1$.

There are different approaches to compute the OWA weights [9, 13, 29, 31]. We will use one of them [9] that will be defined in Section 3.

A key concept for our proposal is the optimism degree of the OWA operator that can be assessed by means of the orness measure. According to the definition of HFLTS, it is a compound of different linguistic terms, and the hesitation among different linguistic terms might imply different importance of such terms. Thus, the orness measure will be used to compute the importance of the linguistic terms of the HFLTS. It is defined as follows:

Definition 6. *[28] The orness measure associated with a weighting vector* $W = (w_1, w_2, \ldots, w_n)^T$ *of an OWA operator is defined as*

$$orness(W) = \sum_{i=1}^{n} w_i \left(\frac{n-i}{n-1} \right). \tag{5}$$

It is noted that $0 \leq orness(W) \leq 1$.

Optimistic (OR-like) OWA operators are those whose orness (W) > 0.5 whereas pessimistic (AND-like) operators have orness (W) < 0.5 [29].

3. A New Fuzzy Envelope for HFLTS

The use of HFLTS provides a flexible and formal way to deal with comparative linguistic expressions in linguistic decision making. To facilitate the CWW processes based on comparative linguistic expressions, we propose a new fuzzy representation. One possible way to represent such expressions is to use a fuzzy membership function, which is similar to the way of representing the linguistic terms by fuzzy membership functions. To achieve such a fuzzy representation, we take into account the following,

1. The hesitation among different linguistic terms usually implies different importance of such terms.
2. The use of a trapezoidal fuzzy membership function is good enough to capture the vagueness of the comparative linguistic expressions [7, 8].
3. The parameters of the trapezoidal fuzzy membership function are computed by using an aggregation operator that aggregate the fuzzy membership functions of the linguistic terms that compound the HFLTS. Meanwhile, the different importance of the linguistic terms of the HFLTS will be reflected by the aggregation operator.

Therefore, here it is introduced a proposal to obtain a fuzzy envelope for HFLTS, which is a trapezoidal fuzzy membership function obtained by aggregating the fuzzy membership functions of the linguistic terms of the HFLTS according to their relevance. To do so, the OWA aggregation operator [28] is used.

Following it is introduced a general process to compute the fuzzy envelope for HFLTS and later on it is further detailed its application to specific comparative linguistic expressions generated from the context-free grammar G_H.

3.1. Fuzzy envelope for HFLTS: General Process

Let $H_S = \{s_i, s_{i+1}, \ldots, s_j\}$ be a HFLTS, where $s_i, s_j \in S = \{s_0, s_1, \ldots, s_g\}$ and $s_i < s_j$. To compute the fuzzy envelope of the HFLTS a four-step process is carried out (see Figure 2).

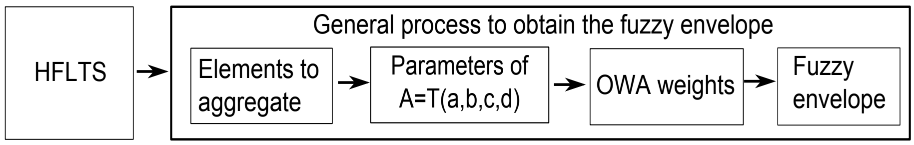

Figure 2: The process to obtain the fuzzy envelope

1. *Obtain the elements to aggregate.*

   To obtain the trapezoidal fuzzy membership function, we need to compute its parameters. In the computational processes, it is reasonable to use all the information contained in the HFLTS, therefore all the linguistic terms in the HFLTS should be considered. We assume that all linguistic terms $s_k \in S$ are defined by trapezoidal (triangular) membership functions $A^k = T(a_L^k, a_M^k, a_M^k, a_R^k),\ k = 0, 1, \ldots, g$. Hence it is straightforward to regard the set of points of all membership functions of the linguistic terms in the HFLTS $H_S = \{s_i, s_{i+1}, \ldots, s_j\}$,

   $$T = \{a_L^i, a_M^i, a_L^{i+1}, a_R^i, a_M^{i+1}, a_L^{i+2}, a_R^{i+1}, \ldots, a_L^j, a_R^{j-1}, a_M^j, a_R^j\}, \quad (6)$$

   as the set of elements to aggregate.

   But for the sake of simplicity, we consider a special case. According to the fuzzy partitions [22], it obtains $a_R^{k-1} = a_M^k = a_L^{k+1},\ k = 1, 2, \ldots, g-1$. In this case, the elements to aggregate are given as

   $$T = \{a_L^i, a_M^i, a_M^{i+1}, \ldots, a_M^j, a_R^j\}. \quad (7)$$

2. *Compute the parameters of the trapezoidal fuzzy membership function.*

   Once obtained the elements to aggregate, we are going to explain how the parameters of the fuzzy membership function are computed.

   Keeping in mind that a trapezoidal fuzzy membership function $A = T(a, b, c, d)$ is used as the representation of the comparative linguistic expressions based on HFLTS H_S, the definition domain of A should be the same as the linguistic terms $s_i, \ldots, s_j \in H_S$. Therefore, we can obtain the left and right limits of A from the left limit of s_i and the right limit of s_j (since $s_i = \min H_S$ and $s_j = \max H_S$). Noting T (see Eq. (6) and (7)) is an ordered set, we use the min and the max operator to compute a and d, i.e.,

- $a = \min\{a_L^i, a_M^i, a_M^{i+1}, \ldots, a_M^j, a_R^j\} = a_L^i$,
- $d = \max\{a_L^i, a_M^i, a_M^{i+1}, \ldots, a_M^j, a_R^j\} = a_R^j$.

The remaining elements $a_M^i, a_M^{i+1}, \ldots, a_M^j \in T$ should contribute to the computation of the parameters b and c. One possible way is to use an aggregation operator to aggregate them. The OWA operator is used because of its aspect of re-ordering and it obtains

- $b = OWA_{W^s}(a_M^i, a_M^{i+1}, \ldots, a_M^j)$,
- $c = OWA_{W^t}(a_M^i, a_M^{i+1}, \ldots, a_M^j)$.

Remark 1. *The OWA weighting vectors for computing b and c are in the form of W^s and W^t respectively, with $s, t = 1, 2,\ s \neq t$ or $s = t$. The latter case implies the same form of the weighting vector but the values of the parameter in the two weighting vectors are different, thus the associated weights are different.*

3. *Obtain the OWA weights.*

As aforementioned, because of the hesitation among the linguistic terms that compound a HFLTS, such terms might have different importance that will be reflected by means of the OWA weights. There are different approaches to compute the OWA weights. We will use the approach presented in [9].

Definition 7. *[9] Let α be a parameter belongs to the unit interval $[0, 1]$. The first kind of OWA weights $W^1 = (w_1^1, w_2^1, \ldots, w_n^1)^T$ is defined as*

$$\begin{aligned}&w_1^1 = \alpha, w_2^1 = \alpha(1-\alpha), w_3^1 = \alpha(1-\alpha)^2, \ldots, w_{n-1}^1 = \alpha(1-\alpha)^{n-2},\\ &w_n^1 = (1-\alpha)^{n-1}.\end{aligned} \tag{8}$$

The second type of OWA weights $W^2 = (w_1^2, w_2^2, \ldots, w_n^2)^T$ *is defined as*

$$w_1^2 = \alpha^{n-1}, w_2^2 = (1-\alpha)\alpha^{n-2}, w_3^2 = (1-\alpha)\alpha^{n-3}, \ldots, w_{n-1}^2 = (1-\alpha)\alpha,\\ w_n^2 = 1-\alpha. \tag{9}$$

There are two reasons that W^1 and W^2 are chosen as the associated weights. One reason is that W^1 and W^2 provide two general classes of OWA weights. Such weights facilitate the computations of the OWA weights with respect to different numbers n if the value of α is known for each n. Thus to determine the OWA weights W^1 and W^2 the value of the parameter α must be determined. The other reason can be seen in Figure 3, in which W^1 and W^2 have the following properties:

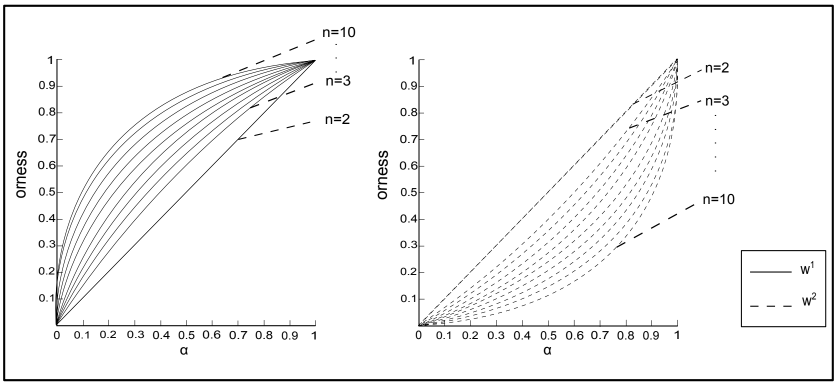

Figure 3: Functional relationship between the orness measure and parameter α of W^1 *and* W^2 for $n = 2, 3, \ldots, 10$ (adapted from [9]).

(a) For a fixed n, the orness measures of W^1 and W^2 increase when α increases;

(b) For a fixed α, the orness measure of W^1 is monotonically increasing with respect to n, while the orness measure of W^2 is monotonically decreasing with respect to n;

(c) For $n = 2$, the orness measures of W^1 and W^2 are equal to α;

(d) For W^1 and W^2, the orness measures approach 0 when α approaches 0, and the orness measures approach 1 when α approaches 1.

Since both of the parameters b and c of the trapezoidal membership function are computed by using the OWA operator, the selection of the weighting vector W^1 or W^2 is also an important aspect. Considering the only difference between W^1 and W^2 is the monotonicity of the orness measure with respect to n, it can be seen that this property will serve as the basis to select the associated weighting vectors for b and c.

4. *Obtain the fuzzy envelope.*

   For a HFLTS H_S, its fuzzy envelope $env_F(H_S)$ can be defined as the trapezoidal fuzzy membership function $T(a, b, c, d)$, i.e.,

$$env_F(H_S) = T(a, b, c, d),$$

   where the parameters of the fuzzy membership function are computed by using the previous steps.

3.2. Fuzzy envelope for comparative linguistic expressions

The general process introduced previously to obtain a fuzzy envelope for HFLTS might be applied to any context-free grammar G, that generates linguistic expressions based on HFLTS. Here we will apply the general process to the comparative linguistic expressions generated by the context-free grammar G_H which are close to common language used by experts in decision making problems.

Definition 8. *Let G_H be a context-free grammar and $S = \{s_0, \ldots, s_g\}$ be a linguistic term set. The elements of $G_H = (V_N, V_T, I, P)$ are defined as follows:*

$V_N = \{\langle \textit{primary term} \rangle,\ \langle \textit{composite term} \rangle,\ \langle \textit{unary relation} \rangle,$
$\quad \langle \textit{binary relation} \rangle,\ \langle \textit{conjunction} \rangle\}$,

$V_T = \{\textit{at most},\ \textit{at least},\ \textit{between},\ \textit{and},\ s_0, \ldots, s_g\}$,

$I \in V_N$.

The production rules are defined in an extended Backus-Naur Form so that the brackets enclose optimal elements and the symbol " | " indicates alternative elements. For the context-free grammar, G_H, the production rules are the following:

$P = \{I ::= \langle \textit{primary term} \rangle | \langle \textit{composite term} \rangle$

$\langle \textit{composite term} \rangle ::= \langle \textit{unary relation} \rangle \langle \textit{primary term} \rangle |\ \langle \textit{binary relation} \rangle$
$\quad \langle \textit{primary term} \rangle \langle \textit{conjunction} \rangle\ \langle \textit{primary term} \rangle$

$\langle \textit{primary term} \rangle ::= s_0 | s_1 | \ldots | s_g$

$\langle$*unary relation*$\rangle$ ::= *at most*|*at least*
$\langle$*binary relation*$\rangle$ ::= *between*
$\langle$*conjunction*$\rangle$ ::= *and*}

The comparative linguistic expressions generated by G_H are transformed into HFLTS by means of the transformation function E_{G_H} as follows,
$E_{G_H}(s_i) = \{s_i | s_i \in S\}$,
$E_{G_H}(\textit{at most } s_i) = \{s_j | s_j \leq s_i \textit{ and } s_j \in S\}$,
$E_{G_H}(\textit{at least } s_i) = \{s_j | s_j \geq s_i \textit{ and } s_j \in S\}$,
$E_{G_H}(\textit{between } s_i \textit{ and } s_j) = \{s_k | s_i \leq s_k \leq s_j \textit{ and } s_k \in S\}$.

3.2.1. Fuzzy envelope for the comparative linguistic expression "at least s_i"

This expression is used by an expert when he/she hesitates among different linguistic terms but he/she is clear about the worst assessment. By using the transformation function, we can obtain the HFLTS as

$$E_{G_H}(\textit{at least } s_i) = \{s_i, s_{i+1}, \ldots, s_g\}.$$

In the following, the general process is applied to obtain the fuzzy envelope $env_F(E_{G_H})$ of the HFLTS, and some properties are then discussed.

1. *Computation of the fuzzy envelope.*

   The fuzzy envelope is computed by using the following steps:

   (a) *Obtain the elements to aggregate.*

   The set of elements to aggregate is

   $$T = \{a_L^i, a_M^i, a_L^{i+1}, a_R^i, a_M^{i+1}, a_L^{i+2}, a_R^{i+1}, \ldots, a_L^g, a_R^{g-1}, a_M^g, a_R^g\}.$$

   Considering $a_R^{k-1} = a_M^k = a_L^{k+1}$, $k = 1, 2, \ldots, g-1$, the elements to aggregate are obtained as

   $$T = \{a_L^i, a_M^i, a_M^{i+1}, \ldots, a_M^g, a_R^g\}.$$

   (b) *Compute the parameters of the trapezoidal fuzzy membership function.*

   In this step, the parameters of the trapezoidal fuzzy membership function $A = T(a, b, c, d)$ are computed as follows.

$$
\begin{aligned}
a &= \min\{a_L^i, a_M^i, a_M^{i+1}, \ldots, a_M^g, a_R^g\} = a_L^i, \\
d &= \max\{a_L^i, a_M^i, a_M^{i+1}, \ldots, a_M^g, a_R^g\} = a_R^g, \\
b &= OWA_{W^2}(a_M^i, a_M^{i+1}, \ldots, a_M^g), \qquad (10) \\
c &= OWA_{W^2}(a_M^i, a_M^{i+1}, \ldots, a_M^g), \qquad (11)
\end{aligned}
$$

being the weights W^2 computed in the next step.

The trapezoidal fuzzy membership function is shown in Figure 4.

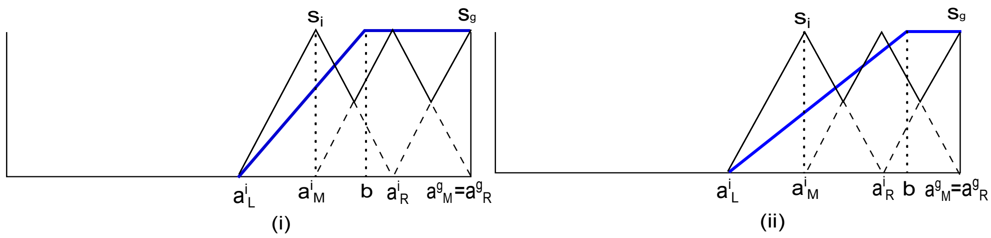

Figure 4: The membership function of $E_{G_H} = \{s_i, s_{i+1}, \ldots, s_g\}$

(c) *The OWA weights.*

The importance of the linguistic terms of the HFLTS obtained from the comparative linguistic expression *at least* s_i will be reflected by the computation of the OWA weights.

The weights used to compute b are in the form of W^2 with $n = g - i + 1$, that is, $W^2 = (w_1^2, w_2^2, \ldots, w_{g-i+1}^2)^T$, where

$$
\begin{aligned}
&w_1^2 = \alpha^{g-i}, w_2^2 = (1-\alpha)\alpha^{g-i-1}, w_3^2 = (1-\alpha)\alpha^{g-i-2}, \ldots, \\
&w_{g-i}^2 = (1-\alpha)\alpha, w_{g-i+1}^2 = 1-\alpha.
\end{aligned} \qquad (12)
$$

On the one hand, the orness measure $orness(W^2) > 0.5$ implies the closeness of b to the maximum value, thus the more importance of the maximum linguistic term s_g in the HFLTS. On the other hand, the orness measure $orness(W^2) < 0.5$ implies the closeness of b to the minimum value, thus the more importance of the minimum linguistic term s_i in the HFLTS.

The weights used to compute c are also in the form of W^2 defined by Eq. (12) with $\alpha = 1$. Therefore $c = a_M^g$.

(d) *The fuzzy envelope.*

For the HFLTS obtained from the comparative linguistic expression *at least* s_i, its fuzzy envelope is defined as the trapezoidal fuzzy membership function $T(a_L^i, b, a_M^g, a_R^g)$, where b is computed by using Eq. (10) with the associated weights W^2 given by Eq. (12).

2. *Discussion of the properties.*

Firstly, it is discussed the properties of the parameter b in the fuzzy envelope $T(a_L^i, b, a_M^g, a_R^g)$, and afterwards it is explained the reason of using W^2 as the associated weighting vector.

Theorem 1. *The parameter* b *defined by Eq. (10) in the fuzzy envelope* $T(a_L^i, b, a_M^g, a_R^g)$*, has the following properties:*

(a) $0 \leq a_M^i \leq b \leq a_M^g = 1$*;*

(b) For a fixed s_i *in the linguistic expression at least* s_i*, if* $\alpha \to 0$*, then* $b \to a_M^i$*; if* $\alpha \gg 0$*, then* $b \gg a_M^i$*; if* $\alpha \to 1$*, then* $b \to a_M^g$*.*

Proof. *(a) Since* $\min\{a_M^i, \ldots, a_M^g\} = a_M^i \geq 0$, $\max\{a_M^i, \ldots, a_M^g\} = a_M^g = 1$ *and the aggregation result of the OWA operator is between the minimum and the maximum of the aggregated values, the result holds.*

(b) If $\alpha \to 0$*, then* $w_1^2 \to 0, \ldots, w_{g-i}^2 \to 0, w_{g-i+1}^2 \to 1$*, and then* $b \to a_M^i$*.*

If $\alpha \gg 0$*, then* $w_1^2, w_2^2, \ldots, w_{g-i}^2 \gg 0, w_{g-i+1}^2 \ll 1$ *and we have* $b \gg a_M^i$*.*

If $\alpha \to 1$*, then* $w_1^2 \to 1, w_2^2 \to 0, \ldots, w_{g-i+1}^2 \to 0$*, and then* $b \to a_M^g$*.* □

Remark 2. *If* $s_i \to s_0$*, then* $\alpha \to 0$ *and* $b \to a_M^0 = 0$*. If* $s_i \to s_g$*, then* $\alpha \to 1$ *and* $b \to a_M^g = 1$*. If* $s_0 < s_i < s_g$*, then* $0 < \alpha < 1$ *and* $a_M^i < b < a_M^g$*. The value* α *increases from 0 to 1 as* s_i *increases from* s_0 *to* s_g*.*

According to Remark 2, the value α depends on the linguistic term s_i, thus it depends on the value $i = index(s_i)$. In order to compute α, we define a function

$$f_1 : [0, g] \to [0, 1], \ \textit{such that} \ \alpha = f_1(i),$$

which satisfies the boundary conditions

$$f_1(0) = 0, \ f_1(g) = 1.$$

For simplicity, we assume that f_1 is a linear function, that is

$$f_1(i) = \beta i + \gamma,$$

where β, γ are unknown parameters. Considering the boundary conditions, we can obtain the form of f_1 as:

$$f_1(i) = \frac{i}{g}.$$

Thus,

$$\alpha = \frac{i}{g} = \frac{i}{(g+1)-1}, \tag{13}$$

where $i = index(s_i)$, and $g+1$ is the granularity of the linguistic term set $S = \{s_0, s_1, \ldots, s_g\}$.

Let us analyse the reason that W^2 is used as the associated weighting vector. To avoid too much uncertainty, the linguistic term s_i in the comparative linguistic term *at least* s_i should satisfy $s_0 \ll s_i < s_g$. From Eq. (13), we see that for a fixed linguistic term set $S = \{s_0, \ldots, s_g\}$, the value of α is determined by i. Considering $s_0 \ll s_i < s_g$ and thus $0 \ll i < g$, it is obtained that $0 \ll \alpha < 1$. From Figure 3, we see that for W^2, when $\alpha \gg 0$, the difference of the orness measure among different values of n is greater than for W^1. Thus, if W^2 is used as the associated weighting vector to compute the points b_1 and b_2 of two trapezoidal fuzzy membership functions $A = T(a_1, b_1, c_1, d_1)$ and $B = T(a_2, b_2, c_2, d_2)$ of two HFLTS, which are generated from two different linguistic expressions *at least* s_{i_1} and *at least* s_{i_2} $(i_1 \neq i_2)$ respectively, the difference between b_1 and b_2, $|b_1 - b_2|$, will be greater than the difference between them if W^1 is used as the associated weighting vector.

3.2.2. Fuzzy envelope for the comparative linguistic expression "at most s_i"

This expression is used when a decision maker hesitates among several linguistic terms but he/she is clear about the best assessment. The HFLTS generated from this linguistic expression is

$$E_{G_H}(at\ most\ s_i) = \{s_0, s_1, \ldots, s_i\}.$$

1. *Computation of the fuzzy envelope.*

   To obtain the fuzzy envelope, it is applied the general process.

   (a) *Obtain the elements to aggregate.*

   The set of elements to aggregate is

$$T = \{a_L^0, a_M^0, a_L^1, a_R^0, a_M^1, a_L^2, a_R^1, \ldots, a_L^i, a_R^{i-1}, a_M^i, a_R^i\}.$$

   Considering $a_R^{k-1} = a_M^k = a_L^{k+1},\ k = 1, 2, \ldots, g$, the elements to aggregate are obtained as

$$T = \{a_L^0, a_M^0, a_M^1, \ldots, a_M^i, a_R^i\}.$$

   (b) *Parameters of the trapezoidal fuzzy membership function.*

   The parameters of the trapezoidal fuzzy membership function $A = T(a, b, c, d)$ are computed as

$$\begin{aligned}
a &= \min\{a_L^0, a_M^0, a_M^1, \ldots, a_M^i, a_R^i\} = a_L^0, \\
d &= \max\{a_L^0, a_M^0, a_M^1, \ldots, a_M^i, a_R^i\} = a_R^i, \\
b &= OWA_{W^1}(a_M^0, a_M^1, \ldots, a_M^i), && (14) \\
c &= OWA_{W^1}(a_M^0, a_M^1, \ldots, a_M^i), && (15)
\end{aligned}$$

   being the weights W^1 in Eq. (14) and Eq. (15) computed by using different parameters in the next step.

   The trapezoidal fuzzy membership function is shown in Figure 5.

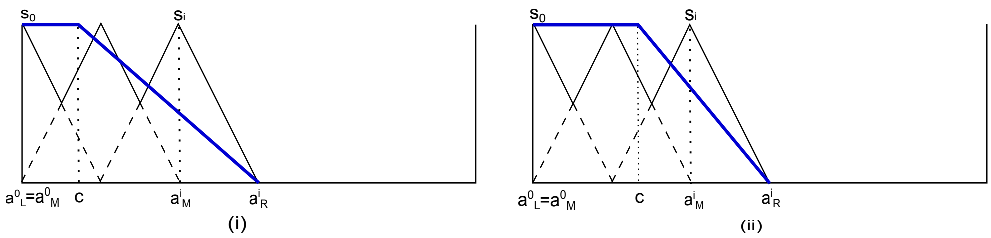

Figure 5: The membership function of $E_{G_H} = \{s_0, s_1, \ldots, s_i\}$

(c) *The OWA weights.*

The linguistic terms of the HFLTS obtained from the comparative linguistic expression *at most* s_i might have different importance which will be reflected by the OWA weights.

The weights used to compute b is W^1 with $n = i + 1$ and $\alpha = 0$, that is, $W^1 = (w_1^1, w_2^1, \ldots, w_{i+1}^1)^T$, where

$$\begin{aligned} &w_1^1 = \alpha, w_2^1 = \alpha(1-\alpha), w_3^1 = \alpha(1-\alpha)^2, \ldots, w_i^1 = \alpha(1-\alpha)^{i-1}, \\ &w_{i+1}^1 = (1-\alpha)^i. \end{aligned} \tag{16}$$

Therefore, $b = a_M^0$.

The weighting vector to compute c is also in the form of W^1 given by Eq. (16).

On the one hand, the orness measure $orness(W^1) > 0.5$ implies the closeness of c to the maximum value, thus the more importance of the maximum linguistic term s_i in the HFLTS. On the other hand, the orness measure $orness(W^1) < 0.5$ implies the closeness of c to the minimum value, thus the more importance of the minimum linguistic term s_0 in the HFLTS.

(d) *The fuzzy envelope.*

For the HFLTS obtained from the comparative linguistic expression *at most* s_i, the fuzzy envelope $env_F(E_{G_H})$ is defined as the trapezoidal fuzzy membership function $T(a_L^0, a_M^0, c, a_R^i)$, where c is computed by using Eq. (15) with the associated weights W^1 given by Eq. (16).

2. *Discussion of the properties.*

Here, it is discussed some properties of the parameter c in the fuzzy envelope $T(a_L^0, a_M^0, c, a_R^i)$, the connection between the comparative linguistic expressions *at least* and *at most*, and the reason to choose W^1 as the associated weights.

Theorem 2. *The parameter c defined by Eq. (15), in the fuzzy envelope* $T(a_L^0, a_M^0, c, a_R^i)$*, has the following properties:*

(a) $0 = a_M^0 \leq c \leq a_M^i \leq 1$*;*

(b) For a fixed s_i*, if* $\alpha \to 0$*, then* $c \to a_M^0$*, if* $\alpha \gg 0$*, then* $c \gg a_M^0$*, if* $\alpha \to 1$*, then* $c \to a_M^i$*.*

The proof of the theorem is similar to Theorem 1.

Remark 3. *If $s_i \to s_0$, then $\alpha \to 0$ and $c \to a_M^0 = 0$. If $s_i \to s_g$, then $\alpha \to 1$ and $c \to a_M^g = 1$. If $s_0 < s_i < s_g$, then $0 < \alpha < 1$ and $a_M^0 < c < a_M^g$. The value α increases from 0 to 1 as s_i increases from s_0 to s_g.*

Considering the Remark 3, we can obtain the value of α in the same way as the comparative linguistic expression *at least*, i.e.,

$$\alpha = \frac{i}{g} = \frac{i}{(g+1)-1}. \tag{17}$$

where $i = index(s_i)$, and $g+1$ is the granularity of the linguistic term set $S = \{s_0, \ldots, s_g\}$.

Let us analyze the reason that W^1 is chosen as the associated weighting vector. In order to avoid too much uncertainty, the linguistic term s_i in the comparative linguistic term *at most* s_i should satisfy $s_0 < s_i \ll s_g$. From Eq. (17), we see that for a fixed linguistic term set $S = \{s_0, \ldots, s_g\}$, the value of α is determined by $i = index(s_i)$. Considering $s_0 < s_i \ll s_g$ and $0 < i \ll g$, it is obtained that $0 < \alpha \ll 1$. From Figure 3, we see that for $\alpha \ll 1$ and W^1, the difference of the orness measure among different values of n is greater than W^2. Thus, if W^1 is used as the associated weighting vector to compute the points c_1 and c_2 of two trapezoidal fuzzy membership functions $A = T(a_1, b_1, c_1, d_1)$ and $B = T(a_2, b_2, c_2, d_2)$ of two HFLTS, which are generated from two linguistic expressions *at most* s_{i_1} and *at most* s_{i_2} ($i_1 \neq i_2$) respectively, the difference between c_1 and c_2, $|c_1 - c_2|$, will be greater than the difference between them if W^2 is used as the associated weighting vector.

The connection between the comparative linguistic expression *at least* and *at most* is shown in the following theorem.

Theorem 3. *Let $S = \{s_0, \ldots, s_g\}$ be a linguistic term set, and $A = T(a_L^i, b, a_M^g, a_R^g)$ be the fuzzy envelope of the HFLTS based on at least s_i, and $B = T(a_L^0, a_M^0, c, a_R^i)$ be the fuzzy envelope of the HFLTS based on at most s_{g-i}, where b and c are computed by Eq. (10) and Eq. (15) respectively. Then b and c satisfy $b + c = 1$.*

Proof. *The associated weighting vector of the OWA aggregation operator to compute b is*

$$W^2 = \left(\alpha_1^{g-i}, (1-\alpha_1)\alpha_1^{g-i-1}, (1-\alpha_1)\alpha_1^{g-i-2}, \ldots, (1-\alpha_1)\alpha_1, 1-\alpha_1\right)^T$$

with $\alpha_1 = i/g$.

The associated weighting vector of the OWA aggregation operator to compute c is

$$W^1 = \left(\alpha_2, \alpha_2(1-\alpha_2), \alpha_2(1-\alpha_2)^2, \ldots, \alpha_2(1-\alpha_2)^{g-i-1}, (1-\alpha_2)^{g-i}\right)^T$$

with $\alpha_2 = (g-i)/g$.

Thus $\alpha_1 + \alpha_2 = 1$ *and*

$$\begin{aligned} b &= \alpha_1^{g-i} a_M^g + (1-\alpha_1)\alpha_1^{g-i-1} a_M^{g-1} + \cdots + (1-\alpha_1)\alpha_1 a_M^{i+1} + (1-\alpha_1) a_M^i, \\ c &= \alpha_2 a_M^{g-i} + \alpha_2(1-\alpha_2) a_M^{g-i-1} + \cdots + \alpha_2(1-\alpha_2)^{g-i-1} a_M^1 + (1-\alpha_2)^{g-i} a_M^0. \end{aligned}$$

Since $a_M^j + a_M^{g-j} = 1,\ j = 0, 1, \ldots, i,$ *then*

$$b + c = \alpha_1^{g-i} + (1-\alpha_1)\alpha_1^{g-i-1} + \cdots + (1-\alpha_1) = 1. \qquad \square$$

3.2.3. Fuzzy envelope for the comparative linguistic expression "between s_i and s_j"

By using the transformation function, we can obtain the HFLTS based on the comparative linguistic expression "*between s_i and s_j*" as

$$E_{G_H}(between\ s_i\ and\ s_j) = \{s_i, s_{i+1}, \ldots, s_j\}.$$

Remark 4. *When $s_i < s_j = s_g$, the expression coincides with at least s_i. When $s_0 = s_i < s_j$, the expression coincides with at most s_j. To avoid these cases, a constraint is given as $s_0 < s_i < s_j < s_g$.*

Firstly, the general process is applied to obtain the fuzzy envelope $env_F(E_{G_H})$ of the HFLTS, and then some properties are discussed.

1. *Computation of the fuzzy envelope.*

   The fuzzy envelope is computed by using the following steps:

(a) *Obtain the elements to aggregate.*

The set of elements to aggregate is

$$T = \{a_L^i, a_M^i, a_L^{i+1}, a_R^i, a_M^{i+1}, a_L^{i+2}, a_R^{i+1}, \ldots, a_L^g, a_R^{g-1}, a_M^g, a_R^g\}.$$

Considering $a_R^{k-1} = a_M^k = a_L^{k+1},\ k = 1, 2, \ldots, g-1$, the elements to aggregate are obtained as

$$T = \{a_L^i, a_M^i, a_M^{i+1}, \ldots, a_M^j, a_R^j\}.$$

(b) *Compute the parameters of the trapezoidal fuzzy membership function.*

In this phase, the parameters a and d of the trapezoidal fuzzy membership function $A = T(a, b, c, d)$ are computed as follows:

$$\begin{aligned} a &= \min\{a_L^i, a_M^i, a_M^{i+1}, \ldots, a_M^j, a_R^j\} = a_L^i, \\ d &= \max\{a_L^i, a_M^i, a_M^{i+1}, \ldots, a_M^j, a_R^j\} = a_R^j. \end{aligned}$$

Meanwhile, the points b and c (see Figure 6) are computed by using the OWA operator and taking into account the number of the linguistic terms in the HFLTS generated by the comparative linguistic expression.

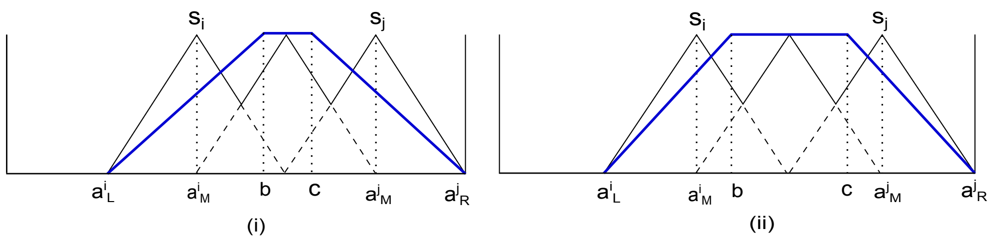

Figure 6: The membership function of $E_{G_H} = \{s_i, s_{i+1}, \ldots, s_j\}$

i. If $i + j$ is odd, then

A. If $i + 1 = j$, then we need not to use the OWA operator to compute b and c. We can obtain $b = a_M^i,\ c = a_M^{i+1}$ directly. In this case, the linguistic terms s_i and s_j are equally important;

B. If $i+1<j$, then

$$b = OWA_{W^2}\left(a_M^i, a_M^{i+1}, \dots, a_M^{\frac{i+j-1}{2}}\right), \tag{18}$$
$$c = OWA_{W^1}\left(a_M^j, a_M^{j-1}, \dots, a_M^{\frac{i+j+1}{2}}\right), \tag{19}$$

being the associated weights further detailed later on.

ii. If $i+j$ is even, then

$$b = OWA_{W^2}\left(a_M^i, a_M^{i+1}, \dots, a_M^{\frac{i+j}{2}}\right), \tag{20}$$
$$c = OWA_{W^1}\left(a_M^j, a_M^{j-1}, \dots, a_M^{\frac{i+j}{2}}\right), \tag{21}$$

being the associated weights introduced later on.

(c) *The OWA weights.*
In this comparative linguistic expression the importance of the linguistic terms of the HFLTS will be reflected by the computation of the OWA weights by using W^1 and W^2. The weights are computed according to the following two cases:

i. If $i+j$ is odd, then the OWA weights in Eq. (18) are $W^2 = (w_1^2, w_2^2, \dots, w_{(j-i+1)/2}^2)^T$, being

$$w_1^2 = \alpha_1^{\frac{j-i-1}{2}},\ w_2^2 = (1-\alpha_1)\alpha_1^{\frac{j-i-3}{2}},\ \dots,\ w_{\frac{j-i-1}{2}}^2 = (1-\alpha_1)\alpha_1,\ w_{\frac{j-i+1}{2}}^2 = 1-\alpha_1. \tag{22}$$

The OWA weights in Eq. (19) are $W^1 = (w_1^1, w_2^1, \dots, w_{(j-i+1)/2}^1)^T$:

$$w_1^1 = \alpha_2,\ w_2^1 = \alpha_2(1-\alpha_2),\ \dots,\ w_{\frac{j-i-1}{2}}^2 = \alpha_2(1-\alpha_2)^{\frac{j-i-3}{2}},\ w_{\frac{j-i+1}{2}}^2 = (1-\alpha_2)^{\frac{j-i-1}{2}}. \tag{23}$$

ii. If $i+j$ is even, then the OWA weights in Eq. (20) are $W^2 = \left(w_1^2, w_2^2, \dots, w_{(j-i+2)/2}^2\right)^T$, where

$$w_1^2 = \alpha_1^{\frac{j-i}{2}}, w_2^2 = (1-\alpha_1)\alpha_1^{\frac{j-i-2}{2}}, \ldots, w_{\frac{j-i}{2}}^2 = (1-\alpha_1)\alpha_1,$$
$$w_{\frac{j-i+2}{2}}^2 = 1-\alpha_1. \tag{24}$$

The OWA weights in Eq. (21) are $W^1 = \left(w_1, w_2^1, \ldots, w_{(j-i+2)/2}^1\right)^T$, where

$$w_1^1 = \alpha_2, w_2^1 = \alpha_2(1-\alpha_2), \ldots, w_{\frac{j-i}{2}}^2 = \alpha_2(1-\alpha_2)^{\frac{j-i-2}{2}},$$
$$w_{\frac{j-i+2}{2}}^2 = (1-\alpha_2)^{\frac{j-i}{2}}. \tag{25}$$

(d) *The fuzzy envelope.*

For the HFLTS obtained from the comparative linguistic expression *between* s_i *and* s_j, its fuzzy envelope $env_F(E_{G_H})$ is defined as the trapezoidal fuzzy membership function $A = T(a_L^i, b, c, a_R^j)$, where b and c are computed by using Eq. (18) and (19), or Eq. (20) and (21).

2. *Discussion of the properties.*

First we discuss the properties of the parameters b and c.

Theorem 4. *The parameters b given by Eq. (18) or Eq. (20), and the parameter c given by Eq. (19) or Eq. (21), have the following properties:*

(a) If $i+j$ is odd, then

i. if $\alpha_1 \to 0$, then $b \to a_M^i$, if $\alpha_1 \to 1$, then $b \to a_M^{(i+j-1)/2}$, if $0 < \alpha_1 < 1$, then $a_M^i < \alpha_1 < a_M^{(i+j-1)/2}$;

ii. if $\alpha_2 \to 0$, then $b \to a_M^{(i+j+1)/2}$, if $\alpha_2 \to 1$, then $b \to a_M^j$, if $0 < \alpha_2 < 1$, then $a_M^{(i+j+1)/2} < \alpha_1 < a_M^j$.

(b) If $i+j$ is even, then

i. if $\alpha_1 \to 0$, then $b \to a_M^i$, if $\alpha_1 \to 1$, then $b \to a_M^{(i+j)/2}$, if $0 < \alpha_1 < 1$, then $a_M^i < \alpha_1 < a_M^{(i+j)/2}$;

ii. *if* $\alpha_2 \to 0$*, then* $b \to a_M^{(i+j)/2}$*, if* $\alpha_2 \to 1$*, then* $b \to a_M^j$*, if* $0 < \alpha_2 < 1$*, then* $a_M^{(i+j)/2} < \alpha_1 < a_M^j$*.*

The proof of this theorem is similar to Theorem 1 and Theorem 2.

Remark 5. *If* $i + j$ *is odd,* $i + 1 < j$*, and* $0 < \alpha_1, \alpha_2 < 1$*, then the linguistic terms* $s_{(i+j-1)/2}$ *and* $s_{(i+j+1)/2}$ *are the most important terms. If* $i + j$ *is even and* $0 < \alpha_1, \alpha_2 < 1$*, then the linguistic term* $s_{(i+j)/2}$ *is the most important term. Thus, the linguistic terms between* s_i *and* s_j *(*$i+1 < j$*) are the most important in the linguistic expression between* s_i *and* s_j*.*

The parameters b and c have a relation shown by the following theorem.

Theorem 5. *If* $\alpha_1 + \alpha_2 = 1$*, then* b *and* c *computed by Eq. (18) and (19) respectively, are symmetric to the middle point of* $a_M^{(i+j-1)/2}$ *and* $a_M^{(i+j+1)/2}$*, i.e.,*

$$b + c = a_M^{\frac{i+j-1}{2}} + a_M^{\frac{i+j+1}{2}}. \tag{26}$$

Proof. *Since* $\alpha_1 + \alpha_2 = 1$*, then* $\alpha_2 = 1 - \alpha_1$*. For simplicity, let* $\delta = 1/g$*, then* $a_M^i = i\delta$*. Thus*

$$\begin{aligned}
b + c &= \alpha_1^{\frac{j-i-1}{2}} a_M^{\frac{i+j-1}{2}} + (1-\alpha_1)\alpha_1^{\frac{j-i-3}{2}} a_M^{\frac{i+j-3}{2}} + \cdots \\
&\quad + (1-\alpha_1)\alpha_1 a_M^{i+1} + (1-\alpha_1) a_M^i + \alpha_2 a_M^j + \alpha_2(1-\alpha_2) a_M^{j-1} \\
&\quad + \cdots + \alpha_2(1-\alpha_2)^{\frac{j-i-3}{2}} a_M^{\frac{i+j+3}{2}} + (1-\alpha_2)^{\frac{j-i-1}{2}} a_M^{\frac{i+j+1}{2}} \\
&= \alpha_1^{\frac{j-i-1}{2}} \left(a_M^{\frac{i+j-1}{2}} + a_M^{\frac{i+j+1}{2}}\right) + (1-\alpha_1)\alpha_1^{\frac{j-i-3}{2}} \left(a_M^{\frac{i+j-3}{2}} + a_M^{\frac{i+j+3}{2}}\right) \\
&\quad + \cdots + (1-\alpha_1)\alpha_1 \left(a_M^{i+1} + a_M^{j-1}\right) + (1-\alpha_1)\left(a_M^i + a_M^j\right) \\
&= \left[\alpha_1^{\frac{j-i-1}{2}} + (1-\alpha_1)\alpha_1^{\frac{j-i-3}{2}} + \cdots + (1-\alpha_1)\right](i+j)\delta \\
&= (i+j)\delta = a_M^{\frac{i+j-1}{2}} + a_M^{\frac{i+j+1}{2}}.
\end{aligned}$$

□

Remark 6. *This theorem indicates that if* $\alpha_1 + \alpha_2 = 1$*, then one value of* b *and* c *can be computed from the other one.*
If $\Delta = a_M^{(i+j-1)/2} - b$*, then* $c = a_M^{(i+j+1)/2} + \Delta$*.*
If $\Delta = c - a_M^{(i+j+1)/2}$*, then* $b = a_M^{(i+j-1)/2} - \Delta$*.*

Similarly, we have the following property:

Theorem 6. *If $\alpha_1 + \alpha_2 = 1$, then b and c computed by Eq. (20) and Eq. (21) respectively, are symmetric to the point $a_M^{(i+j)/2}$, i.e.,*

$$b + c = 2a_M^{\frac{i+j}{2}}. \tag{27}$$

The proof of this theorem is similar to Theorem 4.

Remark 7. *From this theorem it obtains $c = 2a_M^{(i+j)/2} - b$ and $b = 2a_M^{(i+j)/2} - c$.*

Here we introduce the method to compute the values α_1 and α_2 in the OWA weights W^2 and W^1. From Theorem 4 and Theorem 5, we require that $\alpha_1 + \alpha_2 = 1$. Thus we can only discuss α_1 and the value α_2 can be obtained easily. Noting $s_0 < s_i < s_j < s_g$, we have $0 < i < j < g$ and thus $1 \leq j - i < g$. Let us consider two extreme cases.

(a) The first extreme case is that $s_j = s_{i+1}$, i.e., $j - i = 1$. In this case, the OWA weights W^2 are not used because there is only one value to aggregate. But for convenience, it is set $\alpha_1 = 1$ and $W^2 = (\alpha_1)^T = (1)^T$. This assumption does not affect the result as we can see that $b = \alpha_1 \times a_M^i = a_M^i$.

(b) The second extreme case is that $s_i \to s_0$ and $s_j \to s_g$, we have $j - i \to g$ and $\alpha_1 \to 0$.

Thus there exists a function

$$f_2 : [1, g) \to (0, 1], \text{ such that } \alpha_1 = f_2(j - i),$$

which satisfies the boundary conditions $f_2(1) = 1,\ f_2(g) = 0$. Here we also assume that f_2 is a linear function, that is

$$f_2(j - i) = \beta(j - i) + \gamma,$$

where $\beta,\ \gamma$ are unknown parameters. The form of f_2 can be obtained by using the boundary conditions as

$$f_2(j - i) = \frac{g - (j - i)}{g - 1},$$

where $i = index(s_i)$, $j = index(s_j)$, and $g+1$ is the granularity of the linguistic term set $S = \{s_0, \ldots, s_g\}$. Therefore, α_1 is defined as

$$\alpha_1 = \frac{g - (j - i)}{g - 1} \tag{28}$$

and α_2 is defined as

$$\alpha_2 = 1 - \alpha_1 = \frac{(j - i) - 1}{g - 1}. \tag{29}$$

3.3. Computing the fuzzy envelopes

An example to understand the process of obtaining the fuzzy envelope for the comparative linguistic expressions generated by the context-free grammar G_H (see Def. 8) is introduced below.

Let $S = \{s_0 : nothing, s_1 : very\ bad, s_2 : bad, s_3 : medium, s_4 : good, s_5 : very\ good, s_6 : perfect\}$ be a linguistic term set shown in Figure 7. Several fuzzy envelopes for different comparative linguistic expressions are computed as follows.

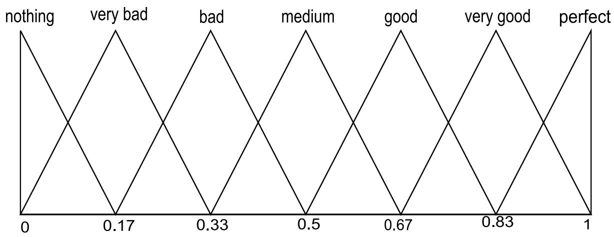

Figure 7: The linguistic term set $S = \{s_0, s_1, \ldots, s_6\}$.

- Fuzzy envelope for HFLTS $H_{s_1} = \{s_4, s_5, s_6\}$ based on $ll_1 = at\ least\ s_4$. The elements to aggregate are

$$T = \{a_L^4, a_M^4, a_L^5, a_R^4, a_M^5, a_L^6, a_R^5, a_M^6, a_R^6\}.$$

Since $a_M^4 = a_L^5$, $a_R^4 = a_M^5 = a_L^6$, and $a_R^5 = a_M^6$, we obtain the elements to aggregate as

$$T = \{a_L^4, a_M^4, a_M^5, a_M^6, a_R^6\}.$$

The points a_1, d_1 of the fuzzy envelope $env_F(H_{S_1}) = T(a_1, b_1, c_1, d_1)$ can be obtained as:

$$a_1 = \min\{a_L^4, a_M^4, a_M^5, a_M^6, a_R^6\} = a_L^4 = 0.5,$$
$$d_1 = \max\{a_L^4, a_M^4, a_M^5, a_M^6, a_R^6\} = a_R^6 = 1.$$

And the parameter c_1 is $c_1 = a_M^6 = 1$.

Since $i = 4$, $g = 6$, it obtains $\alpha = 4/6$ and the associated OWA weighting vector

$$W^2 = \left(\left(\frac{4}{6}\right)^2, \left(1 - \frac{4}{6}\right) \cdot \frac{4}{6}, \left(1 - \frac{4}{6}\right)\right)^T.$$

We use the OWA operator to compute b_1 as:

$$\begin{aligned} b_1 &= \left(\frac{4}{6}\right)^2 \cdot a_M^6 + \left(1 - \frac{4}{6}\right) \cdot \frac{4}{6} \cdot a_M^5 + \left(1 - \frac{4}{6}\right) \cdot a_M^4 \\ &= \left(\frac{4}{6}\right)^2 \cdot 1 + \left(1 - \frac{4}{6}\right) \cdot \frac{4}{6} \cdot 0.83 + \left(1 - \frac{4}{6}\right) \cdot 0.67 \approx 0.85. \end{aligned}$$

Therefore, the fuzzy envelope for H_{s_1} is $env_F(H_{s_1}) = T(0.5, 0.85, 1, 1)$.

- Fuzzy envelope for the HFLTS $H_{s_2} = \{s_0, s_1, s_2\}$ based on $ll_2 = \textit{at most } s_2$.

  The elements to aggregate are

  $$T = \{a_L^0, a_M^0, a_L^1, a_R^0, a_M^1, a_L^2, a_R^1, a_M^2, a_R^2\}.$$

  Since $a_M^0 = a_L^1$, $a_R^0 = a_M^1 = a_L^2$, and $a_R^1 = a_M^2$, we obtain the elements to aggregate as

  $$T = \{a_L^0, a_M^0, a_M^1, a_M^2, a_R^2\}.$$

  The points a_2, d_2 of the fuzzy envelope $env_F(H_{S_2}) = T(a_2, b_2, c_2, d_2)$ can be obtained as:

  $$a_2 = \min\{a_L^0, a_M^0, a_M^1, a_M^2, a_R^2\} = a_L^0 = 0,$$
  $$d_2 = \max\{a_L^0, a_M^0, a_M^1, a_M^2, a_R^2\} = a_R^2 = 0.5.$$

And the parameter b_2 is $b_2 = a_M^0 = 0$.

Since $i = 2, g = 6$, it obtains $\alpha = 2/6$, and the OWA weights to compute c_2 as

$$W^1 = \left(\frac{2}{6}, \frac{2}{6} \cdot \left(1 - \frac{2}{6}\right), \left(1 - \frac{2}{6}\right)^2\right)^T.$$

The value c_2 is computed as

$$\begin{aligned} c_2 &= \frac{2}{6} \cdot a_M^2 + \frac{2}{6} \cdot \left(1 - \frac{2}{6}\right) \cdot a_M^1 + \left(1 - \frac{2}{6}\right)^2 \cdot a_M^0 \\ &= \frac{2}{6} \cdot 0.33 + \frac{2}{6} \cdot \left(1 - \frac{2}{6}\right) \cdot 0.17 + \left(1 - \frac{2}{6}\right)^2 \cdot 0 \approx 0.15. \end{aligned}$$

Therefore, the fuzzy envelope $env_F(H_{s_2}) = T(0, 0, 0.15, 0.5)$. If we use the result of the Theorem 3, the computation can be significantly simplified.

- Fuzzy envelope for the HFLTS $H_{s_3} = \{s_3, s_4, s_5\}$ based on $ll_3 =$ *between* s_3 *and* s_5.

  The elements to aggregate are

  $$T = \{a_L^3, a_M^3, a_L^4, a_R^3, a_M^4, a_L^5, a_R^4, a_M^5, a_R^5\}.$$

  Since $a_M^3 = a_L^4$, $a_R^3 = a_M^4 = a_L^5$, and $a_R^4 = a_M^5$, we obtain the elements to aggregate as

  $$T = \{a_L^3, a_M^3, a_M^4, a_M^5, a_R^5\}.$$

  The points a_3 and d_3 of the fuzzy envelope $env_F(H_{S_3}) = T(a_3, b_3, c_3, d_3)$ can be directly obtained,

  $$\begin{aligned} a_3 &= \min\{a_L^3, a_M^3, a_M^4, a_M^5, a_R^5\} = a_L^3 = 0.33, \\ d_3 &= \max\{a_L^3, a_M^3, a_M^4, a_M^5, a_R^5\} = a_R^5 = 1. \end{aligned}$$

The point b_3 is computed by the OWA operator with $\alpha_1 = (6-(5-3))/(6-1) = 4/5$. Note $3+5$ is even, the associated OWA weighting vector is $W^2 = ((4/5),(1-(4/5)))^T$ and thus

$$b_3 = \frac{4}{5}\cdot a_M^4 + \left(1-\frac{4}{5}\right)\cdot a_M^3 = \frac{4}{5}\cdot 0.67 + \left(1-\frac{4}{5}\right)\cdot 0.5 \approx 0.64.$$

And the point c_3 is computed by means of the point b_3 (see Theorem 5), $c_3 = 2a_M^4 - b_3 = 0.70$.

Therefore, the fuzzy envelope $env_F(H_{s_3}) = T(0.33, 0.64, 0.70, 1)$.

The obtained fuzzy envelopes are plotted in Figure 8.

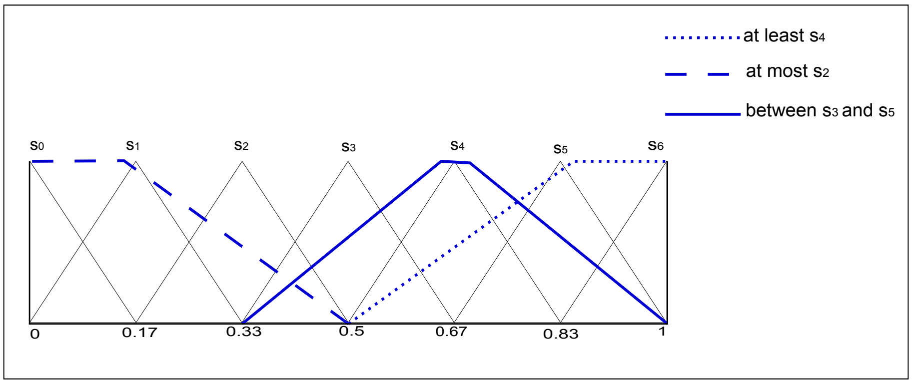

Figure 8: The obtained fuzzy envelopes.

4. Fuzzy TOPSIS Using Comparative Linguistic Expressions

To show the usefulness of the fuzzy envelope proposed for HFLTS, in this section a supplier selection multicriteria decision making problem is solved by using a fuzzy TOPSIS model [2, 5, 26] and follows the scheme depicted in the Figure 9.

Let us suppose that the manager of a company wants to select a material supplier to purchase some key components of a new product. After preliminary screening, four alternatives $X = \{x_1, x_2, x_3, x_4\}$ have remained in the candidate list. The considered criteria are $C = \{c_1 = \textit{quality},\ c_2 = \textit{price},\ c_3 = \textit{business reputation},\ c_4 = \textit{delivery speed}\}$, and the weights of the criteria are $W = (w_1, w_2, w_3, w_4)^T = (0.3, 0.25, 0.15, 0.3)^T$.

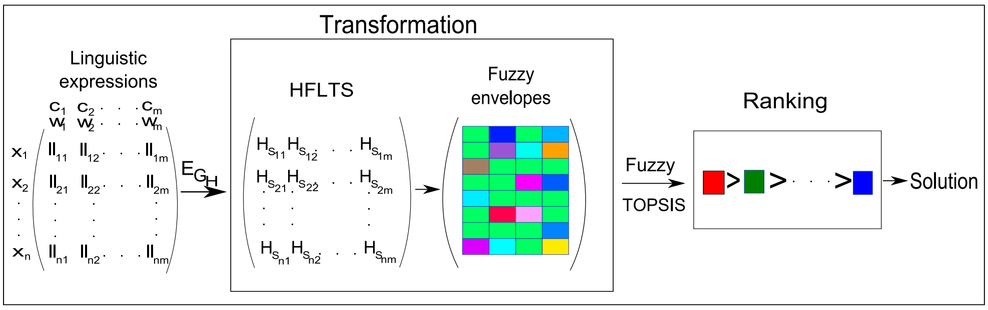

Figure 9: Scheme of the multicriteria decision making model

Sometimes, it is difficult for the manager of the company to provide all the assessments by means of single linguistic terms because of the lack of information and knowledge about the decision making problem. Thus, the manager might hesitate among several linguistic terms and prefer using comparative linguistic expressions close to the natural language used by human beings in decision making problems. To do so, it is used the context-free grammar G_H and the linguistic term set $S = \{s_0 : nothing(N), s_1 : very\ bad(VB), s_2 : bad(B), s_3 : medium(M), s_4 : good(G), s_5 : very\ good(VG), s_6 : perfect(P)\}$.

The assessments provided for this problem are shown in Table 1.

Table 1: Assessments of the problem

	c_1	c_2	c_3	c_4
x_1	*between M and VG*	*M*	*at least G*	*at least VG*
x_2	*at least G*	*at least VG*	*between M and G*	*between B and M*
x_3	*at least VG*	*between B and M*	*between G and VG*	*at least G*
x_4	*between M and G*	*at least VG*	*between G and VG*	*at most B*

To solve the decision problem, we follow the scheme shown in Figure 9.

1. Transform the comparative linguistic expressions into HFLTS and their fuzzy envelopes.

   The corresponding HFLTS $H_{S_{ij}}$, $i, j \in \{1, 2, 3, 4\}$ of the comparative linguistic expressions are shown in Table 2.

   By using the general process proposed in section 3 the fuzzy envelopes of the HFLTS, $env_F(H_{S_{ij}}) = \widetilde{p}_{ij}$, $i, j \in \{1, 2, 3, 4\}$. The fuzzy envelopes are the following ones:

Table 2: HFLTS generated from the comparative linguistic expressions

	c_1	c_2	c_3	c_4
x_1	$\{M, G, VG\}$	$\{M\}$	$\{G, VG, P\}$	$\{VG, P\}$
x_2	$\{G, VG, P\}$	$\{VG, P\}$	$\{M, G\}$	$\{B, M\}$
x_3	$\{VG, P\}$	$\{B, M\}$	$\{G, VG\}$	$\{G, VG, P\}$
x_4	$\{M, G\}$	$\{VG, P\}$	$\{G, VG\}$	$\{N, VB, B\}$

$$\begin{array}{ll}
\widetilde{p}_{11} = T(0.33, 0.64, 0.7, 1), & \widetilde{p}_{12} = T(0.33, 0.5, 0.5, 0.67), \\
\widetilde{p}_{13} = T(0.5, 0.85, 1, 1), & \widetilde{\mathbf{p}}_{\mathbf{14}} = \mathbf{T(0.67, 0.97, 1, 1)}, \\
\widetilde{p}_{21} = T(0.5, 0.85, 1, 1), & \widetilde{p}_{22} = T(0.67, 0.97, 1, 1), \\
\widetilde{\mathbf{p}}_{\mathbf{23}} = \mathbf{T(0.33, 0.5, 0.67, 0.83)}, & \widetilde{p}_{24} = T(0.17, 0.33, 0.5, 0.67), \\
\widetilde{p}_{31} = T(0.67, 0.97, 1, 1), & \widetilde{p}_{32} = T(0.17, 0.33, 0.5, 0.67), \\
\widetilde{p}_{33} = T(0.5, 0.67, 0.83, 1), & \widetilde{p}_{34} = T(0.5, 0.85, 1, 1), \\
\widetilde{p}_{41} = T(0.33, 0.5, 0.67, 0.83), & \widetilde{p}_{42} = T(0.67, 0.97, 1, 1), \\
\widetilde{p}_{43} = T(0.5, 0.67, 0.83, 1), & \widetilde{\mathbf{p}}_{\mathbf{44}} = \mathbf{T(0, 0, 0.15, 0.5)}.
\end{array}$$

In order to clarify the computational processes, we compute several fuzzy envelopes. For example, the comparative linguistic expression ll_{14} : *at least* VG can be transformed into the HFLTS $H_{S_{14}} = \{VG, P\}$. Firstly, the elements to aggregate are

$$T = \{a_L^5, a_M^5, a_M^6, a_R^6\}.$$

The points a_{14}, d_{14} of its fuzzy envelope $T(a_{14}, b_{14}, c_{14}, d_{14})$ can be obtained as:

$$\begin{array}{l}
a_{14} = \min\{a_L^5, a_M^5, a_M^6, a_R^6\} = a_L^5 = 0.67, \\
d_{14} = \max\{a_L^5, a_M^5, a_M^6, a_R^6\} = a_R^6 = 1.
\end{array}$$

And the parameter c_{14} is $c_{14} = a_M^6 = 1$.

Since $i = 5, g = 6$, it obtains $\alpha = 5/6$ and the associated OWA weights $W^2 = (5/6, 1/6)^T$. The point b_{14} is computed as

$$b_{14} = \left(\frac{5}{6}\right) \cdot a_M^6 + \left(\frac{1}{6}\right) \cdot a_M^5 = \left(\frac{5}{6}\right) \cdot 1 + \left(\frac{1}{6}\right) \cdot 0.83 \approx 0.97.$$

Thus the fuzzy envelope is $env_F(H_{S_{14}}) = \widetilde{p}_{14} = T(0.67, 0.97, 1, 1)$.

For the comparative linguistic expression ll_{23} : *between M and G*, it can be transformed into the HFLTS $H_{S_{23}} = \{M, G\}$.

Firstly, the elements to aggregate are

$$T = \{a_L^3, a_M^3, a_M^4, a_R^4\}.$$

The points a_{23}, d_{23} of its fuzzy envelope $T(a_{23}, b_{23}, c_{23}, d_{23})$ can be obtained as:

$$a_{23} = \min\{a_L^3, a_M^3, a_M^4, a_R^4\} = a_L^3 = 0.33,$$
$$d_{23} = \max\{a_L^3, a_M^3, a_M^4, a_R^4\} = a_R^5 = 0.83.$$

Since $i = index(M) = 3$, $j = index(G) = 4$ and $j - i = 1$, the parameters b_{23}, c_{23} can be obtained directly, $a_{23} = a_M^3 = 0.5$, $d_{23} = a_M^4 = 0.67$.

Then its fuzzy envelope is $env_F(H_{S_{23}}) = \widetilde{p}_{23} = T(0.33, 0.5, 0.67, 0.83)$.

For the comparative linguistic expression ll_{44} : *at most B*, it can be transformed into the HFLTS $H_{S_{44}} = \{N, VB, B\}$.

Firstly, the elements to aggregate are

$$T = \{a_L^0, a_M^0, a_M^1, a_M^2, a_R^2\}.$$

The points a_{44}, d_{44} of its fuzzy envelope $T(a_{44}, b_{44}, c_{44}, d_{44})$ can be obtained as:

$$a_{44} = \min\{a_L^0, a_M^0, a_M^1, a_M^2, a_R^2\} = a_L^0 = 0,$$
$$d_{44} = \max\{a_L^0, a_M^0, a_M^1, a_M^2, a_R^2\} = a_R^2 = 0.5.$$

And the parameter b_{44} is $b_{44} = a_M^0 = 0$.

From $i = 2, g = 6$, it obtains $\alpha = 2/6$ and the associated OWA weights $W^1 = (2/6, (2/6) \cdot (4/6), (1 - (2/6)^2))^T$. The point c_{44} is computed as

$$c_{44} = \left(\frac{2}{6}\right) \cdot a_M^2 + \left(\frac{2}{6}\right) \cdot \left(1 - \frac{2}{6}\right) \cdot a_M^1 + \left(1 - \frac{2}{6}\right)^2 \cdot a_M^0 \approx 0.15.$$

Therefore, the fuzzy envelope is $env_F(H_{S_{44}}) = \widetilde{p}_{44} = T(0, 0, 0.15, 0.5)$.

2. Aggregate the assessments represented by fuzzy envelopes.

   In this phase, we use fuzzy TOPSIS method and carry out the following steps:

   (a) Obtain the normalized fuzzy matrix $\widetilde{P}' = (\widetilde{p}'_{ij})_{4\times 4}$. Since $c_1, c_3, c_4 \in B$ (benefit criteria) and $c_1^+ = c_3^+ = c_4^+ = 1$, we have $\widetilde{p}'_{ij} = \widetilde{p}_{ij}$, $i = 1,2,3,4,\ j = 1,3,4$. For $c_2 \in C$ (cost criteria), $c_2^- = 0.17$, we have

   $$\begin{array}{ll} \widetilde{p}'_{12} = T(0.25, 0.34, 0.34, 0.52), & \widetilde{p}'_{22} = T(0.17, 0.17, 0.18, 0.25), \\ \widetilde{p}'_{32} = T(0.25, 0.34, 0.52, 1), & \widetilde{p}'_{42} = T(0.17, 0.17, 0.18, 0.25). \end{array}$$

   (b) Calculate the weighted normalized fuzzy matrix $\widetilde{V} = (\widetilde{v}_{ij})_{4\times 4}$. The results are shown as follows

   $$\begin{array}{ll} \widetilde{v}_{11} = T(0.10, 0.19, 0.21, 0.30), & \widetilde{v}_{12} = T(0.06, 0.09, 0.09, 0.13), \\ \widetilde{v}_{13} = T(0.08, 0.13, 0.15, 0.15), & \widetilde{v}_{14} = T(0.20, 0.29, 0.30, 0.30), \\ \widetilde{v}_{21} = T(0.15, 0.26, 0.30, 0.30), & \widetilde{v}_{22} = T(0.04, 0.04, 0.05, 0.06), \\ \widetilde{v}_{23} = T(0.05, 0.08, 0.10, 0.12), & \widetilde{v}_{24} = T(0.05, 0.10, 0.15, 0.20), \\ \widetilde{v}_{31} = T(0.20, 0.29, 0.30, 0.30), & \widetilde{v}_{32} = T(0.06, 0.09, 0.13, 0.25), \\ \widetilde{v}_{33} = T(0.08, 0.10, 0.12, 0.15), & \widetilde{v}_{34} = T(0.15, 0.26, 0.30, 0.30), \\ \widetilde{v}_{41} = T(0.10, 0.15, 0.20, 0.25), & \widetilde{v}_{42} = T(0.04, 0.04, 0.05, 0.06), \\ \widetilde{v}_{43} = T(0.08, 0.10, 0.12, 0.15), & \widetilde{v}_{44} = T(0, 0, 0.05, 0.15). \end{array}$$

   (c) Identify the fuzzy positive ideal solution and the fuzzy negative ideal solution as

   $$\widetilde{A}^+ = (T(1,1,1,1), T(1,1,1,1), T(1,1,1,1), T(1,1,1,1)),$$
   $$\widetilde{A}^- = (T(0,0,0,0), T(0,0,0,0), T(0,0,0,0), T(0,0,0,0)).$$

   (d) Obtain the distance of each alternative from $\widetilde{A}^+$ and $\widetilde{A}^-$. To do so, we use the geometrical distance [10].

   Definition 9. *Let $A = T(a_1, b_1, c_1, d_1)$ and $B = T(a_2, b_2, c_2, d_2)$ be two trapezoidal fuzzy numbers, the distance between them is defined as*

$$d(A,B) = \begin{cases} \frac{1}{4}\left(|a_1 - a_2|^\lambda + |b_1 - b_2|^\lambda + |c_1 - c_2|^\lambda + |d_1 - d_2|^\lambda\right)^{\frac{1}{\lambda}}, & if\ 1 \le \lambda < \infty \\ \max(|a_1 - a_2|, |b_1 - b_2|, |c_1 - c_2|, |d_1 - d_2|), & if\ \lambda = \infty \end{cases} \tag{30}$$

This distance is actually a kind of Minkowski distance [12].

By using Eq. (30) with $\lambda = 1$, the distances are the following ones:

$$D_1^+ = 3.31,\ D_1^- = 0.69,\ D_2^+ = 3.49,\ D_2^- = 0.51,$$
$$D_3^+ = 3.23,\ D_3^- = 0.77,\ D_4^+ = 3.62,\ D_4^- = 0.38.$$

(e) Finally, it is calculated the closeness coefficient of each alternative.

$$CC_1 = 0.17,\ CC_2 = 0.13,\ CC_3 = 0.19,\ CC_4 = 0.10.$$

3. Ranking phase.

   In this phase, the alternatives are ranked according to the closeness coefficients:

$$x_3 \succ x_1 \succ x_2 \succ x_4.$$

   Therefore, the best alternative of this decision problem is $\{x_3\}$.

5. Concluding Remarks

The use of linguistic terms implies processes of CWW. Usually, experts provide their assessments by using just one linguistic term. However, sometimes experts hesitate among several linguistic terms and need richer linguistic expressions to provide their assessments. Recently it has been introduced the proposal of HFLTS that provides a greater flexibility to elicit comparative linguistic expressions in hesitate situations. To facilitate the CWW processes with HFLTS was introduced the envelope of HFLTS, which is a linguistic interval. The final result of computing with such an envelope loses the initial fuzzy representation.

In this paper has been introduced a fuzzy envelope for HFLTS whose representation is a fuzzy membership function obtained of aggregating the fuzzy membership functions of the linguistic terms of the HFLTS. Such a fuzzy representation facilitates the CWW processes in fuzzy multicriteria decision models. A supplier selection multicriteria decision making problem has been solved with a fuzzy TOPSIS model that deals with comparative linguistic expressions.

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