UNIVERSITY OF JAÉN

School of Engineering and Computing
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TWO DIMENSION 2-TUPLE LINGUISTIC APPROACH FOR MULTI-ATTRIBUTE GROUP DECISION MAKING METHOD UNDER UNCERTAINTY

THESIS MEMORY PRESENTED BY

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TO OBTAIN THE PHD DEGREE IN COMPUTER SCIENCE

SUPERVISORS

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Jaén, Marzo de 2022

The Thesis entitled *Two dimension 2-tuple linguistic approach for multi-attribute group decision making method under uncertainty*, presented by D. Zelin Wang to obtain the PhD degree in Computer Science, has been carried out in the Computer Science Department of the University of Jaén with the supervisors Dr. Luis Martínez López and Dra. Rosa María Rodríguez Domínguez. To be evaluated, this research memory is presented as a set of published articles, according to Article 23, point 3, Regulation of Doctoral Studies of the University of Jaén, approved in March 2022.

Jaén, 25th March 2022

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Acknowledgements

My Ph.D life in UJA was such a memorable time for me. I had fun, had difficulties, and matured, whilst surrounded by friends, colleagues, and supervisors. I could not complete this degree without their help and company. Therefore, I am more than happy to express my sincere gratitude for the things I experienced and the people who helped me with this thesis. Thanks for their teaching, guidance, care and help. They have allowed me to have the transformation that I have today. And they also let me clarify the direction of my future efforts.

First and foremost, I would like to express my sincere thanks to my dear supervisor, Prof. Luis Martínez. He is who provided me great and valuable opportunity to come to Spain and work with and learn from him. During this time, he had let me broaden my horizons, increased my knowledge, and strengthened the scientific spirit. His guidance on my studies gave me a deeper understanding of the attitude and spirit that I should have towards scientific research. The care and help he has given me in life allow me to feel the warmth of my hometown from time to time, and I am happy to study and live in Jaén, not a big city but very cozy. His diligent and hard working has affected me and made me realize that hard working in a right way is very important for one's career. I have learned a lot from him, he often offered me valuable ideas, suggestions and criticisms with his profound knowledge. He is a mentor, a friend, and a role model who provided constant encouragement.

To thank my dear co-supervisor, Prof. Rosa M. Rodríguez. She takes care about me a lot not only in my research, but also in my daily life. She and Luis often invited us to enjoy with them at weekend, and cooked delicious Spanish foods for us, let us know a lot about the Spanish culture and customs. She always revises the paper for me very carefully, and patiently explain to me what I don't understand. When I failed in coming up with ideas, she will make a brainstorming with me and share with me her rich research experience. She is willing to discuss with me anytime available and I have learnt from her a lot not only about thesis writing, but also the professional ethics. I am very much obliged to her efforts of helping me complete the thesis.

To thank my dear friends in Jaén, Álvaro Labella, Wen He, Liang Wang, Diego García Zamora, Flor, Pilar, they not only helped me with my research, but also added a lot of fun to my life. Álvaro Labella, Wen He, Liang Wang and Diego García Zamora have given me their time in listening to me and help me work out my problems during the difficult course of the thesis. Flor and Pilar often take me to taste Spanish food and let me experience the life of the locals and quickly integrate into life in Spain. Every time I think of them is a precious memory.

Last but not the least, my thanks would go to my beloved family for their loving considerations and great confidence in me all through these years. Although we can only keep in touch with each other by the telephone or the Internet, I would like to let them know how much I love them and how much I've appreciated the things they have done for me. They always support my choice, they told me to pursuit my own dreams without hesitation and thanks them for making me an optimistic and cheerful girl.

Hope everything goes as they wish in their life.

Thanks to all!

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Chapter 1

Introduction

1.1 Motivation

Group decision making (GDM) is a decision theory branch that has been widely applied in real world scenarios to solve important and complicated decision problems in a range of domains, such as public health [5], water supply engineering projects [127], foreign policy [8] and so forth. In GDM problems, decision makers (DMs) usually evaluate alternatives based on multiple attributes, leading to multiple attribute group decision making (MAGDM) problems [82]. However, because of the complexity of eliciting assessments and human beings bounded rationality, linguistic terms are easier elicited than crisp numbers for assessing attribute in MAGDM. The concept of linguistic variable was introduced by Zadeh [206], it is a variable whose values are not numbers but words or sentences in natural or artificial language. It turned out to be a useful tool for handling MAGDM problems with qualitative information. Since then, MAGDM approaches dealing with linguistic variables have been widely investigated [53, 108, 111, 117, 177].

When a problem is solved using linguistic information, it is necessary to carry out computing with words (CWW) processes [121, 208, 210] (see Figure 1.1), which is one of the most used methodologies.

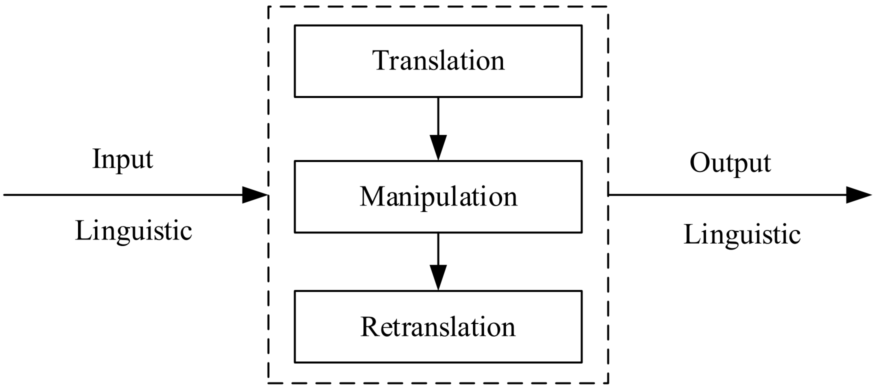

Figure 1.1 Computing with words process

In this process, linguistic outcomes are obtained from linguistic inputs, which are easily understandable and properly represented. Consequently, several linguistic computational models have been developed to accomplish the CWW processes [3, 60, 61, 118, 172, 197]. These models follow the computation scheme depicted by Yager [198, 199] that points out the importance of the translation and retranslation processes in CWW (see Fig. 1.1). However, there are some limitations when fusion processes are performed on linguistic variables. They performed the retranslation step as an approximation process to express the results in the original term set provoking a lack of accuracy [63]. In these approaches, the results usually do not exactly match with any of the initial linguistic terms, then an approximation process must be developed to express the results in the initial expression domain. This produces the consequent loss of information and hence the lack of precision.

To avoid such an inaccuracy in the retranslation step, a 2-tuple linguistic model [60] was proposed. A 2-tuple linguistic representation is composed by a linguistic term and a numerical value called symbolic translation that represents the displacement of the linguistic term. Therefore, it avoids the loss of information and obtains more precise and interpretable results. For this reason, the 2-tuple linguistic model stands out as one of the most widely used in decision making [119, 142].

Furthermore, several 2-tuple linguistic extended models have been proposed within MAGDM problems, such as, the 2-tuple semantic model [1, 163, 164], multi-granular 2-tuple linguistic model [38, 62, 197], proportional 2-tuple linguistic model [172, 173], numerical scale model [34, 36, 37], etc. Based on the extensive and successful research of the 2-tuple linguistic models, Martínez and Herrera [120] provided an overview on these model. The previous 2-tuple linguistic models have been successfully used to elicit the assessments but, the reliability of the assessments is also important for DMs. The extant decision making models based on the 2-tuple linguistic information assume that all assessments have the same confidence level [112], which may be infeasible in practice. Hence, Zhu et al. [225] proposed the concept of two-dimension linguistic information, which includes the reliability information of the subjective assessments. Subsequently, two dimension 2-tuple linguistic (TD2L) [224] was proposed by combing the two-dimension linguistic expression and 2-tuple linguistic information.

Obviously, the information expressed as TD2L is more accurate and reasonable, because the assessment and the reliability of the assessment are provided at the same time. Due to the advantages of eliciting TD2L assessments, several results on MAGDM problems with two-dimension linguistic assessment [98, 99, 220] have been developed, such as:

- *Representation model of TD2L labels.* Generally, the TD2L labels are presented as a binary linguistic term form [223]. The two classes of linguistic information come from two different linguistic term sets respectively. The first term set represents the evaluation assessments provided by DMs. The second term set represents the reliability of the previous assessment, which is also the subjective information provided by DMs [202].

- *Operational and comparison rules of TD2L labels.* Different operators have been developed for different kinds of two-dimension linguistic expression, such as, two-dimension uncertain linguistic aggregation operators [106, 110] used for aggregating the two-dimension linguistic labels under uncertainty, trapezoidal fuzzy two-dimension linguistic power generalized aggregation operators [99] used for aggregating the TD2L labels with the first class linguistic uncertain extended to trapezoidal fuzzy number, etc. Besides, the comparison rules between TD2Ls have been developed based on the traditional comparison rules of 2-tuple linguistic model [60], such as, two-dimension linguistic lattice implication algebra (2DL-LIA) [224] used for expressing and comparing the TD2Ls, the notation of expectation of TD2Ls [110] was proposed for comparing two-dimension uncertain linguistic variables, etc.

- *GDM methods based on TD2L expression.* Since TD2L has unique advantages in modelling information, its research and application combined with these classical GDM methods has attracted attention from scholars. Several GDM methods have been extended under the TD2L environment, such as, PROMETHEE [220], extended TODIM [105], extended VIKOR-QUALIFLEX [98], failure mode and effects analysis [104], extended prospect theory-VIKOR [33],etc.

- *Application of GDM methods based on TD2L labels in real life.* In some real situations, linguistic terms have been considered the most suitable modelling for assessing attributes, such as, emergency decision making [32, 33], quality evaluation of community question answering [97] power plant site selection [185] and risk assessment [186], etc.

Further research in GDM shows that consensus reaching processes (CRPs) have been required to assure the agreement on decision results in GDM problems. However, CRPs generally demand that the original assessments are adjusted if the expected consensus level is not satisfied. In such a situation, the reliability of the adjusted assessments is worth thinking. Obviously, original assessments' reliability could be given by experts in advance, however, the reliability of the adjusted assessments should be derived from an objective measure way.

Despite there are multiple models and approaches to deal with MAGDM and TD2L labels jointly, both theory and practice, it is remarkable that so far these models and approaches are not good enough when they are applied to real world MAGDM problems in which CRPs are applied to. Thus, new difficulties and challenges described below are the main motivations of this research memory:

- *The aggregation of the TD2Ls in MAGDM:* Aggregating the TD2Ls of DMs to rank or sort the alternatives, to select the best option is a necessary process. In MAGDM problems based on TD2L labels, individual DMs' preferences must be aggregated in a collective and well-structured way to make the final decision. The aggregation of the TD2Ls is of great importance in MAGDM because different aggregation operators could lead to different results. However, interpreting and analyzing these DMs' preferences is a complex task. And in the existing methods, no matter which aggregation operator is taken, the two-dimension information of the TD2L labels are taken separately for computing [99, 107, 110, 167, 200]. In fact, when the assessments are not completely reliable, they become random which means the assessment is uncertain. Therefore, an aggregation operator for dealing with the TD2L labels from a stochastic perspective is promising to research.

- *Measuring reliability of the adjusted TD2L assessment:* TD2L labels express the assessment and its reliability. With the advantage of the representation of the TD2L labels, they have been applied to many MAGDM problems [32, 185, 186]. However, by performing a CRP, the initial TD2L labels are modified and the reliability of the adjusted assessment should be recomputed. The reliability of the initial assessment is subjective. However, an objective measurement to improve the use of the TD2L labels in MAGDM is necessary.

- *Determining DMs' weights in MAGDM problems*: The calculation of DMs' weights in the literature can be divided into subjective methods, objective methods and methods combining the objective and subjective approaches [42, 178]. Subjective weight determination methods, such as the analytic hierarchy process (AHP) [146] and Delphi methods [73], assign weights to DMs based on subjective characteristics such as their background, professional levels and experience with the decision making problems. Objective weight determination methods [85], such as the entropy weight [46], technique for order preference by similarity to an ideal solution (TOPSIS) [68] and projection methods [204], etc. Mixed subjective and objective methods for computing DMs' weights combine the subjective and objective weights to obtain comprehensive DMs' weights [116, 147, 176]. When DMs' weights are not given in advance, the objective way to determine the reasonable weights information is important. Therefore, it is a challenge to find out a more effective and suitable way to determine the DMs' weights for MAGDM problems with linguistic assessments.

- *The clustering of large scale number of DMs*: Clustering analysis can effectively simplify the CRP when a large scale number of DMs is involved in MAGDM. Therefore, the clustering analysis has become significant for solving MAGDM problems. Many scholars have focused their attention on clustering method, such as, k-means clustering algorithm [187], a fuzzy c-means based algorithm [151], a hierarchical clustering algorithm [21], etc. Using the clustering method, the DMs can be divided into several small clusters, then DMs' assessment information have higher consistency and a lower degree of conflict for each cluster. However, the existing clustering methods are complex for computing and they ignore the support degree on each alternative of different DMs. Thus, a new clustering method based on the support degree of each alternative of DMs need to be developed so that more information would be obtained during the CRP.

- *The consistency and consensus of DMs' opinions*: Consistency and consensus are other noteworthy challenges in the MAGDM process. Consistency is directly related to the credibility of the MAGDM results. Consensus, on the other hand, means that the agreement of DMs to accept the results of the process. During the CRP, some DMs do not modify at all their opinions, which could happen when there is not enough time to persuade these DMs. DMs agree to modify their preferences to a value that is within their tolerance degree at most. Thus, it might be a challenge to coordinate the stubborn DMs' assessments and the automatic feedback with the consideration of acceptance and tolerance degree of the adjusted opinion for stubborn DMs.

In real world MAGDM problems, previous challenges found in existing MAGDM problems make that current MAGDM approaches need to overcome them in order to better satisfy the situations and needs in decision making. To deeply study the subjects regarding the challenges described above, this research memory conducts comprehensive and deep researches to fill those gaps.

1.2 Objectives

According to the challenges pointed out previously in existing MAGDM approaches based on TD2L labels, this research memory is focused on the improvements of current MAGDM approaches.

Based on such a purpose, the following three research objectives are considered:

1. *To develop a novel TD2L computation model.* It considers two dimensions'

information of TD2L labels from stochastic perspective and then compare the computation models from the general and stochastic perspective by a case study. Additionally, some new aggregation operators and comparison rules will be introduced to improve previous studies.

2. *To consider the reliability degree of the adjusted assessment during CRPs in MAGDM*. Generally, original assessments provided by DMs are linguistic terms, and the adjusted assessments are still linguistic terms or the extension of a linguistic term, such as, 2-tuple linguistic value after the CRP. In such a case, the information of the reliability of the adjusted assessment is usually missing. Thus, another dimension for linguistic information will be obtained for representing the reliability of the adjusted agreed assessment. In this objective it will be considered the minimum adjustment during the CRP, a two-stage minimum adjustment consensus model based on linguistic assessment information will be proposed to show the obtained adjusted assessment and its reliability. Besides, the relations between the reliability of the adjusted assessment and the distance from the original assessment to the adjusted assessment will be discussed.

3. *To define a MAGDM framework.* It is used to solve the problems refer to a large number of DMs and consider the tolerance degree of DMs on changing their opinions. A support degree (SD)-based clustering method is introduced for classifying DMs into several subgroups to make more manageable the large number of DMs. Besides, the tolerance degree of DMs will be considered to improve the reliability of the adjusted opinions, and a minimum adjustment consensus model with two consensus rules will be presented to improve the consensus level (CL) gradually. Eventually, the adjusted assessment will be modelled as TD2L labels. Using the proposed method for comparing TD2Ls, the alternatives ranking could be obtained.

1.3 Structure

To achieve the objectives presented in Section 1.2, and taking into account the article 23, point 3, of the current regulations for Doctoral Studies at the University of Jaén, in accordance with the program established in the RD 99/2011, this research memory will be presented as a compendium of published articles by the PhD student student during her PhD student period.

Two articles have been published in international journals indexed by JCR database, produced by ISI and one International conference contribution was also accepted by IEEE International Conference on Fuzzy Systems 2021 (Ranking in the Core Ranking list of conferences 2020 as CORE A). In summary, the report is composed of a total of two articles which have been published in high quality international journals and one CORE A conference contribution.

The structure of this research memory is briefly described below:

- Chapter 2: Some basic concepts that are used across the research memory to achieve our research goals are revised such as, related concepts of decision making, GDM, MAGDM, MAGDM under uncertainty, MAGDM based on linguistic information. And the methods and models that are used in our proposals, such as, fuzzy linguistic approach, 2-tuple linguistic model, two dimension 2-tuple linguistic label, consensus reaching process, minimum adjustment cost model and so on will be revised in short.

- Chapter 3: The published proposals that compose the research memory are briefly introduced, in addition, discussions of each result obtained is presented in short to clarify the achievements reached in our research.

- Chapter 4: This chapter is the core of this doctoral thesis, which includes the publications obtained as the research results. For each publication, the information about the journals in which the proposals have been published is further indicated.

- Chapter 5: Final conclusions regarding this research and possible promising future works are pointed out.

Chapter 2

Basic Concepts and Methods

This chapter establishes the framework of concepts and tools related to our research memory. Due to the fact that, the different papers that composes this research memory introduce and revise the necessary background for understanding our proposals, in this chapter we have provided a detailed and structured revision of the main necessary concepts related to our proposals including some related concepts about decision making, GDM, MAGDM, CRP and the managing of consensus under uncertainty in GDM by eliciting two dimension 2-tuple linguistic labels. Besides, the methods used for solving MAGDM problems under uncertainty, fuzzy sets, fuzzy linguistic approach, two dimension 2-tuple representation model, linear programming method, are revised. All these concepts, tools and methods are further detailed in each specific paper of the compendium provided in this research memory (see Chapter 4 for further details).

2.1 Decision making

In this section, a brief introduction and a classification of decision making are revised as the basic knowledge of this thesis, which pave the way for our coming researches.

2.1.1 Introduction

Decision making is a complex cognitive process proper of human beings. Within this process, individuals can decide actions based on either personal beliefs or the inference of various factors in various options, or decide the opinions that the individual wants to express. Every decision making process aims at producing the final decision and selecting the final choice [140]. Before making a decision, DMs are often faced with different plans and choices, as well as a certain degree of uncertainty about the consequences of their decisions; DMs need to weigh the pros, cons, and risks of various choices in order to achieve the best decision result.

The decision making process consists of an entire process from asking questions, determining goals, and going through program selection, decision making, and delivery to implementation. It emphasizes the practical significance of decision-making. It is clear that the purpose of decision making is execution, which in turn checks whether the decision is correct and whether the environmental conditions have undergone major changes [115].

In general, decision making is the process of making choices by identifying a decision, gathering information, and assessing alternative resolutions [83]. Seven steps could be considered to help DMs to execute the decision making process as follows [43]:

Step 1: To identify the decision problem: This step determines what the decision problem actually is.

Step 2: To gather relevant information: DMs' preference information is collected before decision making.

Step 3: To identify the alternatives: To list all possible and desirable alternatives.

Step 4: To weight the evidence: To place the alternatives in a priority order based on suitable decision methods.

Step 5: To choose the best possible option: To select the alternative that seems to be the best or even choose a combination of alternatives.

Step 6: To execute the action: The alternative derived from Step 5 is implemented.

Step 7: To review the decision result: The decision result is evaluated and then according to the performance of the alternative to improve next possible decision problems.

To better illustrate the decision making process, a flow chart is shown in Figure 2.1.

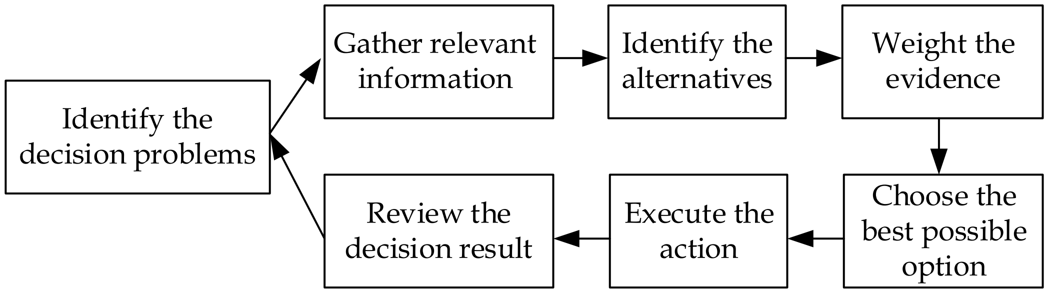

Figure 2.1: General decision making process

2.1.2 Classification

Decision making is a common mankind activity in daily life. Human beings usually face different situations in which there exist several options or alternatives, in some situations, they must choose one among them as the best option or alternative. Such activities widely exist in various fields, such as engineering, technology, economy, management, military, etc.

According to the different situations or contexts in which the decision problem is conducted, decision problems can be classified into different types, such as based on *preference modelling*[113], *number of involved DMs* [126], *decision environment* [84] and so on.

(1) Preference modelling

Considering DMs may choose different types of assessments according to different decision situations, hence decision making could be divided into various types according to the way of modelling the preference assessment, such as: linguistic decision making [56, 129, 219], fuzzy decision making [6, 14, 144], decision making using numerical data [215, 221].

Some researchers deal with decision making problems based on fuzzy sets [207], hesitant fuzzy sets [162], 2-tuple linguistic term sets [60], type-2 fuzzy sets [206], etc. They are frequently conducted in qualitative circumstances because of cognitive limitations and the lack of sufficient information.

(2) Number of involved DMs

According to the involved number of individuals, decision making can be classified into two categories:

- Individual decision making, which means there is only one DM participating in the decision making process and the decision results are completely according to his/her judgment. Individual decision making saves time and cost and usually makes prompt decisions. Moreover, individuals are accountable for their acts by various people. The decision making would be high-quality if the individual has rich experience and excellent professionalism. However, individual is limited in all expertise to some extent and there may not be so many creative solutions generated.

- Group decision making. It is a type of decision making process in which multiple individuals acting collectively, analyze problems or situations, consider and evaluate alternative courses of action, and select from the different alternatives a solution. Group decisions take into account a wider scope of information because each group member may contribute distinct information and expertise. Organization decisions are much more technically and politically complex; hence they usually require GDM [31, 52]. Group members can identify more complete and robust solutions and recommendations through discussing, questioning and collaborative approaches. The classical solving scheme to solve GDM problems is a selection process that consists of two phases (see Figure 2.2) [128]: (1) an aggregation phase, in which individual information is aggregated, and (2) an exploitation phase, in which an alternative or a subset of alternatives is obtained as the solution to the problem.

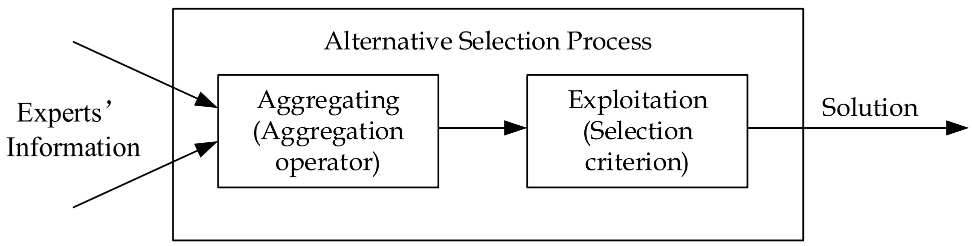

Figure 2.2: Classical scheme of group decision making

(3) Decision environment

According to different decision environments in which the decision problem is carried out, it can be classified into three types of decision problems [13]:

- Decision making under certain environment. It refers to DMs have a very definite comparison of what may happen in the future, such as the alternatives, the attributes, the weights information is definitely sure by DMs. In such a decision environment, the most commonly used decision making methods are linear programming decision making method [94], profit and loss sharing model [131], etc.

- Decision making under risk environment. It is a decision made by DMs based on the probabilities that various natural states may occur and the conditional benefit value of each alternative. The environment for risky decision making is not completely certain, but the probability of its occurrence is known. The commonly used methods for risky decision making are decision-making method based on expectations [182], decision-making method based on maximum probability [109], decision tree method [137], Markov decision process [2], etc.

- Decision making under uncertainty environment. The uncertainty handling has been one of the main concerns of DMs for many years [4]. It refers to a decision in which DMs cannot determine the probability of the occurrence of various natural states in the future. Uncertainty comes from many aspects, such as, incomplete information about the state of the world, practical and theoretical limitations of DMs [84], which means the future environment is unpredictable and everything is in a state of flux. There are various uncertainty handing methods developed for dealing with the decision making under uncertainty environment [155], such as, fuzzy approach [205], information gap decision theory [50], robust optimization [156], interval analysis [124], etc.

2.2 Group Decision Making

In this section, it is revised the GDM problems and its classification according to the number of DMs involved, afterwards different processes and types of methods and models related to the GDM problem and its typology are briefly revised. Such a revision aims at introducing the necessary knowledge for understanding the proposals of this thesis, which pave the way for our novel researches in GDM.

2.2.1 Introduction

Decision making made by a single DM is a process in which only an individual is responsible for defining the problem, assessing the alternatives based on a set of attribute or preference relations and make a final decision [138]. In the context of economics, politics, military, and management, the decision making process is becoming increasingly complex, forcing stakeholders and DMs rely on group wisdom instead of individual judgements. Several DMs with the collective wisdom are more suitable for decision making.

GDM is a common phenomenon in real life, which refers to the selection of the best alternative from a set of feasible alternatives according to the opinions of different DMs. Having more people involved in decision making is beneficial because each individual brings unique information or knowledge to the group, as well as different perspectives on the problem. However, with the increasing of the number of DMs, if the number is larger than 20, then the GDM problem could be large scale GDM [18]. According to the involved number of individuals, GDM can be classified into two categories [126]:

- Classical GDM: To obtain the most satisfactory alternative, a small group of DMs are invited to elicit their preferences. Hence, such decisions are usually taken by a few number of DMs, which can gather collective wisdom compared to individual decision making, which made decision making more reliable and credible. DMs are working together to find a solution for the specific problem. This turns GDM into a more effective and fast process. Groups can take advantage of the GDM to perform certain tasks, such as generating ideas and solutions through the group interaction. It is argued that DMs can enhance their ability to learn and stimulate their cognitive level with the GDM process. The classical GDM solving process is shown in Figure 2.2.

- Large scale group decision making (LSGDM): Unlike classical GDM, LSGDM refers to the selection of the best satisfactory alternative from a set of feasible alternatives, which is predicated on the preferences of a large number of DMs. Solving challenging problems can require a large group of DMs from different fields, the participating DMs are diverse and numerous [19], which has a wide range of applications in areas like earthquake shelter selection [193], urban resettlement [20], internet venture capital [45], financial inclusion [19], social networks [114], and emergency decision-making [95]. The evolution of GDM problems to LSGDM problems, has brought many new challenges, not only regarding the group size but also with regards to other problems such as knowledge distribution, the increase of cost and complexity in the decision making processes.

2.2.2 Consensus reaching process in GDM

In general, at the beginning of the GDM problem, DMs' opinions may differ substantially. The consensus reaching process (CRP) is often a necessity in GDM to achieve a general consensus regarding the selected alternatives [57, 58, 133]. Usually, consensus is defined as the full and unanimous agreement of all the DMs regarding all the feasible alternatives. However, a complete agreement is difficult to achieve in practice, thus "soft" consensus is a common phenomenon in real decision making problems [24, 59, 77]. Reaching consensus implies that DMs should modify their initial opinions throughout different discussion rounds in order to bring them closer to the opinions of the rest of the group.

Consensus can be achieved with or without feedback. The CRPs without feedback achieve consensus by modifying the initial assessments without considering DMs, while CRPs with feedback involve discussions among DMs and they should modify their initial assessments to reach a consensus. Particularly, the feedback process is often guided by a moderator, then the moderator suggests to modify the original assessments far from the collective agreement according to the identification and direction rules [54, 59]. Figure 2.3 shows the general process of consensus reaching.

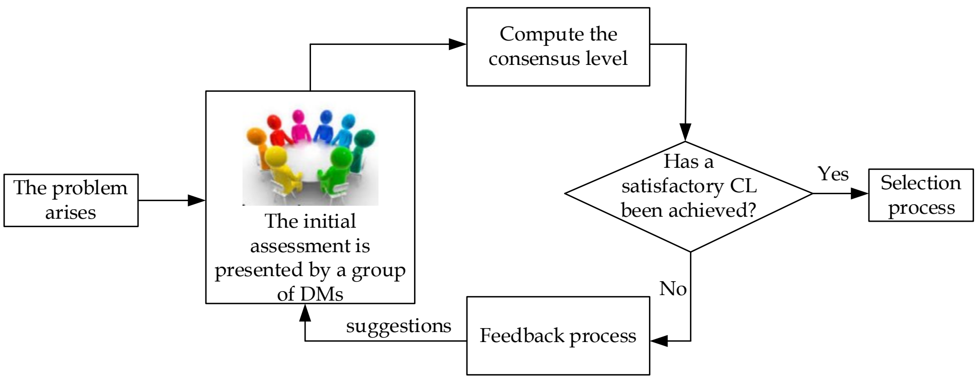

Figure 2.3: Consensus reaching process in group decision making

However, the feedback mechanism has some limitations [217], such as, it is time consuming, it will result in a huge cost consumption and even in deadlocks.

Furthermore, in GDM problems, due to the existence of polarized opinions, the group consensus process is becoming more and more important and worthy of attention. Therefore, the core problems are the assessment adjustment and the consensus cost in CRP. Based on the consideration of consensus cost, the cost of reaching a non-strict consensus is smaller, more effective, and more feasible than strict consensus that is time-consuming and costly. Therefore, the acceptable level of consensus and the coordination cost of reaching a consensus are two very important factors in GDM. Obviously, DMs will prefer a low-cost group consensus process, and the minimum adjustment cost consensus model to solve this problem well.

Since the existing resources are limited, it is expected to spend the least adjustment cost to reach a consensus. The two most common minimum cost consensus models used in the specialized literature to deal with linguistic information are introduced below [9, 35].

（1） Minimum Adjustment Consensus Model (MACM)

The minimum adjustment of this type of model [35] has two core points: one is based on the distance, which aims to minimize the distance between the initial assessment of the DM and the adjusted assessment. The second is based on the number of assessments that need to be adjusted, that is, to minimize the number of changes in the process of reaching a consensus.

Suppose that $E=\{e_k \mid k=1,2,...,m\}$ is a set of DMs, $w=\{w_1,w_2,...,w_m\}$ are the DMs' weights with $\sum_{k=1}^{m} w_k = 1$ and $w_k \in [0,1]$. $S=\{s_0,s_1,...,s_g\}$ is the linguistic term set used for expressing the initial assessment. $O=\{o_1,o_2,...,o_m\}$ and $\bar{O}=\{\bar{o}_1,\bar{o}_2,...,\bar{o}_m\}$ are the initial preferences and adjusted preferences of the DMs, respectively. Usually, o_k is a linguistic term belong to set S, $\bar{o}_k$ is a 2-tuple linguistic value. According to Dong et al. [35], the minimum adjustment cost consensus model in the group consensus process based on linguistic assessment is as follows

$$\min \sum_{k=1}^{m} d(o_k,\bar{o}_k)$$
$$s.t.\begin{cases} d(\bar{o}_k,\bar{o}^c) \le \varepsilon, k=1,2,...,m \\ f(\bar{o}^c) = F_w(f(\bar{o}_1), f(\bar{o}_2),..., f(\bar{o}_m)) \end{cases} \tag{2.1}$$

where f represents the linguistic information conversion function, $d(o_k,\bar{o}_k)$

represents the distance between o_k and $\overline{o}_k$, ε is the given distance threshold and $0 \le \varepsilon \le 1$, $F_w(\cdot)$ is the aggregation function used to obtain the collective preferences of the DMs.

For GDM when the assessments are expressed, both by numerical or linguistic information, the DMs' opinions can be not only elicited in the form of evaluation values in utility vectors, but also in the form of preference relations [37, 54, 55]. Let $R_k = (r_k^{ij})_{n\times n}$ be the preference relation matrix provided by DM e_k and the preference relation r_k^{ij} belong to set S, then the MACM is as follows.

$$\min \sum\nolimits_{k=1}^{m} \sum\nolimits_{j=i+1}^{n} \sum\nolimits_{i=1}^{n-1} \left| f(r_k^{ij}) - f(\overline{r}_k^{ij}) \right|$$
$$s.t. CL \ge \sigma \tag{2.2}$$

where $\overline{r}_k^{ij}$ represents the adjusted preference relation, CL represents the overall consensus level obtained, σ is the CL threshold given in advance, $0 \le \sigma \le 1$.

The consensus level can be considered from the following three aspects [189]:

- The consensus level on each pair of alternatives (x_i, x_j): CL_{ij}, where CL_{ij} is measured by the similarity between the alternative x_i and x_j.
- The consensus level on each alternative x_i : CL_i , where $CL_i = \sum\nolimits_{\substack{j=1 \\ j\ne i}}^{n} CL_{ij} \Big/ (n-1)$.
- The overall consensus level: CL, where $CL = \sum\nolimits_{i=1}^{n} CL_i \Big/ n$, $0 \le CL \le 1$, the closer the CL to 1, the closer the opinions between DMs.

(2) Minimum Cost Consensus Model (MCCM)

Compared with the previous model MACM, this type of model takes into account the cost of persuading each DM to change a unit's point of view, that is, the unit adjustment cost, which was proposed by Ben-Arieh and Easton [9] and Ben-Arieh et al. [10]. In general, the adjustment cost is the unit adjustment cost multiplied by the adjustment distance.

Suppose that $E = \{e_k | k = 1, 2, ..., m\}$ is a set of DMs, $w = \{w_1, w_2, ..., w_m\}$ is the DMs' weights with $\sum\nolimits_{k=1}^{m} w_k = 1$ and $w_k \in [0,1]$. The symbols involved have the same meaning as above. The adjustment cost of adjusting a unit opinion of the DM e_k is recorded as c_k, the MCCM based on linguistic assessment is as follows.

$$\min \sum\nolimits_{k=1}^{m} c_k \left| f(o_k) - f(\overline{o}_k) \right|$$

$$s.t.\left|f(\overline{o}_k)-f(\overline{o}^c)\right|\le\varepsilon, k=1,2,...,m \tag{2.3}$$

where f represents the linguistic information conversion function, $\varepsilon\in[0,1]$ is the distance threshold, $\overline{o}^c$ is the collective opinion of the optimal adjusted opinions.

The solution of the previous model is the optimal adjusted opinion, and then the collective opinion of the optimal adjusted opinion can be obtained. However, there is no explanation in the collective opinion model (2.3) of how to obtain the collective opinion of the optimal adjusted opinions. Therefore, Zhang et al. [213] proposed an extended version of the model (2.3) by considering the operator that aggregates DMs opinions as follows:

$$\min\sum_{k=1}^{m}c_k\left|f(o_k)-f(\overline{o}_k)\right|$$

$$s.t.\begin{cases}\left|f(\overline{o}_k)-f(\overline{o}^c)\right|\le\varepsilon, k=1,2,...,m\\ f(\overline{o}^c)=F_w(f(\overline{o}_1),f(\overline{o}_2),...,f(\overline{o}_m))\end{cases} \tag{2.4}$$

where f represents the linguistic information conversion function, $F_w(\cdot)$ is the aggregation function that obtains the collective opinion of DMs.

The MACM and the MCCM models obtain the adjusted opinions automatically. After achieving the consensus, the selection process is presented to obtain an optimal alternative under agreement. Therefore, a GDM process should including a CRP and a selection process [57, 76, 145].

2.3 Multiple attribute group decision making

To better evaluate a decision making problem, DMs tend to perform the evaluation process from different aspects, which is called multi-attribute group decision making (MAGDM). With the advancement of society and the improvement of technology, more and more real world group decision-making problems are actually modelled as MAGDM problems. Moving from GDM setting to MAGDM setting introduces a great deal of new problems into the analysis, for example, the assessment of the attribute can be provided as different forms.

According to the different expressions of information given by DMs, the decision making can be classified from two different points of view:

- According to the opinions assessment, where DMs considering multiple attributes and give their assessment values on each attribute on different alternatives.

MAGDM refers to selecting the best alternative or ranking the possible alternatives according to several attributes from different DMs' opinions. For a MAGDM problem, let $A=\{A_1, A_2, ..., A_n\}(n \geq 2)$ be a finite set of alternatives, $C=\{c_1, c_2, ..., c_m\}(m \geq 2)$ be a set of attributes and $E=\{e_1, e_2, ..., e_g\}(g \geq 2)$ be a set of DMs. Let $W=\{w_1, w_2, ..., w_m\}$ be the associated weighting vector of DMs, where $w_k \geq 0 (k=1,2,...,g)$ and $\sum_{k=1}^{g} w_k = 1$. Let $X^k = (x_{ij}^k)_{m \times n} (k=1,2,...,g)$ be the evaluation matrix given by DM e_k. The decision problem consists of ranking the alternatives and choosing the best one based on the evaluation matrices X^k, where the assessments provided by DMs are presented as evaluation matrices as follows.

$$
\begin{array}{c}
 \\
\textit{alternative } A_1 \\
\textit{alternative } A_2 \\
\vdots \\
\textit{alternative } A_n
\end{array}
\begin{array}{c}
\begin{array}{cccc}
\textit{attribute } C_1 & \textit{attribute } C_2 & \cdots & \textit{attribute } C_m
\end{array} \\
\begin{pmatrix}
x_{11}^k & x_{12}^k & \cdots & x_{1m}^k \\
x_{21}^k & x_{22}^k & \cdots & x_{2m}^k \\
\vdots & \vdots & \ddots & \vdots \\
x_{n1}^k & x_{n2}^k & \cdots & x_{nm}^k
\end{pmatrix}_{n \times m}
\end{array}
$$

A MAGDM process refers to different opinions provided by several DMs. Owing to the complexity of the decision making problem, quantitative or qualitative information are both used to represent the DMs' opinions on different attributes, such as, 2-tuple linguistic values [60], hesitant fuzzy linguistic term sets [141], interval data [69], grey number [102], real number [152], etc. Usually, multi-attribute evaluation requires DMs to provide the relative importance of the attribute with respect to the overall objective of the problem [30].

- According to the preference structure used to provide the assessments [65]. As each DM has their own ideas, attitudes, motivations and expertise, it is common to see that the different DMs will give their preferences in a different way. Usually, it can be presented in one of the following three ways.

1. *A preference ordering of the alternatives*. In this case, DM e_k gives his preferences on an alternative set A as an individual preference ordering, $O^k = \{o^k(1), o^k(2), ..., o^k(n)\}$, where $o^k(\cdot)$ is a permutation function over the index set $\{1,2,...,n\}$ [23, 149]. Therefore, an ordered vector of alternatives from best to worst is given.

2. *A utility function*. In this case, DM e_k gives his preference on the alternative set A as a set of n utility values, $U^k = \{u_i^k, i = 1, 2, ..., n\}$, where u_i^k represents the utility evaluation given by the DM e_k in terms of alternative x_i [160].

3. *A multiplicative/additive preference relation*. In this case, DM e_k gives his preferences on the alternative set A on the pair of alternatives. Let $R^k = (r_{ij}^k)_{n\times n}(k = 1, 2, ..., g)$ be the preference relations matrix given by DMs e_k, where r_{ij}^k represents the preference relation of alternative x_i in terms of x_j. The decision problem is how to rank the alternatives and choose the best one based on the preference relations matrices R^k, where the preference relations provided by DMs are presented as preference relations matrices as follows.

$$\begin{array}{l} \\ \textit{alternative } A_1 \\ \textit{alternative } A_2 \\ \vdots \\ \textit{alternative } A_n \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & \cdots & A_n \end{array} \\ \begin{pmatrix} r_{11}^k & r_{12}^k & \cdots & r_{1n}^k \\ r_{21}^k & r_{22}^k & \cdots & r_{2n}^k \\ \vdots & \vdots & r_{ij}^k & \vdots \\ r_{n1}^k & r_{n2}^k & \cdots & r_{nn}^k \end{pmatrix}_{n\times n} \end{array}$$

Preference relations are frequently-used structures to reflect DMs' opinions by pairwise comparisons of alternatives. Many kinds of preference relations have been proposed, including fuzzy preference relations [55], intuitionistic fuzzy preference relations [158], hesitant fuzzy preference relations [222], linguistic preference relations [57] and hesitant fuzzy linguistic preference relations (HFLPRs) [168]. For MAGDM problems based on the expression form of preference relations, the consistency checking is the first priority before the selection process.

Despite there are different kinds of MAGDM problems, they share the following common features [68]:

- Multiple attributes: each problem has multiple attributes, which can be evaluated by DMs;
- Assessment values: they are provided by DMs, which could be presented as various expressions and be expressed either as utility vectors or preference relations;

- Incommensurable units: attributes may have different units of measurement;
- Selection: an alternative or subset of alternatives is obtained as the solution to the problem.

2.4 Multiple attribute group decision making under uncertainty

Most of real-world GDM problems are defined under uncertain contexts, this is particularly interesting for MAGDM problems in which fuzzy information expressions have been commonly used for modelling preferences. Therefore, this subsection introduces a basic knowledge about *GDM under uncertainty* and afterwards the methods for dealing with MAGDM problems under uncertainty are presented.

Owing to the fact that in real-world it is often hard to describe something precisely or completely, uncertainty is very common in reality. The uncertainties in decision problems mainly come from three different aspects, including the uncertainty of assessment value, the uncertainty of weights information and the uncertainty of reliability of assessment.

- *The uncertainty of assessment value*. An important phenomenon is that most of decision-making processes are dealing with uncertain and imprecise data. If the vagueness of the mankind process of decision making is ignored, the outcomes could be misleading. Fuzzy set theory [226] can model ambiguity and vagueness, additionally, it provides formalized tools that deal with the imprecision of many problems.
- *The uncertainty of weights information*. The weights information for MAGDM problems refer to the DM weights information and the attribute weights information. The increasing complexity of the decision circumstances makes it hard for DMs to provide the attribute weights or the appropriate DM weights in advance. The weights information just based on the DMs' knowledge and capabilities is not usually enough, many factors should be considered when determine the weights information, such as the similarity of preferences among DMs [78], the incompatibility of attributes [25], the credibility of the evaluations [135], etc.

- *The uncertainty of reliability of assessment*. Due to the complex decision situations and incomplete information that appears in them, DMs tend to trust in the reliability on the original assessment [225], which are often provided as linguistic information given by DMs in advance [224]. However, after the CRP, the original assessments have been usually changed if the desired consensus level has not been achieved. In this situation, the reliability of the adjusted assessment is less than the reliability of the original assessment. Hence, the adjusted assessment implies a greater uncertainty, which is worth to be studied and measured.

Since the process of MAGDM involves human intervention, uncertainty and vagueness are implicit factors. According to different decision environments in which the decision problem is carried out, it can be classified into three types [84]: MAGDM under certain environment [94, 131], MAGDM under risk environment [2, 109, 137, 182] and MAGDM under uncertainty environment [28, 40, 90].

Uncertainty includes randomness, fuzziness, random fuzzy mixing, interval, etc. Uncertain theory is the foundation and tools for studying uncertainty. The existing uncertain theories can be roughly divided into the following categories: random mathematical methods [26], interval mathematical methods [123], fuzzy mathematical methods [79], rough set theory [130], grey system theory [74], etc. *MAGDM under uncertainty* is the main topic discussed in this research memory, common methods for dealing with MAGDM under uncertainty are:

1. *Random mathematical method*. Random mathematical method is one of the earliest methods to deal with uncertainty in real life. It uses probability theory, mathematical statistics, random process and other models and methods in mathematical research to operate on data that follows a probability distribution.

2. *Interval mathematical method*. Accurate values sometimes cannot fully summarize certain data characteristics. Therefore, some scholars use interval numbers to describe certain uncertainties. A variable is represented by an interval number, and the variable can take any value within the value interval [148]. In most cases, the value of a variable satisfies a certain probability distribution in the interval [183]. Probability distributions commonly used could be uniform distributions that include uniform distribution and normal distribution.

3. *Fuzzy mathematical method*. It was developed on the basis of the fuzzy set theory introduced and developed by Prof. Zadeh [205]. According to the ambiguity of the type of set division and the extension of the boundary of the set, Zadeh uses fuzzy sets to expand the classic set. The ambiguity and uncertainty of data are described using the membership function. Fuzzy mathematics method has become one of the most effective methods to deal with uncertain information.

In real life, we will encounter some difficulties in choices inevitably, such as choosing a career, buying a property, choosing a partner, choosing a university, etc. Most of these choices are decision making situations under uncertainty in which multiple attributes describe the different actions of the problem. MAGDM under uncertainty includes five factors [82]: DMs, attributes set, alternatives set, attributes weight and decision making method. The general solving process is shown as Figure 2.4 as follows.

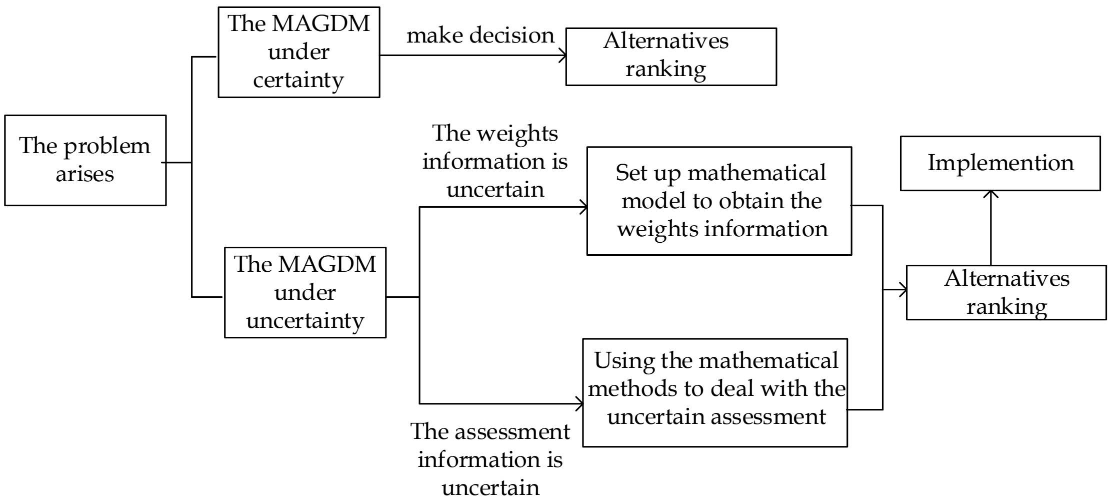

Figure 2.4: The general scheme of MAGDM under uncertainty

Besides, DMs can only predict the possible nature states of each alternative and their corresponding profit and loss values. Due to the lack of decision making information and experience, the probability of each natural state is unknown, therefore the attitude of DMs towards risk should be considered. There are five types of criteria to deal with the decision making problems under uncertainty [66]:

1. Maximum maximum criterion. Making decisions based on the best objective state, then find out the optimal alternative with the best expected effect. This criterion is actually based on the most optimistic estimation of alternative

chosen, which is also the riskiest criterion. Maximum minimum criterion. Looking for the best expected effect alternative based on the worst objective state. The criterion is based on the most pessimistic estimation, thus the criterion is the most conservative criterion.

2. Minimum maximum regret value criterion. Assume that any action taken is a state with the largest regret value, then find the optimal alternative with the smallest regret value. This criterion is based on the worst objective state, which is similar to the maximum and minimum criterion.

3. Equal probability criterion. Assuming that the probability of the occurrence of each natural state is the same, then use a simple arithmetic average method to calculate the average return of each alternative in various natural states, and find the optimal alternative with the largest average return.

4. Hurwice criterion. This is a kind between the maximum maximum criterion and the maximum minimum criterion. When applying this criterion, we must first determine an optimism coefficient indicating the optimism of the DM, then calculate the weighted average of the maximum and minimum benefits of each alternative.

5. Minimize regret criterion: This decision model focuses on the difference between the optimal reward and the actual reward received. It determines the maximum regret for each alternative, and selects the alternative with the minimum value.

Under the *uncertainty* environment, the information about the problem is vague and imprecise, and can be modelled by fuzzy information. In this situation we talk about decision making problems in a fuzzy context or fuzzy decision making [96]. For MAGDM in which preferences are elicited as linguistic assessments, the classical mathematics cannot handle such uncertainty, then fuzzy linguistic approach has been successfully applied but, there are still situations that cannot be properly managed [206]. Especially for TD2L information, there are few literatures study on the CRP based on TD2L information and the analysis of the reliability of assessment for TD2L information is neglected. Therefore, the further studies on TD2L representation and computation model are necessary for MAGDM under uncertainty.

In this memory, the MAGDM problems are studied based on TD2L information, which are MAGDM under uncertainty, the assessment and the reliability of the assessment are both expressed by linguistic information. Different extensions of the 2-tuple linguistic model and TD2L model will be proposed to overcome the difficulties and challenges pointed out in Section 1.1.

2.5 Multiple attribute group decision making based on linguistic information: State of art and limitations

In this section, fuzzy linguistic approach and its use to model the uncertainty in MAGDM problems are briefly revised, besides the limitations in current MAGDM approaches dealing with linguistic information are then pointed out to highlight the importance and necessity of our proposals.

2.5.1 Fuzzy linguistic approach

The fuzzy linguistic approach models the uncertainty by linguistic variables using words or sentences [206]. Most DMs cannot give exact numerical values to express their opinions, more appropriately, measurements are stated as linguistic assessments rather than numerical values. Linguistic MAGDM problems have provided very good results in many fields and applications [27, 117, 129, 188]. The use of linguistic information implies computing with words (CWW) processes [209]. There are different linguistic models for accomplishing such computing processes, one of the most widely-used is the 2-tuple linguistic model [60]. The 2-tuple linguistic model was inspired by the symbolic models used in decision making. Its main application field has been decision analysis and decision making. 2-tuple linguistic model has been widely used as basis for different models. For example, multi-attribute decision making based on 2-tuple linguistic model [134, 170, 175], consensus reaching process based on 2-tuple linguistic model [37, 218], 2-tuple linguistic aggregation operators [169, 171, 174], etc.

Suppose that $S=\{s_0,s_1,...,s_g\}$ is a pre-defined linguistic term set, and the cardinality of S is $g+1$. For any $s_i,s_j \in S$, the following properties should satisfy [62, 64]:

(1) The set is ordered : if $i>j$, then $s_i>s_j$;

(2) Maximum operator: if $s_i > s_j$, then $\max(s_i, s_j) = s_i$;

(3) Minimum operator: if $s_i > s_j$, then $\min(s_i, s_j) = s_j$;

(4) Negation operator: $neg(s_i) = s_{g-i}$.

In general, the cardinality of a linguistic term set S is odd number, more than 5 and less than 9. An example of a linguistic term set S could be:

$$S = \{s_0 = very\ poor, s_1 = poor, s_2 = medium, s_3 = good, s_4 = very\ good\}$$

In order to obtain more accurate results in CWW processes, Herrera and Martínez proposed the 2-tuple linguistic model (s_i, α), where s_i is a linguistic label involved in the set S and α is a numerical value representing the symbolic translation from s_i.

Definition 1 [60] Let $S = \{s_0, s_1, ..., s_g\}$ be a linguistic term set and $\overline{S}$ the 2-tuple set associated with S defined as $\overline{S} = S \times [-0.5, 0.5)$. The 2-tuple linguistic value (s_i, α) is equivalent to β through the function Δ as follows.

$$\Delta : [0, g] \to S \times [-0.5, 0.5) \tag{2.5}$$

$$\Delta(\beta) = (s_i, \alpha),\ with \begin{cases} s_i, i = round(\beta) \\ \alpha = \beta - i, \alpha \in [-0.5, 0.5) \end{cases} \tag{2.6}$$

where $round(\cdot)$ is the usual round operation that assigns to β the closet integer number $i \in \{0, 1, ..., g\}$ to β.

Definition 2 [60] Let $S = \{s_0, s_1, ..., s_g\}$ be a linguistic term set and $(s_i, \alpha) \in \overline{S}$ be a 2-tuple linguistic value. $\beta \in [0, g]$ is equivalent to (s_i, α) through the function Δ^{-1} as follows.

$$\Delta^{-1} : S \times [-0.5, 0.5) \to [0, g] \tag{2.7}$$

$$\Delta^{-1}(s_i, \alpha) = \alpha + i = \beta \tag{2.8}$$

Remark 1 For any two 2-tuple linguistic values (s_i, α_i) and (s_j, α_j), the relations to compare them can be given as follows.

(1) If $i > j$, then $(s_i, \alpha_i) > (s_j, \alpha_j)$;

(2) If $i = j$, then (a) $(s_i, \alpha_i) > (s_j, \alpha_j)$ for $\alpha_i > \alpha_j$;

(b) $(s_i, \alpha_i) < (s_j, \alpha_j)$ for $\alpha_i < \alpha_j$;

(c) $(s_i, \alpha_i) = (s_j, \alpha_j)$ for $\alpha_i = \alpha_j$.

The 2-tuple linguistic values can represent the assessments in MAGDM. However, the real decision making problems may be more complex and uncertain, and it could happen that DMs have to provide not only their evaluations on alternatives, but also elicit the self assessments on the given evaluation results. In this situation, another dimension information is needed to present self-confidence or subjective evaluation on reliability of the given assessments, which is usually expressed as linguistic information. Thus, Zhu et al. [225] first proposed 2-dimension linguistic information as follows.

Definition 3 [225] Let $S=\{s_0,s_1,...,s_g\}$ and $\dot{S}=\{\dot{s}_0,\dot{s}_1,...,\dot{s}_h\}$ be two linguistic label sets, where $g+1$ and $h+1$ are the cardinality of the sets S and $\dot{S}$, respectively. Then a two-dimension linguistic expression is denoted as $(s_u,\dot{s}_v)$, where $s_u \in S$ represents the evaluation about the alternative given by the DM, $\dot{s}_v \in \dot{S}$ represents the self-assessment of DM.

Inspired by the 2-tuple linguistic model [60], Zhu et al. [223] extended the two-dimension linguistic expression to two-dimension 2-tuple linguistic label. It can be seen as an extension of the 2-tuple linguistic model from one dimension to two dimensions.

Definition 4 [223] Let $S=\{s_0,s_1,...,s_g\}$ and $\dot{S}=\{\dot{s}_0,\dot{s}_1,...,\dot{s}_h\}$ be two linguistic term sets. Let $\alpha,\dot{\alpha}\in[-0.5,0.5)$ be two numerical values. Then $\hat{S}=\left((s_u,\alpha),(\dot{s}_v,\dot{\alpha})\right)$ is a TD2L expression, where $s_u \in S$, $\dot{s}_v \in \dot{S}$, (s_u,α) represents the assessment information about the alternative given by DMs, $(\dot{s}_v,\dot{\alpha})$ represents the self-assessment of the DM on reliability of the given assessment result.

Remark 2 If $\alpha=\dot{\alpha}=0$, then $\hat{S}=\left((s_u,\alpha),(\dot{s}_v,\dot{\alpha})\right)$ is simplified as $\hat{S}=(s_u,\dot{s}_v)$, which is exactly the TD2L expression proposed by Zhu et al. [225].

Let $S=\{s_0,s_1,...,s_g\}$ and $\dot{S}=\{\dot{s}_0,\dot{s}_1,...,\dot{s}_h\}$ be two linguistic term sets, $\beta\in[0,g]$ be a numerical value representing the aggregation result of the indexes of the linguistic labels in S, and $\dot{\beta}\in[0,h]$ be the numerical value representing the aggregation result of the indexes of the linguistic labels in $\dot{S}$. According to Definition 2, there exist two functions Δ_1 and Δ_2 such that

$$\Delta_1:[0,g]\to S\times[-0.5,0.5),\beta\to\Delta_1(\beta)=(s_u,\alpha) \tag{2.9}$$

$$\Delta_2:[0,h]\to \dot{S}\times[-0.5,0.5),\dot{\beta}\to\Delta_1(\dot{\beta})=(s_v,\dot{\alpha}) \tag{2.10}$$

where $\alpha=round(\beta)$, $\dot{\alpha}=round(\dot{\beta})$, $\alpha=\beta-u$, $\dot{\alpha}=\dot{\beta}-v$, $\alpha,\dot{\alpha}\in[-0.5,0.5)$, $round(\cdot)$ is the usual round operation.

Definition 5 [223] Let $S=\{s_0,s_1,...,s_g\}$ and $\dot{S}=\{\dot{s}_0,\dot{s}_1,...,\dot{s}_h\}$ be two linguistic term sets. Let $\alpha,\dot{\alpha}\in[-0.5,0.5)$ be two numerical values, β and $\dot{\beta}$ be two numerical values representing the aggregation result of the indexes of the linguistic labels in S and $\dot{S}$, respectively. The function Δ, used to obtain a TD2L, is equivalent to a binary numerical array $(\beta,\dot{\beta})$, and it is defined as

$$\Delta:[0,g]\times[0,h]\to(S\times[-0.5,0.5),\dot{S}\times[-0.5,0.5)) \tag{2.11}$$

$$(\beta,\dot{\beta})\to\Delta(\beta,\dot{\beta})=(\Delta_1(\beta),\Delta_2(\dot{\beta}))=((s_u,\alpha),(\dot{s}_v,\dot{\alpha})) \tag{2.12}$$

Definition 6 [223] Let $S=\{s_0,s_1,...,s_g\}$ and $\dot{S}=\{\dot{s}_0,\dot{s}_1,...,\dot{s}_h\}$ be two linguistic label sets. Let $\alpha,\dot{\alpha}\in[-0.5,0.5)$ be two numerical values, there is a function Δ^{-1}, that maps a TD2L to its equivalent binary numerical number $(\beta,\dot{\beta})$, which is defined as follows.

$$\Delta^{-1}:(S\times[-0.5,0.5),\dot{S}\times[-0.5,0.5))\to[0,g]\times[0,h] \tag{2.13}$$

$$\Delta^{-1}((s_u,\alpha),(\dot{s}_v,\dot{\alpha}))=(\Delta_1^{-1}(s_u,\alpha),\Delta_2^{-1}(\dot{s}_v,\dot{\alpha}))=(u+\alpha,v+\dot{\alpha})=(\beta,\dot{\beta}) \tag{2.14}$$

Remark 3 The general two dimension linguistic label can be represented by two dimension 2-tuple linguistic expression by adding 0 in each linguistic label, that is $(s_u,\dot{s}_v)=((s_u,0),(\dot{s}_v,0))$.

The linguistic term set [60], 2-tuple linguistic representation model [60], and TD2L approach [223] are introduced because they are the main assessment expression way throughout the study.

2.5.2 Multiple attribute group decision making dealing with linguistic assessments

MAGDM problems coping with linguistic information are quite common, because of the advantage of expressing preferences as linguistic information. Through a plenty collection of literature, reading and a comprehensive review, the following main topics related to MAGDM based on linguistic assessment have been discussed in current MAGDM studies.

Using Web of Science Core Collection and Science Citation Index Expanded (SCIE) & Social Sciences Citation Index (SSCI) database, searching "linguistic" and "multiple attribute group decision making" as the title keywords from January 2005 to September 2021, all the publication results of each year are shown in Figure 2.5.

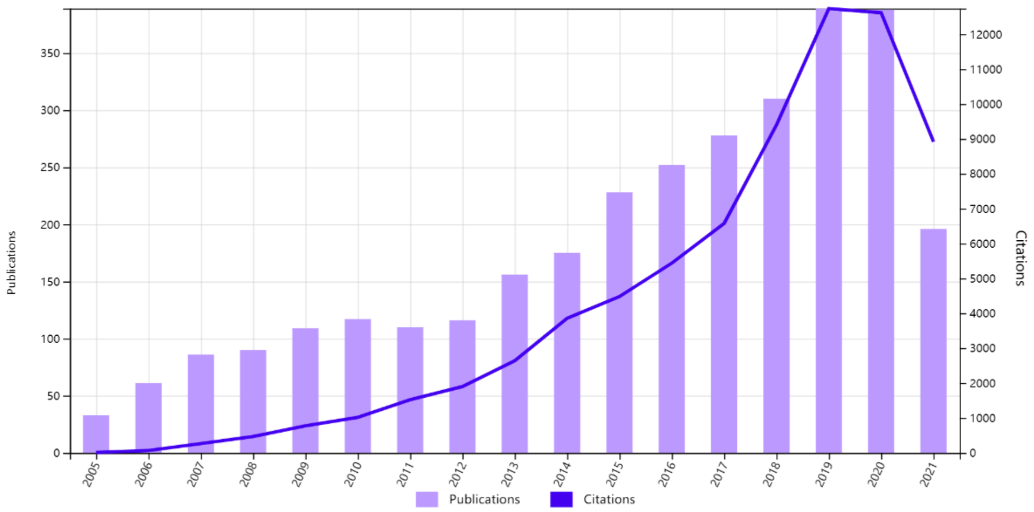

Figure 2.5: Publications of each year on MAGDM based on linguistic assessment

It can be seen clearly that the related studies on MAGDM represents increasing tendency in recent years, and has become one of the active research topics in MAGDM. As the mainstream field in the research of MAGDM methods, the existing research has achieved relatively fruitful results.

Face to multiple alternatives, the joint participation of group DMs is required to evaluate the attribute values under different alternatives. The evaluation presented as linguistic information is a common phenomenon. The solution to such problems is divided into at least two processes: the acquisition of decision making data and the ranking of alternatives.

The acquisition of decision data also includes the acquisition of attribute evaluation values and the acquisition of weights information. Attribute values are the evaluation values directly given by the DMs in the initial stage. In view of different decision making needs and decision making situations, DMs have their own preferences when giving linguistic evaluation values. According to the different manifestations of linguistic information provided by DMs, MAGDM based on fuzzy linguistic assessment is divided into the following main categories:

- **Multi-attribute group decision making based on linguistic terms.**
  Due to the fact that linguistic expression is the standard representation of the concepts used by humans for communication and owing to simply the MAGDM with linguistic information, some certain linguistic terms belong to a set given in advance, then DMs will choose one linguistic term from the certain set to express their preferences. Commonly

MAGDM problems based on linguistic information assess the attributes by linguistic terms [67].

- **Multi-attribute group decision making based on linguistic 2-tuple model.**
  Some authors pointed out that the use of single linguistic terms is not enough to represent the linguistic information because during the CWW processes there is loss of precision [142]. Hence the linguistic 2-tuple model includes a parameter to improve the accuracy of the linguistic computations and the interpretability of the results [60]. The 2-tuple linguistic information is able to represent the linguistic results that do not match with the initial terms of the linguistic term set.

- **Multi-attribute group decision making based on hesitant fuzzy linguistic term sets.**
  When DMs hesitate among different linguistic terms to elicit their opinions, the use of just one linguistic term is not enough to represent such opinions. In these situations, DMs can provide their opinions by using comparative linguistic expressions which are based on hesitant fuzzy linguistic term sets [141].

- **Multi-attribute group decision making based on interval linguistic information.**
  When DMs cannot give specific linguistic evaluation information, but the evaluation value is given in the form of interval linguistic form or the weight information cannot be completely determined, it is necessary to carry out research on multi-attribute group decision-making methods for such uncertain linguistic information.

- **Multi-attribute group decision making based on linguistic distribution evaluation information.**
  When faced with group DMs expressing their opinions alone and unwilling to present them in the collective form, they often choose the expression form of linguistic distribution evaluation information, which can not only present the linguistic term for evaluation, but also reflect the probability information of the evaluation value.

The previous review shows that various studies have examined the characteristic of MAGDM based on linguistic assessment from different perspectives, and have achieved successful results, which has made a significant contribution to the development of MAGDM. However, based on this review, there are still some unresolved problems in the current research, and there are also some limitations. For sake of clarity, the following subsections describe such restrictions in further detail.

2.5.3 Limitations in current multiple attribute group decision making based on linguistic assessment

As mentioned before, the current MAGDM research based on linguistic assessment has some limitations, as listed below:

1. It was previously mentioned that in MAGDM problems could be necessary CRPs for smoothing out conflicts. In such situations, the reliability of the adjusted linguistic preferences after the CRP has not been either studied or evaluated. The reliability of the initial linguistic preferences given by DMs presented as a second term in the TD2L information as a whole [32, 185, 186] is clear because represent DMs' preferences. However, after the CRP, the initial linguistic preferences may be changed [159, 214]. In this situation, the adjusted linguistic preferences are not so reliable, because some automatic adjustments either might not represent or might not be accepted by the DMs [45, 92, 136], thus the study of reliability is necessary for automatic CRP to assure the adjusted linguistic preferences are reliable.

2. In terms of the aggregation of the TD2L labels, the correlation between the two dimensions information has not been considered yet. Existing approaches for dealing with the TD2L labels have considered the two dimensions as independent information without taking into account that the uncertainty of the assessment is related to the reliability degree. Besides, previous studies provided more importance to the assessment than the reliability degree but failed to consider the relative importance degree of the two dimensions [202, 220, 223]. The general aggregation operators of TD2L labels do not reflect the reliability degree of the overall assessment, which may lead to the distortion of

information.

3. For automatic CRP during decision making process, the minimum of the adjustment and the minimum of the consensus cost are also considered during CRP [22, 100, 181, 212]. However, how to make sure the number of the DMs keeping the original assessment as much as possible is an important problem, especially for large scale MAGDM. Besides, face to large scale number of DMs, the suitable way for clustering is the key for better obtaining the collective opinion of the DMs and searching for the deviant opinions. The existing clustering methods [80, 165, 192] are mostly the expansion of fuzzy c-means [11]. These methods usually need to preset several subjective clustering coefficients, which may reduce the objectiveness of the clustering results.

In view of the previous limitations, this research memory will conduct in-depth research on these limitations and related topics to fill these gaps and enrich the theoretical basis and methods of current MAGDM based on linguistic assessment.

2.6 Methods and models

In this subsection, different methods and models used across this research memory are briefly revised, including linear programming, probability theory, stochastic approach and so on. All of them are relevant for the different proposals that will be developed in this research to achieve our goals.

2.6.1 Linear programming

Linear programming is an important branch of operations research that has been studied, developed rapidly and widely used in economic activities such as water supply system development [154], production scheduling [12], social networks [47], nurse rostering problem [157] and so on. It is an indispensable requirement for GDM, and improving economic effects generally taking two ways [44]:

1. The improvement in technology, such as improving the production process, using new equipment and new raw materials.

2. The improvement of production organization and plan, that is, reasonable arrangement of human and material resources.

Generally, the linear programming consists of three elements: variables, objective function and constraints. The problem of finding the maximum or minimum value of a linear objective function under linear constraints is collectively called a linear programming problem. Decision variables, constraints, and objective functions are the three elements of linear programming.

In the process of GDM based on linguistic assessment, linear programming is a common method. Specific applications are reflected in the following aspects.

- Computing weights information. For MAGDM, the weights can be associated to DMs or attribute and they could be provided in advance or unknown. If the weights information is not given in advance, the objective method is needed to obtain the weights information. To construct an optimal model is a common way for obtaining the weights information [7, 29, 203]. The objective function is usually the minimum of the distance among the DMs' opinions or the balance of each alternative from the best or worst alternative.

- Obtaining the adjusted opinions during CRP. The consensus of the DMs' assessment is the prerequisite of further decision making. The adjustment of initial opinions is inevitable if the consensus level is not satisfied. The acquisition of the adjusted opinions is often through the building of a linear programming model [72, 93, 184, 216]. The objective function is usually the minimum adjustment between the original and adjusted opinions or minimum adjustment cost from the original opinion to the adjusted opinion.

2.6.2 Stochastic Approach

Probability theory [39], as the basis of the stochastic approach, is a branch of mathematics that studies the quantitative laws of random phenomena. Since probability theory involves extensive knowledge, here we only introduce the common knowledge often used in MAGDM. According to the category of the stochastic variable, the attribute value could be divided into three parts as:

1) Attribute value is discrete [190, 201]. It is the general distribution of attribute values. Usually, DM gives the assessment of the attribute in advance, then the attribute value is the possible value with possibility value equal to 1.

2) Attribute value is continuous [86]. Usually, the value range will be given in advance. In this situation, the attribute value is presented with probability density function. For the uncertain attribute value, its value is usually views as a stochastic variable that belong to normal distribution or uniform distribution.

- Normal stochastic variable

  Suppose that the probability density function of continuous stochastic variable X is $f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-u)^2}{2\sigma^2}}$, where μ and σ^2 represent the expectation and variance of X, then X is normal stochastic variable, denoted as $X\sim N(\mu,\sigma^2)$. If $Y=aX+b$ with $X\sim N(\mu,\sigma^2)$, a,b are real numbers and $a\neq 0$, according to the knowledge of probability theory and mathematical statistics, Y is still a normal stochastic variable, its probability density function is as

$$g(y)=\frac{1}{|a|\sigma\sqrt{2\pi}}e^{-\frac{[y-(a\mu+b)]^2}{2(a\sigma)^2}} \tag{2.15}$$

  where $Y\sim N(a\mu+b,(a\sigma)^2)$

  Suppose that $X_1,X_2,...,X_n$ are n mutually independent normal random variables, denoted as $X_i\sim N(\mu_i,\sigma_i^2)$. If these stochastic variables are linearly added, which is $Z=c_1X_1+c_2X_2+\cdots+c_nX_n$, where $c_1,c_2,...,c_n$ are real numbers that exist at least one $c_i\neq 0$, then according to the knowledge of probability theory and mathematical statistics [39], Z is still a normal stochastic variable, its probability density function is as

$$h(z)=\frac{1}{\sqrt{2d\pi}}e^{-\frac{(x-c)^2}{2d}} \tag{2.16}$$

  where $c=\sum_{i=1}^{n}c_i\mu_i$, $d=\sum_{i=1}^{n}c_i^2\sigma_i^2$, $Z\sim N(\sum_{i=1}^{n}c_i\mu_i,\sum_{i=1}^{n}c_i^2\sigma_i^2)$.

  Generally, the larger the expectation μ and the smaller the variance σ^2 of a normal stochastic variable X, the greater the X. If $\sigma^2=0$, then X is a real number μ. The comparison rules between any two normal stochastic variables $X_1\sim N(\mu_1,\sigma_1^2)$ and $X_2\sim N(\mu_2,\sigma_2^2)$ are as [39]:

a) If $\mu_1 > \mu_2$ and $\sigma_1^2 < \sigma_2^2$, then $X_1 > X_2$;

b) If $\mu_1 = \mu_2$ and $\sigma_1^2 < \sigma_2^2$, then $X_1 > X_2$;

c) If $\mu_1 > \mu_2$ and $\sigma_1^2 = \sigma_2^2$, then $X_1 > X_2$.

- Uniform stochastic variable

  Suppose that the probability density function of continuous stochastic variable X is $f(x) = \dfrac{1}{b-a}, a < x < b$ and $f(x) = 0, else$, where a and b are the boundary values of x, then X is a uniform stochastic variable, denoted as $X \sim U(a,b)$.

3) Attribute value is non-discrete discontinuous [153]. DM can only make sure the attribute value under certain circumstances, however, in some cases, the attribute value is uncertain [49, 101, 195]. In this situation, the attribute values are the combination of continuous and discrete distribution, then they could be analyzed based on the above two cases.

Stochastic perspective is a common way to deal with uncertainty [41, 89, 91]. When the attribute values are not deterministic, the process of arriving at the weights of objects becomes more complicated [139]. As Honert [166] pointed out that the attributes can be interpreted as stochastic when it is required to deal with a number of values for the same assessment. Thus, stochastic approach can be defined as an approach to handle uncertainty that defines probability distribution for each input value or parameter in the MAGDM process [125]. For example, Tervonen et al. [161] proposed a stochastic method based on stochastic multicritieria acceptability analysis for assessing the stability of the parameters in sorting problems. Celik et al. [15] gave a comprehensive review on stochastic MAGDM applications and approaches. Therefore, the stochastic approach is very useful for the condition when a MAGDM is based on the stochastic initial information.

Chapter 3

Research Results

This chapter provides a summary of the main proposals developed in this research memory. Research findings and results will be discussed for each proposal in short. There are three proposals which are related with the different objectives presented in the Introduction chapter:

- A new representation and computation model of TD2L from stochastic perspective
- The measurement of the reliability of the adjusted preferences modeled by TD2L information.
- A CRP with minimum adjustment in GDM considering the tolerance of DMs for changing their opinions

3.1 A stochastic perspective on a MAGDM method based on TD2L information

In order to achieve the first objective pointed out in Section 1.2, we highlight the operation rules between TD2L labels from the stochastic perspective, and then analyze the limitations in current computation model of TD2L. Afterwards, a MAGDM method with the TD2Ls assessment from the stochastic perspective is proposed and tested on a real life decision making problem.

3.1.1 A new representation and computation model of TD2L

As mentioned previously, TD2L label represents the assessment given by DMs with the reliability of the assessment presented at the same time. However, the reliability degree is a subjective evaluation on reliability of the given assessments and variables due to the limitations in cognitions and the complexity of decision objects. All assessments without total reliability degree are viewed as uncertain ones. Besides, the existing computation model of TD2L considers the two dimensions information as dependent information, in fact, the two dimensions are related to each other, thus the relations should be considered in the process of information transformation.

To address such limitations about the representation and computation of TD2L labels, we have introduced a new proposal that aims to develop a new representation and computation model from a stochastic perspective, and then to propose the new rules for comparison and similarity measure for TD2L labels associated with the relative importance of the two dimensions linguistic information.

3.1.2 MAGDM method based on the new TD2L representation model

This new MAGDM method is mainly based on the new aggregation function of TD2L labels, the new MAGDM method based on the proposed TD2L computation model is then developed accordingly, all of the contributions are enumerated and briefly explained below:

1. This proposal aims at developing a corresponding rule from TD2L label to a stochastic variable and its inverse. Hence, the comparison and similarity measurement between two TD2L labels have been defined with the consideration of the relative importance degree of the two dimensions of information.
2. To deal with the uncertainty of the initial assessment, another dimension linguistic information is added to ensure the reliability of the initial assessment. To further deal with the TD2L information, a TD2L label is

viewed as a stochastic variable with corresponding expectation and variance.

3. To reflect the reliability of the overall assessments accurately, a new aggregation function of TD2L labels is developed. If all DMs provide the same assessment about the object, then the aggregation result obtained by a general aggregation operator is the same with the assessment provided by all DMs, however, the reliability degree is improved by a new aggregation function of TD2Ls, which is more reasonable and interpretable in real life MAGDM.

In addition, for carrying out fair comparisons with other studies, we have described an experimental process on a real world decision making problem about a business angels (BAs) group with rich entrepreneurial experience which desires to select a suitable investment project from four small unlisted target companies.

The article associated to this proposal is the following one:

Z. L. Wang, Y. M. Wang, L. Martínez. A stochastic perspective on a group decision making method based on two-dimension 2-tuple linguistic information. International Journal of Fuzzy Systems, 2022, https://doi.org/10.1007/s40815-021-01199-3.

3.2 A GDM method based on two-stage MACM with the TD2L labels for reliability measure

After applying a minimum cost CRP, the DMs' adjusted opinions are usually different from the original ones. In spite of the original ones were initially reliable, the reliability of the adjusted opinions cannot be guaranteed. Obviously, the reliability of the adjusted opinions is important during the decision process, adjusted opinions with high consensus level but low reliability would be meaningless. Therefore, the adjusted opinions and its reliability should be considered during the GDM solving process. Nevertheless, it has been neglected so far when DMs' opinions are automatically modified without DMs' supervision.

3.2.1 Analysis on the features of MACM and related limitations in current studies

According to the taxonomy presented in [88], CRPs can be classified according to their feedback process into two types: Consensus with feedback and without feedback. Obviously, consensus with feedback improves the level of agreement among DMs, which also leads to increase the reliability on adjusted opinions. However, for some decision making problems, like emergency decision making [33, 191, 194], it is necessary a high-quality decision making within the limited time, and it is not convenient to wait for the adjusted opinions after several rounds feedback, because time is crucial to be effective and successful. To balance the increased reliability of consensus with feedback and the low cost of consensus without feedback, we try to develop an automatic CRP with minimum adjustment considering the reliability of the adjusted opinions. The main results of the analysis and some related outcomes obtained are briefly enumerated:

1. The use of the existing MACMs leads to agreed opinions, by modifying DMs' original ones, very quickly. However, the reliability of the adjusted opinions obtained by these models is not guaranteed, which reduces the reliability of the decision solution. Therefore, an objective detection approach on reliability of adjusted opinions is necessary for GDM.

2. Regarding the adjustment cost, the more DMs' opinions needed to change, the higher the cost of the adjustment. Therefore, the number of the adjusted opinions should be considered. Especially for large scale GDM problems, if too many DMs need to change their initial opinions, then the CRP would be with low execution. A two-stage MACM is proposed, which not only considers the minimum adjustment, but also minimizes the number of adjusted preferences. It includes the following two stages:

   Stage one: To maximize the improvement of consensus level for each pair of alternatives within the minimum adjustment.

   Stage two: To obtain the adjusted preferences with a certain consensus level at the first stage within the minimum adjustment.

3. The relations between the total preference adjustment and the reliability of the

adjusted preferences are discussed. Not only the adjustment distance and the number of the adjusted opinions are considered, but also the reliability of the adjusted opinions is derived from the measure of the distance between the original and adjusted opinions.

3.2.2 A large scale GDM method considering the two-stage MACM with the TD2L labels for reliability measure

As previously mentioned, the decision method for dealing with large number of DMs and the reliability measure of the adjusted opinions after CRP are challenges for large scale GDM, aiming at improve the existing methods, we have proposed a new large scale GDM method that deals with a large number of DMs and a reliability measure of the adjusted opinions after CRP during the decision making process. At the same time, our proposal presents the alternatives ranking with reliability, which illustrates the reliability of one alternative is better or worse than another alternative. The initial assessments provided by DMs are linguistic terms, 2-tuple linguistic information will appear during calculation, while the final decision result is made based on TD2L information. In the process of linguistic transformation from linguistic term to TD2L information, a large scale GDM method based on a two-stage MACM plays a key role. This proposed method has the following novelties.

1. A new support degree (SD)-based clustering method is proposed to classify the large number of DMs into several subgroups for large scale GDM.

2. A novel two-stage minimum adjustment consensus model which is an automatic model is proposed.

3. The relations between the adjustment and the reliability of the adjusted preferences are used to obtain a final reliable solution by using TD2L information.

4. A new selection rule for choosing the best alternative is defined, the new selection rule not only considers the optimal alternative but also considers the reliability of the optimal alternative better than other alternatives.

To highlight the performance, feasibility and validity of our proposal, we have conducted several comparisons with the classical existing methods that are carried out from different perspective.

The contribution associated to this proposal is the following one:

Zelin Wang, Rosa M. Rodríguez, Ying-Ming Wang, Luis Martínez. A two-stage minimum adjustment consensus model for large scale decision making based on reliability modeled by two-dimension 2-tuple linguistic information. Computers & Industrial Engineering, 2021, 151(3): 106973.

3.3 A CRP with MACM in GDM considering the tolerance of DMs for changing their opinions

During our research related to CRP with MACM, it was detected that there are several issues that have not been successfully addressed yet, such as the following ones:

1. Classically many CRPs consider that the minimum distance between original preferences and the adjusted preferences is the key rule for achieving the agreement, but in classical MACM the number of adjusted preferences should be also considered. Zhang et al. [211] proposed a MACM with these two consensus rules, however, they are separately used in the consensus mechanism, which complicates the consensus process.

2. To reach an agreement among DMs, there will be a lot of consensus rules, like, minimize adjustment between the original and adjusted opinions, minimize the number of the original opinions need to be changed, maximize the number of DMs that could stay their original opinions, etc. However, how to balance these consensus rules is also an important factor, which will influence the decision results of GDM.

3. There must be exist an upper and lower limit that DMs could accept or reject the adjusted opinions during the CRP. If the tolerance of DMs for changing their opinions is neglected, then the feedback mechanism is needed, which is contradict with the automatic CRP. Therefore, the tolerance of DMs for changing their opinions is necessary for CRP in GDM.

In order to address previous issues, a new consensus model based on the consideration of tolerance degree of DMs, and two consensus rules are considered: (i) minimum distance between the original and adjusted preferences, and (ii) minimum number of adjusted preferences. Furthermore, the reliability degree detection of adjusted preference is presented. Therefore, the third objective mentioned in Section 1.2 can be reached.

3.3.1 Dealing with the tolerance of DMs on the adjusted opinions

The proposed CRP considers the following two consensus rules: (1) minimize the distance between the original and adjusted preferences. (2) minimize the number of adjusted preferences. In order to balance these two consensus rules, a DM tolerance degree that defines how much is willing the DM to change his original opinion will play an important role, which means DMs only accept the adjusted preferences within tolerance interval.

The adjustment for DMs' preferences is necessary if the overall consensus level is less than the consensus threshold. Hence, DMs' tolerance degree is proposed as the maximum change that DM willing to accept for the adjusted preferences, which need to be considered. The adjusted preferences to be accepted must satisfy the normalized distance between the original and the adjusted preferences less than the tolerance degree of DMs. The tolerance degree ranges from 0 to 1.

If DM does not accept any change of the original preferences, then he/she is a stubborn DM, which means the tolerance degree is 0. If tolerance degree is 1, then DM could accept any change of the original preferences, where he/she is a benevolent DM.

In fact, the consideration of tolerance degree of DMs is a strict view for minimizing the number of the adjusted preferences. If the minimum number of adjusted preferences is the only condition to be considered, then the distance between the original and adjusted preferences may out of the tolerance interval of DMs. In such situation, the minimum number of adjusted preferences is meaningless.

Thus, it is important to consider both DMs' tolerance degree of DMs and the minimum number of adjusted preferences. To simplify the CRP, a consensus mechanism with priority adjustment rule is designed, then a linear programming model with the minimum number of adjusted preferences is developed.

3.3.2 A CRP in GDM based on the reliability measurement considering the tolerance of DMs

It has been already pointed out the lack of considering of the tolerance of DMs for changing their opinions will lead to the unreliability of the adjusted opinions during CRP. For better understanding, Algorithm Ⅰ is designed to obtain the optimal adjusted preference with minimum number of adjusted preferences considering the tolerance interval of DMs.

Algorithm Ⅰ
Input: The preference matrix provided by DMs, the tolerance degree of DMs, the consensus threshed.
Output: The final adjusted preference.
Step 1: Check the overall CL of DMs' preferences based on three consensus levels as described in Section 2.4.2, if whole CL is larger than or equal to the consensus threshed, then the CRP is done, otherwise continue to the next step.
Step 2: Set up consensus model with the first round, if it can be solved by software LINGO 11 and obtain the optimal preference relations. Then the output preferences are as the adjusted preference relations. If the model is unsolved, then go to the next step.
Step 3: Set up consensus model with second round and repeat the process as described in Step 2, if it can be solved, then output the adjusted preferences as the obtained results. If the model is unsolved, then repeat the above steps until the consensus model can be solved.

After using the Algorithm Ⅰ, the adjusted opinions are derived, however, the reliability of the adjusted opinions are not guaranteed. Here we give a reliability model to compute the reliability degree of the adjusted preferences based on the proposed consensus model, where the reliability degree comes from the concept of stability degree of the original preferences. In this subsection, we introduce a concept: stability degree of original preferences. Then, a comparison measure for TD2L is provided in order to facilitate the selection of the best alternative of the GDM problem.

The reliability degree of the adjusted preferences derives from the stability degree of original preferences, which describes the similarity between the original and adjusted preferences after CRP. The more similar the original preference to adjusted preference is, the higher the stability degree of adjusted preferences is. The larger the value of stability degree is, the more stable the original preference is, then the reliability degree of adjusted preference is more likely higher.

With the introduction of the new consensus model and the description of the relations between the reliability of the adjusted preferences and the adjustment, the steps to execute the decision making process are as follows.

Step 1: To use Algorithm Ⅰ to obtain the optimal adjusted opinions.

Step 2: To compute the reliability of the adjusted preferences.

Step 3: To obtain the final assessment information expressed as TD2L labels.

Step 4: To compare the TD2L labels, then obtain the alternative ranking.

Finally, an illustrative example is shown to certificate the effectiveness of the proposed method.

The contribution associated to this proposal is the following one:

Z. L. Wang, R. M. Rodríguez, Y. M. Wang, L. Martínez. A Consensus Reaching Process with Minimum Adjustment in Group Decision Making with Two-dimensional 2-tuple Linguistic Information based on Reliability Measurement. 2021 IEEE International Conference on Fuzzy Systems, Luxembourg, 11th-14th July.

Chapter 4

Publications

By virtue of the provisions of article 25, point 2, of the current regulations for Doctoral Studies at the University of Jaén, corresponding to the RD program. 99/2011, this chapter presents the publications that make up the core of this doctoral thesis.

These publications correspond to two scientific articles published in International Journals indexed by the JCR (Journal Citation Reports) database, produced by Clarivate Analytics and a conference paper indexed in Engineering Village.

4.1 A new presentation and computation model of TD2L from stochastic perspective

- State: Published.
- Title: A stochastic perspective on a group decision making method based on two-dimension 2-tuple linguistic information.
- Authors: Zelin Wang, Ying-Ming Wang, Luis Martínez.
- Journal: International Journal of Fuzzy Systems.
- DOI: https://doi.org/10.1007/s40815-021-01199-3
- ISSN: 1562-2479.
- Impact Factor (JCR 2020): 4.673.
  - Quartiles:
  - * Quartile 1 in Computer Science, Information Systems. Ranking 35/162

Int. J. Fuzzy Syst.
https://doi.org/10.1007/s40815-021-01199-3

A Stochastic Perspective on a Group Decision-Making Method Based on Two-Dimension 2-Tuple Linguistic Information

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Received: 1 September 2020 / Revised: 18 January 2021 / Accepted: 18 February 2021

Abstract The two-dimension 2-tuple linguistic (TD2L) label, based on the traditional 2-tuple linguistic representation model, adds another 2-tuple linguistic term to express the reliability degree of the assessments. However, the reliability degree is a subjective evaluation on reliability of the given assessments and variables due to the limitations in cognitions and the complexity of decision objects. All assessments without total reliability degree are viewed as uncertain ones. Based on this idea, this paper proposes a new TD2L representation model from a stochastic perspective. The assessment expressed by TD2L is a variable that fluctuates around the given linguistic term, and the fluctuation range is decided by the reliability of the assessment. Therefore, the assessments are regarded as stochastic variables, where the expectancy and deviation of the stochastic variable are corresponding to the first dimension and the second dimension information of TD2Ls, respectively. Consequently, two new aggregation functions for aggregating TD2L labels based on the algorithms among stochastic variables are proposed. In addition, the comparison and similarity measure between TD2L labels are developed, which considers the relative importance of the two dimensions of TD2L labels. Finally, the proposed method is applied to an investment decision of medium sized enterprise and alternatives ranking is provided with the probability of superiority. A comparison analysis conducted from three aspects illustrates the effectiveness of the proposed method.

Keywords Multiple attribute group decision-making (MAGDM) · Two-dimension 2-tuple linguistic (TD2L) information · Stochastic analysis · Aggregation function

1 Introduction

Linguistic terms are more easily represented than crisp numbers for attribute assessments in multiple attribute group decision-making (MAGDM). The concept of linguistic variables was introduced by Zadeh [1], which took values expressed by words or sentences in natural languages. It turned out to be a useful tool for handing MAGDM problems with qualitative information. Since then, MAGDM approaches for dealing with linguistic variables have been widely investigated [2–9]. To avoid information loss and reduce the cost of computation complexity in calculation, Herrera and Martínez [10] proposed a 2-tuple linguistic model with respect to computing with words (CWWs). Furthermore, several extensions for the 2-tuple linguistic extension model have been proposed within MAGDM problems which are the 2-tuple semantic model [11–13], multi-granular 2-tuple linguistic model [5, 14, 15], proportional 2-tuple linguistic model [16, 17] and numerical scale model [18–20]. Based on the extensive successful research of 2-tuple linguistic models in CWW, Martínez and Herrera [21] provided an overview on these models.

The previous 2-tuple linguistic models have been successfully used to elicit the assessments but, the reliability

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Published online: 17 January 2022

of the assessments is also important for decision makers (DMs). The extant decision-making models with 2-tuple linguistic assessments assume that all the assessments have the same confidence level [22], which may be infeasible in a practical decision environment. In order to show the evaluation on the reliability of the assessments, Zhu et al. [23] proposed the concept of two-dimension linguistic information, which includes the reliability information of the subjective assessments. Obviously, the information expressed by TD2L is more accurate and reasonable, because the assessment and the reliability of the assessment are provided at the same time. Due to the advantage of expression of TD2L labels, several studies have explored MAGDM problems with two-dimension linguistic assessment [24–36]. Zhu et al. [30] proposed a two-dimension linguistic lattice structure to deal with two-dimension linguistic information, which makes the expression of information more intuitive and comprehensible. Liu et al. [29, 37] proposed some two-dimension uncertain linguistic aggregation operators and applied them to practical MAGDM problems. Wang et al. [33] applied the TD2L representation model to a large scale group decision-making problem. Liu et al. [35] developed an improved failure mode and effects analysis method using two-dimensional uncertain linguistic variables. Zhao et al. [38] combined two-dimension linguistic expression and PROMETHEE methods for multiple attribute decision-making. In addition, it has also been widely used in other fields in real life, such as power plant site selection [31], emergency management [36] and risk assessment [32].

However, some aspects have not been sufficiently analyzed yet. Firstly, existing approaches for dealing with the TD2L labels have considered the two dimensions as independent information without taking into account that the uncertainty of the assessment is related to the reliability degree. Secondly, previous studies attached more importance to the assessment than the reliability degree but failed to consider the relative importance degree of the two dimensions. Thirdly, the general aggregation operators of TD2L labels didn't reflect the reliability degree of the overall assessment, leading to the distortion of information. In this paper, we aim to solve these limitations.

Let $(\dot{s}_{i_1}, \ddot{s}_j)$ and $(\dot{s}_{i_2}, \ddot{s}_j)$ be two TD2L labels, then the reliability degree of the first dimension information $\dot{s}_{i_1}$ and $\dot{s}_{i_2}$ are shown in Fig. 1b, in which the grey shadows represent the uncertainty of the assessment. The assessment is uncertain and adjustable according to the different reliability degree. Figure 1a shows presentation of TD2L labels in previous studies, which regarded the label as previous points, meaning that the first dimension and the second one are two independent measurements. Hence, we process the uncertainty and adjust the assessment according to the reliability degree to propose a new TD2L representation model. With the aid of stochastic analysis, the assessment is regarded as a stochastic variable, where the numerical characters of the stochastic variable are decided by the two-dimensional information of TD2L labels.

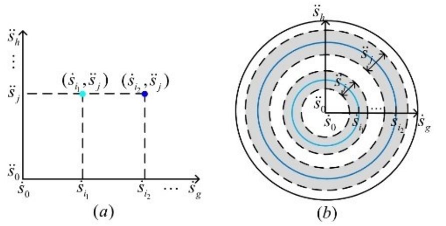

Fig. 1 The different representations of TD2L labels

As pointed out by Honert [39], a judgement can be explained as stochastic when it is required to deal with a number of values for the same judgement [40]. Based on the fuzzy sets theory [41] and probability theory [42], this paper focuses on MAGDM problems with two-dimension linguistic information to overcome the previous limitations and further improve its application research. A novel two-dimension linguistic representation from a stochastic perspective is presented. Then two new aggregation functions are developed based on this, which are different from the traditional aggregation operators. The comprehensive reliability degree will be presented with the aggregation of TD2L labels.

The rest of this paper is arranged as follows: Sect. 2 reviews basic concepts regarding 2-tuple linguistic and two-dimension linguistic representation models. In Sect. 3, a new TD2L model is proposed which makes TD2L label corresponding to a stochastic variable and vice versa, then a comparison and distance measure for TD2L labels are presented. Furthermore, we develop the TD2L aggregation function (TD2LAF) and TD2L ordered aggregation function (TD2LOAF), and related properties of the two functions are studied in details. Section 4 gives a new group decision-making method based on TD2L assessment. In Sect. 5, a practical example is provided to demonstrate the concrete steps and present the results of the proposed method. The comparison analysis shows the flexibility and effectiveness of the proposed method with two-dimension linguistic assessment. The conclusions are drawn in Sect. 6.

2 Preliminaries

In this section, we mainly review the basic concepts of 2-tuple linguistic representation model [10] and two-dimension linguistic label [30], which will provide a basis of the study.

2.1 The 2-Tuple Linguistic Model

Suppose that $S = \{s_0, s_1, \ldots, s_g\}$ is a pre-defined linguistic term set, and the cardinality of S is $g+1$. For any $s_i, s_j \in S$, the following properties should satisfy [10]:

(1) The set is ordered: if $i > j$, then $s_i > s_j$;

(2) Maximum operator: if $s_i > s_j$, then $\max(s_i, s_j) = s_i$;

(3) Minimum operator: if $s_i > s_j$, then $\min(s_i, s_j) = s_j$;

(4) Negation operator: $\text{neg}(s_i) = s_{g-i}$.

In general, the cardinality of linguistic label set S is an odd number, and more than 9 or less than 5 are difficult for DMs to evaluate. Therefore, the cardinalities of linguistic label set S are usually 5, 7 or 9. If S is defined with 7 cardinalities, then it shown as $S = \{s_0 = none, s_1 = verypoor, s_2 = poor, s_3 = medium, s_4 = good, s_5 = verygood, s_6 = perfect\}$.

In order to express linguistic information more exactly, a 2-tuple linguistic term (s_i, α) is proposed by Herrera and Martínez [10], where s_i is a linguistic term involved in set S, α is a numeric number representing the deviation from s_i. Some related notations of 2-tuple linguistic are provided as follows:

Definition 1 [10] Let $S = \{s_0, s_1, \ldots, s_g\}$ be a linguistic term set and $\beta \in [0, g]$ be a value representing the result of aggregation operation. The 2-tuple linguistic term (s_i, α) equivalent to β through the function is Δ as follows:

$$\Delta : [0, g] \rightarrow S \times [-0.5, 0.5) \tag{1}$$

$$\Delta(\beta) = (s_i, \alpha), \text{ with } \begin{cases} s_i, i = \text{round}(\beta) \\ \alpha = \beta - i, \alpha \in [-0.5, 0.5) \end{cases} \tag{2}$$

where *round* is the usual round operation.

Obviously, Δ is a one to one mapping function. Δ has an inverse function Δ^{-1} that $\Delta^{-1} : S \times [-0.5, 0.5) \rightarrow [0, g]$ and $\Delta^{-1}(s_i, \alpha) = \alpha + i = \beta$.

Definition 2 [10] For any two 2-tuple linguistic terms (s_i, α_i) and (s_j, α_j), the relations to compare them can be given as follows:

(1) (1) If $i > j$, then $(s_i, \alpha_i) > (s_j, \alpha_j)$;

(2) If $i = j$, then

(a) $(s_i, \alpha_i) > (s_j, \alpha_j)$ for $\alpha_i > \alpha_j$;

(b) $(s_i, \alpha_i) < (s_j, \alpha_j)$ for $\alpha_i < \alpha_j$;

(c) $(s_i, \alpha_i) = (s_j, \alpha_j)$ for $\alpha_i = \alpha_j$

2.2 Two-Dimension 2-Tuple Linguistic Label

2-tuple linguistic variable only express the assessment of attribute provided by DMs, however, in real life, DMs not only want to know the objective assessment but also want to obtain the evaluation on the reliability of assessment. Therefore, Zhu et al. [23] introduced the definition of two-dimension linguistic label as follows.

Definition 3 [23] Let $S = \{\dot{s}_0, \dot{s}_1, \ldots, \dot{s}_g\}$ and $S^* = \{\ddot{s}_0, \ddot{s}_1, \ldots, \ddot{s}_h\}$ be two linguistic term sets, where the cardinality of S and S^* is $g+1$ and $h+1$, respectively. Then $T = (\dot{s}_i, \ddot{s}_j)$ is a two-dimension lingusitic label, in which $\dot{s}_i \in S$ used for describing the assessment of alternatives provided by DMs. $\ddot{s}_j \in S^*$ describes the self-assessment of DMs on the reliability of the given evaluation result.

In order to express more possibilities of two-dimension linguistic information in the calculation process, Zhu et al. [30] extended the discrete two-dimension linguistic label into continuous, which is defined as follows.

Definition 4 [30] Let $S = \{\dot{s}_0, \dot{s}_1, \ldots, \dot{s}_g\}$ and $S^* = \{\ddot{s}_0, \ddot{s}_1, \ldots, \ddot{s}_h\}$ be two linguistic term sets, $\dot{\beta} \in [0, g]$ be a number value representing the aggregation results of the linguistic labels in S, and $\ddot{\beta} \in [0, h]$ be a number value representing the aggregation resluts of the linguistic lables in S^*. A TD2L label expressed by two 2-tuple linguistic labels equivalent to a binary numerical array $(\dot{\beta}, \ddot{\beta})$ through the function Λ is defined as:

$$\Lambda : [0, g] \times [0, h] \rightarrow (S \times [-0.5, 0.5), S^* \times [-0.5, 0.5)) \tag{3}$$

$$\Lambda\left(\left(\dot{\beta}, \ddot{\beta}\right)\right) = ((\dot{s}_i, \alpha_i), (\ddot{s}_j, \alpha_j)), \text{ with } \begin{cases} \dot{s}_i, i = round\left(\dot{\beta}\right) \\ \alpha_i = \dot{\beta} - i, \alpha_i \in [-0.5, 0.5) \\ \ddot{s}_j, j = round\left(\ddot{\beta}\right) \\ \alpha_j = \ddot{\beta} - j, \alpha_j \in [-0.5, 0.5) \end{cases} \tag{4}$$

where *round* is the usual round operation.

The two-dimension linguistic label defined in Definition 3 can be represented by TD2L label defined in Definition 4 by adding 0 in each linguistic label, that is $(\dot{s}_i, \ddot{s}_j) = ((\dot{s}_i, 0), (\ddot{s}_j, 0))$.

3 A New Two-Dimension 2-Tuple Linguistic Computation Model

This section proposes a TD2L representation model from a stochastic perspective, and then develops a comparison and similarity measure for TD2L labels associated with the relative importance of two dimensions. Furthermore, two TD2L aggregation functions and their superiority are described.

3.1 Two-Dimension 2-Tuple Representation from a Stochastic Perspective

Inspired by symbolic translation α of 2-tuple linguistic label (s_i, α), the second dimension information $(\ddot{s}_j, \alpha_j)$ for a TD2L label $((\dot{s}_i, \alpha_i), (\ddot{s}_j, \alpha_j))$ can be regarded as the deviation degree from the first dimension information $(\dot{s}_i, \alpha_i)$. A new TD2L model with the aid of stochastic analysis and some related notations are provided as follows.

Definition 5 Let $S = \{\dot{s}_0, \dot{s}_1, \ldots, \dot{s}_g\}$ and $S^* = \{\ddot{s}_0, \ddot{s}_1, \ldots, \ddot{s}_h\}$ be two linguistic term sets, $\widehat{S} = ((\dot{s}_i, \alpha_i), (\ddot{s}_j, \alpha_j))$ be a TD2L label. $\dot{s}_i \in S$, $\ddot{s}_j \in S^*$, $\alpha_i, \alpha_j \in [-0.5, 0.5)$. $a \in [0, g]$ and $b \in [0, g^2/9]$ are the values representing the symbolic aggregation of linguistic labels in S and S^*, respectively. The function φ, used to obtain a TD2L label represented by two 2-tuples that is equivalent to a binary numerical array (a, b), is defined as:

$$\varphi : (a, b) \to (S \times [-0.5, 0.5), S^* \times [-0.5, 0.5)) \tag{5}$$

$$\varphi((a, b)) = ((\dot{s}_i, \alpha_i), (\ddot{s}_j, \alpha_j)) \tag{6}$$

Obviously, φ is a one to one mapping. Accordingly, φ has an inverse function with $\varphi^{-1} : (S \times [-0.5, 0.5), S^* \times [-0.5, 0.5)) \to (a, b)$, where $\varphi^{-1}(\widehat{S}) = \varphi^{-1}((\dot{s}_i, \alpha_i), (\ddot{s}_j, \alpha_j)) = (a, b)$.

Different from Definition 4, here a TD2L label corresponds to a stochastic variable and the value of binary numerical array (a, b) is regarded as the numerical characteristics of the stochastic variable, while a represents the expectancy of stochastic variable and b represents the variance of stochastic variable. The value rules of a and b are defined as follows.

Let X be a stochastic variable with the expectancy and deviation being $E(X)$ and $D(X)$, respectively. (a, b) is a binary numerical array equivalent to a TD2L label $\widehat{S} = ((\dot{s}_i, \alpha_i), (\ddot{s}_j, \alpha_j))$. $\widehat{S}$ corresponds to a stochastic variable X with $E(X) = a$ and $D(X) = b$, respectively. Then the relations among $\widehat{S}$, (a, b) and X are shown in Fig. 2 as follows.

In real life, one of the most common forms is that attribute values follow or approximately follow normal distribution [43]. In this paper, we consider that a TD2L label corresponding to a stochastic variable X in normal distribution and $X \sim N(a, b)$, then the probability of x between $a - 3\sqrt{b}$ and $a + 3\sqrt{b}$ is 99.74% according to 3σ principle of normal distribution [42], where x is a possible value of X and represents the assessment of attribute value and $x \in [0, g]$. Therefore, $x \in [max\{a - 3\sqrt{b}, 0\}, min\{a + 3\sqrt{b}, g\}]$. The development of two-dimension 2-tuple linguistic information and the corresponding relations between $((\dot{s}_i, \alpha_i), (\ddot{s}_j, \alpha_j))$ and (a, b) are described in Fig. 3 as follows.

Obviously, $\widehat{S} = ((\dot{s}_i, \alpha_i), (\ddot{s}_0, 0))$ is a TD2L label with the lowest reliability degree of assessment. Let (a, b) be the binary numerical array equivalent to $\widehat{S}$, then the value of b is maximum, which represents the lowest reliability of the assessment $(\dot{s}_i, \alpha_i)$. The value of a represents the expectancy of stochastic variable X corresponded to $\widehat{S}$, $a = \Delta^{-1}(\dot{s}_i, \alpha_i)$. In the following part, we study the rules of valuing b.

The range of assessment value is $[0, g]$ and the range of the value of X corresponded to $\widehat{S}$ is $[a - 3\sqrt{b}, a + 3\sqrt{b}]$. Considering that the assessment value could be any one belonging to interval $[0, g]$ for $\widehat{S} = ((\dot{s}_i, \alpha_i), (\ddot{s}_0, 0))$. If $a - 3\sqrt{b} = 0$ or $a + 3\sqrt{b} = g$, then the overlap between intervals $[0, g]$ and $[a - 3\sqrt{b}, a + 3\sqrt{b}]$ is the largest, which reduces the uncertainty of the assessment. Based on this, we consider two cases as follows.

(1) For $a - 3\sqrt{b} = 0$, then $x \in [0, min\{2a, g\}]$. If $0 \le a < g/2$, then $x \in [0, 2a] \subseteq [0, g]$. If $g/2 \le a \le g$, then $x \in [0, g]$. Therefore, $g/2 \le a \le g$ with $a - 3\sqrt{b} = 0$.The probability density function of X is shown in Fig. 4 as follows.

(2) For $a + 3\sqrt{b} = g$, then $x \in [max\{2a - g, 0\}, g]$. If $0 \le a < g/2$, $x \in [0, g]$. If $g/2 \le a \le g$, then $x \in [2a - g, g] \subseteq [0, g]$. Therefore, $0 \le a < g/2$ with

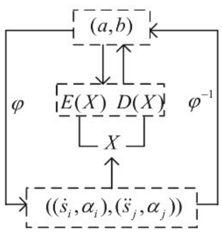

Fig. 2 The relations between a TD2L label and its corresponding stochastic variable

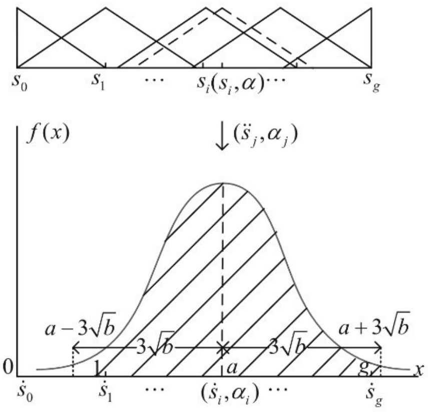

Fig. 3 The corresponding relations between $((\dot{s}_i,\alpha_i),(\ddot{s}_j,\alpha_j))$ and (a,b)

$a+3\sqrt{b}=g$. The probability density function of X is shown in Fig. 5 as follows.

Based on the above statement, for the TD2L label $\widehat{S}=((\dot{s}_i,\alpha_i),(\ddot{s}_0,0))$ with the minimum reliability degree, if $g/2\le a\le g$, then $b=a^2/9$. If $0\le a<g/2$, then $b=(g-a)^2/9$.

Then, we explore the rules of valuing b of general TD2L label $\widehat{S}=((\dot{s}_i,\alpha_i),(\ddot{s}_j,\alpha_j))$. Suppose that the terms in set S^* are uniformly and symmetrically distributed as the general lingusitic terms set proposed by Herrera and Martínez [10]. Likewise, the rules of valuing b with $g/2\le a\le g$ and $0\le a<g/2$ are shown in Figs. 6 and 7 as follows.

where Fig. 6-(a) represents the corresponding rules between the first dimension information $(\dot{s}_i,\alpha_i)$ of TD2L label and numerical number a. Fig. 6-(b) represents the

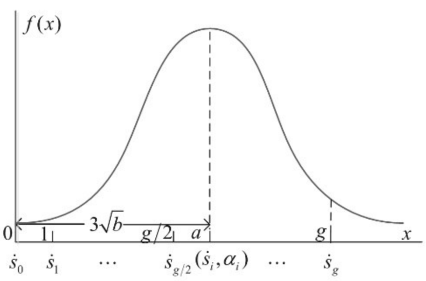

Fig. 4 The probability density function of X with $a-3\sqrt{b}=0$

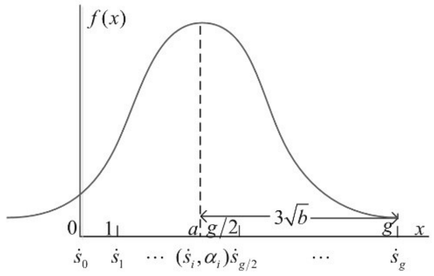

Fig. 5 The probability density function of X with $a+3\sqrt{b}=g$

corresponding rules between the second dimension information $(\ddot{s}_j,\alpha_j)$ of TD2L label and numerical number $a-3\sqrt{b}$.

where Fig. 7-(a) represents the corresponding rules between the first dimension information $(\dot{s}_i,\alpha_i)$ of TD2L label and numerical number a. Fig. 7-(b) represents the corresponding rules between the second dimension information $(\ddot{s}_j,\alpha_j)$ of TD2L label and numerical number $a+3\sqrt{b}$.

Therefore, for any TD2L labels, the rules of valuing (a,b) that are equivalent to a TD2L label $\widehat{S}=((\dot{s}_i,\alpha_i),(\ddot{s}_j,\alpha_j))$ are provided as follows.

Definition 6 Let $S=\{\dot{s}_0,\dot{s}_1,\ldots,\dot{s}_g\}$ and $S^*=\{\ddot{s}_0,\ddot{s}_1,\ldots,\ddot{s}_h\}$ be two linguistic term sets, (a,b) be a binary numerical array. $a\in[0,g]$, $b\in[0,g^2/9]$. Let $\widehat{S}=((\dot{s}_i,\alpha_i),(\ddot{s}_j,\alpha_j))$ be a TD2L label. The functions χ_1 and ψ_1 used to obtain $(\dot{s}_i,\alpha_i)$ and $(\ddot{s}_j,\alpha_j)$, respectively, where $\widehat{S}$ equivalent to (a,b), are defined as:

$$\chi_1(a)=(\dot{s}_i,\alpha_i) \tag{7}$$

$$\psi_1(a,b)=(\ddot{s}_j,\alpha_j) \tag{8}$$

where (1) If $g/2\le a\le g$, then $i=round(a)$, $\alpha_i=a-i$, $j=round((1-3\sqrt{b}/a)h)$, $\alpha_j=(1-3\sqrt{b}/a)h-j$. (2) If $0\le a<g/2$, then $i=round(a)$, $\alpha_i=a-i$, $j=round((1-3\sqrt{b}/(g-a))h)$, $\alpha_j=(1-3\sqrt{b}/(g-a))h-j$.

In special, if $b=0$, then the corresponding TD2L label is $((\dot{s}_i,\alpha_i),(\ddot{s}_h,0))$. If $b=g^2/9$, then the corresponding TD2L label is $((\dot{s}_i,\alpha_i),(\ddot{s}_0,0))$.

Obviously, the function χ_1 is the same as function Δ proposed by Herrera and Martínez [10], and there exists an inverse function χ_2. As such from the first dimension of TD2L label it returns its equivalent numerical value $a\in[0,g]$. The function χ_2 is provided as follows.

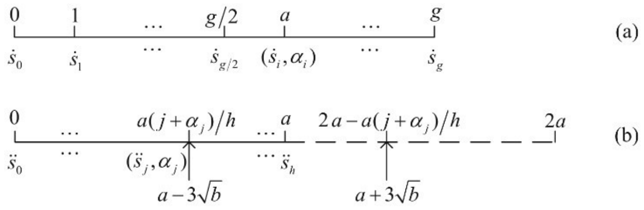

Fig. 6 The numerical scales of two dimensions linguistic terms with $g/2 \le a \le g$

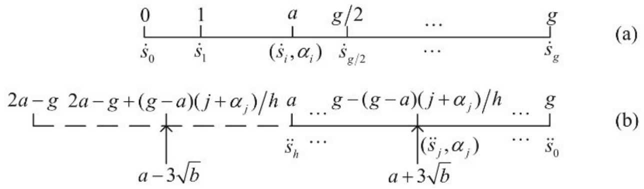

Fig. 7 The numerical scales of two dimensions linguistic terms with $0 \le a < g/2$

$$\chi_2((\dot{s}_i, \alpha_i)) = a \tag{9}$$

where $a = i + \alpha_i$. Similarly, there exist a function ψ_2, as such, from a TD2L label it returns its equivalent numerical value $b \in [0, g^2/9]$. The function ψ_2 is provided as follows.

$$\psi_2\big((\dot{s}_i, \alpha_i), (\ddot{s}_j, \alpha_j)\big) = b \tag{10}$$

where $a = \chi_2((\dot{s}_i, \alpha_i))$. If $g/2 \le a \le g$, then $b = \big(a\big(1 - (j+\alpha_j)/h\big)/3\big)^2$; If $0 \le a < g/2$, then $b = \big((g-a)\big(1 - (j+\alpha_j)/h\big)/3\big)^2$.

Property 1 The value of b in the range from 0 to$g^2/9$. In special, if $g/2 \le a \le g$, then$0 \le b \le a^2/9$. If$0 \le a < g/2$, then$0 \le b \le (g-a)^2/9$.

Proof If $g/2 \le a \le g$, then $b = \big(a\big(1 - (j+\alpha_j)/h\big)/3\big)^2$. Since $(j+\alpha_j)/h \in [0,1]$, then $1 - (j+\alpha_j)/h \in [0,1]$, $a\big(1 - (j+\alpha_j)/h\big)/3 \in [0, a/3]$, $b \in [0, a^2/9] \subseteq [0, g^2/9]$. If $0 \le a < g/2$, then $b = \big((g-a)\big(1 - (j+\alpha_j)/h\big)/3\big)^2$. Since $(j+\alpha_j)/h \in [0,1]$, then $1 - (j+\alpha_j)/h \in [0,1]$, $(g-a)\ \ \big(1 - (j+\alpha_j)/h\big)/3 \in [0,\ \ (g-a)/3]$,$b \in [0,(g-a)^2/9\big] \subseteq [0, g^2/9]$, which completes the proof of Property 1.

Based on Definitions 5 and 6, the relations among functions $\varphi, \varphi^{-1}, \chi_1, \chi_2, \psi_1, \psi_2$ are concluded as follows.

$$\varphi((a,b)) = \big(\chi_1((\dot{s}_i, \alpha_i)), \psi_1\big((\dot{s}_i, \alpha_i), (\ddot{s}_j, \alpha_j)\big)\big) \tag{11}$$

$$\varphi^{-1}\big(\big((\dot{s}_i, \alpha_i), (\ddot{s}_j, \alpha_j)\big)\big) = \big(\chi_2((\dot{s}_i, \alpha_i)), \psi_2\big((\dot{s}_i, \alpha_i), (\ddot{s}_j, \alpha_j)\big)\big) \tag{12}$$

As shown in Fig. 1, a TD2L label can transform to and form a binary numerical array through functions φ and φ^{-1}, respectively. A more detailed description of the relations among TD2L label, binary numerical array and stochastic variable $X \sim N(\mu, \sigma^2)$ is shown in Fig. 8 as follows.

In order to illustrate the value of a, $a - 3\sqrt{b}$, $a + 3\sqrt{b}$ that (a,b) equivalent to TD2L labels, an example is provided as follows for a better the understanding of the rules.

Example 1 Let $\widehat{S}_1 = ((\dot{s}_4, 0.2), (\ddot{s}_3, 0.1))$ and $\widehat{S}_2 = ((\dot{s}_3, -0.2), (\ddot{s}_3, -0.2))$ be two TD2L labels with $g = 6, h = 4$, then the value of a_1, $a_1 - 3\sqrt{b_1}$, $a_1 + 3\sqrt{b_1}$ that (a_1, b_1) equivalent to $\widehat{S}_1$ are shown in Fig. 9. The value of a_2, $a_2 - 3\sqrt{b_2}$, $a_2 + 3\sqrt{b_2}$ that (a_2, b_2) equivalent to $\widehat{S}_2$ are shown in Fig. 10 as follows.

Example 2 Let $S = \{\dot{s}_0, \dot{s}_1, \ldots, \dot{s}_6\}$ and $S^* = \{\ddot{s}_0, \ddot{s}_1, \ldots, \ddot{s}_4\}$ be two linguistic term sets, $\widehat{S} = ((\dot{s}_3, 0.4), (\ddot{s}_3, 0.2))$ be a TD2L label, then the equivalent binary numerical array to $\widehat{S}$ is $(3.4, 0.0529)$, where $a = 3 + 0.4 = 3.4$;
$b = (3.4 \times (1 - (3+0.2)/4)/3)^2 = 0.0529$. Inversely, let $(3.4, 0.0529)$ be a binary numerical array, then the equivalent TD2L label is $((\dot{s}_3, 0.4), (\ddot{s}_3, 0.2))$, where $i = round(3.4) = 3$, $\alpha_i = 3.4 - 3 = 0.4$; $j = round\big(\big(1 - 3 \times \sqrt{0.0529}/3.4\big) \times 4\big) = 3$, $\alpha_j = (1 - 3 \times \sqrt{0.0529}/3.4) \times 4 - 3 = 0.2$.

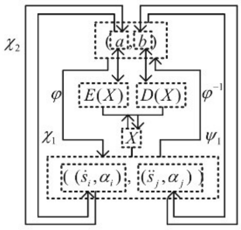

Fig. 8 The transformations between a TD2L label and its equivalent binary numerical array

3.2 Comparison of Two TD2L Labels

The method of comparing attribute information is very important in decision-making and constitutes the fundament of data analysis. In the following part, a comparison that takes the relative importance of two dimensions of TD2L labels into account is proposed. The comparison results show the probability of the superior of one TD2L label over another.

Definition 7 Let $\widehat{S}_1=((\dot{s}_{i_1},\alpha_{i_1}),(\ddot{s}_{j_1},\alpha_{j_1}))$ and $\widehat{S}_2=((\dot{s}_{i_2},\alpha_{i_2}),(\ddot{s}_{j_2},\alpha_{j_2}))$ be two TD2L labels, (a_1,b_1) and (a_2,b_2) be two binary numerical arrays equivalent to $\widehat{S}_1$, $\widehat{S}_2$, respectively. X and Y are two normal stochastic variables with $X\sim N(a_1,b_1)$ and $Y\sim N(a_2,b_2)$. Then the probability of $X<Y$ is defined as follows:

$$P(X<Y)=\int_0^g\int_x^g f(x)f(y)dydx \tag{13}$$

where $f(x)=\frac{1}{\sqrt{2\pi b_1}}e^{-\frac{(x-a_1)^2}{2b_1}}$, $f(y)=\frac{1}{\sqrt{2\pi b_2}}e^{-\frac{(y-a_2)^2}{2b_2}}$.

To make a comprehensive comparison, we consider the symmetric binary numerical array of (a,b), which is denoted as (a',b'). The definitions of (a',b') are provided as follows.

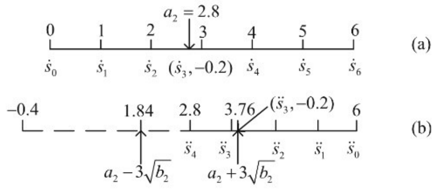

Fig. 10 The numerical scales of two dimensions linguistic terms of $\widehat{S}_2=((\dot{s}_3,-0.2),(\ddot{s}_3,-0.2))$

Definition 8 Let $S=\{\dot{s}_0,\dot{s}_1,\ldots,\dot{s}_g\}$ and $S^*=\{\ddot{s}_0,\ddot{s}_1,\ldots,\ddot{s}_h\}$ be two linguistic term sets, $\widehat{S}=((\dot{s}_i,\alpha_i),(\ddot{s}_j,\alpha_j))$ be a TD2L label. (a,b) is the binary numerical array equivalent to $\widehat{S}$. Then the symmetric binary numerical array of (a,b) is (a',b'), where $a'=\chi_2((\ddot{s}_j,\alpha_j))$. If $h/2\le a'\le h$, then $b'=(a'(1-(i+\alpha_i)/g)/3)^2$; If $0\le a'<h/2$, then $b'=((g-a')(1-(i+\alpha_i)/g)/3)^2$.

Example 3 Let $S=\{\dot{s}_0,\dot{s}_1,\ldots,\dot{s}_6\}$ and $S^*=\{\ddot{s}_0,\ddot{s}_1,\ldots,\ddot{s}_4\}$ be two linguistic term sets, $\widehat{S}=((\dot{s}_3,0.2),(\ddot{s}_4,-0.4))$ be a TD2L label, then the binary numerical array equivalent to $\widehat{S}$ is $(3.2,0.0114)$, the symmetric binary numerical array of $(3.2,0.0114)$ is $(3.6,0.3136)$, where $a'=4-0.4=3.6$, $b'=(3.6\times(1-3.2/6)/3)^2=0.3136$.

Let $\widehat{S}_1=((\dot{s}_{i_1},\alpha_{i_1}),(\ddot{s}_{j_1},\alpha_{j_1}))$ and $\widehat{S}_2=((\dot{s}_{i_2},\alpha_{i_2}),(\ddot{s}_{j_2},\alpha_{j_2}))$ be two TD2L labels, (a_1,b_1) and (a_2,b_2) be two binary numerical arrays equivalent to $\widehat{S}_1$, $\widehat{S}_2$, respectively. $X'\sim N(a'_1,b'_1)$ and $Y'\sim N(a'_2,b'_2)$, where (a'_1,b'_1) and (a'_2,b'_2) are symmetric binary numerical arrays of (a_1,b_1) and (a_2,b_2), respectively. Then the probability of $X'<Y'$ is as follows:

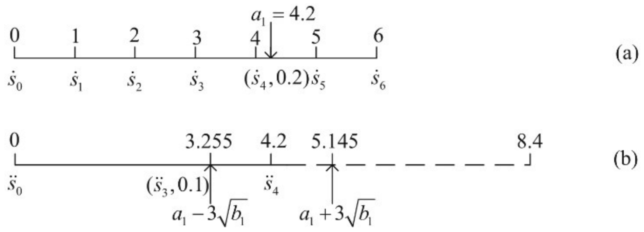

Fig. 9 The numerical scales of two dimensions linguistic terms of $\widehat{S}_1=((\dot{s}_4,0.2),(\ddot{s}_3,0.1))$

$$P\left(X' < Y'\right) = \int_0^h \int_x^h f(x)f(y)dydx \tag{14}$$

where $f(x) = \frac{1}{\sqrt{2\pi b_1'}} e^{-\frac{(x-a_1')^2}{2b_1'}}$, $f(y) = \frac{1}{\sqrt{2\pi b_2'}} e^{-\frac{(y-a_2')^2}{2b_2'}}$.

The comparison results of two TD2L labels cannot be decided only by the probability of $X < Y$. It is obtained by the comprehensive consideration of the probabilities of $X < Y$ and $X' < Y'$. The probability of the superior of one TD2L label over another is as follows.

Definition 9 Let $\widehat{S}_1 = \left((\dot{s}_{i_1}, \alpha_{i_1}), (\ddot{s}_{j_1}, \alpha_{j_1})\right)$ and $\widehat{S}_2 = \left((\dot{s}_{i_2}, \alpha_{i_2}), (\ddot{s}_{j_2}, \alpha_{j_2})\right)$ be two TD2L labels. Then the probability of the superior of $\widehat{S}_2$ over $\widehat{S}_1$ is as follows

$$P\left(\widehat{S}_1 \prec \widehat{S}_2\right) = k_1 P(X < Y) + k_2 P\left(X' < Y'\right) \tag{15}$$

where $0 \le k_1, k_2 \le 1$, $k_1 + k_2 = 1$.

The comparison rules between $\widehat{S}_1$ and $\widehat{S}_2$ are as follows.

(1) If $0 \le P\left(\widehat{S}_1 \prec \widehat{S}_2\right) < 0.5$, then $\widehat{S}_1$ is superior to $\widehat{S}_2$ with the probability $1 - P\left(\widehat{S}_1 \prec \widehat{S}_2\right)$, denoted as $\hat{S}_1 \overset{\succ}{1 - P(\hat{S}_1 \prec \hat{S}_2)} \hat{S}_2$.

(2) If $0.5 < P\left(\widehat{S}_1 \prec \widehat{S}_2\right) \le 1$, then $\widehat{S}_2$ is superior to $\widehat{S}_1$ with the probability $P\left(\widehat{S}_1 \prec \widehat{S}_2\right)$, denoted as $\hat{S}_2 \overset{\succ}{P(\hat{S}_1 \prec \hat{S}_2)} \hat{S}_1$.

(3) If $P\left(\widehat{S}_1 \prec \widehat{S}_2\right) = P\left(\widehat{S}_2 \prec \widehat{S}_2\right) = 0.5$, then $\widehat{S}_1$ is equivalent to $\widehat{S}_2$, denoted as $\widehat{S}_1 \sim \widehat{S}_2$.

Remark 1 k_1 and k_2 are the parameters representing the relative importance assigned by DMs of the two dimensions of 2D2L label. If DMs give more importance to the first dimension information of TD2L label, then $k_1 > k_2$. If DMs give more importance to the second dimension information of TD2L label, then $k_1 < k_2$. If the first dimension is of equal importance with the second dimension, then $k_1 = k_2 = 0.5$. Unless otherwise specified, $k_1 = k_2 = 0.5$. Specially, if $k_1 = 1, k_2 = 0$, then the comparison results are decided by the first dimension information of TD2L labels, if $k_1 = 0, k_2 = 1$, then the comparison results are decided by the second dimension information of TD2L labels.

Example 4 Suppose that $\widehat{S}_1 = ((\dot{s}_5, -0.2), (\ddot{s}_3, 0.4))$ and $\widehat{S}_2 = ((\dot{s}_4, 0.3), (\ddot{s}_3, 0.3))$, $g = 6$, $h = 4$, $k_1 = k_2 = 0.5$, then $P(X < Y) = \int_0^6 \int_x^6 \frac{1}{\sqrt{2\pi}} \times 0.0576 e^{-\frac{(x-4.8)^2}{2\times 0.0576}} \frac{1}{\sqrt{2\pi}} \times 0.0629 e^{-\frac{(x-4.3)^2}{2 \times 0.0629}} dydx = 0.0749$. $P\left(X' < Y'\right) = \int_0^4 \int_x^4 \frac{1}{\sqrt{2\pi}} \times 0.0514 e^{-\frac{(x-3.4)^2}{2\times 0.0514}} \frac{1}{\sqrt{2\pi \times 0.0971}} e^{-\frac{(x-3.3)^2}{2\times 0.0971}} dydx = 0.3853$.
$P\left(\widehat{S}_1 \prec \widehat{S}_2\right) = 0.5 \times 0.0749 + 0.5 \times 0.3853 = 0.2301$.
Therefore, $\widehat{S}_1$ is superior to $\widehat{S}_2$ with the probability 0.7699.

3.3 Similarity Measures Between Two TD2L Labels

Kullback Leibler (KL) divergence is usually used to measure the difference between two probability distributions [44]. Here, we use KL divergence to measure the difference between two 2D2L labels. The KL divergence between two TD2L labels is defined as follows.

Definition 10 Let $\widehat{S}_1 = \left((\dot{s}_{i_1}, \alpha_{i_1}), (\ddot{s}_{j_1}, \alpha_{j_1})\right)$ and $\widehat{S}_2 = \left((\dot{s}_{i_2}, \alpha_{i_2}), (\ddot{s}_{j_2}, \alpha_{j_2})\right)$ be two TD2L labels, (a_1, b_1) and (a_2, b_2) be two binary numerical arrays equivalent to $\widehat{S}_1$ and $\widehat{S}_2$, respectively. $X \sim N(a_1, b_1)$ and $Y \sim N(a_2, b_2)$ are stochastic variables corresponding to $\widehat{S}_1$ and $\widehat{S}_2$, respectively. The KL divergence between $\widehat{S}_1$ and $\widehat{S}_2$ is derived from the KL divergence between X and Y, which is defined as follows.

$$KL(X \| Y) = \int_0^g f_1(x) log \frac{f_1(x)}{f_2(x)} dx \tag{16}$$

where $f_1(x)$ and $f_2(x)$ are probability density function of X and Y, respectively. $f_1(x) = \frac{1}{\sqrt{2\pi b_1}} e^{-\frac{(x-a_1)^2}{2b_1}}$, $f_2(x) = \frac{1}{\sqrt{2\pi b_2}} e^{-\frac{(x-a_2)^2}{2b_2}}$.

Similarly, KL divergence between $X' \sim N\left(a_1', b_1'\right)$ and $Y' \sim N\left(a_2', b_2'\right)$ is as follows.

$$KL\left(X' \| Y'\right) = \int_0^g f_1'(x) log \frac{f_1'(x)}{f_2'(x)} dx \tag{17}$$

where $\left(a_1', b_1'\right)$ and $\left(a_2', b_2'\right)$ are symmetric binary numerical arrays of (a_1, b_1) and (a_2, b_2), respectively. (a_1, b_1) and (a_2, b_2) are the same as before mentioned. $f_1'(x) = \frac{1}{\sqrt{2\pi b_1'}} e^{-\frac{(x-a_1')^2}{2b_1'}}$ and $f_2'(x) = \frac{1}{\sqrt{2\pi b_2'}} e^{-\frac{(x-a_2')^2}{2b_2'}}$ are probability density functions of X' and Y', respectively.

However, KL divergence is a distribution-wise asymmetric measure. In order to overcome the disadvantage, Jensen-Shannon (JS) divergence [45] was proposed by its advantage of symmetric and bounded. Following the extension notations of JS divergence of TD2L labels are provided.

Definition 11 Let $\widehat{S}_1 = ((\dot{s}_{i_1}, \alpha_{i_1}), (\ddot{s}_{j_1}, \alpha_{j_1}))$ and $\widehat{S}_2 = ((\dot{s}_{i_2}, \alpha_{i_2}), (\ddot{s}_{j_2}, \alpha_{j_2}))$ be two TD2L labels, X and Y are stochastic variables corresponding to $\widehat{S}_1$ and $\widehat{S}_2$, respectively. The Jensen-Shannon (JS) divergence between $\widehat{S}_1$ and $\widehat{S}_2$ is derived from the JS divergence between X and Y, which is defined as follows.

$$JS(X,Y) = \frac{1}{2}KL\left(X\|\frac{X+Y}{2}\right) + \frac{1}{2}KL\left(Y\|\frac{X+Y}{2}\right) \tag{18}$$

where $KL\left(X\|\frac{X+Y}{2}\right) = \int_0^g f_1(x)log\frac{2f_1(x)}{f_1(x)+f_2(x)}dx, KL\left(Y\|\frac{X+Y}{2}\right) = \int_0^g f_2(x)log\frac{2f_2(x)}{f_1(x)+f_2(x)}dx$, $f_1(x) = \frac{1}{\sqrt{2\pi b_1}}e^{-\frac{(x-a_1)^2}{2b_1}}, f_2(x) = \frac{1}{\sqrt{2\pi b_2}}e^{-\frac{(x-a_2)^2}{2b_2}}$.

Theorem 1 *JS divergence between any two TD2L labels $\widehat{S}_1$ and $\widehat{S}_2$ satisfies*: $0 \le JS\left(\widehat{S}_1, \widehat{S}_2\right) \le 1$.

Proof The logarithmic base e is used throughout this correspondence unless otherwise stated. Therefore, $KL\left(X\|\frac{X+Y}{2}\right) = \int_0^g f_1(x)log\frac{2f_1(x)}{f_1(x)+f_2(x)}dx = \int_0^g f_1(x)log\frac{f_1(x)}{f_1(x)+f_2(x)}dx + \int_0^g f_1(x)log2dx$. Obviously, $0 \le \int_0^g f_1(x)log2dx \le 1$. $\int_0^g f_1(x)log\frac{f_1(x)}{f_1(x)+f_2(x)}dx < \int_0^g f_1(x)log\frac{f_1(x)+f_2(x)}{f_1(x)+f_2(x)}dx = 0$. Then $KL\left(X\|\frac{X+Y}{2}\right) \le 1$. In special, if $f_2(x) = 0$, then $KL\left(X\|\frac{X+Y}{2}\right) = 1$. Owing to the convexity of KL divergence, it follows that $KL\left(X\|\frac{X+Y}{2}\right) = \int_0^g f_1(x)log\left(\frac{2f_1(x)}{f_1(x)+f_2(x)}\right)dx = -\int_0^g log\left(\frac{f_1(x)+f_2(x)}{2f_1(x)}\right)f_1(x)dx \ge -log\int_0^g \frac{f_1(x)+f_2(x)}{f_1(x)}f_1(x)dx = -log\int_0^g f_1(x)+f_2(x)dx = 0$. Specially, if $f_1(x) = f_2(x)$, then $KL\left(X\|\frac{X+Y}{2}\right) = 0$. Based on above statement, $0 \le KL\left(X\|\frac{X+Y}{2}\right) \le 1$. Similarly, $0 \le KL\left(Y\|\frac{X+Y}{2}\right) \le 1$. Thus, $0 \le JS\left(\widehat{S}_1, \widehat{S}_2\right) \le 1$, which completes the proof of Theorem 1.

Theorem 2 *JS divergence and variational distance satisfies*: $2JS(X,Y) \le \int_0^g |f_1(x) - f_2(x)|dx$

Proof Since $2JS(X,Y) = KL\left(X\|\frac{X+Y}{2}\right) + KL\left(Y\|\frac{X+Y}{2}\right)$

$$= \int_0^g f_1(x)log\frac{2f_1(x)}{f_1(x)+f_2(x)}dx + \int_0^g f_2(x)log\frac{2f_2(x)}{f_1(x)+f_2(x)}dx$$

$$= \int_0^g f_1(x)log\frac{2f_1(x)}{f_1(x)+f_2(x)} + f_2(x)log\frac{2f_2(x)}{f_1(x)+f_2(x)}dx$$

$$= \int_0^g (f_1(x)+f_2(x))\left(\frac{f_1(x)}{f_1(x)+f_2(x)}log\frac{2f_1(x)}{f_1(x)+f_2(x)} + \frac{f_2(x)}{f_1(x)+f_2(x)}log\frac{2f_2(x)}{f_1(x)+f_2(x)}\right)dx$$

Let $\frac{f_1(x)}{f_1(x)+f_2(x)} = A$, $\frac{f_2(x)}{f_1(x)+f_2(x)} = B$, then $A + B = 1, 2JS(X,Y) = \int_0^g (f_1(x)+f_2(x))\ (Alog2A + Blog2B)dx$. Using the inequality $logZ \le Z - 1 (Z \in R)$, it follows that $Alog2A + Blog2B \le A(2A-1) + B(2B-1) = A(2A-1) + (1-A)(2-2A-1) = (2A-1)^2 = (A-B)^2 = (|A-B|)^2$. Since $A, B \in [0,1]$, then $|A-B| \in [0,1]$, $(|A-B|)^2 \le |A-B|$. Thus $Alog2A + Blog2B \le |A-B|$. $2JS(X,Y) \le \int_0^g (f_1(x)+f_2(x))\left|\frac{f_1(x)}{f_1(x)+f_2(x)} - \frac{f_2(x)}{f_1(x)+f_2(x)}\right|dx = \int_0^g |f_1(x) - f_2(x)|dx$, which completes the proof of Theorem 2.

Similarly, JS divergence between $X' \sim N(a_1', b_1')$ and $Y' \sim N(a_2', b_2')$ is as follows.

$$JS\left(X', Y'\right) = \frac{1}{2}KL\left(X'\|\frac{X'+Y'}{2}\right) + \frac{1}{2}KL\left(Y'\|\frac{X'+Y'}{2}\right) \tag{19}$$

where (a_1', b_1') and (a_2', b_2') are symmetric binary numerical arrays of (a_1, b_1) and (a_2, b_2), respectively. (a_1, b_1) and (a_2, b_2) are the same as before mentioned. $KL\left(\widehat{S}_1'\|\frac{\widehat{S}_1'+\widehat{S}_2'}{2}\right) = \int_0^h f_1'(x)log\frac{2f_1'(x)}{f_1'(x)+f_2'(x)}dx$, $KL\left(\widehat{S}_2'\|\frac{\widehat{S}_1'+\widehat{S}_2'}{2}\right) = \int_0^h f_2'(x)log\frac{2f_2'(x)}{f_1'(x)+f_2'(x)}dx$. $f_1'(x) = \frac{1}{\sqrt{2\pi b_1'}}e^{-\frac{(x-a_1')^2}{2b_1'}}, f_2'(x) = \frac{1}{\sqrt{2\pi b_2'}}e^{-\frac{(y-a_2')^2}{2b_2'}}$.

Similar to the comparison of two TD2L labels, a similarity measure between two TD2L labels considering the two dimensions information comprehensively is provided as follows.

Definition 12 Let $\widehat{S}_1 = ((\dot{s}_{i_1}, \alpha_{i_1}), (\ddot{s}_{j_1}, \alpha_{j_1}))$ and $\widehat{S}_2 = ((\dot{s}_{i_2}, \alpha_{i_2}), (\ddot{s}_{j_2}, \alpha_{j_2}))$ be two TD2L labels. Then the similarity between $\widehat{S}_1$ and $\widehat{S}_2$ is defined as:

$$Sim\left(\widehat{S}_1, \widehat{S}_2\right) = k_1 JS(X,Y) + k_2 JS\left(X', Y'\right) \tag{20}$$

where $X \sim N(a_1, b_1)$ and $Y \sim N(a_2, b_2)$ are stochastic variables corresponding to $\widehat{S}_1$ and $\widehat{S}_2$, respectively.

$X' \sim N\left(a_1', b_1'\right)$ and $Y' \sim N\left(a_2', b_2'\right)$ are the same as before. k_1 and k_2 represent the relative importance of the two dimensions of TD2L label.

The meaning of k_1 and k_2 is the same as in **Definition 11**. Unless otherwise specified, $k_1 = k_2 = 0.5$. Obviously, $0 \le Sim\left(\widehat{S}_1, \widehat{S}_2\right) \le 1$. The larger the value of $Sim\left(\widehat{S}_1, \widehat{S}_2\right)$ is, the smaller the distance between $\widehat{S}_1$ and $\widehat{S}_2$ is.

Example 5 Suppose that $\widehat{S}_1 = ((\dot{s}_4, -0.2), (\ddot{s}_3, 0.4))$ and $\widehat{S}_2 = ((\dot{s}_4, 0.2), (\ddot{s}_3, 0.1))$, $g = 6$, $h = 4$, $k_1 = k_2 = 0.5$. Then $KL(X \| \frac{X+Y}{2}) = 0.4340$, $KL(Y \| \frac{X+Y}{2}) = 0.5037$, $KL\left(X' \| \frac{X'+Y'}{2}\right) = 0.0267$, $KL\left(Y' \| \frac{X'+Y'}{2}\right) = 0.1122$, $JS(X, Y) = 0.5 \times 0.4340 + 0.5 \times 0.5037 = 0.4689$, $JS(X', Y') = 0.5 \times 0.0267 + 0.5 \times 0.1122 = 0.0695$, $Sim\left(\widehat{S}_1, \widehat{S}_2\right) = 0.5 \times 0.4689 + 0.5 \times 0.0695 = 0.2692$.

Based on the above statement, the comparison and the similarity measure between any two TD2L labels is available through the JS divergence between two stochastic variables corresponded to the TD2L labels.

3.4 Two-Dimension 2-Tuple Linguistic Aggregation Functions

Here, two TD2L aggregation functions are introduced, with the advantage of dependent additivity of stochastic variables with normal distribution, including TD2LAF and TD2LOAF, which are defined as follows.

Definition 13 Let $S = \left\{\widehat{S}_1, \widehat{S}_1, \ldots, \widehat{S}_n\right\}$ be a TD2L label set, $\widehat{S}_t = \left((\dot{s}_{i_t}, \alpha_{i_t}), (\ddot{s}_{j_t}, \alpha_{j_t})\right), t = 1, 2, \ldots, n$. $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weight vector of S. Then TD2LAF is given by the following expression.

$$\mathrm{TD2LAF}\left(\widehat{S}_1, \widehat{S}_2, \ldots, \widehat{S}_n\right) = \left((\dot{s}_{i_p}, \alpha_{i_p}), \left(\ddot{s}_{j_p}, \alpha_{j_p}\right)\right) \tag{21}$$

where $i_p = round\left(\sum_{t=1}^n \omega_t a_t\right)$, $\alpha_{i_p} = \sum_{t=1}^n \omega_t a_t - i_p$. The value of j_p, α_{j_p} satisfies the following conditions.

If $g/2 \le \sum_{t=1}^n \omega_t a_t \le g$, then

$$\begin{cases} j_p = round\left(\left(1 - 3\sqrt{\sum_{t=1}^n \omega_t^2 b_t} / \sum_{t=1}^n \omega_t a_t\right)h\right) \\ \alpha_{j_p} = \left(1 - 3\sqrt{\sum_{t=1}^n \omega_t^2 b_t} / \sum_{t=1}^n \omega_t a_t\right)h - j \end{cases};$$

If $0 \le \sum_{t=1}^n \omega_t a_t < g/2$, then

$$\begin{cases} j_p = round\left(\left(1 - 3\sqrt{\sum_{t=1}^n \omega_t^2 b_t} / \left(g - \sum_{t=1}^n \omega_t a_t\right)\right)h\right) \\ \alpha_{j_p} = \left(1 - 3\sqrt{\sum_{t=1}^n \omega_t^2 b_t} / \left(g - \sum_{t=1}^n \omega_t a_t\right)\right)h - j \end{cases},$$

where (a_t, b_t) is the equivalent binary numerical array of $\widehat{S}_t$, $t = 1, 2, \cdots n$.

Suppose that X_t is the normal stochastic variable corresponding to $\widehat{S}_t$, $t = 1, 2, \ldots, n$. X_p is the normal stochastic variable corresponding to $\mathrm{TD2LAF}\left(\widehat{S}_1, \widehat{S}_2, \ldots, \widehat{S}_n\right)$, then X_p has the same numerical characteristics as $X_1 + X_2 + \ldots + X_n$.

Theorem 3 Let $S = \left\{\widehat{S}_1, \widehat{S}_1, \ldots, \widehat{S}_n\right\}$ be a TD2L label set, $\widehat{S}_t = \left((\dot{s}_i, \alpha_i), (\ddot{s}_{j_t}, \alpha_{j_t})\right)$, and $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the weight vector of S. $\widehat{S} = \left((\dot{s}_i, \alpha_i), (\ddot{s}_j, \alpha_j)\right)$ is a TD2L label with $j + \alpha_j = \sum_{t=1}^n \omega_t \left(j_t + \alpha_{j_t}\right)$, then.

(1) $\mathrm{TD2LAF}\left(\widehat{S}_1, \widehat{S}_2, \ldots, \widehat{S}_n\right) \succ \widehat{S}$;

(2) $\mathrm{TD2LAF}\left(\widehat{S}_1, \widehat{S}_2, \ldots, \widehat{S}_n\right) = \widehat{S}_0$, where $\widehat{S}_0 = \left((\dot{s}_{i_0}, \alpha_{i_0}), (\ddot{s}_{j_0}, \alpha_{j_0})\right)$ with $(\dot{s}_{i_0}, \alpha_{i_0}) = (\dot{s}_i, \alpha_i)$, the value of j_0 and α_{j_0} are as follows.

If $g/2 \le i + \alpha_i \le g$, then

$$\begin{cases} j_0 = round((1 - 3\sqrt{\sum_{t=1}^n \omega_t^2 b_t} / a_t)h) \\ \alpha_{j_0} = \left(1 - 3\sqrt{\sum_{t=1}^n \omega_t^2 b_t} / a_t\right)h - j_0 \end{cases};$$

If $0 \le i + \alpha_i < g/2$, then

$$\begin{cases} j_0 = round((1 - 3\sqrt{\sum_{t=1}^n \omega_t^2 b_t} / (g - a_t))h) \\ \alpha_{j_0} = (1 - 3\sqrt{\sum_{t=1}^n \omega_t^2 b_t} / (g - a_t))h - j_0 \end{cases}.$$

In special, if $\widehat{S}_1 = \widehat{S}_1 = \ldots = \widehat{S}_n$, then $\mathrm{TD2LAF}\left(\widehat{S}_1, \widehat{S}_2, \ldots, \widehat{S}_n\right) \succ \widehat{S}_1$; $\mathrm{TD2LAF}\left(\widehat{S}_1, \widehat{S}_2, \ldots, \widehat{S}_n\right) = \left((\dot{s}_i, \alpha_i), (\ddot{s}_{j_0}, \alpha_{j_0})\right)$, where the value of j_0 and α_{j_0} satisfies the following: If $g/2 \le a \le g$, then $j_0 = round((1 - 3\sqrt{\sum_{t=1}^n \omega_t^2 b} / a)h)$, $\alpha_{j_0} = \left(1 - 3\sqrt{\sum_{t=1}^n \omega_t^2 b} / a\right)h - j_0$. If $0 \le a < g/2$, then $j_0 = round((1 - 3\sqrt{\sum_{t=1}^n \omega_t^2 b} / (g - a))h)$, $\alpha_{j_0} = (1 - 3\sqrt{\sum_{t=1}^n \omega_t^2 b} / (g - a))h - j$, (a, b) is the binary numerical array corresponding to $\widehat{S}_1$.

Proof (1) If $g/2 \le a \le g$, then

$$\begin{aligned} j_0 + \alpha_{j_0} &= \left(1 - 3\sqrt{\sum_{t=1}^n \omega_t^2 b_t / a}\right)h \\ &= \left(1 - 3\sqrt{\sum_{t=1}^n \omega_t^2 \left(a\left(1 - \left(j_t + \alpha_{j_t}\right)/h\right)/3\right)^2 / a}\right)h \\ &= \left(1 - 3\sqrt{\sum_{t=1}^n \left(\omega_t a\left(1 - \left(j_t + \alpha_{j_t}\right)/h\right)/3\right)^2 / a}\right)h \\ &= \left(1 - \sqrt{\sum_{t=1}^n \left(\omega_t a\left(1 - \left(j_t + \alpha_{j_t}\right)/h\right)\right)^2 / a}\right)h \\ &= \left(1 - \sqrt{\sum_{t=1}^n \left(\omega_t\left(1 - \left(j_t + \alpha_{j_t}\right)/h\right)\right)^2}\right)h \end{aligned}$$

Since $0 \le \omega_t \le 1$, $0 \le 1-(j_t+\alpha_{j_t})/h \le 1$, then $0 \le \omega_t(1-(j_t+\alpha_{j_t})/h) \le 1$, then $\sum_{t=1}^{n}(\omega_t(1-(j_t+\alpha_{j_t})/h))^2 \le \sum_{t=1}^{n}\omega_t(1-(j_t+\alpha_{j_t})/h)$, $\sum_{t=1}^{n}(\omega_t(1-(j_t+\alpha_{j_t})/h))^2 \le 1-\sum_{t=1}^{n}(j_t+\alpha_{j_t})/h$, $j_0 + \alpha_{j_0} \ge (1-(1-\sum_{t=1}^{n}(j_t+\alpha_{j_t})/h))h = \sum_{t=1}^{n}(j_t+\alpha_{j_t})$. Therefore, $\mathrm{TD2LAF}(\widehat{S}_1, \widehat{S}_2, \ldots, \widehat{S}_n) \succ \widehat{S}$.

(2) If $0 \le a < g/2$, then

$$j_0 + \alpha_{j_0} = \left(1 - 3\sqrt{\sum_{t=1}^{n}\omega_t^2 b_t/(g-a)}\right)h$$

$$= \left(1 - 3\sqrt{\sum_{t=1}^{n}\omega_t^2((g-a)(1-(j_t+\alpha_{j_t})/h)/3)^2/(g-a)}\right)h$$

$$= \left(1 - \sqrt{\sum_{t=1}^{n}(\omega_t(g-a)(1-(j_t+\alpha_{j_t})/h))^2/(g-a)}\right)h$$

$$= \left(1 - \sqrt{\sum_{t=1}^{n}(\omega_t(1-(j_t+\alpha_{j_t})/h))^2}\right)h$$

where it is the same with the situation of $g/2 \le a \le g$. Therefore, $\mathrm{TD2LAF}(\widehat{S}_1, \widehat{S}_2, \ldots, \widehat{S}_n) \succ \widehat{S}$, which compeletes the proof of (1) of Theorem 3.

Example 6Let $\widehat{S}_1 = ((\dot{s}_3, 0.4), (\ddot{s}_2, 0.3))$, $\widehat{S}_2 = ((\dot{s}_4, -0.2), (\ddot{s}_3, 0.2))$, $\widehat{S}_3 = ((\dot{s}_3, -0.1), (\ddot{s}_3, -0.2))$, $\widehat{S}_4 = ((\dot{s}_4, 0.4), (\ddot{s}_3, 0.1))$ be four TD2L labels. $g = 6$, $h = 4$. $\omega = (0.2, 0.3, 0.1, 0.4)$ is weight vector. By **Definition 13**, the TD2LAF of $\widehat{S}_1$, $\widehat{S}_2$, $\widehat{S}_3$, $\widehat{S}_4$ is obtained as $\mathrm{TD2LAF}(\widehat{S}_1, \widehat{S}_2, \widehat{S}_3, \widehat{S}_4) = ((\dot{s}_4, -0.13), (\ddot{s}_3, 0.42))$. (The binary numerical arrays equivalent to $\widehat{S}_1, \widehat{S}_2, \widehat{S}_3, \widehat{S}_4$ are $(3.4, 0.2320), (3.8, 0.0642), (2.9, 0.0961), (4.4, 0.1089)$, respectively. The binary numerical array corresponding to $\mathrm{TD2LOAF}(\widehat{S}_1, \widehat{S}_2, \widehat{S}_3, \widehat{S}_4)$ is (a_0, b_0), where $a_0 = 0.2 \times 3.4 + 0.3 \times 3.8 + 0.1 \times 2.9 + 0.4 \times 4.4 = 3.87$, $b_0 = 0.2^2 \times 0.2320 + 0.3^2 \times 0.0642 + 0.1^2 \times 0.0961 + 0.4^2 \times 0.1089 = 0.0334$。 $((\dot{s}_4, -0.13), (\ddot{s}_3, 0.42))$ is the TD2L label equivalent to binary numerical array $(3.87, 0.0334)$).

The main difference between TD2LAF and the general aggregation operators of linguistic information is that TD2LAF can reflect reliability degree of overall assessments accurately. Specially, if all DMs provide the same assessment about the object, then aggregation result by general aggregation operators is the same with the assessment provided by all DMs, however, the reliability degree is improved by TD2LAF, which is more reasonable and interpretable in real life MAGDM.

Example 7 Suppose that there are three DMs E_1, E_2, E_3 evaluating three *PhD* candidate students PhD_1, PhD_2 and PhD_3. Every DM is of equal importance. The evaluation information and the result are shown in Table 1 as follows.

As shown in Table 1, the three students cannot be differentiated by TD2LWAA. However, $PhD_2 \succ PhD_3 \succ PhD_1$ is concluded by TD2LAF. Suppose that the assessment value of $PhD_m (m = 1, 2, 3)$ provided by DM $E_q (q = 1, 2, 3)$ is $\widehat{S}_{mq}$. Then the reason of $PhD_2 \succ PhD_3$ is that $1 - sim(\widehat{S}_{23}, \widehat{S}_{33}) > \frac{1}{2}\left[\left(1 - sim(\widehat{S}_{22}, \widehat{S}_{32})\right) + \left(1 - sim(\widehat{S}_{21}, \widehat{S}_{31})\right)\right]$, the reason of $PhD_3 \succ PhD_1$ is that $1 - sim(\widehat{S}_{32}, \widehat{S}_{12}) > \frac{1}{2}\left[\left(1 - sim(\widehat{S}_{31}, \widehat{S}_{11})\right) + \left(1 - sim(\widehat{S}_{33}, \widehat{S}_{13})\right)\right]$. Therefore, TD2LAF is more effective to explain the aggregation result.

Owing to the complexity and uncertainty of decision-making process, the weight information is not always specified as given, To consider the position weight, the ordered aggregation is provided as follows.

Definition 14 Let $S = \{\widehat{S}_1, \widehat{S}_1, \cdots, \widehat{S}_n\}$ be a TD2L label set, $\widehat{S}_t = ((\dot{s}_{i_t}, \alpha_{i_t}), (\ddot{s}_{j_t}, \alpha_{j_t}))$,$t = 1, 2, \cdots n$. $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weight vector of S. Then TD2L ordered aggregation function (TD2LOAF) is given by the following expression.

$$\mathrm{TD2LOAF}(\widehat{S}_1, \widehat{S}_2, \ldots, \widehat{S}_n) = \left((\dot{s}_{i_q}, \alpha_{i_q}), (\ddot{s}_{j_p}, \alpha_{j_q})\right) \quad (22)$$

where $i_q = round(\sum_{t=1}^{n}\omega_t a_{\tau(t)})$, $\alpha_{i_q} = \sum_{t=1}^{n}\omega_t a_{\tau(t)} - i_q$. The value of j_q, α_{j_q} satisfies the following conditions.

Table 1 The aggregation results in different ways

Alternative	E_1	E_2	E_3	Aggregation result (TD2LWAA)	Aggregation result (TD2LAF)
PhD_1	$((\dot{s}_4, 0), (\ddot{s}_2, 0))$	$((\dot{s}_4, 0), (\ddot{s}_3, 0))$	$((\dot{s}_4, 0), (\ddot{s}_4, 0))$	$((\dot{s}_4, 0), (\ddot{s}_3, 0))$	$((\dot{s}_4, 0), (\ddot{s}_3, 0.25))$
PhD_2	$((\dot{s}_3, 0), (\ddot{s}_2, 0))$	$((\dot{s}_4, 0), (\ddot{s}_3, 0))$	$((\dot{s}_5, 0), (\ddot{s}_4, 0))$	$((\dot{s}_4, 0), (\ddot{s}_3, 0))$	$((\dot{s}_4, 0), (\ddot{s}_4, -0.01))$
PhD_3	$((\dot{s}_5, 0), (\ddot{s}_2, 0))$	$((\dot{s}_5, 0), (\ddot{s}_3, 0))$	$((\dot{s}_2, 0), (\ddot{s}_4, 0))$	$((\dot{s}_4, 0), (\ddot{s}_3, 0))$	$((\dot{s}_4, 0), (\ddot{s}_3, -0.31))$

If $g/2 \le \sum_{t=1}^{n} \omega_t a_{\tau(t)} \le g$, then $\begin{cases} j_q = round\left(\left(1 - 3\sqrt{\sum_{t=1}^{n} \omega_t^2 b_{\tau(t)}} / \sum_{t=1}^{n} \omega_t a_{\tau(t)}\right)h\right) \\ \alpha_{j_q} = \left(1 - 3\sqrt{\sum_{t=1}^{n} \omega_t^2 b_{\tau(t)}} / \sum_{t=1}^{n} \omega_t a_{\tau(t)}\right)h - j_q \end{cases}$;
If $0 \le \sum_{t=1}^{n} \omega_t a_{\tau(t)} < g/2$, then $\Big\{ j_q = round\left(\left(1 - 3\sqrt{\sum_{t=1}^{n} \omega_t^2 b_{\tau(t)}} / \left(g - \sum_{t=1}^{n} \omega_t a_{\tau(t)}\right)\right)h\right) \alpha_{j_q} = \left(1 - 3\sqrt{\sum_{t=1}^{n} \omega_t^2 b_{\tau(t)}} / \left(g - \sum_{t-1}^{n} \omega_t a_{\tau(t)}\right)\right)h - j_q$., where (a_t, b_t) is the equivalent binary numerical array to $\widehat{S}_t$, $t = 1, 2, \ldots, n$. $(\tau(1), \tau(2), \ldots, \tau(n))$ is a permutation of $(1, 2, \ldots, n)$ such that $\widehat{S}_{\tau(t)}$ is superior to $\widehat{S}_{\tau(t+1)}$.

Suppose that X_t is the normal stochastic variable corresponding to $\widehat{S}_t$, X_q is the normal stochastic variable corresponding to $\left(\left(\dot{s}_{i_q}, \alpha_{i_q}\right), \left(\ddot{s}_{j_q}, \alpha_{j_q}\right)\right)$, then X_q has the same numerical characteristics as $X_1 + X_2 + \ldots + X_n$.

Example 8 Let $\widehat{S}_1$, $\widehat{S}_2$, $\widehat{S}_3$, $\widehat{S}_4$ be as the same in Example 6, then $\text{TD2LOAF}\left(\widehat{S}_1, \widehat{S}_2, \widehat{S}_3, \widehat{S}_4\right) = ((\dot{s}_4, -0.48), (\ddot{s}_3, 0.42))$, where the binary numerical arrays equivalent to $\widehat{S}_1, \widehat{S}_2, \widehat{S}_3, \widehat{S}_4$ are (3.4,0.2320),(3.8,0.0642),(2.9,0.0961),(4.4,0.1089), respectively. The binary numerical array corresponding to $\text{TD2LOAF}\left(\widehat{S}_1,\widehat{S}_2,\widehat{S}_3,\widehat{S}_4\right)$ is (a_0', b_0'), where $a_0' = 0.2\times4.4+0.3\times3.8+0.1\times3.4+0.4\times2.9 = 3.52$, $b_0' = 0.2^2\times0.1089+0.3^2\times0.0642+0.1^2\times0.2320+0.4^2\times0.0961 = 0.0033$. $((\dot{s}_4,-0.48),(\ddot{s}_3,0.42))$ is the TD2L label equivalent to binary numerical array (3.52,0.0033).

4 Multi-Attribute Group Decision-Making with TD2L Assessment

Consider a MAGDM problem in which the performance of M decision alternatives are evaluated according to N attributes by K DMs based on TD2L labels $\widehat{S}_{mn}^{k} = \left(\dot{s}_{i_{mn}^k}, \ddot{s}_{j_{mn}^k}\right)$. The linguistic terms are provided by $S = \{\dot{s}_0, \dot{s}_1, \cdots, \dot{s}_g\}$ and $S^* = \{\ddot{s}_0, \ddot{s}_1, \ldots, \ddot{s}_h\}$. The decision matrix R_k provided by DM $E_k (k = 1, 2, \ldots, K)$ is given as follows.

$$
\begin{array}{cc}
 & \begin{array}{cccc} C_1 & C_2 & \cdots & C_N \end{array} \\
R_k = \begin{array}{c} A_1 \\ A_2 \\ \vdots \\ A_M \end{array} &
\begin{bmatrix}
\left((\dot{s}_{i_{11}^k},0),(\ddot{s}_{j_{11}^k},0)\right) & \left((\dot{s}_{i_{12}^k},0),(\ddot{s}_{j_{12}^k},0)\right) & \cdots & \left((\dot{s}_{i_{1N}^k},0),(\ddot{s}_{j_{1N}^k},0)\right) \\
\left((\dot{s}_{i_{21}^k},0),(\ddot{s}_{j_{21}^k},0)\right) & \left((\dot{s}_{i_{22}^k},0),(\ddot{s}_{j_{22}^k},0)\right) & \cdots & \left((\dot{s}_{i_{2N}^k},0),(\ddot{s}_{j_{2N}^k},0)\right) \\
\vdots & \vdots & \ddots & \vdots \\
\left((\dot{s}_{i_{M1}^k},0),(\ddot{s}_{j_{M1}^k},0)\right) & \left((\dot{s}_{i_{M2}^k},0),(\ddot{s}_{j_{M2}^k},0)\right) & \cdots & \left((\dot{s}_{i_{MN}^k},0),(\ddot{s}_{j_{MN}^k},0)\right)
\end{bmatrix}
\end{array}
$$

Let $\omega = (\omega_1, \omega_2, \ldots, \omega_N)^T$ be the weight vector of the attributes and $e = (e_1, e_2, \ldots, e_K)^T$ be the weight vector of the DMs, where $0 \le \omega_n \le 1$, $\sum_{n=1}^{N} \omega_n = 1$, $0 \le e_k \le 1$, $\sum_{k=1}^{K} e_k = 1$. We propose an efficient method for ranking alternatives by the following steps. The general framework for choosing the optimal alternative of an MAGDM problem with TD2L information described in Fig. 11 as follows.

As we know, attributes include benefit attributes and cost ones. For sake of calculation simplicity, we suppose that the larger the value of $\widehat{S}_{mn}^{k}$, the better the corresponding object is. The detailed steps of decision-making proposed in this paper are as follows.

Step 1 Transform the TD2L label $\widehat{S}_{mn}^{k} = \left(\dot{s}_{i_{mn}^k}, \ddot{s}_{j_{mn}^k}\right)$ into corresponding stochastic variable $X_{mn}^{k} \sim N\left(a_{mn}^{k}, b_{mn}^{k}\right)$ according to Eqs. (9) and (10), and transform $\left(a_{mn}^{k}, b_{mn}^{k}\right)$ into its symmetric binary numerical arrays $\left(a'^{k}_{mn}, b'^{k}_{mn}\right)$ according to Definition 8. Then the stochastic variables decision matrices $\overline{R}_k$ and $\overline{R}_k'$ are as follows, respectively.

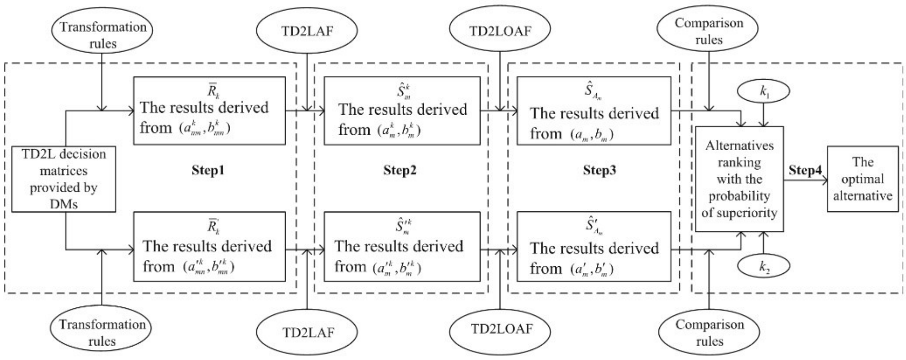

Fig. 11 Framework for MAGDM based on TD2L assessment

$$
\bar{R}_k = \begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_M \end{array}
\begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \dots & C_N \end{array} \\
\begin{bmatrix}
X^k_{11}\sim N(a^k_{11}, b^k_{11}) & X^k_{12}\sim N(a^k_{12}, b^k_{12}) & \dots & X^k_{1N}\sim N(a^k_{1N}, b^k_{1N}) \\
X^k_{21}\sim N(a^k_{21}, b^k_{21}) & X^k_{22}\sim N(a^k_{22}, b^k_{22}) & \dots & X^k_{2N}\sim N(a^k_{2N}, b^k_{2N}) \\
\vdots & \vdots & \ddots & \vdots \\
X^k_{M1}\sim N(a^k_{M1}, b^k_{M1}) & X^k_{M2}\sim N(a^k_{M2}, b^k_{M2}) & \dots & X^k_{MN}\sim N(a^k_{MN}, b^k_{MN})
\end{bmatrix}
\end{array}
$$

$$
\bar{R}'_k = \begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_M \end{array}
\begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \dots & C_N \end{array} \\
\begin{bmatrix}
X'^k_{11}\sim N(a'^k_{11}, b'^k_{11}) & X'^k_{12}\sim N(a'^k_{12}, b'^k_{12}) & \dots & X'^k_{1N}\sim N(a'^k_{1N}, b'^k_{1N}) \\
X'^k_{21}\sim N(a^{k\prime}_{21}, b'^k_{21}) & X'^k_{22}\sim N(a'^k_{22}, b'^k_{22}) & \dots & X'^k_{2N}\sim N(a'^k_{2N}, b'^k_{2N}) \\
\vdots & \vdots & \ddots & \vdots \\
X'^k_{M1}\sim N(a^{k\prime}_{M1}, b'^k_{M1}) & X'^k_{M2}\sim N(a'^k_{M2}, b'^k_{M2}) & \dots & X'^k_{MN}\sim N(a'^k_{MN}, b'^k_{MN})
\end{bmatrix}
\end{array}
$$

where $\overline{R}_k = \left(X^k_{mn} \sim N\left(a^k_{mn}, b^k_{mn}\right)\right)_{M\times N}$, $\bar{R}'_k = \left(X'^k_{mn} \sim N\left(a'^k_{mn}, b'^k_{mn}\right)\right)_{M\times N}$. $m = 1, 2, \ldots, M$, $n = 1, 2, \ldots, N$, $k = 1, 2, \ldots, K$.

Step 2 Utilize weight vector of attributes $(\omega_1, \omega_2, \ldots, \omega_N)^T$ and decision matrices $\overline{R}_k, \overline{R}'_k$ as above, the individual overall attribute values of the alternative A_m provided by the DM E_k can be obtained by TD2LAF as follows.

$$\widehat{S}^k_m = \left(\left(\dot{s}_{i^k_m}, \alpha_{i^k_m}\right), \left(\ddot{s}_{j^k_m}, \alpha_{j^k_m}\right)\right) \tag{23}$$

$$\widehat{S}'^k_m = \left(\left(\dot{s}_{i'^k_m}, \alpha_{i'^k_m}\right), \left(\ddot{s}_{j'^k_m}, \alpha_{j'^k_m}\right)\right) \tag{24}$$

where $m = 1, 2, \ldots, M$, $k = 1, 2, \ldots, K$. $\widehat{S}^k_m = \mathrm{TD2LAF}\left(\widehat{S}^k_{m1}, \widehat{S}^k_{m2}, \ldots, \widehat{S}^k_{mN}\right)$ with the corresponding stochastic variable is $X^k_m \sim N\left(a^k_m, b^k_m\right)$, $a^k_m = \sum_{n=1}^N \omega_n a^k_{mn}$, $b^k_m = \sum_{n=1}^N \omega_n^2 b^k_{mn}$. $X'^k_m \sim N\left(a'^k_m, b'^k_m\right)$ is the stochastic variable corresponding to $\hat{S}'^k_m = \mathrm{TD2LAF}\left(\hat{S}'^k_{m1}, \hat{S}'^k_{m2}, \ldots, \hat{S}'^k_{mN}\right)$, $a'^k_m = \sum_{n=1}^N \omega_n a'^k_{mn}$, $b'^k_m = \sum_{n=1}^N \omega_n^2 b'^k_{mn}$. The value of $i^k_m, \alpha_{i^k_m}, j^k_m, \alpha_{j^k_m}, i'^k_m, \alpha_{i'^k_m}, j'^k_m, \alpha_{j'^k_m}$ are as follows.

If $g/2 \le \alpha_{j^k_m} \le g$, then $\begin{cases} i^k_m = round\left(a^k_m\right) \\ \alpha_{i^k_m} = a^k_m - i^k_m \end{cases}$ and

$$\begin{cases} j^k_m = round((1 - 3\sqrt{b^k_m/a^k_m})h) \\ \alpha_{j^k_m} = \left(1 - 3\sqrt{b^k_m/a^k_m}\right)h - j^k_m \end{cases};$$

If $0 \leq \alpha_{j_m^k} < g/2$, then $\begin{cases} i_m^k = round\left(a_m^k\right) \\ \alpha_{i_m^k} = a_m^k - i_m^k \end{cases}$ and $\begin{cases} j_m^k = round((1 - 3\sqrt{b_m^k}/(g - a_m^k))h) \\ \alpha_{j_m^k} = \left(1 - 3\sqrt{b_m^k}/(g - a_m^k)\right)h - j_m^k \end{cases}$.

If $g/2 \leq \alpha_{j'^k_m} \leq g$, then $\begin{cases} i'^k_m = round\left(a'^k_m\right) \\ \alpha_{i'^k_m} = a'^k_m - i'^k_m \end{cases}$ and $\begin{cases} j'^k_m = round((1 - 3\sqrt{b'^k_m}/a'^k_m)h) \\ \alpha_{j'^k_m} = \left(1 - 3\sqrt{b'^k_m}/a'^k_m\right)h - j'^k_m \end{cases}$;

If $0 \leq \alpha_{j'^k_m} < g/2$, then $\begin{cases} i'^k_m = round\left(a'^k_m\right) \\ \alpha_{i'^k_m} = a'^k_m - i'^k_m \end{cases}$ and $\begin{cases} j'^k_m = round((1 - 3\sqrt{b'^k_m}/\left(g - a'^k_m\right))h) \\ \alpha_{j'^k_m} = \left(1 - 3\sqrt{b'^k_m}/\left(g - a'^k_m\right)\right)h - j'^k_m \end{cases}$.

Step 3 Utilize weight vector of DMs $e = (e_1, e_2, \ldots, e_K)^T$ and the value of $\widehat{S}_m^k, \widehat{S}'^k_m$ as above, the overall assessment of alternative value $\widehat{S}_{A_m}$ and $\widehat{S}'_{A_m}$ can be derived by TD2LOAF as follows, which is from collective suggestions of overall DMs.

$$\widehat{S}_{A_m} = ((\dot{s}_{i_m}, \alpha_{i_m}), (\ddot{s}_{j_m}, \alpha_{j_m})) \tag{25}$$

$$\widehat{S}'_{A_m} = \left(\left(\dot{s}_{i'_m}, \alpha_{i'_m}\right), \left(\ddot{s}_{j'_m}, \alpha_{j'_m}\right)\right) \tag{26}$$

where $m = 1, 2, \ldots, M$. $X_m \sim N(a_m, b_m)$ is the corresponding normal stochastic variable of $\widehat{S}_{A_m}$. $\widehat{S}_{A_m} = \text{TD2LOAF}\left(\widehat{S}_m^1, \widehat{S}_m^2, \cdots \widehat{S}_m^K\right) = ((\dot{s}_{i_m}, \alpha_{i_m}), (\ddot{s}_{j_m}, \alpha_{j_m}))$. $X'_m \sim N\left(a'_m, b'_m\right)$ is the corresponding normal stochastic variable of $\widehat{S}'_{A_m}$. $\widehat{S}'_{A_m} = \text{TD2LOAF}\left(\widehat{S}'^1_m, \widehat{S}'^2_m, \ldots, \widehat{S}'^K_m\right) = \left(\left(\dot{s}_{i'_m}, \alpha_{i'_m}\right), \left(\ddot{s}_{j'_m}, \alpha_{j'_m}\right)\right)$. $a_m = \sum_{k=1}^K e_k a_m^k$, $b_m = \sum_{k=1}^K e_k^2 b_m^k$, $a'_m = \sum_{k=1}^K e_k a'^k_m$, $b'_m = \sum_{k=1}^K e_k^2 b'^k_m$. The value of i_m, α_{i_m}, j_m, α_{j_m}, i'_m, $\alpha_{i'_m}$, j'_m, $\alpha_{j'_m}$ are as follows.

If $g/2 \leq \alpha_{i_m} \leq g$, then $\begin{cases} i_m = round(a_m) \\ \alpha_{i_m} = a_m - i_m \end{cases}$ and $\begin{cases} j_m = round((1 - 3\sqrt{b_m}/a_m)h) \\ \alpha_{i_m} = (1 - 3\sqrt{b_m}/a_m)h - j_m \end{cases}$;

If $0 \leq \alpha_{i_m} < g/2$, then $\begin{cases} i_m = round(a_m) \\ \alpha_{i_m} = a_m - i_m \end{cases}$ and $\begin{cases} j_m = round((1 - 3\sqrt{b_m}/(g - a_m))h) \\ \alpha_{i_m} = (1 - 3\sqrt{b_m}/(g - a_m))h - j_m \end{cases}$.

If $g/2 \leq \alpha_{j'_m} \leq g$, then $\begin{cases} i'_m = round\left(a'_m\right) \\ \alpha_{i'_m} = a'_m - i'_m \end{cases}$ and $\begin{cases} j'_m = round((1 - 3\sqrt{b'_m}/a'_m)h) \\ \alpha_{j'_m} = \left(1 - 3\sqrt{b'_m}/a'_m\right)h - j'_m \end{cases}$;

If $0 \leq \alpha_{j'_m} < g/2$, then $\begin{cases} i'_m = round\left(a'_m\right) \\ \alpha_{i'_m} = a'_m - i'_m \end{cases}$ and $\begin{cases} j'_m = round((1 - 3\sqrt{b'_m}/(g - a'_m))h) \\ \alpha_{j'_m} = \left(1 - 3\sqrt{b'_m}/(g - a'_m)\right)h - j'_m \end{cases}$.

Step 4 Compute the probability of $\widehat{S}_{m_1}$ superior to $\widehat{S}_{m_2}$, where $m_1, m_2 = 1, 2, \ldots, M$, $m_1 \neq m_2$. The comparison results are calculated as $\hat{S}_{A_{\tau(1)}} \overset{\succ}{P}_{\tau(1)\tau(2)} \hat{S}_{A_{\tau(2)}} \overset{\succ}{P}_{\tau(2)\tau(3)} \cdots \overset{\succ}{P}_{\tau(M-1)\tau(M)} \hat{S}_{A_{\tau(M)}}$, where $(\tau(1), \tau(2), \ldots, \tau(M))$ is a permutation of $(1, 2, \ldots, M)$ such that $\widehat{S}_{A_{\tau(m)}}$ is superior to $\widehat{S}_{A_{\tau(m+1)}}$, $P_{\tau(m)\tau(m+1)}$ is the probability of $\widehat{S}_{A_{\tau(m)}}$ superior to $\widehat{S}_{A_{\tau(m+1)}}$, Then the alternatives ranking is $A_{\tau(1)} \overset{\succ}{P}_{\tau(1)\tau(2)} A_{\tau(2)} \overset{\succ}{P}_{\tau(2)\tau(3)} \cdots \overset{\succ}{P}_{\tau(M-1)\tau(M)} A_{\tau(M)}$.

Step 5 Choose the optimal alternative $A_{\tau(1)}$.

5 Illustrative Example

In this section, an illustrative example is provided to show the application of the proposed method to the investment selection.

5.1 Method Implementation

Suppose a business angels (BAs) group with rich entrepreneurial experience desires to select a suitable investment project from four small unlisted target companies (A_1,A_2,A_3 and A_4). An angel investment group including three members (E_1, E_2 and E_3) are invited to make investment decisions and choose the optimal start-up. Then, with the help of the three members of BAs group, six attributes are considered: market impact (C_1), competitive edge (C_2), potential returns (C_3), team (C_4), technology and services (C_5), exit plan (C_6). The evaluation matrices are provided as $\widehat{S}_{mn}^k = \left(\dot{s}_{i_{mn}^k}, \ddot{s}_{j_{mn}^k}\right)$, $m = 1, 2, 3, 4$, $n = 1, 2, 3, 4, 5, 6$, $k = 1, 2, 3$. $S = \{\dot{s}_0 = poor,\ \dot{s}_1 = medium,\ \dot{s}_2 = littlegood,\ \dot{s}_3 = particalgood,\ \dot{s}_4 = good,\ \dot{s}_5 = verygood,\ \dot{s}_6 = perfect\}$ and $S^* = \{\ddot{s}_0 = littlefamilier,\ \ddot{s}_1 = particalfamilier,\ \ddot{s}_2 = familier,\ \ddot{s}_3 = veryfamilier,\ \ddot{s}_4 = highlyfamilier\}$ are two linguistic evaluation term sets. Let $\omega = (0.15, 0.1, 0.25, 0.15, 0.2, 0.15)^T$ be weight vector of attributes and $e = (0.35, 0.4, 0.25)^T$ be weight vector of DMs. $k_1 = k_2 = 0.5$. The TD2L assessment decision-making matrices $R_k = \left(\widehat{S}_{mn}^{k}\right)_{4\times 6}$ provided by three DMs as follows.

$$R_1 = \begin{array}{cc} & \begin{array}{cccccc} C_1 & C_2 & C_3 & C_4 & C_5 & C_6 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} & \begin{bmatrix} ((\dot{s}_3,0),(\ddot{s}_2,0)) & ((\dot{s}_3,0),(\ddot{s}_2,0)) & ((\dot{s}_4,0),(\ddot{s}_2,0)) & ((\dot{s}_3,0),(\ddot{s}_3,0)) & ((\dot{s}_3,0),(\ddot{s}_2,0)) & ((\dot{s}_4,0),(\ddot{s}_3,0)) \\ ((\dot{s}_4,0),(\ddot{s}_3,0)) & ((\dot{s}_5,0),(\ddot{s}_3,0)) & ((\dot{s}_3,0),(\ddot{s}_3,0)) & ((\dot{s}_3,0),(\ddot{s}_2,0)) & ((\dot{s}_3,0),(\ddot{s}_4,0)) & ((\dot{s}_3,0),(\ddot{s}_4,0)) \\ ((\dot{s}_4,0),(\ddot{s}_2,0)) & ((\dot{s}_3,0),(\ddot{s}_3,0)) & ((\dot{s}_3,0),(\ddot{s}_3,0)) & ((\dot{s}_4,0),(\ddot{s}_2,0)) & ((\dot{s}_4,0),(\ddot{s}_3,0)) & ((\dot{s}_4,0),(\ddot{s}_2,0)) \\ ((\dot{s}_3,0),(\ddot{s}_4,0)) & ((\dot{s}_4,0),(\ddot{s}_4,0)) & ((\dot{s}_3,0),(\ddot{s}_4,0)) & ((\dot{s}_4,0),(\ddot{s}_3,0)) & ((\dot{s}_3,0),(\ddot{s}_3,0)) & ((\dot{s}_3,0),(\ddot{s}_3,0)) \end{bmatrix} \end{array}$$

$$R_2 = \begin{array}{cc} & \begin{array}{cccccc} C_1 & C_2 & C_3 & C_4 & C_5 & C_6 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} & \begin{bmatrix} ((\dot{s}_4,0),(\ddot{s}_2,0)) & ((\dot{s}_3,0),(\ddot{s}_3,0)) & ((\dot{s}_3,0),(\ddot{s}_3,0)) & ((\dot{s}_5,0),(\ddot{s}_2,0)) & ((\dot{s}_5,0),(\ddot{s}_2,0)) & ((\dot{s}_3,0),(\ddot{s}_3,0)) \\ ((\dot{s}_3,0),(\ddot{s}_4,0)) & ((\dot{s}_3,0),(\ddot{s}_3,0)) & ((\dot{s}_3,0),(\ddot{s}_4,0)) & ((\dot{s}_3,0),(\ddot{s}_3,0)) & ((\dot{s}_4,0),(\ddot{s}_3,0)) & ((\dot{s}_2,0),(\ddot{s}_3,0)) \\ ((\dot{s}_3,0),(\ddot{s}_2,0)) & ((\dot{s}_2,0),(\ddot{s}_3,0)) & ((\dot{s}_4,0),(\ddot{s}_2,0)) & ((\dot{s}_4,0),(\ddot{s}_3,0)) & ((\dot{s}_3,0),(\ddot{s}_2,0)) & ((\dot{s}_3,0),(\ddot{s}_2,0)) \\ ((\dot{s}_4,0),(\ddot{s}_3,0)) & ((\dot{s}_3,0),(\ddot{s}_3,0)) & ((\dot{s}_3,0),(\ddot{s}_3,0)) & ((\dot{s}_3,0),(\ddot{s}_4,0)) & ((\dot{s}_2,0),(\ddot{s}_3,0)) & ((\dot{s}_4,0),(\ddot{s}_3,0)) \end{bmatrix} \end{array}$$

$$R_3 = \begin{array}{cc} & \begin{array}{cccccc} C_1 & C_2 & C_3 & C_4 & C_5 & C_6 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} & \begin{bmatrix} ((\dot{s}_3,0),(\ddot{s}_3,0)) & ((\dot{s}_4,0),(\ddot{s}_2,0)) & ((\dot{s}_3,0),(\ddot{s}_2,0)) & ((\dot{s}_5,0),(\ddot{s}_2,0)) & ((\dot{s}_3,0),(\ddot{s}_3,0)) & ((\dot{s}_4,0),(\ddot{s}_2,0)) \\ ((\dot{s}_5,0),(\ddot{s}_3,0)) & ((\dot{s}_3,0),(\ddot{s}_4,0)) & ((\dot{s}_4,0),(\ddot{s}_4,0)) & ((\dot{s}_3,0),(\ddot{s}_3,0)) & ((\dot{s}_3,0),(\ddot{s}_4,0)) & ((\dot{s}_3,0),(\ddot{s}_3,0)) \\ ((\dot{s}_2,0),(\ddot{s}_3,0)) & ((\dot{s}_4,0),(\ddot{s}_3,0)) & ((\dot{s}_3,0),(\ddot{s}_1,0)) & ((\dot{s}_4,0),(\ddot{s}_3,0)) & ((\dot{s}_5,0),(\ddot{s}_1,0)) & ((\dot{s}_3,0),(\ddot{s}_2,0)) \\ ((\dot{s}_3,0),(\ddot{s}_3,0)) & ((\dot{s}_3,0),(\ddot{s}_4,0)) & ((\dot{s}_3,0),(\ddot{s}_3,0)) & ((\dot{s}_3,0),(\ddot{s}_4,0)) & ((\dot{s}_3,0),(\ddot{s}_3,0)) & ((\dot{s}_3,0),(\ddot{s}_4,0)) \end{bmatrix} \end{array}$$

.

We implement the method proposed in this paper again in the illustrative example to choose the optimal SME. The steps are as follows.

Step 1 By Eqs. (9) and (10) and Definition 8, the TD2L decision matrices $R_k = \left(\widehat{S}_{mn}^{k}\right)_{4\times 6}$ turned into the matrices

$$\bar{R}_1 = \begin{array}{cc} & \begin{array}{cccccc} C_1 & C_2 & C_3 & C_4 & C_5 & C_6 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} & \begin{bmatrix} (3,0.50^2) & (3,0.50^2) & (4,0.67^2) & (3,0.25^2) & (3,0.50^2) & (4,0.33^2) \\ (4,0.33^2) & (5,0.42^2) & (3,0.25^2) & (3,0.50^2) & (3,0^2) & (3,0^2) \\ (4,0.67^2) & (3,0.25^2) & (3,0.25^2) & (4,0.67^2) & (4,0.33^2) & (4,0.67^2) \\ (3,0^2) & (4,0^2) & (3,0^2) & (4,0.33^2) & (3,0.25^2) & (3,0.25^2) \end{bmatrix} \end{array}$$

$$\bar{R}_2 = \begin{array}{cc} & \begin{array}{cccccc} C_1 & C_2 & C_3 & C_4 & C_5 & C_6 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} & \begin{bmatrix} (4,0.67^2) & (3,0.25^2) & (3,0.25^2) & (5,0.83^2) & (5,0.83^2) & (3,0.25^2) \\ (3,0^2) & (3,0.25^2) & (3,0^2) & (3,0.25^2) & (4,0.33^2) & (2,0.33^2) \\ (3,0.5^2) & (2,0.17^2) & (4,0.67^2) & (4,0.33^2) & (3,0.50^2) & (3,0.50^2) \\ (4,0.33^2) & (3,0.25^2) & (3,0.25^2) & (3,0^2) & (2,0.33^2) & (4,0.33^2) \end{bmatrix} \end{array}$$

$$\bar{R}_3 = \begin{array}{cc} & \begin{array}{cccccc} C_1 & C_2 & C_3 & C_4 & C_5 & C_6 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} & \begin{bmatrix} (3,0.25^2) & (4,0.67^2) & (3,0.5^2) & (5,0.83^2) & (3,0.25^2) & (4,0.67^2) \\ (5,0.42^2) & (3,0^2) & (4,0^2) & (3,0.25^2) & (3,0^2) & (3,0.25^2) \\ (2,0.33^2) & (4,0.33^2) & (3,0.75^2) & (4,0.33^2) & (5,1.25^2) & (3,0.50^2) \\ (3,0.25^2) & (3,0^2) & (3,0.25^2) & (3,0^2) & (3,0.25^2) & (3,0^2) \end{bmatrix} \end{array}$$

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$$\bar{R}'_1 = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cccccc} C_1 & C_2 & C_3 & C_4 & C_5 & C_6 \end{array} \\ \begin{bmatrix} (2,0.33^2) & (2,0.33^2) & (2,0.22^2) & (3,0.50^2) & (2,0.33^2) & (3,0.33^2) \\ (3,0.33^2) & (3,0.17^2) & (3,0.50^2) & (2,0.33^2) & (4,0.67^2) & (4,0.67^2) \\ (2,0.22^2) & (3,0.50^2) & (3,0.50^2) & (2,0.22^2) & (3,0.33^2) & (2,0.22^2) \\ (4,0.67^2) & (4,0.44^2) & (4,0.67^2) & (3,0.33^2) & (3,0.50^2) & (3,0.50^2) \end{bmatrix} \end{array}$$

$$\bar{R}'_2 = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cccccc} C_1 & C_2 & C_3 & C_4 & C_5 & C_6 \end{array} \\ \begin{bmatrix} (2,0.22^2) & (3,0.50^2) & (3,0.50^2) & (2,0.11^2) & (2,0.11^2) & (3,0.50^2) \\ (4,0.67^2) & (3,0.50^2) & (4,0.67^2) & (3,0.50^2) & (3,0.33^2) & (3,0.67^2) \\ (2,0.33^2) & (3,0.67^2) & (2,0.22^2) & (3,0.33^2) & (2,0.33^2) & (2,0.33^2) \\ (3,0.33^2) & (3,0.50^2) & (3,0.50^2) & (4,0.67^2) & (3,0.67^2) & (3,0.33^2) \end{bmatrix} \end{array}$$

$$\bar{R}'_3 = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cccccc} C_1 & C_2 & C_3 & C_4 & C_5 & C_6 \end{array} \\ \begin{bmatrix} (3,0.50^2) & (2,0.22^2) & (2,0.33^2) & (2,0.11^2) & (3,0.50^2) & (2,0.22^2) \\ (3,0.17^2) & (4,0.67^2) & (4,0.44^2) & (3,0.50^2) & (2,0.67^2) & (3,0.50^2) \\ (3,0.67^2) & (3,0.33^2) & (1,0.50^2) & (3,0.33^2) & (1,0.17^2) & (2,0.33^2) \\ (3,0.50^2) & (4,0.67^2) & (3,0.50^2) & (4,0.67^2) & (3,0.50^2) & (4,0.67^2) \end{bmatrix} \end{array}$$

$\overline{R}_k$ and $\overline{R}'_k$, respectively as follows.

where $m = 1,2,3,4$, $n = 1,2,3,4,5,6$, $k = 1,2,3$. $\overline{R}_k = \left((a^k_{mn}, b^k_{mn})\right)_{M\times N}$ represents $\overline{R}_k = \left(X^k_{mn} \sim N \ (a^k_{mn}, b^k_{mn})\right)_{M\times N}$, $\overline{R}'_k = \left((a'^k_{mn}, b'^k_{mn})\right)_{M\times N}$ represents $\overline{R}'_k = \left(X'^k_{mn} \sim N \ (a'^k_{mn}, b'^k_{mn})\right)_{M\times N}$.

Step 2 Utilize attributes weight $\omega = (0.15, 0.1, 0.25, 0.15, 0.2, 0.15)^T$ and decision matrices $\overline{R}_k$, $\overline{R}'_k$ as above, the individual overall attribute value of the alternative A_m provided by DM E_k obtained, respectively, as follows by Eqs. (23) and (24).

$\widehat{S}^1_1 = ((\dot{s}_3, 0.40), (\ddot{s}_3, 0.21))$, $\widehat{S}^1_2 = ((\dot{s}_3, 0.35), (\ddot{s}_4, -0.42))$,
$\widehat{S}^1_3 = ((\dot{s}_4, -0.35), (\ddot{s}_3, 0.35))$, $\widehat{S}^1_4 = ((\dot{s}_3, 0.25), (\ddot{s}_4, -0.30))$,
$\widehat{S}^2_1 = ((\dot{s}_4, -0.15), (\ddot{s}_3, 0.24))$, $\widehat{S}^2_2 = ((\dot{s}_3, 0.05), (\ddot{s}_4, -0.37))$,
$\widehat{S}^2_3 = ((\dot{s}_3, 0.30), (\ddot{s}_3, 0.17))$, $\widehat{S}^2_4 = ((\dot{s}_3, 0.10), (\ddot{s}_4, -0.46))$,
$\widehat{S}^3_1 = ((\dot{s}_4, -0.45), (\ddot{s}_3, 0.25))$, $\widehat{S}^3_2 = ((\dot{s}_4, -0.45), (\ddot{s}_4, -0.28))$,
$\widehat{S}^3_3 = ((\dot{s}_4, -0.5), (\ddot{s}_3, -0.12))$, $\widehat{S}^3_4 = ((\dot{s}_3, 0), (\ddot{s}_4, -0.35))$;
$\widehat{S}'^1_1 = ((\ddot{s}_2, 0.30), (\dot{s}_5, -0.09))$, $\widehat{S}'^1_2 = ((\ddot{s}_3, 0.20), (\dot{s}_5, -0.24))$,
$\widehat{S}'^1_3 = ((\ddot{s}_3, -0.45), (\dot{s}_5, -0.14))$, $\widehat{S}'^1_4 = ((\ddot{s}_4, -0.50), (\dot{s}_5, -0.24))$,
$\widehat{S}'^2_1 = ((\ddot{s}_3, -0.50), (\dot{s}_5, -0.15))$, $\widehat{S}'^2_2 = ((\ddot{s}_3, 0.40), (\dot{s}_5, -0.30))$,
$\widehat{S}'^2_3 = ((\ddot{s}_2, 0.25), (\dot{s}_5, -0.12))$, $\widehat{S}'^2_4 = ((\ddot{s}_3, 0.15), (\dot{s}_5, -0.29))$,
$\widehat{S}'^3_1 = ((\ddot{s}_2, 0.35), (\dot{s}_5, -0.20))$, $\widehat{S}'^3_2 = ((\ddot{s}_4, -0.45), (\dot{s}_5, -0.09))$,
$\widehat{S}'^3_3 = ((\ddot{s}_2, -0.05), (\dot{s}_5, -0.67))$, $\widehat{S}'^3_4 = ((\ddot{s}_3, 0.40), (\dot{s}_5, -0.25))$

Step 3 Utilize weight vector of DMs $e = (0.35, 0.4, 0.25)^T$ and the value of $\widehat{S}^k_m, \widehat{S}'^k_m$ as above, the overall assessment of alternative value $\widehat{S}_{A_m}$ and $\widehat{S}'_{A_m}$ derived, respectively, as follows by Eqs. (25) and (26), which is from collective suggestions of overall DMs.

$\widehat{S}_{A_1} = ((\dot{s}_4, -0.38), (\ddot{s}_4, -0.45))$, $\widehat{S}_{A_2} = ((\dot{s}_3, 0.35), (\ddot{s}_4, -0.21))$,
$\widehat{S}_{A_3} = ((\dot{s}_3, 0.47), (\ddot{s}_4, -0.49))$, $\widehat{S}_{A_4} = ((\dot{s}_3, 0.11), (\ddot{s}_4, -0.21))$;
$\widehat{S}'_{A_1} = ((\ddot{s}_2, 0.39), (\dot{s}_5, 0.32))$, $\widehat{S}'_{A_2} = ((\ddot{s}_3, 0.37), (\dot{s}_5, 0.30))$,
$\widehat{S}'_{A_3} = ((\ddot{s}_2, 0.28), (\dot{s}_5, 0.28))$, $\widehat{S}'_{A_4} = ((\ddot{s}_3, 0.37), (\dot{s}_5, 0.26))$

Step 4 Compute the probability of $\widehat{S}_{A_{m_1}}$ superior to $\widehat{S}_{A_{m_2}}$, where $m_1, m_2 = 1,2,3,4$, $m_1 \neq m_2$. The comparison results are calculated as $\hat{S}_2 \overset{\succ}{0.75} \hat{S}_4 \overset{\succ}{0.5} \hat{S}_1 \overset{\succ}{0.79} \hat{S}_3$, Then the alternatives ranking is $A_2 \overset{\succ}{0.75} A_4 \overset{\succ}{0.50} A_1 \overset{\succ}{0.79} A_3$.

Step 5 Choose the optimal alternative A_2.

5.2 Comparative Analysis

In this subsection, we perform comparative analyses from three aspects to show the advantage of proposed method in handling the MAGDM problems with TD2L assessment.

(1) Considering the influence of the second dimension information of TD2L label on decision-making results, the following three matrices R_4, R_5, R_6 are provided, while the

matrices R_4, R_5, R_6 have the same first information of TD2L labels with matrices R_1, R_2, R_3, respectively.

$$R_4 = \begin{array}{c c} & \begin{array}{cccccc} C_1 & C_2 & C_3 & C_4 & C_5 & C_6 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} & \begin{bmatrix} (s_3,0) & (s_3,0) & (s_4,0) & (s_3,0) & (s_3,0) & (s_4,0) \\ (s_4,0) & (s_5,0) & (s_3,0) & (s_3,0) & (s_3,0) & (s_3,0) \\ (s_4,0) & (s_3,0) & (s_3,0) & (s_4,0) & (s_4,0) & (s_4,0) \\ (s_3,0) & (s_4,0) & (s_3,0) & (s_4,0) & (s_3,0) & (s_3,0) \end{bmatrix} \end{array}$$

$$R_5 = \begin{array}{c c} & \begin{array}{cccccc} C_1 & C_2 & C_3 & C_4 & C_5 & C_6 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} & \begin{bmatrix} (s_4,0) & (s_3,0) & (s_3,0) & (s_5,0) & (s_5,0) & (s_3,0) \\ (s_3,0) & (s_3,0) & (s_3,0) & (s_3,0) & (s_4,0) & (s_2,0) \\ (s_3,0) & (s_2,0) & (s_4,0) & (s_4,0) & (s_3,0) & (s_3,0) \\ (s_4,0) & (s_3,0) & (s_3,0) & (s_3,0) & (s_2,0) & (s_4,0) \end{bmatrix} \end{array}$$

$$R_6 = \begin{array}{c c} & \begin{array}{cccccc} C_1 & C_2 & C_3 & C_4 & C_5 & C_6 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} & \begin{bmatrix} (s_3,0) & (s_4,0) & (s_3,0) & (s_5,0) & (s_3,0) & (s_4,0) \\ (s_5,0) & (s_3,0) & (s_4,0) & (s_3,0) & (s_3,0) & (s_3,0) \\ (s_2,0) & (s_4,0) & (s_3,0) & (s_4,0) & (s_5,0) & (s_3,0) \\ (s_3,0) & (s_3,0) & (s_3,0) & (s_3,0) & (s_3,0) & (s_3,0) \end{bmatrix} \end{array}$$

Implement the decision-making process with the general weighted aggregation operator and ordered weighted aggregation operator, then get the alternatives ranking as $A_1 \succ A_3 \succ A_2 \succ A_4$. Therefore, alternative A_1 is the optimal alternative, which illustrates the second dimension information of TD2L label has certain influences on the decision-making results. Therefore, we have to consider the reliability degree of assessment when processing the attribute values.

Then study the influence of different reliability degree of assessment on decision-making results. Consider the following three matrices R_7, R_8, R_9. The matrices R_1 and R_7, R_2 and R_8, R_3 and R_9, these three pairs TD2L label matrices have the same first dimension information and different second dimension information.

$$R_7 = \begin{array}{c c} & \begin{array}{cccccc} C_1 & C_2 & C_3 & C_4 & C_5 & C_6 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} & \begin{bmatrix} ((\dot{s}_3,0),(\ddot{s}_3,0)) & ((\dot{s}_3,0),(\ddot{s}_2,0)) & ((\dot{s}_4,0),(\ddot{s}_3,0)) & ((\dot{s}_3,0),(\ddot{s}_3,0)) & ((\dot{s}_3,0),(\ddot{s}_2,0)) & ((\dot{s}_4,0),(\ddot{s}_3,0)) \\ ((\dot{s}_4,0),(\ddot{s}_3,0)) & ((\dot{s}_5,0),(\ddot{s}_2,0)) & ((\dot{s}_3,0),(\ddot{s}_2,0)) & ((\dot{s}_3,0),(\ddot{s}_2,0)) & ((\dot{s}_3,0),(\ddot{s}_2,0)) & ((\dot{s}_3,0),(\ddot{s}_2,0)) \\ ((\dot{s}_4,0),(\ddot{s}_4,0)) & ((\dot{s}_3,0),(\ddot{s}_3,0)) & ((\dot{s}_3,0),(\ddot{s}_3,0)) & ((\dot{s}_4,0),(\ddot{s}_3,0)) & ((\dot{s}_4,0),(\ddot{s}_3,0)) & ((\dot{s}_4,0),(\ddot{s}_3,0)) \\ ((\dot{s}_3,0),(\ddot{s}_4,0)) & ((\dot{s}_4,0),(\ddot{s}_3,0)) & ((\dot{s}_3,0),(\ddot{s}_4,0)) & ((\dot{s}_4,0),(\ddot{s}_3,0)) & ((\dot{s}_3,0),(\ddot{s}_3,0)) & ((\dot{s}_3,0),(\ddot{s}_3,0)) \end{bmatrix} \end{array}$$

$$R_8 = \begin{array}{c c} & \begin{array}{cccccc} C_1 & C_2 & C_3 & C_4 & C_5 & C_6 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} & \begin{bmatrix} ((\dot{s}_4,0),(\ddot{s}_3,0)) & ((\dot{s}_3,0),(\ddot{s}_3,0)) & ((\dot{s}_3,0),(\ddot{s}_3,0)) & ((\dot{s}_5,0),(\ddot{s}_3,0)) & ((\dot{s}_3,0),(\ddot{s}_3,0)) & ((\dot{s}_3,0),(\ddot{s}_3,0)) \\ ((\dot{s}_3,0),(\ddot{s}_3,0)) & ((\dot{s}_3,0),(\ddot{s}_2,0)) & ((\dot{s}_3,0),(\ddot{s}_3,0)) & ((\dot{s}_3,0),(\ddot{s}_2,0)) & ((\dot{s}_4,0),(\ddot{s}_2,0)) & ((\dot{s}_2,0),(\ddot{s}_3,0)) \\ ((\dot{s}_3,0),(\ddot{s}_4,0)) & ((\dot{s}_2,0),(\ddot{s}_3,0)) & ((\dot{s}_4,0),(\ddot{s}_2,0)) & ((\dot{s}_4,0),(\ddot{s}_3,0)) & ((\dot{s}_3,0),(\ddot{s}_2,0)) & ((\dot{s}_3,0),(\ddot{s}_2,0)) \\ ((\dot{s}_4,0),(\ddot{s}_3,0)) & ((\dot{s}_3,0),(\ddot{s}_4,0)) & ((\dot{s}_3,0),(\ddot{s}_4,0)) & ((\dot{s}_3,0),(\ddot{s}_3,0)) & ((\dot{s}_2,0),(\ddot{s}_3,0)) & ((\dot{s}_4,0),(\ddot{s}_3,0)) \end{bmatrix} \end{array}$$

$$R_9 = \begin{array}{c c} & \begin{array}{cccccc} C_1 & C_2 & C_3 & C_4 & C_5 & C_6 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} & \begin{bmatrix} ((\dot{s}_3,0),(\ddot{s}_3,0)) & ((\dot{s}_4,0),(\ddot{s}_3,0)) & ((\dot{s}_3,0),(\ddot{s}_2,0)) & ((\dot{s}_5,0),(\ddot{s}_2,0)) & ((\dot{s}_3,0),(\ddot{s}_3,0)) & ((\dot{s}_4,0),(\ddot{s}_2,0)) \\ ((\dot{s}_5,0),(\ddot{s}_2,0)) & ((\dot{s}_3,0),(\ddot{s}_2,0)) & ((\dot{s}_4,0),(\ddot{s}_3,0)) & ((\dot{s}_3,0),(\ddot{s}_3,0)) & ((\dot{s}_3,0),(\ddot{s}_2,0)) & ((\dot{s}_3,0),(\ddot{s}_2,0)) \\ ((\dot{s}_2,0),(\ddot{s}_3,0)) & ((\dot{s}_4,0),(\ddot{s}_3,0)) & ((\dot{s}_3,0),(\ddot{s}_4,0)) & ((\dot{s}_4,0),(\ddot{s}_4,0)) & ((\dot{s}_5,0),(\ddot{s}_3,0)) & ((\dot{s}_3,0),(\ddot{s}_4,0)) \\ ((\dot{s}_3,0),(\ddot{s}_4,0)) & ((\dot{s}_3,0),(\ddot{s}_4,0)) & ((\dot{s}_3,0),(\ddot{s}_3,0)) & ((\dot{s}_3,0),(\ddot{s}_3,0)) & ((\dot{s}_3,0),(\ddot{s}_4,0)) & ((\dot{s}_3,0),(\ddot{s}_4,0)) \end{bmatrix} \end{array}$$

Table 2 The trend of alternatives ranking with different k_1, k_2

Alternative	Ranking	Trend	The best position	The worst position
A_1			1	3
A_2			1	3
A_3			2	4
A_4			2	4

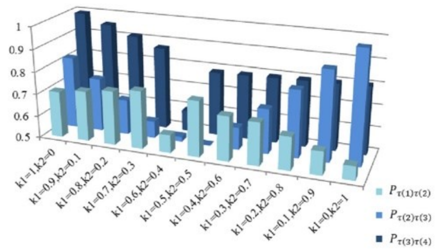

Fig. 12 The probability of superiority with different k_1, k_2. where represents probability of the optimal alternative $A_{\tau(1)}$ superior to the second alternative $A_{\tau(2)}$, represents probability of the second alternative $A_{\tau(2)}$ superior to the third alternative $A_{\tau(3)}$, represents probability of the third alternative $A_{\tau(3)}$ superior to the worst alternative $A_{\tau(4)}$

Perform the decision-making process again as above and then get the alternatives ranking $A_3 \overset{\succ}{0.52} A_4 \sim A_1 \overset{\succ}{0.99} A_2$. Therefore, alternative A_3 is the optimal alternative, which illustrates the different reliability degree of assessment result in different optimal alternative.

(2) Discuss the impact on decision-making results caused by the changes of k_1 and k_2. The trend of alternatives ranking with the decreasing of k_1 and the increasing of k_2 is shown in Table 2 as follows.where in Table 2 represents the position of alternative A_1 in ranking with the decreasing of k_1 and the increasing of k_2 (k_1, k_2=0, 0.1, ..., 1). If k_1=0.7, 0.8, 0.9, 1, then A_1 is the optimal option. If k_2=0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, then A_1 is at the third position. $\Downarrow$ represents the descending of the ranking position and $\Uparrow$ represents the ascending of the ranking position. The explanation of alternatives A_2, A_3, A_4 is similar.

The probability of $A_{\tau(1)}$ superior to $A_{\tau(2)}$, $A_{\tau(2)}$ superior to $A_{\tau(3)}$, $A_{\tau(3)}$ superior to $A_{\tau(4)}$ with different k_1 and k_2 are shown, respectively, in Fig. 12 as follows.

where represents probability of the optimal alternative $A_{\tau(1)}$ superior to the second alternative $A_{\tau(2)}$, represents probability of the second alternative $A_{\tau(2)}$ superior to the third alternative $A_{\tau(3)}$, represents probability of the third alternative $A_{\tau(3)}$ superior to the worst alternative $A_{\tau(4)}$.

(3) Compare the proposed method with the existing methods based on TD2L assessment, the results are presented in Table 3 as follows.

From Fig. 12 and Tables 2–3, the following observations are highlighted:

(1) With the decreasing of k_1 and the increasing of k_2, alternatives ranking and the probability of superiority are different. If $k_1 \in [0.7, 1]$, then $A_1 \succ A_3 \succ A_2 \succ A_4$, which is the same with the method in [28, 29]. It suggests that the methods in [28, 29] consider the first dimension information more important than the second dimension information of TD2L labels. In this situation, the probability of A_1 superior to A_3 is increasing because of $\widehat{S}'_{A_1} \succ \widehat{S}'_{A_3}$, the probability of A_2 superior to A_4 is decreasing because of $\widehat{S}_{A_2} \succ \widehat{S}_{A_4}$ in the situation of $k_1 \in [0.7, 1]$. Similarly, the probability of A_2 superior to A_4 is decreasing because of $\widehat{S}_{A_2} \succ \widehat{S}_{A_4}$, the probability of A_4 superior to A_1 because of

Table 3 The ranking results with different methods

Method	Alternatives ranking	The probability of superiority
[28]	$A_1 \succ A_3 \succ A_2 \succ A_4$	Incalculable
[29]	$A_1 \succ A_3 \succ A_2 \succ A_4$	Incalculable
[30]	$A_1 \succ A_3$, $A_2 \succ A_4$ (A_1 and A_2, A_1 and A_4, A_2 and A_3, A_3 and A_4 is incomparable)	Incalculable
The proposed method	$A_2 \overset{\succ}{0.75} A_4 \overset{\succ}{0.50} A_1 \overset{\succ}{0.79} A_3$	Calculable

$\widehat{S}'_{A_4} \succ \widehat{S}'_{A_1}$, the probability of A_1 superior to A_3 because of $\widehat{S}'_{A_1} \succ \widehat{S}'_{A_3}$ in the situation of $k_1 \in [0.5, 0]$. If $k_1 \in [0.6, 0.7)$, then $A_1 \succ A_2 \succ A_3 \succ A_4$. If $k_1 \in [0, 0.5]$, then $A_2 \succ A_4 \succ A_1 \succ A_3$. Alternative A_3 changes from the second position to the third position then to the worst option with the decreasing of k_1. Alternative A_2 changes from the third position to the second position then to the optimal option with the decreasing of k_1. However, alternative A_4 changes from the worst option to the second position and alternative A_1 from the optimal option to the third position. Therefore, there must be $k_1 \in (0.5, 0.6)$ that makes alternative A_1 the second and alternative A_4 the third option.

(2) Yu et.al [28] consider TD2L labels quantified by using a generalized triangular fuzzy number (TFN), which makers TD2L labels transform to TFNs and vice versa. The comparison results depend on the superiority of the first dimension information, which ignores the importance of the reliability degree of assessment. Therefore, the ranking of alternatives is the same with the one not considering the second dimension information of TD2L labels. By applying the approach in [29], the second dimension information of TD2L label convert to the weight of DMs, which can be involved in operations and avoid information distortion. However, the aggregation of the second dimension information is minimal operator, which magnifies the real aggregation results. In [30], Zhu et al. construct two-dimension linguistic lattice implication algebra as linguistic evaluation set. It can express any continuous TD2L label obtained in the aggregation process without information loss, however, for comparing two TD2L labels, parameter δ ($\delta < 1$ is a parameter that provided by DMs in advance) is set to decide the relations between two TD2L labels, leading to the incomparability of the aggregation results. For example, A_1 and A_2, A_2 and A_3, A_3 and A_4 are incomparable as shown in Table 3. Therefore, alternatives ranking is not always the total order and the final order is adjusted by a predefined parameter.

(3) Using the proposed approach in this study, the assessment is adjustable with the aid of stochastic variable. The importance degree of two dimensions is flexible by DMs. The main advantages of the method proposed in this paper are: (i) A novel TD2L computation model is proposed considering the adjustment of assessment according to the TD2L labels provided by DMs with the aid of stochastic analysis. (ii) The relative importance degree of two dimensions are considered, which makes the decision-making results more effective and flexible. iii) For MAGDM problems based on TD2L assessment, alternatives ranking is provided with the probability of the superiority of one alternative over another, depending on the relative importance degree of two dimensions of TD2L labels. It provides more useful decision-making information for DMs. For example, A_4 is narrowly superior to A_1, and DMs prefer to A_1 instead of A_4 with the decreasing of competitive edge of A_4 during periods of market volatility.

6 Conclusions

Because of the complexity and diversity of decision-making environment, the reliability degree of assessment has increasingly drawn the attention from researchers in MAGDM problems based on linguistic assessments [22, 46–50]. The TD2L model facilitates the modelling of reliability of assessments by introducing one dimension to describe the reliability degree of the assessment. MAGDM problems with TD2L assessment have been applied to a wide range of areas [31, 32, 36]. In spite of the successful application of the two-dimension linguistic information to deal with the representation and computation of two-dimension linguistic labels [28–30, 51], the analysis of the uncertainty of assessments according to the second dimension information has not been explored. In this paper, a corresponding rule from TD2L label to a stochastic variable and its inverse have been presented. Hence, the comparison and similarity measurement between two TD2L labels have been developed with the consideration of the relative importance degree of the two dimensions of information. Furthermore, the influence of the second dimension information on the final decision results and the impacts of different methods on the final decision results have been discussed as well.

MAGDM is a complex problem under linguistic assessment especially for large scale group decision-making. In future, we will further study the application of TD2L representation and computation model from stochastic perspective in large scale group decision-making problems.

Funding The work is supported by the National Nature Science Foundation of China (Grant No. 61773123) and by the Spanish National Research Project PGC2018-099402-B-I00.

Declarations

Conflict of interest The authors declare that they have no conflicts of interest.

Ethical Approval This article does not contain any studies with human participants or animals performed by any of the authors.

References

1. Zadeh, L.A.: The concept of a linguistic variable and its application to approximate reasoning—I. Inf. Sci. **8**, 199–249 (1975)
2. Cid-López, A., Hornos, M.J., Carrasco, R.A., Herrera-Viedma, E., Chiclana, F.: Linguistic multi-criteria decision-making model with output variable expressive richness. Expert Syst. Appl. **83**, 350–362 (2017)
3. Gong, Z.W., Forrest, J., Yang, Y.J.: The optimal group consensus models for 2-tuple linguistic preference relations. Knowl. Based Syst. **37**, 427–437 (2013)
4. Li, C.C., Rodríguez, R.M., Martínez, L., Dong, Y.C., Herrera, F.: Personalized individual semantics based on consistency in hesitant linguistic group decision making with comparative linguistic expressions. Knowl. Based Syst. **145**, 156–165 (2018)
5. Xu, Z.S.: A method based on linguistic aggregation operators for group decision making with linguistic preference relations. Inf. Sci. **166**, 19–30 (2004)
6. Xu, Z.S.: An interactive approach to multiple attribute group decision making with multigranular uncertain linguistic information. Group Decis. Negot. **18**, 119–145 (2009)
7. Zeng, S., Qiyas, M., Arif, M., Mahmood, T.: Extended version of linguistic picture fuzzy TOPSIS method and its applications in enterprise resource planning systems. Math. Probl. Eng. **2019**, 1–8 (2019)
8. Khan, A.A., Qiyas, M., Abdullah, S., Luo, J., Bano, M.: Analysis of robot selection based on 2-tuple picture fuzzy linguistic aggregation operators. Mathematics **7**, 1000 (2019)
9. Qiyas, M., Abdullah, S., Ashraf, S., Khan, S., Khan, A.: Triangular picture fuzzy linguistic induced ordered weighted aggregation operators and its application on decision making problems. Math. Found. Comput. **2**, 183 (2019)
10. Herrera, F., Martínez, L.: A 2-tuple fuzzy linguistic representation model for computing with words. IEEE Trans. Fuzzy Syst. **8**, 746–752 (2000)
11. Abchir, M.A., Truck, I.: Towards an extension of the 2-tuple linguistic model to deal with unbalanced linguistic term sets. Kybernetika **49**, 164–180 (2013)
12. Truck, I.: Comparison and links between two 2-tuple linguistic models for decision making. Knowl. Based Syst. **87**, 61–68 (2015)
13. Truck, I., Akdag, H.: Manipulation of qualitative degrees to handle uncertainty: formal models and applications. Knowl. Inf. Syst. **9**, 385–411 (2006)
14. Dong, Y.C., Wu, Y.Z., Zhang, H.J., Zhang, G.Q.: Multi-granular unbalanced linguistic distribution assessments with interval symbolic proportions. Knowl. Based Syst. **82**, 139–151 (2015)
15. Herrera, F., Martinez, L.: A model based on linguistic 2-tuples for dealing with multigranular hierarchical linguistic contexts in multi-expert decision-making. IEEE Trans Syst Man Cybern B **31**, 227–234 (2001)
16. Wang, J.H., Hao, J.Y.: A new version of 2-tuple fuzzy linguistic representation model for computing with words. IEEE Trans. Fuzzy Syst. **14**, 435–445 (2006)
17. Wang, J.H., Hao, J.Y.: An approach to computing with words based on canonical characteristic values of linguistic labels. IEEE Trans. Fuzzy Syst. **15**, 593–604 (2007)
18. Dong, Y.C., Xu, Y.F., Yu, S.: Computing the numerical scale of the linguistic term set for the 2-tuple fuzzy linguistic representation model. IEEE Trans. Fuzzy Syst. **17**, 1366–1378 (2009)
19. Dong, Y.C., Zhang, G.Q., Hong, W.C., Yu, S.: Linguistic computational model based on 2-tuples and intervals. IEEE Trans. Fuzzy Syst. **21**, 1006–1018 (2013)
20. Dong, Y.C., Li, C.C., Xu, Y.F., Gu, X.: Consensus-based group decision making under multi-granular unbalanced 2-tuple linguistic preference relations. Group Decis. Negot. **24**, 217–242 (2015)
21. Martínez, L., Herrera, F.: An overview on the 2-tuple linguistic model for computing with words in decision making: extensions, applications and challenges. Inf. Sci. **207**, 1–18 (2012)
22. Liu, W.Q., Dong, Y.C., Chiclana, F., Cabrerizo, F.J., Herrera-Viedma, E.: Group decision-making based on heterogeneous preference relations with self-confidence. Fuzzy Optim. Decis. Mak. **16**, 429–447 (2017)
23. Zhu, W.D., Zhou, G.Z., Yang, S.L.: An approach to group decision making based on 2-dimension linguistic assessment information. Syst. Eng. **27**, 113–118 (2009)
24. Verma, R., Merigó, J.M.: Multiple attribute group decision making based on 2-dimension linguistic intuitionistic fuzzy aggregation operators. Soft Comput. 1–24 (2020).
25. Sun, Y., Zhang Q., Li, Z., Ren, Y., Zhang, Y., Yuan, Z.: Multi-attribute group decision making method based on 2-dimension 2-tuple linguistic information. In: 2020 IEEE International Conference on Artificial Intelligence and Information Systems (ICAIIS), IEEE, 2020, pp. 44–48.
26. Liu, Y., Wang, X.-K., Wang, J.-Q., Li, L., Cheng, P.-F.: Cloud model-based PROMETHEE method under 2D uncertain linguistic environment, J. Intell. Fuzzy Syst. 1–19 (2020).
27. Liu, P.D.: An approach to group decision making based on 2-dimension uncertain linguistic information. Technol. Econ. Dev. Econ. **18**, 424–437 (2012)
28. Yu, X.H., Xu, Z.S., Liu, S.S., Chen, Q.: Multicriteria decision making with 2-dimension linguistic aggregation techniques. Int. J. Intell. Syst. **27**, 539–562 (2012)
29. Liu, P.D., Yu, X.C.: 2-Dimension uncertain linguistic power generalized weighted aggregation operator and its application in multiple attribute group decision making. Knowl. Based Syst. **57**, 69–80 (2014)
30. Zhu, H., Zhao, J.B., Xu, Y.: 2-Dimension linguistic computational model with 2-tuples for multi-attribute group decision making. Knowl. Based Syst. **103**, 132–142 (2016)
31. Wu, Y.N., Sun, X.K., Lu, Z.M., Zhou, J.L., Xu, C.B.: Optimal site selection of straw biomass power plant under 2-dimension uncertain linguistic environment. J. Clean Prod. **212**, 1179–1192 (2019)
32. Wu, Y.N., Xu, C.B., Li, L.W., Wang, Y., Chen, K.F., Xu, R.H.: A risk assessment framework of PPP waste-to-energy incineration projects in China under 2-dimension linguistic environment. J. Clean Prod. **183**, 602–617 (2018)
33. Wang, Z.L., Rodríguez, R.M., Wang, Y.-M., Martínez, L.: A two-stage minimum adjustment consensus model for large scale decision making based on reliability modeled by two-dimension 2-tuple linguistic information. Comput. Ind. Eng. (2020). https://doi.org/10.1016/j.cie.2020.106973
34. Zhao, J., Zhu, H., Li, H.: 2-Dimension linguistic PROMETHEE methods for multiple attribute decision making. Expert Syst. Appl. **127**, 97–108 (2019)
35. Liu, H.-C., Hu, Y.-P., Wang, J.-J., Sun, M.: Failure mode and effects analysis using two-dimensional uncertain linguistic variables and alternative queuing method. IEEE Trans. Reliab. **68**, 554–565 (2018)
36. Ding, X.-F., Liu, H.-C.: A 2-dimension uncertain linguistic DEMATEL method for identifying critical success factors in emergency management. Appl. Soft Comput. **71**, 386–395 (2018)
37. Liu, P.D., Qi, X.F.: Some generalized dependent aggregation operators with 2-dimension linguistic information and their

application to group decision making. J. Intell. Fuzzy Syst. **27**, 1761–1773 (2014)
38. J. Zhao, H. Zhu, H. Li, 2-dimension linguistic PROMETHEE methods for multiple attribute decision making. Expert Syst. Appl. (2019).
39. Van den Honert, R.: Stochastic group preference modelling in the multiplicative AHP: a model of group consensus. Eur. J. Oper. Res. **110**, 99–111 (1998)
40. Ramanathan, R.: Stochastic decision making using multiplicative AHP. Eur. J. Oper. Res. **97**, 543–549 (1997)
41. Zadeh, L.A.: Fuzzy sets. Inf. Control. **3**, 338–353 (1965)
42. Ross, S.M.: Introduction to Probability Models. Academic Press, New York (2014)
43. Bertsimas, D., Freund, R.M.: Data, Models, and Decisions: The Fundamentals of Management Science. South-Western College Press, Cambridge (2002)
44. Kullback, S., Leibler, R.A.: On information and sufficiency. Ann. Math. Stat. **22**, 79–86 (1951)
45. Lin, J.H.: Divergence measures based on the Shannon entropy. IEEE Trans. Inf. Theory **37**, 145–151 (1991)
46. Fenton, N., Wang, W.: Risk and confidence analysis for fuzzy multicriteria decision making. Knowl. Based Syst. **19**, 430–437 (2006)
47. Fu, C., Yang, J.B., Yang, S.L.: A group evidential reasoning approach based on expert reliability. Eur. J. Oper. Res. **246**, 886–893 (2015)
48. Guha, D., Chakraborty, D.: Fuzzy multi attribute group decision making method to achieve consensus under the consideration of degrees of confidence of experts' opinions. Comput. Ind. Eng. **60**, 493–504 (2011)
49. Liu, W.Q., Zhang, H.J., Chen, X., Yu, S.: Managing consensus and self-confidence in multiplicative preference relations in group decision making. Knowl. Based Syst. **162**, 62–73 (2018)
50. Ureña, R., Chiclana, F., Fujita, H., Herrera-Viedma, E.: Confidence-consistency driven group decision making approach with incomplete reciprocal intuitionistic preference relations. Knowl. Based Syst. **89**, 86–96 (2015)
51. Liu, P.D., Teng, F.: An extended TODIM method for multiple attribute group decision-making based on 2-dimension uncertain linguistic variable. Complexity **21**, 20–30 (2016)

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4.2 The measurement of the reliability of the adjusted preferences modeled by TD2L information

- State: Published.
- Title: A two-stage minimum adjustment consensus model for large scale decision making based on reliability modeled by two-dimension 2-tuple linguistic information
- Authors: Zelin Wang, Rosa M. Rodríguez, Ying-Ming Wang, Luis Martínez.
- Journal: Computers & Industrial Engineering.
- Volume: 151. Page: 106973. Year: 2021.
- DOI: 10.1016/j.cie.2020.106973
- ISSN: 0360-8352.
- Impact Factor (JCR 2020): 5.431.
  - Quartiles:
  - ∗ Quartile 1 in Computer Science, Interdisciplinary. Ranking 21/112.

Computers & Industrial Engineering 151 (2021) 106973

Contents lists available at ScienceDirect

Computers & Industrial Engineering

journal homepage: www.elsevier.com/locate/caie

A two-stage minimum adjustment consensus model for large scale decision making based on reliability modeled by two-dimension 2-tuple linguistic information

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ARTICLE INFO

Keywords:
Minimum adjustment consensus
Large scale group decision making (LSGDM)
Reliability degree
Two-dimension 2-tuple linguistic (TD2L) expression

ABSTRACT

Consensus reaching processes (CRPs) have been required to assure the consensus in large scale group decision making (LSGDM). Opinion reliability detection has been demanded to ensure the trustworthiness of the original information and different information modeling approaches have facilitated it in which two dimensional linguistic (TDL) information has an outstanding place. The reliability degree of original opinions elicited by TDL expressions is often given in advance as subjective evaluation, and after adjustment during CRP, the reliability of the adjusted opinions is often neglected especially for automatic CRP, which may lead to unreliable decisions. In real decision making, considering the interest of decision makers (DMs) themselves, the self-assessment of the DMs on the reliability of the given opinions could be easily manipulated by DMs. To reduce the subjectivity of the decision making, we propose a method to obtain objectively the reliability of the adjusted opinions through a two-stage minimum cost consensus model based on 2-tuple linguistic additive preference relations. Firstly, a support degree (SD)-based clustering method will be developed for classifying DMs into several subgroups to make more manageable the large number of DMs. Subsequently, a two-stage minimum adjustment consensus model will be presented to improve the consensus level (CL) gradually. Eventually, the adjusted opinions will be presented as two-dimension 2-tuple linguistic (TD2L) information. A comparative performance analysis of this CRP based LSGDM approach will be provided to show its effectiveness.

1. Introduction

Group decision-making (GDM) refers to the selection of the best alternative from a set of feasible alternatives according to the opinions of different decision makers (DM). Generally, a GDM process includes two parts: Consensus process and selection process (Herrera-Viedma, Martinez, Mata, & Chiclana, 2005; Kacprzyk, Fedrizzi, & Nurmi, 1992; Rodríguez, Labella, De Tré, & Martínez, 2018). The consensus process aims at maximizing the agreement among DMs, which is usually controlled by a moderator, who helps the DMs involved while changing their opinions towards consensus (Labella, Liu, Rodríguez, & Martínez, 2018; Palomares & Martínez, 2014; Rodríguez et al., 2018). The increasing complexity of decision making environments makes the unanimous and complete agreement regarding the collective solution hard to achieve. Therefore, "soft" consensus approaches in GDM problems have received widespread attention (Herrera-Viedma, Javier Cabrerizo, Kacprzyk, & Pedrycz, 2014; Kacprzyk et al., 1992; Zhang, Kou, & Peng, 2019).

It can be seen that there are several problems related to consensus reaching process (CRP) in GDM. A CRP is usually defined as an iterative process with several rounds for improving the agreement among DMs. In a CRP, DMs discuss and modify their preferences to make them closer to each other with the aim of achieving an acceptable consensus level (CL), and then obtain a satisfactory agreed solution with higher agreement among DMs for the GDM problems (Dong & Xu, 2016; Herrera-Viedma et al., 2014; Iván Palomares, Rodríguez, & Martínez, 2013).

For real-life GDM problems, large scale group decision making (LSGDM) is common because of the societal and technological trends (Labella et al., 2018; Rodríguez et al., 2018; Wu, Chiclana, Fujita, & Herrera-Viedma, 2018; Zhang, Dong, & Herrera-Viedma, 2017; Zhang

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https://doi.org/10.1016/j.cie.2020.106973
Received 23 May 2020; Received in revised form 5 November 2020; Accepted 6 November 2020
Available online 13 November 2020

et al., 2020). In LSGDM problems, CRP is more important than classical GDM problems because of polarize opinions (Dong, Zhou, & Martínez, 2019; Labella et al., 2018). Based on the consideration of consensus cost, acceptable agreements achieved on soft consensus (Kacprzyk & Fedrizzi, 1988; Kacprzyk et al., 1992) are less costly, more effective and desirable than time-consuming and expensive unanimous agreements. Therefore, both acceptable CL and consensus cost are important factors in LSGDM. Clearly, it is preferable low cost CRPs, thus minimum cost consensus models (MCCMs) have been introduced to achieve such a target (Ben-Arieh & Easton, 2007; Ben-Arieh, Easton, & Evans, 2009; Labella, Liu, Rodríguez, & Martínez, 2020). In general, there are three kinds of minimum cost consensus models in CRP:

1) The minimum cost consensus model (MCCM). The main idea is to minimize the direct costs of adjusting opinion deviations among DMs (Ben-Arieh & Easton, 2007; Cheng, Zhou, Cheng, Zhou, & Xie, 2018; Y. C. Dong et al., 2010; Zhang et al., 2019).

2) The maximum DMs' consensus model (MECM). The model aims at maximizing the number of DMs within CL under the given cost budget (Zhang, Dong, & Xu, 2013).

3) The minimum adjustments consensus model (MACM). There are two main ideas, one is based on distance, which aims at minimizing the adjustment between individual original opinions and adjusted opinions (Ben-Arieh et al., 2009; Zhang, Liang, & Zhang, 2018). Another is based on number of adjusted values, which aims at minimizing the number of adjusted preference values when reaching consensus (Zhang & Dong, 2013).

After applying a minimum cost CRP, the experts' adjusted opinions are usually different from the original ones. In spite of the original ones were initially reliable, the reliability of the adjusted opinions cannot be guaranteed. At present, few studies on the reliability of the adjusted opinions have been reported. Numerous approaches have been developed for measuring and improving the consensus based on different information domains: Numerical (Dong et al., 2019), Interval-valued (Fu & Yang, 2012), linguistic (Li, Dong, Herrera, Herrera-Viedma, & Martínez, 2017), however, linguistic descriptors are often used for expressing assessments in many situations in real-world LSGDM (Martínez & Herrera, 2012). Considering the subjective evaluation on the reliability of the assessments, the use of the two-dimension linguistic (TDL) was proposed by Zhu, Zhao, and Xu (2016) that extends fuzzy linguistic labels (Lotfi Asker Zadeh, 1975; Lotfi A Zadeh, 1975a, 1975b; Zadeh, 1983) by adding reliability of the assessment as the second dimension information. Furthermore, based on the TDL expression, Zhu et al. (2016) proposed two-dimension 2-tuple linguistic (TD2L) expression. The reliability of TDL and TD2L information is considered as subjective evaluation given in advance, which is in fact with the same meaning as self-confidence, being proposed by Liu, Dong, Chiclana, Cabrerizo, and Herrera-Viedma (2017). Self-confidence of assessments given by DMs in advance is reasonable, however, subjective evaluation on the reliability of the assessments could be biased by DMs, which may reduce the trustworthiness of the solution. Therefore, an objective detection on reliability of the assessment is necessary (Liu, Xu, Montes, Ding, & Herrera, 2019). Usually, objective detection on the reliability refers to the original opinions provided by DMs, it concerns on two aspects (Liu et al., 2019). The one is contradictory of DMs' opinions (Gong, Guo, Herrera-Viedma, Gong, & Wei, 2020; Li, Rodríguez, Martínez, Dong, & Herrera, 2018; Xu, Patnayakuni, & Wang, 2013). The another is the large deviation of individual opinion and the collective opinion (Dong, Zhao, Zhang, Chiclana, & Herrera-Viedma, 2018; Ivan Palomares et al., 2013). However, after adjustment during CRP, there is not any measurement of the reliability of the adjusted opinions, especially for automatic CRP. Hence, it seems necessary to measure in an objective fashion the reliability of adjusted opinions to avoid unreliable decisions.

According to the taxonomy presented in Palomares, Estrella, Martínez, and Herrera (2014), CRPs can be classified according to their feedback process into two types: Consensus with feedback and without feedback. Obviously, consensus with feedback considers DMs' will if they would accept the adjusted opinion, which lead to increased reliability on adjusted opinions. However, for some decision making, like emergency decision making (Wang, Wang, & Martínez, 2017; Yu & Lai, 2011), demand high-quality decision making within the limited time, it is not convenient to wait for the adjusted opinions after several rounds feedback, because time is crucial to be effective and successful. To balance the increased reliability of consensus with feedback and the low cost of consensus without feedback, we try to develop an automatic CRP with minimum adjustment considering the reliability of the adjusted opinions.

Obviously, the reliability of the adjusted opinions is important during the decision process, adjusted opinions with high CL but low reliability would be meaningless. Therefore, the adjusted opinions and its reliability should be considered during the LSGDM solving process. Nevertheless, it has been neglected so far when DMs' opinions are automatically modified without DMs' supervision (Ben-Arieh & Chen, 2006; Wu & Xu, 2012; Zhang, Dong, Xu, & Li, 2011). In this paper, we aim at measuring the reliability of adjusted opinions in a formal objective way to obtain reliable decision based on the TD2L information.

Consequently, this paper constructs a two-stage minimum adjustment consensus model that supports agreed decisions based on the reliability detection of adjusted opinions. Such a consensus LSGDM method consists of the following three parts:

(1) The large number of DMs of the LSGDM are classified into several subgroups according to a new support degree (SD)-based clustering method.
(2) A novel two-stage minimum adjustment consensus model which is an automatic model is proposed.
(3) The relations between the adjustment and the reliability of the adjusted preferences are used to obtain a final reliable solution by using TD2L information.

The rest of this paper is set out as follows. In Section 2, some basic concepts about 2-tuple linguistic representation model, two-dimension linguistic expressions and minimum adjustment consensus model are reviewed. The main steps of the proposed method for solving LSGDM problems with two-stage minimum adjustment consensus model and the measurement of the reliability of the adjusted preferences are presented in Section 3. Section 4 presents an illustrative example and conducts a comparative analysis. Some concluding remarks are finally drawn in Section 5.

2. Preliminaries

In this section, we mainly revise some basic knowledge about 2-tuple linguistic representation model, two-dimension linguistic expressions and consensus models with minimum adjustment, which provide a basis for the study.

2.1. 2-tuple linguistic representation model

Suppose that $S = \{s_0, s_1, \cdots, s_g\}$ is a pre-defined linguistic term set, and cardinality of S is $g+1$. For any $s_i, s_j \in S$, the following properties should satisfy:

(1) The set is ordered: if $i > j$, then $s_i > s_j$;
(2) Maximum operator: if $s_i > s_j$, then $\max(s_i, s_j) = s_i$;
(3) Minimum operator: if $s_i > s_j$, then $\min(s_i, s_j) = s_j$;
(4) Negation operator: $\mathrm{neg}(s_i) = s_{g-i}$.

In general, the cardinality of linguistic label set S is odd number, more than 5 and less than 9 (Miller, 1956). An example of S linguistic terms could be:

$S = \{s_0 = verypoor, s_1 = poor, s_2 = medium, s_3 = good, s_4 = verygood\}$

In order to obtain more accurate results in computing with words, Herrera and Martínez proposed the 2-tuple linguistic model (s_i, α) (Herrera & Martínez, 2000), where s_i is a linguistic label involved in the set S and α is a numerical value representing the symbolic translation from s_i.

Definition 1.. ((*Herrera & Martínez, 2000*).) *Let $S = \{s_0, s_1, \cdots, s_g\}$ be a linguistic term set and $\overline{S}$ the 2-tuple set associated with S defined as $\overline{S} = S \times [-0.5, 0.5)$. The 2-tuple linguistic value (s_i, α) is equivalent to β through the function Δ as follows*:

$$\Delta : [0, g] \rightarrow S \times [-0.5, 0.5)$$

$\Delta(\beta) = (s_i, \alpha)$, with $\begin{cases} s_i, i = round(\beta) \\ \alpha = \beta - i, \alpha \in [-0.5, 0.5) \end{cases}$ where $round(\cdot)$ is the usual round operation that assigns to β the closet integer number $i \in \{0, 1, \cdots, g\}$ to β.

Definition 2.. ((*Herrera & Martínez, 2000*).) *). Let $S = \{s_0, s_1, \cdots, s_g\}$ be a linguistic term set and $(s_i, \alpha) \in \overline{S}$ be a 2-tuple linguistic value. $\beta \in [0, g]$ is equivalent to (s_i, α) through the function Δ^{-1} as follows*:

$$\Delta^{-1} : S \times [-0.5, 0.5) \rightarrow [0, g]$$

$$\Delta^{-1}(s_i, \alpha) = \alpha + i = \beta$$

Remark 1.. ((*Herrera & Martínez, 2000*).) *For any two 2-tuple linguistic values (s_i, α_i) and (s_j, α_j), the relations to compare them can be given as follows*:

(1) If $i > j$, then $(s_i, \alpha_i) > (s_j, \alpha_j)$;
(2) If $i = j$, then (a) $(s_i, \alpha_i) > (s_j, \alpha_j)$ for $\alpha_i > \alpha_j$;
(b) $(s_i, \alpha_i) < (s_j, \alpha_j)$ for $\alpha_i < \alpha_j$;
(c) $(s_i, \alpha_i) = (s_j, \alpha_j)$ for $\alpha_i = \alpha_j$.

2.2. Two-dimension linguistic expressions

2-tuple linguistic labels as a kind of information can express the assessments in GDM, but in real life, another dimension information is often needed to present self-confidence (Liu et al., 2017) or subjective evaluation on reliability of the given assessments, which is usually expressed as well as linguistic information.

Definition 3.. ((*Zhu et al., 2016*).) *Let $S = \{s_0, s_1, \cdots, s_g\}$ and $\dot{S} = \{\dot{s}_0, \dot{s}_1, \cdots, \dot{s}_h\}$ be two linguistic label sets. Let $\alpha, \dot{\alpha} \in [-0.5, 0.5)$ be two numerical numbers. Then $\widehat{S} = \left((s_u, \alpha), \left(\dot{s}_v, \dot{\alpha}\right)\right)$ is a TD2L expression, where $s_u \in S$, $\dot{s}_v \in \dot{S}$, (s_u, α) represents the assessment information about the alternative given by DMs, $\left(\dot{s}_v, \dot{\alpha}\right)$ represents the self-assessment of the DM on the reliability of the given assessment result.*

Remark 2.. ((*Zhu et al., 2016*).) *If $\alpha = \dot{\alpha} = 0$, then $\widehat{S} = \left((s_u, \alpha), \left(\dot{s}_v, \dot{\alpha}\right)\right)$ is simplified as $\widehat{S} = \left(s_u, \dot{s}_v\right)$, which is exactly the TDL expression proposed by* Zhu, Zhou, and Yang (2009).

In some extent, the DM's self-assessment on the reliability of the given assessment result is her/his self-confidence for the assessment. Liu et al. (2017) proposed a concept of self-confidence on preferences, which is defined as follows.

Definition 4.. ((*Liu et al., 2017*).) Let $A = (a_{ij})_{n \times n}$, $a_{ij} \in [1/9, 9]$ be a multiplicative preference relation and a_{ij} the preference of alternative x_i over alternative x_j. Let $S = \{s_0, s_1, \cdots, s_g\}$ be a linguistic label set, $s_{ij} \in S$ representing the self-confidence level associated to a_{ij}. Then (a_{ij}, s_{ij}) is a two-dimension linguistic expression, where a_{ij} represents the preference relation and s_{ij} represents the self-confidence level of preference relation.

Based on Definitions 3 and 4, the self-assessment (or self-confidence level) on the given assessment is subjective information provided by DMs. In this paper, we aim at taking advantage of two-dimension linguistic expression to introduce an objective value (reliability degree) that show how reliable is the preference elicited by two-dimension linguistic expression.

2.3. Minimum adjustment consensus model

Here, the classical MACM is revised (Y. Dong et al., 2010). Let $E = \{e_1, e_2, \cdots, e_m\}$ be the set of DMs, $\lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_m\}$ be the weight vector over E, $O = \{o_1, o_2, \cdots, o_m\}$ and $\overline{O} = \{\overline{o}_1, \overline{o}_2, \cdots, \overline{o}_m\}$ be the original and the adjusted opinions of DMs. Then the MACM can be presented using an

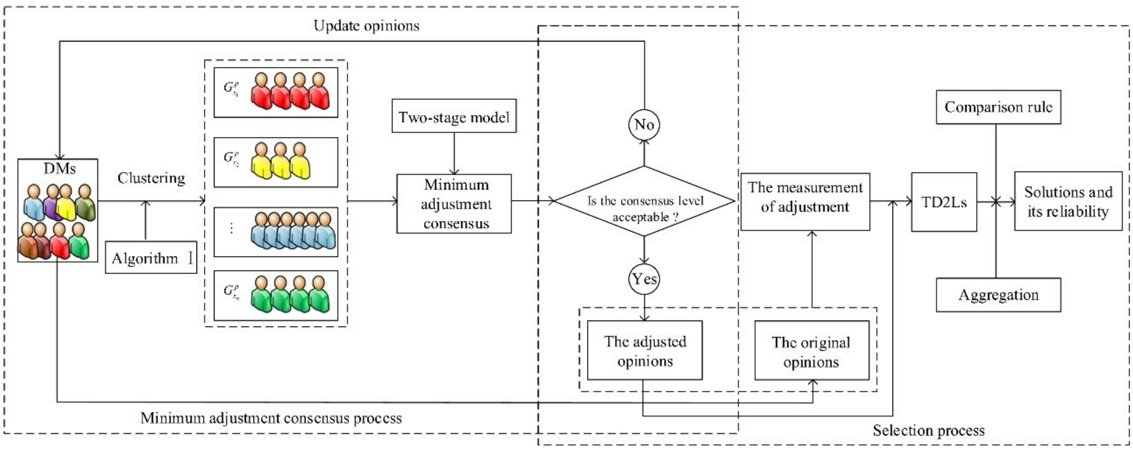

Fig. 1. The flow chart of the proposed CRP-LSGDM method.

optimization model (Y. Dong et al., 2010), i.e.,

$$\begin{cases} \min \sum_{k=1}^{m} d(o_k - \bar{o}_k) \\ s.t. \left|\bar{o}_k - \bar{o}^c\right| \le \varepsilon, k = 1, 2, \cdots, m \\ \bar{o}^c = F_\lambda\left(\bar{o}_1, \bar{o}_2, \cdots, \bar{o}_m\right) \end{cases}$$

where $d(o_k - \bar{o}_k)$ is the distance between o_k and $\bar{o}_k$. $\varepsilon \ge 0$ is the consensus threshold. $F_\lambda\left(\bar{o}_1, \bar{o}_2, \cdots, \bar{o}_m\right) = \sum_{k=1}^{m} \lambda_i \bar{o}_i$ is an aggregation function to obtain the collective adjusted opinion.

In LSGDM problems, DMs' opinions can be presented as fuzzy preference relations. Let $R^k = \left(r_{ij}^k\right)_{n \times n}$ be the preference relation given by the DM e_k. In this situation, the MACM is as (Zhang, Dong, & Xu, 2012):

$$\begin{cases} min \sum_{k=1}^{m} \sum_{j=i+1}^{n} \sum_{i=1}^{n-1} \left| r_{ij}^k - \bar{r}_{ij}^k \right| \\ s.t. CL \ge \sigma \end{cases}$$

where CL represent the overall consensus level, σ is the given CL threshold.

The use of these models lead to agreed opinions, by modifying DMs' original ones, very quickly. However, the reliability of the adjusted opinions obtained by these models is not guaranteed, which reduce the reliability of the decision solution. Therefore, in this paper, we aim at presenting an objective detection approach on reliability of adjusted opinions.

3. A novel two-stage minimum adjustment consensus method for LSGDM with TD2L expressions

This section introduces a new CRP-LSGDM method with a two-stage minimum adjustment consensus model based on TD2L expressions. First, it is provided the structure of CRP-LSGDM with minimum adjustment consensus based on TD2L expressions. The framework of the two-stage consensus model in LSGDM with TD2L expressions is shown in Fig. 1 whose main steps are sketched below:

A. *Minimum adjustment consensus process*: it consists of two steps.

(1) A SD-based clustering method (see **Algorithm I**), in which DMs are divided into n clusters, where n is the number of alternatives.

(2) A minimum adjustment consensus control. Based on the principle of minimum adjustment and preservation of the original preference values as much as possible, a novel two-stage consensus model with minimum adjustment is presented. (see **Models (15)- (16)**).

B. *Selection process to obtain an agreed solution of the LSGDM problems.*

(1) A new measure for computing the reliability degree of agreed preferences obtained by a minimum adjustment consensus model dealing with TD2L information is presented. (see Sections 3.3 and 3.4).

(2) A ranking of alternatives based on TD2L expressions and its reliability is introduced. (see Section 3.5).

Keeping in mind that our focus is on LSGDM problems based on two-stage minimum adjustment consensus model where DMs use 2-tuple

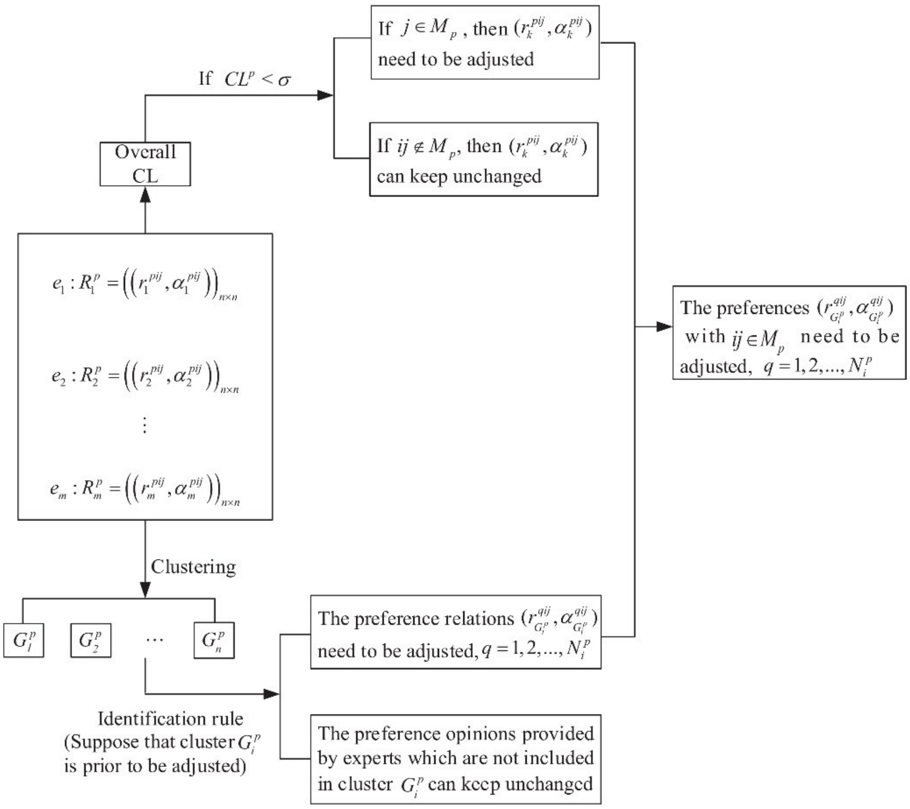

Fig. 2. Identification of the preferences need to be adjusted.

linguistic fuzzy preference relations to express their opinion. For the sake of simplicity, the elements involved in the problem are noted as follows.

- A group of m DMs $E = \{e_k | k = 1, 2, \cdots, m\}$.
- A set of n alternatives $X = \{x_i | i = 1, 2, \cdots, n\}$.
- The assessment linguistic term set $S = \{s_0, s_1, \cdots, s_g\}$.
- The reliability degree expression linguistic term set $\dot{S} = \left\{\dot{s}_0, \dot{s}_1, \cdots, \dot{s}_h\right\}$.
- The TD2L expressions $\left((s_u, \alpha), \left(\dot{s}_v, \dot{\alpha}\right)\right)$, $s_u \in S$, $\dot{s}_v \in \dot{S}$, α, $\dot{\alpha} \in [-0.5, 0.5)$.
- The threshold for CL σ, $\sigma > 0$.
- The iteration rounds p, $p = 0, 1, \cdots P$, where P is the final iteration round that satisfies the acceptable CL.
- The additive 2-tuple linguistic preference relation matrix $R_k^p = \left(\left(r_k^{pij}, \alpha_k^{pij}\right)\right)_{n\times n}$, $r_k^{pij} \in S$, $\Delta^{-1}\left(r_k^{pij}, \alpha_k^{pij}\right) + \Delta^{-1}\left(r_k^{pji}, \alpha_k^{pji}\right) = g$, $\alpha_k^{pij} \in [-0.5, 0.5)$. $\left(r_k^{pij}, \alpha_k^{pij}\right)$ is a 2-tuple linguistic term represents the preference of alternative x_i over alternative x_j provided by DM e_k in round p.
- The collective 2-tuple linguistic preference relation matrix in round p, $R_c^p = \left(\left(r_c^{pij}, \alpha_c^{pij}\right)\right)_{n\times n}$.
- The adjusted individual preference is $\left(\overline{r}_k^{pij}, \overline{\alpha}_k^{pij}\right)$, where $\overline{r}_k^{pij} = r_k^{p-1ij}$, $\overline{\alpha}_k^{pij} = \alpha_k^{p+1ij}$, $p = 0, 1, \cdots, P-1$.
- The adjusted collective preference is $\left(\overline{r}_c^{pij}, \overline{\alpha}_c^{pij}\right)$, where $\overline{r}_c^{pij} = r_c^{p-1ij}$, $\overline{\alpha}_c^{pij} = \alpha_c^{p+1ij}$, $p = 0, 1, \cdots, P-1$.

3.1. Support degree (SD)-based clustering method

In a group with few DMs is easier to discuss and improve the preference information elicited by them. However, in LSGDM the discussion and improvement is much more complex. Therefore, sub-groups detection is necessary for simplifying LSGDM. In order to save original information as much as possible, we propose a priority adjustment rule such that, the cluster which includes the less DMs is adjusted first. Therefore, a novel clustering method is proposed with the SD on each alternative of DMs. Assume that all DMs are divided into n clusters because we can find the clusters of DMs supporting each different alternative.

The SD on alternative x_i of DM e_k is derived from two aspects: (1) The SD of DM e_k on alternative x_i with respect to other DMs. (2) The SD of DM e_k on alternative x_i regarding other alternatives. The details of SD are defined as follows.

Definition 5.. *Let $R_k^p = \left(\left(r_k^{pij}, \alpha_k^{pij}\right)\right)_{n\times n}$ be the preference matrix provided by DM e_k as before, then SD of DM e_k in terms of alternative x_i in round p is defined as*

$$\mathrm{SD}^p(e_k, x_i) = \mathrm{Pr}^p_{e_k}(e_k, x_i) \times \mathrm{Pr}^p_{x_i}(e_k, x_i)$$

where $\mathrm{Pr}^p_{e_k}(e_k, x_i)$ represents the proportion of DM e_k's preference on alternative x_i among the sum of all DMs' preference on x_i, which is expressed as

$$\mathrm{Pr}^p_{e_k}(e_k, x_i) = \frac{\sum_{j=1, j\neq i}^{n} \Delta^{-1}\left(r_k^{pij}, \alpha_k^{pij}\right)}{\sum_{k=1}^{m}\sum_{j=1, j\neq i}^{n} \Delta^{-1}\left(r_k^{pij}, \alpha_k^{pij}\right)}$$

$\mathrm{Pr}^p_{x_i}(e_k, x_i)$ represents the proportion of DM e_k's preference on alternative x_i among the sum of e_k's preference on all alternatives, which is computed as

$$\mathrm{Pr}^p_{x_i}(e_k, x_i) = \frac{\sum_{j=1, j\neq i}^{n} \Delta^{-1}\left(r_k^{pij}, \alpha_k^{pij}\right)}{\sum_{i=1}^{n}\sum_{j=1, j\neq i}^{n} \Delta^{-1}\left(r_k^{pij}, \alpha_k^{pij}\right)}$$

Obviously, $\mathrm{Pr}^p_{e_k}(e_k, x_i) \in [0,1]$, $\mathrm{Pr}^p_{x_i}(e_k, x_i) \in [0,1]$, $\sum_{k=1}^{m}\mathrm{Pr}^p_{e_k}(e_k, x_i) = 1$ and $\sum_{i=1}^{n}\mathrm{Pr}^p_{e_k}(e_k, x_i) = 1$. Then $\mathrm{SD}^p(e_k, x_i) \in [0,1]$, $\sum_{i=1}^{n}\sum_{k=1}^{m}\mathrm{SD}^p(e_k, x_i) = 1$.

Then according to the value of $\mathrm{SD}^p(e_k, x_i)$, DMs can be assigned into corresponding cluster $G_i^p (i = 1, 2, \cdots, n)$. All DMs in cluster G_i^p support alternative x_i most, which satisfy $\mathrm{SD}^p(e_k, x_i) = max\{\mathrm{SD}^p(e_k, x_1), \mathrm{SD}^p(e_k, x_2), \cdots, \mathrm{SD}^p(e_k, x_n)\}$ for $k = 1, 2, \cdots, m$. Denote DMs in cluster G_i^p as $e^1_{G_i^p}, e^2_{G_i^p}, \ldots, e^{N_i^p}_{G_i^p}$, where N_i^p is the number of DMs in cluster G_i^p. All DMs $e_k (k = 1, 2, \cdots, m)$ can be expressed as $e^q_{G_i^p} (q = 1, 2, \cdots, N_i^p)$ after clustering. In other words, for $e^q_{G_i^p} (q = 1, 2, \cdots, N_i^p)$ in cluster G_i^p, it satisfy $\mathrm{SD}^p\left(e^q_{G_i^p}, x_i\right) = max\left\{\mathrm{SD}^p\left(e^q_{G_i^p}, x_1\right), \mathrm{SD}^p\left(e^q_{G_i^p}, x_2\right), \cdots, \mathrm{SD}^p\left(e^q_{G_i^p}, x_n\right)\right\}$, $q = 1, 2, \cdots, N_i^p$. Specially, for alternative x_i, there may exist $\mathrm{SD}^p(e_k, x_i) \neq max\{\mathrm{SD}^p(e_1, x_i), \mathrm{SD}^p(e_2, x_i), \cdots, \mathrm{SD}^p(e_m, x_i)\}$ for any $k = 1, 2, \cdots, m$, which means there is no DM in cluster G_i^p, but it is possible for $\mathrm{SD}^{p^*}(e_k, x_i) = max\{\mathrm{SD}^{p^*}(e_1, x_i), \mathrm{SD}^{p^*}(e_2, x_i), \cdots, \mathrm{SD}^{p^*}(e_m, x_i)\}$, $p^* \neq p, p^* = 0, 1, \cdots, P$. That implies there may exist DMs in cluster $G_i^{p^*} (p^* \neq p, p^* = 0, 1, \cdots, P)$. Therefore, it is reasonable to set n clusters. **Algorithm I** is used for describing clustering method provided below.

Algorithm I

Input: The preference matrix provided by DM e_k is expressed as $R_k^p = \left(r_k^{pij}, \alpha_k^{pij}\right)_{n\times n}$, $k = 1, 2, \cdots, m$, $p = 0, 1, \cdots, P$, $i, j = 1, 2, \cdots, n$.
Output: All DMs are divided into n clusters $G_1^p, G_2^p, \ldots, G_n^p$.
Step 1: Based on preference matrices, compute the value of $\mathrm{SD}^p(e_k, x_i)$ according to Eqs. (7)-(9). $k = 1, 2, \cdots, m$, $i = 1, \cdots, n$.
Step 2: According to the value of $\mathrm{SD}^p(e_k, x_i)$ for $i = 1, 2, \cdots, n$. Assign DM e_k into cluster G_i^p, where $\mathrm{SD}^p(e_k, x_i)$ satisfies $\mathrm{SD}^p(e_k, x_i) = max\{\mathrm{SD}^p(e_k, x_1), \mathrm{SD}^p(e_k, x_2), \cdots, \mathrm{SD}^p(e_k, x_n)\}$
Output SD-based clusters $G_1^p, G_2^p, \ldots, G_n^p$.

After DMs' clustering process, their weights will be computed based on clusters' size (Rodríguez et al., 2018; Zhang et al., 2017). Therefore, the weight of cluster G_i^p is $w_{G_i^p} = \frac{N_i^p}{m}$.

For the DMs' weights in the same cluster, we set up an optimization model to obtain it based on the deviation between any two DMs in the same cluster. In this paper, we use Manhattan distance to measure the deviation between any two preferences, the model is as follows.

$$\text{Min} \sum_{q_2=q_1+1}^{N_i^p}\sum_{q_1=1}^{N_i^p-1} d\left(\left(r^{q_1}_{G_i^p}, \alpha^{q_1}_{G_i^p}\right), \left(r^{q_2}_{G_i^p}, \alpha^{q_2}_{G_i^p}\right)\right)$$

$$s.t.\begin{cases} d\left(\left(r^{q_1}_{G_i^p}, \alpha^{q_1}_{G_i^p}\right), \left(r^{q_2}_{G_i^p}, \alpha^{q_2}_{G_i^p}\right)\right) \\ = \left|\lambda^{q_1}_{G_i^p}\Delta^{-1}\left(r^{q_1}_{G_i^p}, \alpha^{q_1}_{G_i^p}\right) - \lambda^{q_2}_{G_i^p}\Delta^{-1}\left(r^{q_2}_{G_i^p}, \alpha^{q_2}_{G_i^p}\right)\right| \\ \sum_{q=1}^{N_i^p}\lambda^q_{G_i^p} = 1 \\ 0 < \lambda^q_{G_i^p} < 1, q = 1, 2, \cdots, N_i^p \end{cases}$$

where $\left(r^{q_1}_{G_i^p}, \alpha^{q_1}_{G_i^p}\right)$ and $\left(r^{q_2}_{G_i^p}, \alpha^{q_2}_{G_i^p}\right)$ are 2-tuple linguistic values that represent the preference of q_1th and q_2th DM in cluster G_i^p, respectively. $\lambda^q_{G_i^p}$ represents the weight of the qth DM in cluster $G_{x_i}^p$, $q = 1, 2, \cdots, N_i^p$.

We use LINGO 11 to solve model (10) and obtain the DMs' weights in cluster $G^p_{x_i}$: $\lambda_{G^p_i} = \left(\lambda^1_{G^p_i}, \lambda^2_{G^p_i}, \cdots, \lambda^{N^p_i}_{G^p_i}\right)^T$.

Finally, based on the two previous weights, the DMs' weights are $\omega = (\omega_1, \omega_2, \cdots, \omega_m)$, where $\omega_k = w_{G^p_i} \cdot \lambda^q_{G^p_i}$, DM e_k is the *pth* DM in cluster G^p_i.

During the CRP, the classification of DMs is changeable. Until the acceptable CL is achieved, then we obtain the final clusters: G^p_1, G^p_2,…,G^p_n, which are the basis to obtain the collective solution with high agreement.

3.2. Two-stage minimum adjustment consensus model

Here, it is presented a novel automatic two-stage minimum adjustment consensus model. At the beginning, we compute the CL of all DMs, if the CL is not acceptable, then it is necessary to make adjustment. In existing literature, the weights are not considered into the computation of overall CL, however, DMs with different weights contribute different to the consensus (Labella et al., 2020). Therefore, we consider DMs' weights to obtain the CL of all DMs, where it is based on the following three levels (Wu and Xu, 2018):

(1) *CL for each pair of alternatives.* The CL for each pair of alternatives is based on the similarities among preferences.

$$CL^p_{ij} = sim^p_{ij}$$

Considering that DMs have different importance in the CRP (Labella et al., 2020), the similarity can be aggregated by means of the weighted average operator as follows.

$$sim^p_{ij} = 1 - \sum_{k_1=1}^{m-1}\sum_{k_2=k_1+1}^{m} \frac{\omega_{k_1}+\omega_{k_2}}{m-1} \left|\Delta^{-1}\left(r^{pij}_{k_1}, \alpha^{pij}_{k_1}\right) - \Delta^{-1}\left(r^{pij}_{k_2}, \alpha^{pij}_{k_2}\right)\right| \Big/ g$$

where $sim^p_{ij} \in [0, 1]$.

(2) *CL for each alternative.* Based on the results obtained in Eq. (12), a CL matrix is presented as $\left(CL^p_{ij}\right)_{n\times n}$. The CL for each alternative $CL^p_{x_i}$ is obtained by averaging each row of consensus matrix, which is expressed as

$$CL^p_{x_i} = \frac{1}{n-1}\sum_{\substack{j=1\\ j\neq i}}^{n} CL^p_{ij}, i = 1, 2, \cdots, n$$

(3) *Overall CL for all preferences.* The overall CL for all preferences in round p is obtained by

$$CL^p = \frac{1}{n}\sum_{i=1}^{n} CL^p_{x_i}$$

Therefore, we can make a comparison between overall CL and the given consensus threshold value σ. If $CL^p \geq \sigma$, then the CRP is over. If $CL^p < \sigma$, then two steps are performed immediately as:

1. First step identifying which cluster is needed to be adjusted first. The rule for deciding which cluster is chosen to be adjusted first is as: If $\lambda_{G^p_i} = min\left\{\lambda_{G^p_1}, \lambda_{G^p_2}, \cdots, \lambda_{G^p_n}\right\}$, then preference matrices provided by DMs in cluster $G^p_{x_i}$ are prior to be adjusted.

2. Second step identifying which preferences in the identified cluster need to be adjusted.

Obviously, only when the CL for each pair of alternatives improves, the overall CL improves. Different from the existing MACMs, we focus on the improvement of CL for each pair of alternatives in the first stage. For CL for each pair of alternatives, not all CL^p_{ij} with $i, j = 1, 2, \cdots, n$ are less than σ, therefore, we center on the pair of alternatives which satisfy $CL^p_{ij} < \sigma$. For simplify, we denote *ij* as a position. In the position *ij*, DMs given their preferences on alternative x_i in terms of alternative x_j. Owing to the preferences information are presented as symmetric matrix, only the upper triangular elements in matrix are considered is reasonable. Therefore, we only consider positions *ij* with $i < j$. Let M_p be a set, all elements in set M_p are positions *ij*, where *ij* satisfy $CL^p_{ij} < \sigma$. If $ij \in M_p$, then CL^p_{ij} need to be improved. If $ij \notin M_p$, then CL^p_{ij} can keep unchanged, which means the preference $\left(r^{pij}_k, \alpha^{pij}_k\right)$ for $ij \notin M_p$ keep unchanged.

Based on these two steps, the preferences need to be adjusted can be decided. In this way, the number of the preferences need to be adjusted is least. In order to better understanding, the principle of identifying the preferences need to be adjusted is shown in Fig. 2.

After the identification, we set up a two-stage minimum adjustment consensus model to adjust the preferences $\left(r^{pij}_k, \alpha^{pij}_k\right)$ provided by DMs in cluster G^p_i. The details of the model are as follows.

The first stage maximizes the improvement of CL for each pair of alternatives within minimum number of adjusted preferences.

Stage one

$$\text{Max}\sum_{ij\subset M_p}\gamma^{pij}$$

$$s.t.\begin{cases} \overline{CL}^p_{ij} \geq CL^p_{ij} + \gamma^{pij} \\ \overline{CL}^p_{ij} = 1 - \sum_{k_1=1}^{m-1}\sum_{k_2=1}^{m}\frac{\omega_{k_1}+\omega_{k_2}}{m-1}\times \\ \left(\left|\Delta^{-1}\left(\overline{r}^{pij}_{k_1}, \overline{\alpha}^{pij}_{k_1}\right) - \Delta^{-1}\left(\overline{r}^{pij}_{k_2}, \overline{\alpha}^{pij}_{k_2}\right)\right| \Big/ g\right) \\ CL^p_{ij} = 1 - \sum_{k_1=1}^{m-1}\sum_{k_2=1}^{m}\frac{\omega_{k_1}+\omega_{k_2}}{m-1}\times \\ \left(\left|\Delta^{-1}\left(r^{pij}_{k_1}, \alpha^{pij}_{k_1}\right) - \Delta^{-1}\left(r^{pij}_{k_2}, \alpha^{pij}_{k_2}\right)\right|/g\right) \\ \Delta^{-1}\left(\overline{r}^{pij}_k, \overline{\alpha}^{pij}_k\right)\Big\langle g, k = 1, 2, \cdots, m \\ \Delta^{-1}\left(\overline{r}^{pij}_k, \overline{\alpha}^{pij}_k\right)\Big\rangle 0, k = 1, 2, \cdots, m \end{cases}$$

where γ^{pij} is the distance between CL^p_{ij} and $\overline{CL}^p_{ij}$. $\overline{CL}^p_{ij}$ is the CL for each pair of alternatives with adjusted preferences.

The condition $\overline{CL}^p_{ij} \geq CL^p_{ij} + \gamma^{pij}$ in model (15) means the improvement of the CL for each pair of alternatives after adjustment. The larger the value of γ^{pij} is, the larger the improvement of the CL for each pair of alternatives.

Use LINGO 11 to obtain the optimal solution of γ^p is $\gamma^{p^*} = \left\{\gamma^{pij^*} | ij \in M_p\right\}$ and CL for each pair of alternatives with optimal adjusted preferences $\overline{CL}^{p^*} = \left\{\overline{CL}^{p^*}_{ij} \middle| ij \in M_p\right\}$.

The second stage consists of constructing a minimum adjustment cost model with the maximum improvement of the CL for $ij \in M_p$.

Stage two

$$\text{Min}\sum_{q=1}^{N^p_i}\sum_{ij\in M_p} d\left(\left(r^{qij}_{G^p_i}, \alpha^{qij}_{G^p_i}\right), \left(\overline{r}^{qij}_{G^p_i}, \overline{\alpha}^{qij}_{G^p_i}\right)\right)$$

$$s.t.\begin{cases} \overline{CL}_{ij}^{p} \geq \theta \cdot \overline{CL}_{ij}^{p^*}, ij \in M_p \\ \overline{CL}_{ij}^{p} = 1 - \sum_{k_1=1}^{m-1}\sum_{k_2=1}^{m} \frac{\omega_{k_1}+\omega_{k_2}}{m-1} \times \\ \left(\left| \Delta^{-1}\left(\overline{r}_{k_1}^{pij}, \overline{\alpha}_{k_1}^{pij}\right) - \Delta^{-1}\left(\overline{r}_{k_2}^{pij}, \overline{\alpha}_{k_2}^{pij}\right) \right| \right) / g \\ \Delta^{-1}\left(\overline{r}_{G_i^p}^{qij}, \overline{\alpha}_{G_i^p}^{qij}\right) \langle g, q = 1,2,\cdots,N_i^p \\ \Delta^{-1}\left(\overline{r}_{G_i^p}^{qij}, \overline{\alpha}_{G_i^p}^{qij}\right) \rangle 0, q = 1,2,\cdots,N_i^p \end{cases}$$

where θ is a numerical value close to 1 and $\theta \leq 1$.

Remark 3.. *$0 \leq |\theta - 1| \leq \xi$, where ξ is a small number. In Stage Two, CL for each pair of alternatives with $ij \in M_p$ and adjusted preferences $\overline{CL}_{ij}^{p}$ would be closer to the optimal solution $\overline{CL}_{ij}^{p^*}$ as much as possible solved in Stage One while the adjustment cost is minimum. In fact, the constraint $\overline{CL}_{ij}^{p} \geq \theta \cdot \overline{CL}_{ij}^{p^*}$ means CL for each pair of alternatives with $ij \in M_p$ and adjusted preferences $\overline{CL}_{ij}^{p}$ is almost equal to the optimal solution $\overline{CL}_{ij}^{p^*}$ at Stage One.*

Use LINGO 11 to solve model (16) we obtain the optimal adjusted preferences provided by DMs in cluster $G_{x_i}^p$:

$$\left(\overline{r}_{G_i^p}^{ij}, \overline{\alpha}_{G_i^p}^{ij}\right) = \left\{ \left(\overline{r}_{G_i^p}^{1ij}, \overline{\alpha}_{G_i^p}^{1ij}\right), \left(\overline{r}_{G_i^p}^{2ij}, \overline{\alpha}_{G_i^p}^{2ij}\right), \cdots, \left(\overline{r}_{G_i^p}^{N_i^p ij}, \overline{\alpha}_{G_i^p}^{N_i^p ij}\right) \right\}, ij \in M_p$$

Based on the solution (17), check if the overall CL satisfy $CL^p > \sigma$. If $CL^p > \sigma$, then the CRP stop, if not, replace preferences with adjusted preferences obtained in Eq.(17), then restart the CRP again. The steps for obtaining agreed adjusted preferences are as follows.

Step 1: Obtain n clusters by **Algorithm I**: G_1^p, G_2^p,...,G_n^p. The corresponding weights is as: $w_{G_i^p} = \left(w_{G_1^p}, w_{G_2^p}, \cdots, w_{G_n^p}\right)$, DMs' weights in cluster $G_{x_i}^p$ is as: $\lambda_{G_i^p} = \left(\lambda_{G_i^p}^1, \lambda_{G_i^p}^2, \cdots, \lambda_{G_i^p}^{N_i^p}\right)^T$.

Step 2: Compute the value of CL^p according to Eqs. (11)-(14). If $CL^p \geq \sigma$, then the acceptable consensus is reached. If not, continue to the next step for cluster G_i^p with position $ij \in M_p$, where cluster G_i^p satisfies the identification rules described in Section 3.2.

Step 3: Construct two-stage minimum adjustment consensus model as model (15) and (16) to obtain the adjusted preference $\left(\overline{r}_{G_i^p}^{pij}, \overline{\alpha}_{G_i^p}^{pij}\right)$. After adjustment, the overall CL denoted as $\overline{CL}^P$.

Step 4: Check the value of $\overline{CL}^P$, if $\overline{CL}^P \geq \sigma$, then the acceptable consensus is reached. If not, then return to step 1 until $\overline{CL}^P \geq \sigma$.

3.3. Measurement of the reliability of adjusted preferences

In automatic CRPs without feedback for DMs, it is assumed by default that all DMs accept the adjusted preferences. It makes the CRP easier to reach. However, in real-world decision making, there exist some of DMs do not accept the adjusted preferences. In order to save the advantage of automatic CRP and make the adjusted preferences more reliable, we try to analyze the reliability of the adjusted preferences. In general, the higher the total adjustment cost, the more likely the DM do not accept the adjusted preferences, therefore, the reliability of the adjusted preferences is lower.

In fact, there is a connection between the adjustment cost and reliability of adjusted preferences. For the adjusted preferences, which are the same as the original preferences, the adjustment cost is zero, obviously the reliability of these unchanged preferences will be maximum. Conversely, the larger the adjustment cost, the more change from the original preferences, therefore, the lower the reliability of adjusted preferences.

In this study, we assume that the original preferences are logical and focus on the measurement of the reliability of the adjusted preferences. Suppose $\dot{S} = \left\{\dot{s}_0, \dot{s}_1, \cdots, \dot{s}_h\right\}$ is the set used for expressing the reliability degree of the adjusted preferences. Similar to the linguistic terms set used for describing the preference relations can be of different cardinality, the cardinality of linguistic term set used for describing the level of reliability degree is usually between 3 and 5 (Liu and Yu, 2014; Y.N. Wu et al., 2018; Yu, Xu, Liu, and Chen, 2012; Zhu et al., 2016). Let $\dot{S}$ be as before, if $h = 2$, the linguistic terms set could be $\left\{\dot{s}_0 = littlehigh, \dot{s}_1 = high, \dot{s}_2 = veryhigh\right\}$ Y. N. Wu et al. (2018); Yu et al. (2012); Zhu et al. (2016). If $h = 4$, the linguistic term set could be $\left\{\dot{s}_0 = verylow, \dot{s}_1 = low, \dot{s}_2 = medium, \dot{s}_3 = high, \dot{s}_4 = veryhigh\right\}$ Liu and Yu (2014). We use that $h = 4$.

The reliability degree is derived from the total adjustment between original and adjusted preferences. The total adjustment of each pair of alternative a_k^{ij} is the change from the original preference $\left(r_k^{0ij}, \alpha_k^{0ij}\right)$ to the final adjusted preference $\left(r_k^{Pij}, \alpha_k^{Pij}\right)$, which is expressed as

$$a_k^{ij} = d\left(\left(r_k^{0ij}, \alpha_k^{0ij}\right), \left(r_k^{Pij}, \alpha_k^{Pij}\right)\right)$$

where $d\left(\left(r_k^{0ij}, \alpha_k^{0ij}\right), \left(r_k^{Pij}, \alpha_k^{Pij}\right)\right)$ is the distance between the original preference $\left(r_k^{0ij}, \alpha_k^{0ij}\right)$ and the final adjusted preference $\left(r_k^{Pij}, \alpha_k^{Pij}\right)$.

After the known of total adjustment of each preference, we need to compare the total adjustment with the same pair of alternatives but different DMs' preferences, however, in order to make the comparison more accurate, the total adjustment value should be normalized. Min-max normalization method is a common way used for normalization, then the normalized total adjustment $\overline{a}_k^{ij}$ is expressed as

$$\overline{a}_k^{ij} = \frac{a_k^{ij} - \min_{k=1,2,\cdots,m}\left\{a_k^{ij}\right\}}{\max_{k=1,2,\cdots,m}\left\{a_k^{ij}\right\} - \min_{k=1,2,\cdots,m}\left\{a_k^{ij}\right\}}, k = 1,2,\cdots,m$$

where $i,j = 1,2,\cdots,n$. $\overline{a}_k^{ij} \in [0,1]$.

Thereafter, utilizing a function to obtain the reliability degree of the adjusted preferences through the normalized adjustment cost $\overline{a}_k^{ij}$. Following the same idea of the function Δ in the linguistic 2-tuple model that transforms a numerical value into a linguistic 2-tuple value, we suppose a function φ similar to function Δ that makes $\overline{a}_k^{ij} \in [0,1]$ transform into a linguistic 2-tuple value, which is the reliability degree of adjusted preferences expressed as

$$\varphi\left(\overline{a}_k^{ij}\right) = \left(\dot{s}_{v_k^{ij}}, \dot{\alpha}_k^{ij}\right)$$

where $\left(\dot{s}_{v_k^{ij}}, \dot{\alpha}_k^{ij}\right) = \Delta\left(\left(1 - \overline{a}_k^{ij}\right)h\right)$, $v_k^{ij} = round\left(\left(1 - \overline{a}_k^{ij}\right)h\right)$, $\dot{\alpha}_k^{ij} = \left(1 - \overline{a}_k^{ij}\right)h - v_k^{ij}$, *round* is the usual round operation. $\left(\dot{s}_{v_k^{ij}}, \dot{\alpha}_k^{ij}\right)$ represents the reliability degree of the adjusted preferences.

Finally, TD2L expression is presented as adjusted preferences combined with the corresponding reliability degree, which is presented as

$\left(\left(r_k^{Pi_1^* i_2^*}, \alpha_k^{Pi_1^* i_2^*}\right), \left(\dot{s}_{v_k^{ij}}, \dot{\alpha}_k^{ij}\right)\right)$. The first dimension information $\left(r_k^{Pi_1^* i_2^*}, \alpha_k^{Pi_1^* i_2^*}\right)$ is used for representing the preference value and, the second dimension information represents the reliability degree of the preference.

Example 3.. *Suppose that there are a large number of DMs* $(e_1, e_2, \cdots, e_m)$ *given their opinions as preference matrices* $R_k^0 = \left(\left(r_k^{0ij}, \alpha_k^{0ij}\right)\right)_{n\times n}$, $k = 1, 2, \cdots, m$. *The preference matrix provided by DM* e_k *is as* $\begin{pmatrix} s_3 & s_0 & s_6 & s_3 \\ s_6 & s_3 & s_4 & s_4 \\ s_0 & s_2 & s_3 & s_1 \\ s_3 & s_2 & s_5 & s_3 \end{pmatrix}$. *After the two-stage minimum adjustment consensus, the final adjusted preference matrix is* $\begin{pmatrix} (s_3, 0) & (s_3, -0.370) & (s_5, -0.480) & (s_3, 0) \\ (s_3, 0.370) & (s_3, 0) & (s_4, 0) & (s_4, -0.402) \\ (s_1, 0.480) & (s_2, 0) & (s_3, 0) & (s_3, -0.309) \\ (s_3, 0) & (s_2, 0.402) & (s_3, 0.309) & (s_3, 0) \end{pmatrix}$, *take position row* 1 *column* (13) *as an example, the adjustment between* $\left(r_k^{013}, \alpha_k^{013}\right)$ *and* $\left(r_k^{P13}, \alpha_k^{P13}\right)$*for such a position is* 1.48. *We assume that after the normalization, the normalization adjustment of* 1.48 *is* 0.63, *then according to Eq.* (20), *the reliability degree of adjusted preference* $(s_3, -0.370)$ *is* $\varphi(0.63) = \left(\dot{s}_3, -0.48\right)$. *Therefore, the two-dimension 2-tuple expression for alternative* x_1 *in terms of* x_3 *is* $\left(\left(s_3, -0.370\right), \left(\dot{s}_3, -0.48\right)\right)$.

3.4. The adjusted collective preferences for each alternative

Based on the TD2L expressions for each pair of alternatives previously obtained, the adjusted collective preferences for each pair of alternatives is obtained by *TD2LWA* operator (Zhu et al., 2016) as follows:

$$\left(\left(s_{u_c}^{ij}, \alpha_c^{ij}\right), \left(\dot{s}_{v_c}^{ij}, \dot{\alpha}_c^{ij}\right)\right) = TD2LWA_\omega\left(\left(\left(s_{u_k}^{ij}, \alpha_k^{ij}\right), \left(\dot{s}_{u_k}^{ij}, \dot{\alpha}_k^{ij}\right)\right) \middle| k = 1, 2, \cdots, m\right)$$

where $\left(\left(s_{u_k}^{ij}, \alpha_k^{ij}\right), \left(\dot{s}_{u_k}^{ij}, \dot{\alpha}_k^{ij}\right)\right)$ is a TD2L representing the preference of DM e_k for each pair of alternatives. $\omega = (\omega_1, \omega_2, \cdots, \omega_m)$ is DMs' weights referring to Section 3.1.

Then the preference for each alternative is derived from the aggregation of the row of the TD2L matrix as follows.

$$\left(\left(s_{u_{x_i}}, \alpha_{x_i}\right), \left(\dot{s}_{v_{x_i}}, \dot{\alpha}_{x_i}\right)\right) = TD2LWA_\psi\left(\left(\left(s_{u_c}^{ij}, \alpha_c^{ij}\right), \left(\dot{s}_{u_c}^{ij}, \dot{\alpha}_c^{ij}\right)\right) \middle| j = 1, 2, \cdots, n, j \neq i\right)$$

where $\psi = \left(\frac{1}{n-1}, \frac{1}{n-1}, \cdots, \frac{1}{n-1}\right)$ is the weights for aggregating the row of the TD2L expressions of collective preferences matrix. $\left(\left(s_{u_c}^{ij}, \alpha_c^{ij}\right), \left(\dot{s}_{u_c}^{ij}, \dot{\alpha}_c^{ij}\right)\right)$ represents the collective preference for alternative x_i in terms of alternative x_j. $\left(\left(s_{u_{x_i}}, \alpha_{x_i}\right), \left(\dot{s}_{v_{x_i}}, \dot{\alpha}_{x_i}\right)\right)$ represents the collective opinion for alternative x_i, denoted as T_{x_i}, $i = 1, 2, \cdots, n$.

3.5. Ranking of alternatives and its reliability

Once the collective opinions have been obtained for each alternative, and expressed by TD2Ls, the ranking of alternatives could be computed from the comparison among those TD2L expressions. In consideration of the comprehensiveness of comparison rules, reliability of the ranking of alternatives is needed as the supplementary information to ensure the high-quality decisions.

Let $S = \{s_0, s_1, \cdots, s_g\}$, $\dot{S} = \left\{\dot{s}_0, \dot{s}_1, \cdots, \dot{s}_h\right\}$ be two linguistic term sets as before, $s_{u_1}, s_{u_2} \in S$, $\dot{s}_{v_1}, \dot{s}_{v_1} \in \dot{S}$, $\alpha_1, \alpha_2, \dot{\alpha}_1, \dot{\alpha}_2 \in [-0.5, 0.5)$, $\widehat{S}_1 = \left((s_{u_1}, \alpha_1), \left(\dot{s}_{v_1}, \dot{\alpha}_1\right)\right)$ and $\widehat{S}_2 = \left((s_{u_2}, \alpha_2), \left(\dot{s}_{v_2}, \dot{\alpha}_2\right)\right)$ be two TD2L expressions. Then the comparison rules (Zhu et al., 2016) are as

(1) If $\Delta^{-1}(s_{u_1}, \alpha_1) \rangle \Delta^{-1}(s_{u_2}, \alpha_2)$ and $\Delta^{-1}\left(s_{v_1}, \dot{\alpha}_1\right) \rangle \Delta^{-1}\left(s_{v_2}, \dot{\alpha}_2\right)$, then $\widehat{S}_1 \succ \widehat{S}_2$.

(2) If $\Delta^{-1}(s_{u_1}, \alpha_1) \langle \Delta^{-1}(s_{u_2}, \alpha_2)$ and $\Delta^{-1}\left(s_{v_1}, \dot{\alpha}_1\right) \langle \Delta^{-1}\left(s_{v_2}, \dot{\alpha}_2\right)$, then $\widehat{S}_1 \prec \widehat{S}_2$.

If $\Delta^{-1}(s_{u_1}, \alpha_1) \rangle \Delta^{-1}(s_{u_2}, \alpha_2)$ and $\Delta^{-1}\left(s_{v_1}, \dot{\alpha}_1\right) \langle \Delta^{-1}\left(s_{v_2}, \dot{\alpha}_2\right)$ or $\Delta^{-1}(s_{u_1}, \alpha_1) \langle \Delta^{-1}(s_{u_2}, \alpha_2)$ and $\Delta^{-1}\left(s_{v_1}, \dot{\alpha}_1\right) \rangle \Delta^{-1}\left(s_{v_2}, \dot{\alpha}_2\right)$, then the comparison result between $\widehat{S}_1$ and $\widehat{S}_2$ is difficult to describe (Zhu et al., 2016). For solving this problem, we propose a comparison way that the result is shown with reliability. The reliability of $\widehat{S}_1 < \widehat{S}_2$ is as follows.

$$r_{\widehat{S}_1 < \widehat{S}_2} = k_1 \int_0^g \int_x^g f_1(x) f_1(y) dy dx + k_2 \int_0^h \int_x^h f_2(x) f_2(y) dy dx$$

where $f_1(x) = \frac{1}{\sqrt{2\pi b_1}} e^{-\frac{(x-a_1)^2}{2b_1}}$, $f_1(y) = \frac{1}{\sqrt{2\pi b_2}} e^{-\frac{(y-a_2)^2}{2b_2}}$, $f_2(x) = \frac{1}{\sqrt{2\pi b_1^{'}}} e^{-\frac{\left(x-a_1^{'}\right)^2}{2b_1^{'}}}$, $f_2(y) = \frac{1}{\sqrt{2\pi b_2^{'}}} e^{-\frac{\left(y-a_2^{'}\right)^2}{2b_2^{'}}}$, $a_1 = \Delta^{-1}(s_{u_1}, \alpha_1)$, $a_1^{'} = \Delta^{-1}(s_{u_2}, \alpha_2)$. If $g/2 \leq a_1 \leq g$, then $b_1 = \left(a_1\left(1 - \left(v_1 + \dot{\alpha}_{v_1}\right)\Big/h\right)\Big/3\right)^2$, $b_1 = \left((g - a_1)\left(1 - \left(v_1 + \dot{\alpha}_{v_1}\right)\Big/h\right)\Big/3\right)^2$ if $0 \leq a_1 \leq g/2$, the value of b_2, $b_1^{'}, b_2^{'}$ are in a similar way as b_1.

Remark 4.. k_1*and* k_2 *are the parameters representing the relative importance of the two-dimensions of TD2Ls provide by DMs.* $k_1 + k_2 = 1$. *Unless otherwise specified, we consider* $k_1 = k_2 = 0.5$.

In fact, we treat a TD2L $\left((s_u, \alpha), \left(\dot{s}_v, \dot{\alpha}\right)\right)$ as a normal stochastic variable $X\, N\left(a, b^2\right)$, the value of a comes from the first dimension information (s_u, α) of TD2L, the value of b comes from the second dimension information $\left(\dot{s}_v, \dot{\alpha}\right)$ of TD2L.

Example 4.. *we compare* $\widehat{S}_1 = \left((s_3, 0.4), \left(\dot{s}_4, -0.2\right)\right)$ *and* $\widehat{S}_2 = \left((s_4, -0.2), \left(\dot{s}_3, 0.4\right)\right)$, *where the cardinalities of the two linguistic terms sets are the same as before. Then the reliability of* $\widehat{S}_1 < \widehat{S}_2$ *is* $r_{\widehat{S}_1 < \widehat{S}_2} = \frac{1}{2}\int_0^6 \int_x^6 f_1(x) f_1(y) dy dx + \frac{1}{2}\int_0^4 \int_x^4 f_2(x) f_2(y) dy dx = (0.979 + 0.225)/2 = 0.602$, *where* $f_1(x) = \frac{1}{\sqrt{2\pi \times 0.0032}} e^{-\frac{(x-3.4)^2}{2\times 0.0032}}$, $f_1(y) = \frac{1}{\sqrt{2\pi \times 0.0361}} e^{-\frac{(y-3.8)^2}{2\times 0.00361}}$, $f_2(x) = \frac{1}{\sqrt{2\pi \times 0.3013}} e^{-\frac{(x-3.8)^2}{2\times 0.3013}}$, $f(y) = \frac{1}{\sqrt{2\pi \times 0.1727}} e^{-\frac{(y-3.4)^2}{2\times 0.1727}}$. *The reliability of* $\widehat{S}_1 < \widehat{S}_2$ *is* 0.602.

Suppose that alternatives ranking is as $x_{\tau(1)} > x_{\tau(2)} > \ldots > x_{\tau(n)}$, where $\tau(1), \tau(2), \cdots, \tau(n)$ is the permutation of $1, 2, \cdots, n$. Then there have $n - 1$ pairs of alternatives need to be compared to obtain the ranking. The reliability of alternatives ranking is as:

$r_R = \prod_{l=1}^{n-1} r_l$

where $r_l(l = 1, 2, \cdots, n-1)$ is the reliability of $x_{\tau(l)} > x_{\tau(l+1)}$. $r_l \in [0.5, 1]$ for $l = 1, 2, \cdots, n-1$.

Remark 5.. *r_R is an important indicator that is necessary for describing the reliability of solution. Obviously, the larger the reliability of solution, the model designed for solving this GDM problem is better. If the reliability of solution with this method is lower, then the ranking of alternatives is meaningless. If the reliability of solution with this method is higher, then the proposed method is feasible and effective.*

Example 5.. *Let $S = \{s_0, s_1, s_2, s_3, s_4, s_5, s_6\}$ and $\dot{S} = \left\{\dot{s}_0, \dot{s}_1, \dot{s}_2, \dot{s}_3, \dot{s}_4\right\}$ be two linguistic term sets, $\widehat{S}_1 = \left((s_3, 0), \left(\dot{s}_4, -0.1\right)\right)$, $\widehat{S}_2 = \left((s_5, 0.4), \left(\dot{s}_4, -0.4\right)\right)$, $\widehat{S}_3 = \left((s_3, 0.4), \left(\dot{s}_4, -0.2\right)\right)$ and $\widehat{S}_4 = \left((s_5, 0.2), \left(\dot{s}_3, 0.4\right)\right)$ be four TD2L expressions that represent the four alternatives' collective opinion x_1, x_2, x_3, x_4, respectively. Then the alternatives ranking is $\widehat{S}_2 > \widehat{S}_4 > \widehat{S}_3 > \widehat{S}_1$ with reliability of 0.368, where $0.368 = 0.593 \times 0.620 \times 1$, $r_1 = 0.593$, $r_2 = 0.620$, $r_3 = 1$. Therefore, $x_2 > x_4 > x_3 > x_1$ with the reliability of 0.368.*

4. Numerical example and simulation analysis

In this section, we provide a numerical example and a simulation experiment to illustrate the use of two-stage minimum adjustment consensus model with TD2L expressions in LSGDM.

Here, we provide a LSGDM problem, which includes a set of 20 DMs, $E = \{e_1, e_2, \cdots, e_{20}\}$ and a set of 4 alternatives, $X = \{x_1, x_2, \cdots, x_4\}$. Although the number of decision makers in the example is small for LSGDM problems, it is enough to illustrate the proposed consensus model (Chen & Liu, 2006). Let $\{s_0 = extremelypoor, \ s_1 = verypoor, \ s_2 = poor, \ s_3 = fair, \ s_4 = good, \ s_5 = verygood, \ s_6 = extremelygood\}$ be a linguistic term set S. Suppose the acceptable CL is $\sigma = 0.8$. All DMs express their preference as additive linguistic fuzzy preference relations matrix $R_k(k = 1, 2, \cdots, 20)$, which are generated by AFRYCA (Palomares et al., 2014) as follows.

$$R_1 = \begin{pmatrix} s_3 & s_3 & s_5 & s_3 \\ - & s_3 & s_4 & s_3 \\ - & - & s_3 & s_2 \\ - & - & - & s_3 \end{pmatrix} \quad R_2 = \begin{pmatrix} s_3 & s_1 & s_2 & s_4 \\ - & s_3 & s_3 & s_4 \\ - & - & s_3 & s_4 \\ - & - & - & s_3 \end{pmatrix}$$

$$R_3 = \begin{pmatrix} s_3 & s_4 & s_4 & s_5 \\ - & s_3 & s_4 & s_2 \\ - & - & s_3 & s_2 \\ - & - & - & s_3 \end{pmatrix}$$

$$R_4 = \begin{pmatrix} s_3 & s_0 & s_6 & s_3 \\ - & s_3 & s_4 & s_4 \\ - & - & s_3 & s_1 \\ - & - & - & s_3 \end{pmatrix} \quad R_5 = \begin{pmatrix} s_3 & s_2 & s_4 & s_4 \\ - & s_3 & s_4 & s_5 \\ - & - & s_3 & s_6 \\ - & - & - & s_3 \end{pmatrix}$$

$$R_6 = \begin{pmatrix} s_3 & s_3 & s_3 & s_4 \\ - & s_3 & s_3 & s_4 \\ - & - & s_3 & s_4 \\ - & - & - & s_3 \end{pmatrix}$$

$$R_7 = \begin{pmatrix} s_3 & s_1 & s_5 & s_6 \\ - & s_3 & s_6 & s_6 \\ - & - & s_3 & s_1 \\ - & - & - & s_3 \end{pmatrix} \quad R_8 = \begin{pmatrix} s_3 & s_3 & s_6 & s_2 \\ - & s_3 & s_4 & s_0 \\ - & - & s_3 & s_2 \\ - & - & - & s_3 \end{pmatrix}$$

$$R_9 = \begin{pmatrix} s_3 & s_5 & s_5 & s_6 \\ - & s_3 & s_1 & s_0 \\ - & - & s_3 & s_2 \\ - & - & - & s_3 \end{pmatrix}$$

$$R_{10} = \begin{pmatrix} s_3 & s_3 & s_4 & s_5 \\ - & s_3 & s_4 & s_4 \\ - & - & s_3 & s_5 \\ - & - & - & s_3 \end{pmatrix} \quad R_{11} = \begin{pmatrix} s_3 & s_3 & s_6 & s_6 \\ - & s_3 & s_4 & s_4 \\ - & - & s_3 & s_3 \\ - & - & - & s_3 \end{pmatrix}$$

$$R_{12} = \begin{pmatrix} s_3 & s_4 & s_4 & s_2 \\ - & s_3 & s_5 & s_0 \\ - & - & s_3 & s_2 \\ - & - & - & s_3 \end{pmatrix}$$

$$R_{13} = \begin{pmatrix} s_3 & s_5 & s_6 & s_2 \\ - & s_3 & s_4 & s_0 \\ - & - & s_3 & s_1 \\ - & - & - & s_3 \end{pmatrix} \quad R_{14} = \begin{pmatrix} s_3 & s_4 & s_1 & s_0 \\ - & s_3 & s_0 & s_1 \\ - & - & s_3 & s_1 \\ - & - & - & s_3 \end{pmatrix}$$

$$R_{15} = \begin{pmatrix} s_3 & s_2 & s_0 & s_2 \\ - & s_3 & s_1 & s_3 \\ - & - & s_3 & s_5 \\ - & - & - & s_3 \end{pmatrix}$$

$$R_{16} = \begin{pmatrix} s_3 & s_1 & s_4 & s_0 \\ - & s_3 & s_5 & s_3 \\ - & - & s_3 & s_2 \\ - & - & - & s_3 \end{pmatrix} \quad R_{17} = \begin{pmatrix} s_3 & s_6 & s_4 & s_3 \\ - & s_3 & s_2 & s_2 \\ - & - & s_3 & s_5 \\ - & - & - & s_3 \end{pmatrix}$$

$$R_{18} = \begin{pmatrix} s_3 & s_5 & s_5 & s_3 \\ - & s_3 & s_2 & s_0 \\ - & - & s_3 & s_5 \\ - & - & - & s_3 \end{pmatrix}$$

$$R_{19} = \begin{pmatrix} s_3 & s_2 & s_1 & s_5 \\ - & s_3 & s_6 & s_4 \\ - & - & s_3 & s_5 \\ - & - & - & s_3 \end{pmatrix} \quad R_{20} = \begin{pmatrix} s_3 & s_5 & s_0 & s_1 \\ - & s_3 & s_2 & s_1 \\ - & - & s_3 & s_4 \\ - & - & - & s_3 \end{pmatrix}$$

4.1. Illustration of proposed method

In this subsection, we show how to apply the two-stage minimum adjustment model with TD2L expressions in LSGDM. The steps for solving the problem are as follows.

Step 1: Clustering process

Four clusters are obtained by **Algorithm I** as:

G_1^0: e_1, e_3, e_9, e_{10}, e_{11}, e_{17}, e_{18}

G_2^0: e_2, e_4, e_5, e_7, e_{16}, e_{19}

G_3^0: e_6, e_{15}, e_{20}

G_4^0: e_8, e_{12}, e_{13}, e_{14}

Clusters' weights are $w_{G_l^0} = (0.35, 0.3, 0.15, 0.2)$ based on Section 3.1. DMs' weights in the same cluster are obtained from the optimization models (***see model (10)***). Taking DMs' weight in cluster G_1^0 as an example, the model is as

$$\text{Min} \sum_{q_2=q_1+1}^{7} \sum_{q_1=1}^{6} d\left(\left(r_{G_1^0}^{q_1}, \alpha_{G_1^0}^{q_1}\right), \left(r_{G_1^0}^{q_2}, \alpha_{G_1^0}^{q_2}\right)\right)$$

$$s.t. \begin{cases} d\left(\left(r_{G_1^0}^{q_1}, \alpha_{G_1^0}^{q_1}\right), \left(r_{G_1^0}^{q_2}, \alpha_{G_1^0}^{q_2}\right)\right) \\ = \left|\lambda_{G_1^0}^{q_1} \Delta^{-1}\left(r_{G_1^0}^{q_1}, \alpha_{G_1^0}^{q_1}\right) - \lambda_{G_1^0}^{q_2} \Delta^{-1}\left(r_{G_1^0}^{q_2}, \alpha_{G_1^0}^{q_2}\right)\right| \\ \sum_{q=1}^{7} \lambda_{G_1^0}^{q} = 1 \\ 0 < \lambda_{G_1^0}^{q} < 1, q = 1, 2, \cdots, 7 \end{cases}$$

Use LINGO 11 to solve model (25) and obtain the results as:

$\lambda_{G_1^0} = (0.15, 0.15, 0.125, 0.15, 0.125, 0.15, 0.15)^T$

Similar, DMs' weights in clusters G_2^0, G_3^0, G_4^0 are $\lambda_{G_2^0} = (0.186, 0.186, 0.149, 0.124, 0.207, 0.148)^T$, $\lambda_{G_3^0} = (0.261, 0.348, 0.391)^T$, $\lambda_{G_4^0} = (0.143, 0.143, 0.143, 0.571)^T$, respectively.

Step 2: Two-stage minimum adjustment consensus model

Compute the three kinds of CLs as the way mentioned in Section 3.2:

(1) The CL for each pair of alternatives.

$$CL_{ij}^{0} = \begin{pmatrix} 1 & 0.678 & 0.619 & 0.620 \\ - & 1 & 0.672 & 0.647 \\ - & - & 1 & 0.676 \\ - & - & - & 1 \end{pmatrix}$$

(2) The CL for each alternative.

$$CL_{x_i}^{0} = (0.639, 0.666, 0.656, 0.648)$$

(3) The overall CL for all preferences.

$$CL^{0} = 0.652$$

We can see that $CL^{0} = 0.652 < \sigma$, the acceptable consensus is not reached in this problem, then the two-stage minimum adjustment consensus model is constructed in the following.

$M_0 = \{12, 13, 14, 23, 24, 34\}$ owing to CL_{12}^{0}, CL_{13}^{0}, CL_{14}^{0}, CL_{23}^{0}, CL_{24}^{0}, $CL_{34}^{0} < \sigma$, The preferences for all pairs of alternatives need to be adjusted in the first stage. According to the identification rules described in Section 3.2, Cluster G_3^0 is chosen to be adjusted first. The optimization model for obtaining the maximum improvement of the CL for each pair of alternatives is constructed as follows.

Stage One

$$\text{Max} \sum_{ij \in M_0} \gamma^{0ij}$$

$$s.t. \begin{cases} \overline{CL}_{ij}^{0} \geq CL_{ij}^{0} + \gamma^{0ij} \\ \overline{CL}_{ij}^{0} = 1 - \sum_{k_1-1}^{19} \sum_{k_2-1}^{20} \dfrac{\omega_{k_1}^{0} + \omega_{k_2}^{0}}{19} \left| \Delta^{-1}\left(\overline{r}_{k_1}^{0ij}, \overline{\alpha}_{k_1}^{0ij}\right) - \Delta^{-1}\left(\overline{r}_{k_2}^{0ij}, \overline{\alpha}_{k_2}^{0ij}\right) \right| \Big/ 6 \\ CL_{ij}^{0} = 1 - \sum_{k_1-1}^{19} \sum_{k_2-1}^{20} \dfrac{\omega_{k_1}^{0} + \omega_{k_2}^{0}}{19} \left| \Delta^{-1}\left(r_{k_1}^{0ij}, \alpha_{k_1}^{0ij}\right) - \Delta^{-1}\left(r_{k_2}^{0ij}, \alpha_{k_2}^{0ij}\right) \right| / 6 \\ \Delta^{-1}\left(\overline{r}_{k}^{0ij}, \overline{\alpha}_{k}^{0ij}\right) \Big\langle 6, k = 6, 15, 20 \\ \Delta^{-1}\left(\overline{r}_{k}^{0ij}, \overline{\alpha}_{k}^{0ij}\right) \Big\rangle 0, k = 6, 15, 20 \end{cases}$$

where $ij \in M_0$. Use LINGO 11 to solve Model (26), then obtain the optimal vector of γ^0 is $\gamma^{0^*} = (0.021, 0.102, 0.030, 0.043, 0.014, 0.029)^T$ and CL for each pair of alternatives with adjusted preferences is $\overline{CL}_{ij}^{0^*} = (0.700, 0.721, 0.650, 0.715, 0.661, 0.705)^T$.

Next, based on the maximum improvement, a minimum adjustment cost model is constructed as the second stage according to model (16) as follows.

Stage Two

$$\text{Min} \sum_{q-1}^{3} \sum_{ij \in M_0} d\left(\left(r_{G_3^0}^{qij}, \alpha_{G_3^0}^{qij}\right), \left(\overline{r}_{G_3^0}^{qij}, \overline{\alpha}_{G_3^0}^{qij}\right)\right)$$

$$s.t. \begin{cases} \overline{CL}_{ij}^{0} \geq \theta \cdot \overline{CL}_{ij}^{0^*}, ij \in M_0 \\ \overline{CL}_{ij}^{0} = 1 - \sum_{k_1=1}^{m-1} \sum_{k_2=1}^{m} \left(\dfrac{\omega_{k_1} + \omega_{k_2}}{m-1} \times \left| \Delta^{-1}\left(\overline{r}_{k_1}^{0ij}, \overline{\alpha}_{k_1}^{0ij}\right) - \Delta^{-1}\left(\overline{r}_{k_2}^{0ij}, \overline{\alpha}_{k_2}^{0ij}\right) \right| \Big/ 6\right) \\ \Delta^{-1}\left(\overline{r}_{G_3^0}^{qij}, \overline{\alpha}_{G_3^0}^{qij}\right) \Big\langle 6, q = 1, 2, 3 \\ \Delta^{-1}\left(\overline{r}_{G_3^0}^{qij}, \overline{\alpha}_{G_3^0}^{qij}\right) \Big\rangle 0, q = 1, 2, 3 \\ c_q = 1, q = 1, 2, 3 \end{cases}$$

where θ is a numerical number close to 1, here we assume that $\theta = 0.99$. $\omega = (\omega_1, \omega_2, \cdots, \omega_m)^T$ is the same as in **Step 1**.

Use LINGO 11 to solve model (27), then obtain the adjusted preferences of DMs in G_3^0:

$$\overline{R}_{G_3^0}^{1} = \begin{pmatrix} (s_3, 0) & (s_3, 0) & (s_4, -0.287) & (s_4, 0) \\ - & (s_3, 0) & (s_4, -0.496) & (s_4, -0.204) \\ - & - & (s_3, 0) & (s_3, -0.093) \\ - & - & - & (s_3, 0) \end{pmatrix}$$

$$\overline{R}_{G_3^0}^{2} = \begin{pmatrix} (s_3, 0) & (s_3, -0.370) & (s_4, -0.297) & (s_3, -0.274) \\ - & (s_3, 0) & (s_4, -0.496) & (s_2, 0.045) \\ - & - & (s_3, 0) & (s_3, -0.093) \\ - & - & - & (s_3, 0) \end{pmatrix}$$

$$\overline{R}_{G_3^0}^{3} = \begin{pmatrix} (s_3, 0) & (s_4, 0) & (s_4, -0.287) & (s_3, -0.274) \\ - & (s_3, 0) & (s_4, -0.496) & (s_2, 0) \\ - & - & (s_3, 0) & (s_3, -0.093) \\ - & - & - & (s_3, 0) \end{pmatrix}$$

where $\overline{R}_{G_3^0}^{1}$, $\overline{R}_{G_3^0}^{2}$, $\overline{R}_{G_3^0}^{3}$ represent the adjusted preference matrix of $1th$, $2th$, $3th$ DM in cluster G_3^0, respectively. In fact, they are the adjusted preference matrices of R_6, R_{15}, R_{20}.

Step 3: Consensus reaching process

After the first round adjustment, the overall CL is $CL^1 = 0.685 < \sigma$, which does not reach the acceptable CL. We continue using the two-stage CRP model until $CL^P \geq \sigma$. After 5 rounds, the overall CL is acceptable, the final clustering and the adjusted preferences are as

(1) The clustering results:

G_1^5: e_3, e_9, e_{10}, e_{11}, e_{17}, e_{18}

G_2^5: e_2, e_5, e_7

G_3^5: e_4, e_6, e_8, e_{12}, e_{13}, e_{14}, e_{15}, e_{20}

G_4^5: e_1, e_{16}, e_{19}

(2) The adjusted preferences:

$$\overline{R}_{4}^{5} = \begin{pmatrix} (s_3, 0) & (s_3, -0.370) & (s_5, -0.480) & (s_3, 0) \\ - & (s_3, 0) & (s_4, 0) & (s_4, -0.402) \\ - & - & (s_3, 0) & (s_3, -0.309) \\ - & - & - & (s_3, 0) \end{pmatrix}$$

$$\overline{R}_{6}^{5} = \begin{pmatrix} (s_3, 0) & (s_3, -0.370) & (s_4, 0) & (s_4, -0.484) \\ - & (s_3, 0) & (s_4, -0.283) & (s_3, 0.359) \\ - & - & (s_3, 0) & (s_3, -0.093) \\ - & - & - & (s_3, 0) \end{pmatrix}$$

$$\overline{R}_{8}^{5} = \begin{pmatrix} (s_3, 0) & (s_3, 0) & (s_5, -0.478) & (s_4, -0.484) \\ - & (s_3, 0) & (s_4, 0) & (s_3, 0.359) \\ - & - & (s_3, 0) & (s_3, -0.308) \\ - & - & - & (s_3, 0) \end{pmatrix}$$

$$\overline{R}_{12}^5 = \begin{pmatrix} (s_3,0) & (s_3,0.061) & (s_4,0) & (s_4,-0.484) \\ - & (s_3,0) & (s_4,0) & (s_3,0.359) \\ - & - & (s_3,0) & (s_3,-0.308) \\ - & - & - & (s_3,0) \end{pmatrix}$$

$$\overline{R}_{13}^5 = \begin{pmatrix} (s_3,0) & (s_3,0.061) & (s_5,-0.478) & (s_4,-0.484) \\ - & (s_3,0) & (s_4,0) & (s_3,0.359) \\ - & - & (s_3,0) & (s_3,-0.308) \\ - & - & - & (s_3,0) \end{pmatrix}$$

$$\overline{R}_{14}^5 = \begin{pmatrix} (s_3,0) & (s_3,0) & (s_4,0) & (s_4,-0.5) \\ - & (s_3,0) & (s_4,-0.389) & (s_3,0.359) \\ - & - & (s_3,0) & (s_3,-0.308) \\ - & - & - & (s_3,0) \end{pmatrix}$$

$$\overline{R}_{15}^5 = \begin{pmatrix} (s_3,0) & (s_3,-0.370) & (s_4,0) & (s_3,0) \\ - & (s_3,0) & (s_4,-0.496) & (s_3,0.241) \\ - & - & (s_3,0) & (s_3,-0.093) \\ - & - & - & (s_3,0) \end{pmatrix}$$

$$\overline{R}_{19}^5 = \begin{pmatrix} (s_3,0) & (s_3,-0.370) & (s_4,0) & (s_5,0) \\ - & (s_3,0) & (s_4,0) & (s_4,0) \\ - & - & (s_3,0) & (s_3,3) \\ - & - & - & (s_3,0) \end{pmatrix}$$

$$\overline{R}_{20}^5 = \begin{pmatrix} (s_3,0) & (s_3,0.061) & (s_4,0) & (s_4,-0.484) \\ - & (s_3,0) & (s_4,-0.293) & (s_3,0.359) \\ - & - & (s_3,0) & (s_3,-0.093) \\ - & - & - & (s_3,0) \end{pmatrix}$$

where $\overline{R}_k^5 = R_k^0$ for $k = 1,2,3,5,7,9,10,11,16,17,18$.

Step 4: Measurement of reliability of adjusted preferences

Based on the computation results in Step 3 and Section 3.3, the preference adjustment matrices for $k = 4,6,8,12,13,14,15,19,20$ are as:

$$A_4 = \begin{pmatrix} 0 & 2.63 & 1.48 & 0 \\ - & 0 & 0 & 0 \\ - & - & 0 & 1.691 \\ - & - & - & 0 \end{pmatrix} \quad A_6 = \begin{pmatrix} 0 & 0.37 & 1 & 0.484 \\ - & 0 & 0.717 & 0.641 \\ - & - & 0 & 1.093 \\ - & - & - & 0 \end{pmatrix}$$

$$A_8 = \begin{pmatrix} 0 & 0 & 1.478 & 1.516 \\ - & 0 & 0 & 3.359 \\ - & - & 0 & 0.692 \\ - & - & - & 0 \end{pmatrix} \quad A_{12} = \begin{pmatrix} 0 & 0.939 & 0 & 1.516 \\ - & 0 & 1 & 3.359 \\ - & - & 0 & 0.692 \\ - & - & - & 0 \end{pmatrix}$$

$$A_{13} = \begin{pmatrix} 0 & 1.939 & 1.478 & 1.516 \\ - & 0 & 0 & 3.359 \\ - & - & 0 & 1.692 \\ - & - & - & 0 \end{pmatrix} \quad A_{14} = \begin{pmatrix} 0 & 1 & 3 & 3.5 \\ - & 0 & 3.611 & 2.359 \\ - & - & 0 & 1.692 \\ - & - & - & 0 \end{pmatrix}$$

$$A_{15} = \begin{pmatrix} 0 & 0.63 & 4 & 1 \\ - & 0 & 2.504 & 0.241 \\ - & - & 0 & 2.093 \\ - & - & - & 0 \end{pmatrix} \quad A_{19} = \begin{pmatrix} 0 & 0.63 & 3 & 0 \\ - & 0 & 2 & 0 \\ - & - & 0 & 2 \\ - & - & - & 0 \end{pmatrix}$$

$$A_{20} = \begin{pmatrix} 0 & 1.939 & 4 & 2.516 \\ - & 0 & 1.717 & 3.359 \\ - & - & 0 & 1.093 \\ - & - & - & 0 \end{pmatrix}$$

where the preference adjustment matrices A_k for $k = 1,2,3,5,7,9,10,11,16,17,18$ are 0. That means they keep unchanged their preferences.

The final opinions are presented as TD2L expressions based on Eq. (20) as shown in matrices. Take R_4^* as an example, R_4^* is a preference matrix of DM e_4 that represents the agreed preferences with its reliability degree for each pair of alternatives, which are as follows.

$$R_4^* = \left(\left((s_3,0),\left(\dot{s}_4,0\right)\right) \quad \left((s_3,-0.370),\left(\dot{s}_0,0\right)\right) \right.$$

$$\left((s_5,-0.480),\left(\dot{s}_3,-0.48\right)\right)\left((s_3,0),\left(\dot{s}_4,0\right)\right)$$

$$-\left((s_3,0),\left(\dot{s}_4,0\right)\right)\left((s_4,0),\left(\dot{s}_4,0\right)\right)\left((s_4,-0.402),\left(\dot{s}_4,0\right)\right)$$

$$--\left((s_3,0),\left(\dot{s}_4,0\right)\right)\left((s_3,-0.309),\left(\dot{s}_1,1.148\right)\right)--$$

$$\left. -\left((s_3,0),\left(\dot{s}_4,0\right)\right) \right)$$

where the matrices $R_k^*(k = 1,2,\cdots,20)$ are listed in **Appendix A.**

Step 5: Alternatives ranking based on TD2L expressions

According to Section 3.3, DMs' weighs are $\omega = (0.055, 0.063,$

$0.058, 0.057,$

$0.048, 0.057, 0.040, 0.046, 0.048,$

$0.048, 0.048, 0.049, 0.038, 0.061, 0.046, 0.055,$

$0.048, 0.048, 0.041, 0.046)^T$

The final collective opinion matrix with reliability degree is as

$$\left(\left((s_3,0),\left(\dot{s}_4,0\right)\right)\left((s_3,-0.036),\left(\dot{s}_3,0.246\right)\right) \right.$$

$$\left((s_4,0.234),\left(\dot{s}_3,0.060\right)\right)\left((s_4,-0.200),\left(\dot{s}_3,0.309\right)\right)$$

$$-\left((s_3,0),\left(\dot{s}_4,0\right)\right)\left((s_3,-0.340),\left(\dot{s}_3,0.351\right)\right)\left((s_3,0.197),\left(\dot{s}_3,0.056\right)\right)$$

$$\left. --\left((s_3,0),\left(\dot{s}_4,0\right)\right)\left((s_3,0.108),\left(\dot{s}_3,0.100\right)\right)---\left((s_3,0),\left(\dot{s}_4,0\right)\right) \right)$$

The collective opinion of each alternative expressed as TD2L expressions as

$$T_{x_1} = \left((s_4,-0.334),\left(\dot{s}_3,0.205\right)\right)$$

$$T_{x_2} = \left((s_3,0.274),\left(\dot{s}_3,0.218\right)\right)$$

Table 1
The simulation results of alternatives ranking with different methods.

Method	Collective preferences of each alternative	Alternatives ranking	The reliability of alternatives ranking	The number of DMs need to be adjusted.
Zhang et al. (2011)	$x_1:(s_3,0.208)$,$x_2:(s_3,0.057)$, $x_3:(s_3,0.488)$,$x_4:(s_3,-0.008)$	$x_3>x_1>x_2>x_4$	–	14
Wu and L. F., Chiclana, F., Fujita, H., & Herrera-Viedma, E. (2018)	$x_1:(s_3,0.288)$,$x_2:(s_3,-0.062)$, $x_3:(s_3,0.269)$,$x_4:(s_3,-0.019)$	$x_1>x_3>x_4>x_2$	–	14
Proposed method	$x_1:\left((s_4,-0.334),\left(\dot{s}_3,0.205\right)\right)$, $x_2:\left((s_3,0.274),\left(\dot{s}_3,0.218\right)\right)$, $x_3:\left((s_3,-0.333),\left(\dot{s}_3,0.170\right)\right)$,$x_4:\left((s_3,0.368),\left(\dot{s}_3,0.155\right)\right)$	$x_1>x_3>x_4>x_2$	0.531 ($x_3>x_4>x_2$ with the reliability of 1 and x_1 is slightly better than x_3)	9

$$T_{x_3}=\left((s_3,-0.333),\left(\dot{s}_3,0.170\right)\right)$$

$$T_{x_4}=\left((s_3,0.368),\left(\dot{s}_3,0.155\right)\right)$$

According to the comparison rules proposed in Section 3.5, the alternatives ranking is $x_1>x_3>x_4>x_2$ with the reliability of 0.531, x_1 is the optimal alternative.

4.2. Comparisons and discussions

To further illustrate the characteristics of the proposed model, we make comparisons with two methods. One of them is classical MCCM proposed by Zhang et al. (2011). The other one is a MACM with feedback mechanism (Wu et al., 2018). To compare these models in the same condition, we assume that the unit cost of all DMs in Zhang et al.'s model is unit one. The comparison details are as follows.

(1) Solving the above problem with MCCM without reliability (Zhang et al., 2011).

Without considering the reliability degree of adjusted preferences, and the unit adjustment cost of all DMs are unit one, the general MACM is set up as follows.

$$\mathrm{Min}\sum_{k=1}^{20}\sum_{i=1}^{4}\sum_{j=1}^{4}d\left(\left(r_k^{ij},\alpha_k^{ij}\right),\left(\overline{r}_k^{ij},\overline{\alpha}_k^{ij}\right)\right)$$

$$s.t.\begin{cases}1-\frac{1}{4\times4}\sum_{i=1}^{4}\sum_{j=1}^{4}d\left(\left(\overline{r}_k^{ij},\overline{\alpha}_k^{ij}\right),\left(\overline{r}^{ij},\overline{\alpha}^{ij}\right)\right)\geq\sigma,k=1,2,\cdots,20\\ \left(\overline{r}^{ij},\overline{\alpha}^{ij}\right)=F\left(\left(\overline{r}_1^{ij},\overline{\alpha}_1^{ij}\right),\left(\overline{r}_2^{ij},\overline{\alpha}_2^{ij}\right),\cdots,\left(\overline{r}_{20}^{ij},\overline{\alpha}_{20}^{ij}\right)\right)\end{cases}$$

where $F(\cdot)$ is a weighted aggregation function, in this situation, we consider all DMs have the same importance.

Use LINGO 11 to solve model (28), the adjusted preferences of each DM are obtained as in Appendix B. Then the overall opinions of each alternative are as $x_1:(s_3,0.208)$, $x_2:(s_3,0.057)$, $x_3:(s_3,0.488)$, $x_4:(s_3,-0.008)$. Obviously, alternatives ranking in this situation is $x_3>x_1>x_2>x_4$.

(2) Solve the above problem with MACM and feedback (Wu et al., 2018).

Based on the minimum adjustment feedback mechanism proposed in Wu et al. (2018), the adjusted preferences is decided by the recommendation advice. There is a feedback mechanism parameter δ to control the accepting degree of recommendation advice. The adjusted preferences are different when the value of δ changes. The model for solving the accepting recommendation advice is as follows.

$$\mathrm{Min}\sum_{k=1}^{20}\sum_{i=1}^{4}\sum_{j=1}^{4}\delta d\left(\left(r_k^{ij},\alpha_k^{ij}\right),\left(r^{ij},\alpha^{ij}\right)\right)$$

$$s.t.\begin{cases}\overline{CL}_k\geq\sigma,k=1,2,\cdots,20\\ \overline{CL}_k=\frac{1}{4}\sum_{i=1}^{4}\overline{CL}_k^i\\ \overline{CL}_k^i=\frac{1}{3}\sum_{j=1,\,j\neq i}^{4}\overline{CL}_k^{ij},i=1,2,3,4\\ \overline{CL}_k^{ij}=1-d\left(\left(\overline{r}_k^{ij},\overline{\alpha}_k^{ij}\right),\left(\overline{r}^{ij},\overline{\alpha}^{ij}\right)\right)\\ \left(r^{ij},\alpha^{ij}\right)=F\left(\left(r_1^{ij},\alpha_1^{ij}\right),\left(r_2^{ij},\alpha_2^{ij}\right),\cdots,\left(r_{20}^{ij},\alpha_{20}^{ij}\right)\right)\end{cases}$$

where $\Delta^{-1}\left(\overline{r}_k^{ij},\overline{\alpha}_k^{ij}\right)=(1-\delta)\Delta^{-1}\left(r_k^{ij},\alpha_k^{ij}\right)+\delta\Delta^{-1}\left(\overline{r}^{ij},\overline{\alpha}^{ij}\right)$ for $CL_k<\sigma$ and $\left(\overline{r}_k^{ij},\overline{\alpha}_k^{ij}\right)=\left(r_k^{ij},\alpha_k^{ij}\right)$ for $CL_k\geq\sigma$.

Use LINGO 11 to solve model (29), use $\delta=0.55$ (Wu et al., 2018), the recommendation advice is as in Appendix C. In this situation, there are 14 DMs that need to adjust their original preferences. After accepting the recommendation advice, the alternatives ranking is as $x_1>x_3>x_4>x_2$.

Based on the above comparison, we make a summary to show the difference with different methods as in Table 1.

With method Zhang et al. (2011), we can see that large number of DMs need to change their preferences. For example, preference values r_8^{014}, and r_8^{023} keep unchanged with the proposed method, however, r_8^{0ij} for $ij=12,13,14,23,24,34$ need to be adjusted (see Appendix B). However, it is not practical in real decision making, the number of DMs is a huge number. In order to keep more original preferences unchanged and adjust the preferences with accepting recommendation mechanism, Wu et al. (2018) proposed a feedback mechanism with automatic parameter adjustment. The adjusted preferences are acceptable but the reliability degree of the adjusted preferences is still missing. We consider the reliability degree of adjusted preferences, which are key factor to ensure the accuracy of the decision making.

The collective opinions of each alternative with the methods (Wu et al., 2018; Zhang et al., 2011) are 2-tuple linguistic values, however, in our proposal, the collective opinions are presented as TD2L, which adds the reliability degree of adjusted preferences. The alternatives ranking are both $x_1>x_3>x_4>x_2$ with method Wu et al. (2018) and the proposed method, however, $x_3>x_1>x_2>x_4$ with the method Zhang et al. (2011). It shows that with the consideration of reliability of adjusted opinions, the accuracy of the decision making is ensured as the same as the involvement of the feedback (Wu et al., 2018). In addition, with our proposal, CRP is time-saving and more suitable for real life, like emergency decision making problems. Finally, the reliability of alternatives ranking can be used as an indicator to evaluate methods. The final solution obtained with the consideration of reliability of alternatives ranking is more reasonable. Even though the reliability of the alternatives ranking derived from the proposed method is not high, which is 0.531, the reason is that the reliability of $x_1 > x_3$ is 0.531, which means x_1 is barely superior to x_3. The reliability of alternatives ranking could be as important information for decision making. Based on the above statement, the proposed method is effective and has an important significance in this field.

5. Conclusion

For automatic CRP, the reliability of the adjusted preferences is often neglected because all adjusted preferences are accepted by DMs by default. Based on this observation, this paper proposes a two-stage minimum adjustment consensus model for LSGDM problems based on TD2L expressions with the consideration of reliability degree of the adjusted preferences.

The main contributions in this article are: First, a two-stage minimum adjustment consensus model is proposed, which not only considers the minimum adjustment, but also minimizes the number of adjusted preferences. The first stage is to maximize the improvement of CL for each pair of alternatives within minimum adjustment. The second stage is to obtain the adjusted preferences with the certain CL at the first stage within minimum adjustment. Second, we further study TD2L information and provide a mathematical way to obtain the second dimension information as the reliability of the adjusted preferences. Third, we find out the relations between the total preference adjustment and the reliability degree of the adjusted preferences. Finally, alternatives ranking is presented with its reliability. The effectiveness of the proposed model has been illustrated by a numerical example in which comparisons with other CRP-LSGDM models has been done.

Consensus reaching is such a complex problem in LSGDM, during the decision making process, determination of the unit cost of each DM should be considered, in this paper, we consider the uniform unit cost. In the future, we will further study this reliability model based on TD2L information in LSGDM with the consideration of the determination of unit cost.

CRediT authorship contribution statement

Zelin Wang: Conceptualization, Data curation, Formal analysis, Writing - original draft, Writing - review & editing. **Rosa M. Rodríguez:** Conceptualization, Data curation, Formal analysis, Writing - original draft, Writing - review & editing. **Ying-ming Wang:** Conceptualization, Data curation, Formal analysis, Writing - review & editing. **Luis Martínez:** Conceptualization, Data curation, Formal analysis, Writing - original draft, Writing - review & editing.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

The work is supported by the National Nature Science Foundation of China (Grant No. 61773123), the Spanish Ministry of Economy and Competitiveness through the Spanish National Research PGC2018-099402-B-I00, and the Postdoctoral Fellowship Ramón y Cajal (RYC-2017-21978).

Appendix A. The final opinions solved by the proposed method are presented as TD2L expressions as follows

$$R_1^* = \begin{pmatrix} \left((s_3,0),\left(\dot{s}_4,0\right)\right) & \left((s_3,0),\left(\dot{s}_4,0\right)\right) & \left((s_5,0),\left(\dot{s}_4,0\right)\right) & \left((s_3,0),\left(\dot{s}_4,0\right)\right) \\ - & \left((s_3,0),\left(\dot{s}_4,0\right)\right) & \left((s_4,0),\left(\dot{s}_4,0\right)\right) & \left((s_3,0),\left(\dot{s}_4,0\right)\right) \\ - & - & \left((s_3,0),\left(\dot{s}_4,0\right)\right) & \left((s_2,0),\left(\dot{s}_4,0\right)\right) \\ - & - & - & \left((s_3,0),\left(\dot{s}_4,0\right)\right) \end{pmatrix}$$

$$R_2^* = \begin{pmatrix} \left((s_3,0),\left(\dot{s}_4,0\right)\right) & (s_1,0),\left(\left(\dot{s}_4,0\right)\right) & \left((s_2,0),\left(\dot{s}_3,0\right)\right) & \left((s_4,0),\left(\dot{s}_4,0\right)\right) \\ - & \left((s_3,0),\left(\dot{s}_4,0\right)\right) & \left((s_3,0),\left(\dot{s}_4,0\right)\right) & \left((s_4,0),\left(\dot{s}_4,0\right)\right) \\ - & - & \left((s_3,0),\left(\dot{s}_4,0\right)\right) & \left((s_4,0),\left(\dot{s}_4,0\right)\right) \\ - & - & - & \left((s_3,0),\left(\dot{s}_4,0\right)\right) \end{pmatrix}$$

$$R_3^* = \begin{pmatrix} \left((s_3,0),\left(\dot{s}_4,0\right)\right) & \left((s_4,0),\left(\dot{s}_4,0\right)\right) & \left((s_4,0),\left(\dot{s}_4,0\right)\right) & \left((s_5,0),\left(\dot{s}_4,0\right)\right) \\ - & \left((s_3,0),\left(\dot{s}_4,0\right)\right) & \left((s_4,0),\left(\dot{s}_4,0\right)\right) & \left((s_2,0),\left(\dot{s}_4,0\right)\right) \\ - & - & \left((s_3,0),\left(\dot{s}_4,0\right)\right) & \left((s_2,0),\left(\dot{s}_4,0\right)\right) \\ - & - & - & \left((s_3,0),\left(\dot{s}_4,0\right)\right) \end{pmatrix}$$

$$R_4^* = \begin{pmatrix} \left((s_3,0),\left(\dot{s}_4,0\right)\right) & \left((s_3,-0.370),\left(\dot{s}_0,0\right)\right) & \left((s_5,-0.480),\left(\dot{s}_3,-0.48\right)\right) & \left((s_3,0),\left(\dot{s}_4,0\right)\right) \\ - & \left((s_3,0),\left(\dot{s}_4,0\right)\right) & \left((s_4,0),\left(\dot{s}_4,0\right)\right) & \left((s_4,-0.402),\left(\dot{s}_4,0\right)\right) \\ - & - & \left((s_3,0),\left(\dot{s}_4,0\right)\right) & \left((s_3,-0.309),\left(\dot{s}_1,1.148\right)\right) \\ - & - & - & \left((s_3,0),\left(\dot{s}_4,0\right)\right) \end{pmatrix}$$

$$R_5^* = \begin{pmatrix} \left((s_3,0),\left(\dot{s}_4,0\right)\right) & \left((s_2,0),\left(\dot{s}_4,0\right)\right) & \left((s_4,0),\left(\dot{s}_4,0\right)\right) & \left((s_4,0),\left(\dot{s}_4,0\right)\right) \\ - & \left((s_3,0),\left(\dot{s}_4,0\right)\right) & \left((s_4,0),\left(\dot{s}_4,0\right)\right) & \left((s_5,0),\left(\dot{s}_4,0\right)\right) \\ - & - & \left((s_3,0),\left(\dot{s}_4,0\right)\right) & \left((s_6,0),\left(\dot{s}_4,0\right)\right) \\ - & - & - & \left((s_3,0),\left(\dot{s}_4,0\right)\right) \end{pmatrix}$$

$$R_6^* = \begin{pmatrix} \left((s_3,0),\left(\dot{s}_4,0\right)\right) & \left((s_3,-0.370),\left(\dot{s}_3,0.437\right)\right) & \left((s_4,0),\left(\dot{s}_3,0\right)\right) & \left((s_4,-0.484),\left(\dot{s}_3,0.447\right)\right) \\ - & \left((s_3,0),\left(\dot{s}_4,0\right)\right) & \left((s_4,-0.283),\left(\dot{s}_3,0.206\right)\right) & \left((s_3,0.359),\left(\dot{s}_3,0.237\right)\right) \\ - & - & \left((s_3,0),\left(\dot{s}_4,0\right)\right) & \left((s_3,-0.093),\left(\dot{s}_3,-0.145\right)\right) \\ - & - & - & \left((s_3,0),\left(\dot{s}_4,0\right)\right) \end{pmatrix}$$

$$R_7^* = \begin{pmatrix} \left((s_3,0),\left(\dot{s}_4,0\right)\right) & \left((s_1,0),\left(\dot{s}_4,0\right)\right) & \left((s_5,0),\left(\dot{s}_4,0\right)\right) & \left((s_6,0),\left(\dot{s}_4,0\right)\right) \\ - & \left((s_0,0),\left(\dot{s}_4,0\right)\right) & \left((s_6,0),\left(\dot{s}_4,0\right)\right) & \left((s_6,0),\left(\dot{s}_4,0\right)\right) \\ - & - & \left((s_3,0),\left(\dot{s}_4,0\right)\right) & \left((s_1,0),\left(\dot{s}_4,0\right)\right) \\ - & - & - & \left((s_3,0),\left(\dot{s}_4,0\right)\right) \end{pmatrix}$$

$$R_8^* = \begin{pmatrix} \left((s_3,0),\left(\dot{s}_4,0\right)\right) & \left((s_3,0),\left(\dot{s}_4,0\right)\right) & \left((s_5,-0.478),\left(\dot{s}_3,-0.478\right)\right) & \left((s_4,-0.484),\left(\dot{s}_3,-0.733\right)\right) \\ - & \left((s_3,0),\left(\dot{s}_4,0\right)\right) & \left((s_4,0),\left(\dot{s}_4,0\right)\right) & \left((s_3,0.359),\left(\dot{s}_0,0\right)\right) \\ - & - & \left((s_3,0),\left(\dot{s}_4,0\right)\right) & \left((s_3,-0.308),\left(\dot{s}_4,0\right)\right) \\ - & - & - & \left((s_3,0),\left(\dot{s}_4,0\right)\right) \end{pmatrix}$$

$$R_9^* = \begin{pmatrix} \left((s_3,0),\left(\dot{s}_4,0\right)\right) & \left((s_5,0),\left(\dot{s}_4,0\right)\right) & \left((s_5,0),\left(\dot{s}_4,0\right)\right) & \left((s_6,0),\left(\dot{s}_4,0\right)\right) \\ - & \left((s_3,0),\left(\dot{s}_4,0\right)\right) & \left((s_1,0),\left(\dot{s}_4,0\right)\right) & \left((s_0,0),\left(\dot{s}_4,0\right)\right) \\ - & - & \left((s_3,0),\left(\dot{s}_4,0\right)\right) & \left((s_2,0),\left(\dot{s}_4,0\right)\right) \\ - & - & - & \left((s_3,0),\left(\dot{s}_4,0\right)\right) \end{pmatrix}$$

$$R_{10}^* = \begin{pmatrix} \left((s_3,0),\left(\dot{s}_4,0\right)\right) & \left((s_3,0),\left(\dot{s}_4,0\right)\right) & \left((s_4,0),\left(\dot{s}_4,0\right)\right) & \left((s_5,0),\left(\dot{s}_4,0\right)\right) \\ - & \left((s_3,0),\left(\dot{s}_4,0\right)\right) & \left((s_4,0),\left(\dot{s}_4,0\right)\right) & \left((s_4,0),\left(\dot{s}_4,0\right)\right) \\ - & - & \left((s_3,0),\left(\dot{s}_4,0\right)\right) & \left((s_5,0),\left(\dot{s}_4,0\right)\right) \\ - & - & - & \left((s_3,0),\left(\dot{s}_4,0\right)\right) \end{pmatrix}$$

$$R_{11}^{*}=\begin{pmatrix} \left((s_3,0),\left(\dot{s}_4,0\right)\right) & \left((s_3,0),\left(\dot{s}_4,0\right)\right) & \left((s_6,0),\left(\dot{s}_4,0\right)\right) & \left((s_6,0),\left(\dot{s}_4,0\right)\right) \\ - & \left((s_3,0),\left(\dot{s}_4,0\right)\right) & \left((s_4,0),\left(\dot{s}_4,0\right)\right) & \left((s_4,0),\left(\dot{s}_4,0\right)\right) \\ - & - & \left((s_3,0),\left(\dot{s}_4,0\right)\right) & \left((s_3,0),\left(\dot{s}_4,0\right)\right) \\ - & - & - & \left((s_3,0),\left(\dot{s}_4,0\right)\right) \end{pmatrix}$$

$$R_{12}^{*}=\begin{pmatrix} \left((s_3,0),\left(\dot{s}_4,0\right)\right) & \left((s_3,0.061),\left(\dot{s}_3,-0.428\right)\right) & \left((s_4,0),\left(\dot{s}_4,0\right)\right) & \left((s_4,-0.484),\left(\dot{s}_2,0.267\right)\right) \\ - & \left((s_3,0),\left(\dot{s}_4,0\right)\right) & \left((s_4,0),\left(\dot{s}_3,-0.108\right)\right) & \left((s_3,0.359),\left(\dot{s}_0,0\right)\right) \\ - & - & \left((s_3,0),\left(\dot{s}_4,0\right)\right) & \left((s_3,-0.308),\left(\dot{s}_4,0\right)\right) \\ - & - & - & \left((s_3,0),\left(\dot{s}_4,0\right)\right) \end{pmatrix}$$

$$R_{13}^{*}=\begin{pmatrix} \left((s_3,0),\left(\dot{s}_4,0\right)\right) & \left((s_3,0.061),\left(\dot{s}_1,0.051\right)\right) & \left((s_5,-0.478),\left(\dot{s}_3,-0.478\right)\right) & \left((s_4,-0.484),\left(\dot{s}_2,0.267\right)\right) \\ - & \left((s_3,0),\left(\dot{s}_4,0\right)\right) & \left((s_4,0),\left(\dot{s}_4,0\right)\right) & \left((s_3,0.359),\left(\dot{s}_0,0\right)\right) \\ - & - & \left((s_3,0),\left(\dot{s}_4,0\right)\right) & \left((s_3,-0.308),\left(\dot{s}_1,0.145\right)\right) \\ - & - & - & \left((s_3,0),\left(\dot{s}_4,0\right)\right) \end{pmatrix}$$

$$R_{14}^{*}=\begin{pmatrix} \left((s_3,0),\left(\dot{s}_4,0\right)\right) & \left((s_3,0),\left(\dot{s}_2,0.479\right)\right) & \left((s_4,0),\left(\dot{s}_1,0\right)\right) & \left((s_4,-0.5),\left(\dot{s}_0,0\right)\right) \\ - & \left((s_3,0),\left(\dot{s}_4,0\right)\right) & \left((s_4,-0.389),\left(\dot{s}_0,0\right)\right) & \left((s_3,0.359),\left(\dot{s}_1,0.191\right)\right) \\ - & - & \left((s_3,0),\left(\dot{s}_4,0\right)\right) & \left((s_3,-0.308),\left(\dot{s}_1,0.145\right)\right) \\ - & - & - & \left((s_3,0),\left(\dot{s}_4,0\right)\right) \end{pmatrix}$$

$$R_{15}^{*}=\begin{pmatrix} \left((s_3,0),\left(\dot{s}_4,0\right)\right) & \left((s_3,-0.370),\left(\dot{s}_3,0.042\right)\right) & \left((s_4,0),\left(\dot{s}_0,0\right)\right) & \left((s_3,0),\left(\dot{s}_3,-0.143\right)\right) \\ - & \left((s_3,0),\left(\dot{s}_4,0\right)\right) & \left((s_4,-0.496),\left(\dot{s}_1,0.226\right)\right) & \left((s_3,0.241),\left(\dot{s}_4,-0.287\right)\right) \\ - & - & \left((s_3,0),\left(\dot{s}_4,0\right)\right) & \left((s_3,-0.093),\left(\dot{s}_0,0\right)\right) \\ - & - & - & \left((s_3,0),\left(\dot{s}_4,0\right)\right) \end{pmatrix}$$

$$R_{16}^{*}=\begin{pmatrix} \left((s_3,0),\left(\dot{s}_4,0\right)\right) & \left((s_1,0),\left(\dot{s}_4,0\right)\right) & \left((s_4,0),\left(\dot{s}_4,0\right)\right) & \left((s_0,0),\left(\dot{s}_4,0\right)\right) \\ - & \left((s_3,0),\left(\dot{s}_4,0\right)\right) & \left((s_5,0),\left(\dot{s}_4,0\right)\right) & \left((s_3,0),\left(\dot{s}_4,0\right)\right) \\ - & - & \left((s_3,0),\left(\dot{s}_4,0\right)\right) & \left((s_2,0),\left(\dot{s}_4,0\right)\right) \\ - & - & - & \left((s_3,0),\left(\dot{s}_4,0\right)\right) \end{pmatrix}$$

$$R_{17}^{*}=\begin{pmatrix} \left((s_3,0),\left(\dot{s}_4,0\right)\right) & \left((s_6,0),\left(\dot{s}_4,0\right)\right) & \left((s_4,0),\left(\dot{s}_4,0\right)\right) & \left((s_3,0),\left(\dot{s}_4,0\right)\right) \\ - & \left((s_3,0),\left(\dot{s}_4,0\right)\right) & \left((s_2,0),\left(\dot{s}_4,0\right)\right) & \left((s_2,0),\left(\dot{s}_4,0\right)\right) \\ - & - & \left((s_3,0),\left(\dot{s}_4,0\right)\right) & \left((s_5,0),\left(\dot{s}_4,0\right)\right) \\ - & - & - & \left((s_3,0),\left(\dot{s}_4,0\right)\right) \end{pmatrix}$$

$$R_{18}^{*}=\begin{pmatrix} \left((s_3,0),\left(\dot{s}_4,0\right)\right) & \left((s_5,0),\left(\dot{s}_4,0\right)\right) & \left((s_5,0),\left(\dot{s}_4,0\right)\right) & \left((s_3,0),\left(\dot{s}_4,0\right)\right) \\ - & \left((s_3,0),\left(\dot{s}_4,0\right)\right) & \left((s_2,0),\left(\dot{s}_4,0\right)\right) & \left((s_0,0),\left(\dot{s}_4,0\right)\right) \\ - & - & \left((s_3,0),\left(\dot{s}_4,0\right)\right) & \left((s_5,0),\left(\dot{s}_4,0\right)\right) \\ - & - & - & \left((s_3,0),\left(\dot{s}_4,0\right)\right) \end{pmatrix}$$

$$R_{19}^{*}=\begin{pmatrix} \left((s_3,0),\left(\dot{s}_4,0\right)\right) & \left((s_3,-0.370),\left(\dot{s}_3,0.042\right)\right) & \left((s_4,0),\left(\dot{s}_1,0\right)\right) & \left((s_5,0),\left(\dot{s}_4,0\right)\right) \\ - & \left((s_3,0),\left(\dot{s}_4,0\right)\right) & \left((s_4,0),\left(\dot{s}_2,-0.215\right)\right) & \left((s_4,0),\left(\dot{s}_4,0\right)\right) \\ - & - & \left((s_3,0),\left(\dot{s}_4,0\right)\right) & \left((s_3,0),\left(\dot{s}_0,0.266\right)\right) \\ - & - & - & \left((s_3,0),\left(\dot{s}_4,0\right)\right) \end{pmatrix}$$

$$R_{20}^{*}=\begin{pmatrix} \left((s_3,0),\left(\dot{s}_4,0\right)\right) & \left((s_3,0.061),\left(\dot{s}_1,0.051\right)\right) & \left((s_4,0),\left(\dot{s}_0,0\right)\right) & \left((s_4,-0.484),\left(\dot{s}_1,0.125\right)\right) \\ - & \left((s_3,0),\left(\dot{s}_4,0\right)\right) & \left((s_4,-0.283),\left(\dot{s}_2,0.098\right)\right) & \left((s_3,0.359),\left(\dot{s}_0,0\right)\right) \\ - & - & \left((s_3,0),\left(\dot{s}_4,0\right)\right) & \left((s_3,-0.093),\left(\dot{s}_3,-0.144\right)\right) \\ - & - & - & \left((s_3,0),\left(\dot{s}_4,0\right)\right) \end{pmatrix}$$

Appendix B. The adjusted preferences obtained from model (28) are as follows

$$\overline{R}_1=\begin{pmatrix} (s_3,0) & (s_3,0) & (s_5,0) & (s_3,0) \\ - & (s_3,0) & (s_4,0) & (s_3,0) \\ - & - & (s_3,0) & (s_2,0) \\ - & - & - & (s_3,0) \end{pmatrix} \quad \overline{R}_2=\begin{pmatrix} (s_3,0) & (s_1,0) & (s_2,0) & (s_4,0) \\ - & (s_3,0) & (s_3,0) & (s_4,0) \\ - & - & (s_3,0) & (s_4,0) \\ - & - & - & (s_3,0) \end{pmatrix}$$

$$\overline{R}_3=\begin{pmatrix} (s_3,0) & (s_4,-0.053) & (s_4,-0.295) & (s_5,-0.002) \\ - & (s_3,0) & (s_4,-0.107) & (s_2,-0.042) \\ - & - & (s_3,0) & (s_2,-0.233) \\ - & - & - & (s_3,0) \end{pmatrix} \quad \overline{R}_4=\begin{pmatrix} (s_3,0) & (s_0,0) & (s_6,0) & (s_3,0) \\ - & (s_3,0) & (s_4,0) & (s_4,0) \\ - & - & (s_3,0) & (s_1,0) \\ - & - & - & (s_3,0) \end{pmatrix}$$

$$\overline{R}_5=\begin{pmatrix} (s_3,0) & (s_2,-0.179) & (s_3,-0.156) & (s_3,-0.160) \\ - & (s_3,0) & (s_4,-0.024) & (s_3,-0.210) \\ - & - & (s_3,0) & (s_6,-0.005) \\ - & - & - & (s_3,0) \end{pmatrix} \quad \overline{R}_6=\begin{pmatrix} (s_3,0) & (s_3,-0.049) & (s_3,-0.033) & (s_4,-0.194) \\ - & (s_3,0) & (s_3,-0.081) & (s_4,-0.184) \\ - & - & (s_3,0) & (s_4,-0.192) \\ - & - & - & (s_3,0) \end{pmatrix}$$

$$\overline{R}_7=\begin{pmatrix} (s_3,0) & (s_1,-0.233) & (s_2,-0.380) & (s_6,0) \\ - & (s_3,0) & (s_5,0.09) & (s_6,0) \\ - & - & (s_3,0) & (s_1,-0.206) \\ - & - & - & (s_3,0) \end{pmatrix} \quad \overline{R}_8=\begin{pmatrix} (s_3,0) & (s_4,-0.354) & (s_6,0) & (s_2,0.103) \\ - & (s_3,0) & (s_4,0.046) & (s_0,0) \\ - & - & (s_3,0) & (s_2,0.071) \\ - & - & - & (s_3,0) \end{pmatrix}$$

$$\overline{R}_9=\begin{pmatrix} (s_3,0) & (s_5,0) & (s_5,0) & (s_6,0) \\ - & (s_3,0) & (s_1,0) & (s_0,0) \\ - & - & (s_3,0) & (s_2,0) \\ - & - & - & (s_3,0) \end{pmatrix} \quad \overline{R}_{10}=\begin{pmatrix} (s_3,0) & (s_3,-0.003) & (s_3,0.082) & (s_3,0.081) \\ - & (s_3,0) & (s_3,0.080) & (s_3,0.080) \\ - & - & (s_3,0) & (s_5,-0.054) \\ - & - & - & (s_3,0) \end{pmatrix}$$

$$\overline{R}_{11}=\begin{pmatrix} (s_3,0) & (s_3,0) & (s_4,-0.214) & (s_4,-0.146) \\ - & (s_3,0) & (s_4,-0.001) & (s_4,-0.007) \\ - & - & (s_3,0) & (s_2,-0.364) \\ - & - & - & (s_3,0) \end{pmatrix} \quad \overline{R}_{12}=\begin{pmatrix} (s_3,0) & (s_4,0.014) & (s_4,0.058) & (s_2,0.116) \\ - & (s_3,0) & (s_5,0.046) & (s_0,0) \\ - & - & (s_3,0) & (s_3,-0.367) \\ - & - & - & (s_3,0) \end{pmatrix}$$

$$\overline{R}_{13}=\begin{pmatrix} (s_3,0) & (s_5,0) & (s_6,0) & (s_2,-0.024) \\ - & (s_3,0) & (s_4,0) & (s_0,0) \\ - & - & (s_3,0) & (s_1,0) \\ - & - & - & (s_3,0) \end{pmatrix} \quad \overline{R}_{14}=\begin{pmatrix} (s_3,0) & (s_6,0) & (s_3,-0.470) & (s_0,0) \\ - & (s_3,0) & (s_0,0) & (s_3,0.340) \\ - & - & (s_3,0) & (s_6,0) \\ - & - & - & (s_3,0) \end{pmatrix}$$

$$\overline{R}_{15}=\begin{pmatrix} (s_3,0) & (s_2,0.168) & (s_0,0) & (s_2,0.167) \\ - & (s_3,0) & (s_5,-0.486) & (s_3,0.018) \\ - & - & (s_3,0) & (s_6,0) \\ - & - & - & (s_3,0) \end{pmatrix} \quad \overline{R}_{16}=\begin{pmatrix} (s_3,0) & (s_1,0.187) & (s_4,0.272) & (s_0,0) \\ - & (s_3,0) & (s_6,0) & (s_3,3.418) \\ - & - & (s_3,0) & (s_3,-0.010) \\ - & - & - & (s_3,0) \end{pmatrix}$$

$$\overline{R}_{17}=\begin{pmatrix} (s_3,0) & (s_5,0.459) & (s_3,0.227) & (s_3,-0.126) \\ - & (s_3,0) & (s_2,-0.085) & (s_2,-0.086) \\ - & - & (s_3,0) & (s_5,-0.123) \\ - & - & - & (s_3,0) \end{pmatrix} \quad \overline{R}_{18}=\begin{pmatrix} (s_3,0) & (s_5,0) & (s_5,0) & (s_3,0) \\ - & (s_3,0) & (s_2,0) & (s_0,0) \\ - & - & (s_3,0) & (s_5,0) \\ - & - & - & (s_3,0) \end{pmatrix}$$

$$\overline{R}_{19}=\begin{pmatrix} (s_3,0) & (s_0,0) & (s_1,-0.154) & (s_5,-0.056) \\ - & (s_3,0) & (s_6,0) & (s_4,-0.457) \\ - & - & (s_3,0) & (s_5,-0.066) \\ - & - & - & (s_3,0) \end{pmatrix} \quad \overline{R}_{20}=\begin{pmatrix} (s_3,0) & (s_6,0) & (s_1,-0.406) & (s_1,0.228) \\ - & (s_3,0) & (s_3,-0.124) & (s_1,0.226) \\ - & - & (s_3,0) & (s_6,-0.057) \\ - & - & - & (s_3,0) \end{pmatrix}$$

where $\overline{R}_k = R_k$ for $k = 1,2,4,9,13,18$.

Appendix C. The accepting recommendation advices solved by model (29) are as follows.

$$\overline{R}_4 = \begin{pmatrix} (s_3,0) & (s_1,0.472) & (s_5,-0.173) & (s_3,0.109) \\ - & (s_3,0) & (s_4,0.401) & (s_1,0.469) \\ - & - & (s_3,0) & (s_2,0.021) \\ - & - & - & (s_3,0) \end{pmatrix} \quad \overline{R}_5 = \begin{pmatrix} (s_3,0) & (s_3,-0.463) & (s_4,-0.242) & (s_4,-0.358) \\ - & (s_3,0) & (s_4,-0.401) & (s_4,-0.107) \\ - & - & (s_3,0) & (s_5,-0.317) \\ - & - & - & (s_3,0) \end{pmatrix}$$

$$\overline{R}_7 = \begin{pmatrix} (s_3,0) & (s_2,0.004) & (s_4,0.291) & (s_5,-0.294) \\ - & (s_3,0) & (s_5,-0.337) & (s_4,0.425) \\ - & - & (s_3,0) & (s_2,0.021) \\ - & - & - & (s_3,0) \end{pmatrix} \quad \overline{R}_8 = \begin{pmatrix} (s_3,0) & (s_3,0.069) & (s_5,-0.177) & (s_3,-0.423) \\ - & (s_3,0) & (s_4,-0.401) & (s_1,0.230) \\ - & - & (s_3,0) & (s_3,-0.447) \\ - & - & - & (s_3,0) \end{pmatrix}$$

$$\overline{R}_9 = \begin{pmatrix} (s_3,0) & (s_4,0.134) & (s_4,0.290) & (s_4,-0.294) \\ - & (s_3,0) & (s_2,0.001) & (s_1,0.230) \\ - & - & (s_3,0) & (s_3,-0.447) \\ - & - & - & (s_3,0) \end{pmatrix} \quad \overline{R}_{11} = \begin{pmatrix} (s_3,0) & (s_3,0.069) & (s_5,-0.177) & (s_5,-0.294) \\ - & (s_3,0) & (s_4,-0.401) & (s_3,0.360) \\ - & - & (s_3,0) & (s_3,0.086) \\ - & - & - & (s_3,0) \end{pmatrix}$$

$$\overline{R}_{12} = \begin{pmatrix} (s_3,0) & (s_4,-0.399) & (s_4,-0.242) & (s_3,-0.423) \\ - & (s_3,0) & (s_4,0.131) & (s_1,0.230) \\ - & - & (s_3,0) & (s_3,-0.447) \\ - & - & - & (s_3,0) \end{pmatrix} \quad \overline{R}_{13} = \begin{pmatrix} (s_3,0) & (s_4,0.134) & (s_5,-0.177) & (s_3,-0.423) \\ - & (s_3,0) & (s_4,-0.401) & (s_1,0.230) \\ - & - & (s_3,0) & (s_2,0.021) \\ - & - & - & (s_3,0) \end{pmatrix}$$

$$\overline{R}_{14} = \begin{pmatrix} (s_3,0) & (s_4,-0.399) & (s_2,0.161) & (s_2,-0.488) \\ - & (s_3,0) & (s_1,0.469) & (s_2,-0.237) \\ - & - & (s_3,0) & (s_2,0.021) \\ - & - & - & (s_3,0) \end{pmatrix} \quad \overline{R}_{15} = \begin{pmatrix} (s_3,0) & (s_3,-0.463) & (s_2,-0.444) & (s_3,-0.423) \\ - & (s_3,0) & (s_2,0.001) & (s_3,-0.172) \\ - & - & (s_3,0) & (s_4,0.150) \\ - & - & - & (s_3,0) \end{pmatrix}$$

$$\overline{R}_{16} = \begin{pmatrix} (s_3,0) & (s_2,0.004) & (s_4,-0.242) & (s_2,-0.488) \\ - & (s_3,0) & (s_4,0.131) & (s_3,-0.172) \\ - & - & (s_3,0) & (s_3,-0.447) \\ - & - & - & (s_3,0) \end{pmatrix} \quad \overline{R}_{18} = \begin{pmatrix} (s_3,0) & (s_4,0.134) & (s_4,0.290) & (s_3,0.109) \\ - & (s_3,0) & (s_3,-0.466) & (s_1,0.230) \\ - & - & (s_3,0) & (s_4,0.150) \\ - & - & - & (s_3,0) \end{pmatrix}$$

$$\overline{R}_{19} = \begin{pmatrix} (s_3,0) & (s_3,-0.463) & (s_2,0.161) & (s_4,0.174) \\ - & (s_3,0) & (s_5,-0.337) & (s_3,0.360) \\ - & - & (s_3,0) & (s_4,0.150) \\ - & - & - & (s_3,0) \end{pmatrix} \quad \overline{R}_{20} = \begin{pmatrix} (s_3,0) & (s_4,0.134) & (s_2,-0.362) & (s_2,0.044) \\ - & (s_3,0) & (s_3,-0.466) & (s_2,-0.237) \\ - & - & (s_3,0) & (s_4,-0.362) \\ - & - & - & (s_3,0) \end{pmatrix}$$

where $\overline{R}_k = R_k$ for $k = 1, 2, 3, 6, 10, 17$.

References

Ben-Arieh, D., & Chen, Z. (2006). Linguistic-labels aggregation and consensus measure for autocratic decision making using group recommendations. *IEEE Transactions on Systems Man and Cybernetics Part A-Systems and Humans, 36*(3), 558–568.

Ben-Arieh, D., & Easton, T. (2007). Multi-criteria group consensus under linear cost opinion elasticity. *Decision Support Systems, 43*(3), 713–721.

Ben-Arieh, D., Easton, T., & Evans, B. (2009). Minimum cost consensus with quadratic cost functions. *IEEE Transactions on Systems Man and Cybernetics Part A-Systems and Humans, 39*(1), 210–217.

Chen, X. H., & Liu, R. (2006). Improved clustering algorithm and its application in complex huge group decision-making. *Systems Engineering and Electronics, 28*(11), 1695–1699.

Cheng, D., Zhou, Z. L., Cheng, F. X., Zhou, Y. F., & Xie, Y. J. (2018). Modeling the minimum cost consensus problem in an asymmetric costs context. *European Journal of Operational Research, 270*(3), 1122–1137.

Dong, Q. X., Zhou, X., & Martínez, L. (2019). A hybrid group decision making framework for achieving agreed solutions based on stable opinions. *Information Sciences, 490*, 227–243.

Dong, Y. C., Xu, Y. F., Li, H. Y., & Feng, B. (2010). The OWA-based consensus operator under linguistic representation models using position indexes. *European Journal of Operational Research, 203*(2), 455–463.

Dong, Y. C., & Xu, J. P. (2016). *Consensus building in group decision making*. Singapore: Springer.

Dong, Y. C., Zhao, S. H., Zhang, H. J., Chiclana, F., & Herrera-Viedma, E. (2018). A self-management mechanism for noncooperative behaviors in large-scale group consensus reaching processes. *IEEE Transactions on Fuzzy Systems, 26*(6), 3276–3288.

Fu, C., & Yang, S. L. (2012). An evidential reasoning based consensus model for multiple attribute group decision analysis problems with interval-valued group consensus requirements. *European Journal of Operational Research, 223*(1), 167–176.

Gong, Z. W., Guo, W. W., Herrera-Viedma, E., Gong, Z. J., & Wei, G. (2020). Consistency and consensus modeling of linear uncertain preference relations. *European Journal of Operational Research, 283*(1), 290–307.

Herrera-Viedma, E., Cabrerizo, F. J., Kacprzyk, J., & Pedrycz, W. (2014). A review of soft consensus models in a fuzzy environment. *Information Fusion, 17*, 4–13.

Herrera-Viedma, E., Martinez, L., Mata, F., & Chiclana, F. (2005). A consensus support system model for group decision-making problems with multigranular linguistic preference relations. *IEEE Transactions on fuzzy Systems, 13*(5), 644–658.

Herrera, F., & Martínez, L. (2000). A 2-tuple fuzzy linguistic representation model for computing with words. *IEEE Transactions on Fuzzy Systems, 8*(6), 746–752.

Kacprzyk, J., & Fedrizzi, M. (1988). A 'soft'measure of consensus in the setting of partial (fuzzy) preferences. *European Journal of Operational Research, 34*(3), 316–325.

Kacprzyk, J., Fedrizzi, M., & Nurmi, H. (1992). Group decision making and consensus under fuzzy preferences and fuzzy majority. *Fuzzy Sets and Systems, 49*(1), 21–31.

Labella, Á., Liu, H. B., Rodríguez, R. M., & Martínez, L. (2020). A cost consensus metric for consensus reaching processes based on a comprehensive minimum cost model. *European Journal of Operational Research, 281*(2), 316–331.

Labella, Á., Liu, Y. Y., Rodríguez, R. M., & Martínez, L. (2018). Analyzing the performance of classical consensus models in large scale group decision making: A comparative study. *Applied Soft Computing, 67*, 677–690.

Li, C.-C., Dong, Y. C., Herrera, F., Herrera-Viedma, E., & Martínez, L. (2017). Personalized individual semantics in computing with words for supporting linguistic group decision making. An application on consensus reaching. *Information Fusion, 33*, 29–40.

Li, C.-C., Rodríguez, R. M., Martínez, L., Dong, Y., & Herrera, F. (2018). Consistency of hesitant fuzzy linguistic preference relations: An interval consistency index. *Information Sciences, 432*, 347–361.

Liu, P. D., & Yu, X. C. (2014). 2-Dimension uncertain linguistic power generalized weighted aggregation operator and its application in multiple attribute group decision making. *Knowledge-based Systems, 57*, 69–80.

Liu, W. Q., Dong, Y. C., Chiclana, F., Cabrerizo, F. J., & Herrera-Viedma, E. (2017). Group decision-making based on heterogeneous preference relations with self-confidence. *Fuzzy Optimization and Decision Making, 16*(4), 429–447.

Liu, X., Xu, Y. J., Montes, R., Ding, R.-X., & Herrera, F. (2019). Alternative ranking-based clustering and reliability index-based consensus reaching process for hesitant fuzzy large scale group decision making. *IEEE Transactions on Fuzzy Systems, 27*(1), 159–171.

Martínez, L., & Herrera, F. (2012). An overview on the 2-tuple linguistic model for computing with words in decision making: Extensions, applications and challenges. *Information Sciences, 207*, 1–18.

Miller, G. A. (1956). The magical number seven, plus or minus two: Some limits on our capacity for processing information. *Psychological review, 63*(2), 81–97.

Palomares, I., Estrella, F. J., Martínez, L., & Herrera, F. (2014). Consensus under a fuzzy context: Taxonomy, analysis framework AFRYCA and experimental case of study. *Information Fusion, 20*, 252–271.

Palomares, I., & Martínez, L. (2014). A semisupervised multiagent system model to support consensus-reaching processes. *IEEE Transactions on Fuzzy Systems, 22*(4), 762–777.

Palomares, I., Martinez, L., & Herrera, F. (2013). A consensus model to detect and manage noncooperative behaviors in large-scale group decision making. *IEEE Transactions on Fuzzy Systems, 22*(3), 516–530.

Palomares, I., Rodríguez, R. M., & Martínez, L. (2013). An attitude-driven web consensus support system for heterogeneous group decision making. *Expert Systems with Applications, 40*, 139–149.

Rodríguez, R. M., Labella, Á., De Tré, G., & Martínez, L. (2018). A large scale consensus reaching process managing group hesitation. *Knowledge-Based Systems, 159*, 86–97.

Wang, L., Wang, Y.-M., & Martínez, L. (2017). A group decision method based on prospect theory for emergency situations. *Information Sciences, 418*, 119–135.

Wu, Dai, L. F., Chiclana, F., Fujita, H., & Herrera-Viedma, E. (2018). A minimum adjustment cost feedback mechanism based consensus model for group decision making under social network with distributed linguistic trust. Information Fusion, 41, 232-242.

Wu, Y. N., Xu, C. B., Li, L. W., Wang, Y., Chen, K. F., & Xu, R. H. (2018). A risk assessment framework of PPP waste-to-energy incineration projects in China under 2-dimension linguistic environment. *Journal of Cleaner Production, 183*, 602–617.

Wu, Z. B., & Xu, J. P. (2012). A consistency and consensus based decision support model for group decision making with multiplicative preference relations. *Decision Support Systems, 52*(3), 757–767.

Wu, Z. B., & Xu, J. P. (2018). A consensus model for large-scale group decision making with hesitant fuzzy information and changeable clusters. *Information Fusion, 41*, 217–231.

Xu, Y. J., Patnayakuni, R., & Wang, H. M. (2013). The ordinal consistency of a fuzzy preference relation. *Information Sciences, 224*, 152–164.

Yu, L., & Lai, K. K. (2011). A distance-based group decision-making methodology for multi-person multi-criteria emergency decision support. *Decision Support Systems, 51* (2), 307–315.

Yu, X. H., Xu, Z. S., Liu, S. S., & Chen, Q. (2012). Multicriteria decision making with 2-dimension linguistic aggregation techniques. *International Journal of Intelligent Systems, 27*(6), 539–562.

Zadeh, L. A. (1975a). The concept of a linguistic variable and its application to approximate reasoning—I. *Information sciences, 8*(3), 199–249.

Zadeh, L. A. (1975b). The concept of a linguistic variable and its application to approximate reasoning—II. *Information Sciences, 8*(4), 301–357.

Zadeh, L. A. (1975c). The concept of a linguistic variable and its application to approximate reasoning—III. *Information Sciences, 9*(1), 43–80.

Zadeh, L. A. (1983). A computational approach to fuzzy quantifiers in natural languages. *Computers and Mathematics with Applications, 9*, 149–184.

Zhang, B. W., & Dong, Y. C. (2013). Consensus rules with minimum adjustments for multiple attribute group decision making. *Procedia Computer Science, 17*, 473–481.

Zhang, B. W., Dong, Y. C., & Xu, Y. F. (2013). Maximum expert consensus models with linear cost function and aggregation operators. *Computers & Industrial Engineering, 66* (1), 147–157.

Zhang, B. W., Liang, H. M., & Zhang, G. Q. (2018). Reaching a consensus with minimum adjustment in MAGDM with hesitant fuzzy linguistic term sets. *Information Fusion, 42*, 12–23.

Zhang, G., Dong, Y., & Xu, Y. (2012). Linear optimization modeling of consistency issues in group decision making based on fuzzy preference relations. *Expert Systems with Applications, 39*(3), 2415–2420.

Zhang, G. Q., Dong, Y. C., Xu, Y. F., & Li, H. Y. (2011). Minimum-cost consensus models under aggregation operators. *IEEE Transactions on Systems, Man, and Cybernetics-Part A: Systems and Humans, 41*(6), 1253–1261.

Zhang, H. H., Kou, G., & Peng, Y. (2019). Soft consensus cost models for group decision making and economic interpretations. *European Journal of Operational Research, 277* (3), 964–980.

Zhang, H. J., Dong, Y. C., & Herrera-Viedma, E. (2017). Consensus building for the heterogeneous large-scale GDM with the individual concerns and satisfactions. *IEEE Transactions on Fuzzy Systems, 26*(2), 884–898.

Zhang, H. J., Zhao, S. H., Kou, G., Li, C.-C., Dong, Y. C., & Herrera, F. (2020). An overview on feedback mechanisms with minimum adjustment or cost in consensus reaching in group decision making: Research paradigms and challenges. *Information Fusion, 60*, 65–79.

Zhu, H., Zhao, J. B., & Xu, Y. (2016). 2-dimension linguistic computational model with 2-tuples for multi-attribute group decision making. *Knowledge-Based Systems, 103*, 132–142.

Zhu, W. D., Zhou, G. Z., & Yang, S. L. (2009). An approach to group decision making based on 2-dimension linguistic assessment information. *Systems engineering, 27*(2), 113–118.

4.3 A CRP with minimum adjustment in GDM considering the tolerance of DMs for changing their opinions

- State: Published.
- Title: A consensus reaching process with minimum adjustment in group decision making with two-dimensional 2-tuple linguistic information based on reliability measurement.
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- Journal: 2021 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE) Luxembourg, 2021, 11^{th}-14^{th}, July.
- DOI:10.1109/FUZZ45933.2021.9494397
- Quality indicator:

  Ranking Core: Core A

A Consensus Reaching Process with Minimum Adjustment in Group Decision Making with Two-dimensional 2-tuple Linguistic Information based on Reliability Measurement

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2021 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE) | 978-1-6654-4407-1/21/$31.00 ©2021 IEEE | DOI: 10.1109/FUZZ45933.2021.9494397

***Abstract*—This paper designs a consensus reaching process (CRP) with minimum adjustment considering that decision makers (DMs) have a tolerance for changing their preferences, which means DMs only accept the adjustment if such adjustment fall within their tolerance interval. In addition, two consensus rules are considered: minimum distance between original and adjusted preferences, and minimum number of adjusted preferences. Unlike to classical CRP, our proposal includes a reliability detection of adjusted preferences, in order to avoid biased solutions because of the inherent subjectivity of group decision-making problems. To obtain the agreed adjusted opinions and their reliability, the two-dimensional 2-tuple linguistic (TD2L) information is used in which, the first dimension of a TD2L is 2-tuple linguistic value that assess the adjusted preference and, the second dimension information is another 2-tuple linguistic value assessing the reliability degree of the adjusted preferences, which is derived according to similarity between adjusted and original preferences. Then a novel comparison method for TD2Ls is developed. Finally, an illustrative example is given to verify the proposed method and the results show that the approach is feasible and effective.**

Keywords*—*consensus reaching process, minimum adjustment, group decision making, reliability degree, tolerance degree, two-dimensional 2-tuple

I. INTRODUCTION

Group decision making (GDM) refers to the selection of the best alternative(s) from a set of feasible alternatives according to the preferences of different decision makers (DMs). Due to the complexity and uncertainty of objective reality and the ambiguity of human thinking, Zadeh [1] introduced the fuzzy linguistic approach and fuzzy linguistic variables for eliciting linguistic information for assessing DM's preferences in real-life GDM, among the different fuzzy based linguistic models the 2-tuple linguistic model [2] has been one of the most useful and widely used linguistic representation model in recent years [3-5].

The 2-tuple linguistic model only considers the evaluation of the object and ignores the reliability of the evaluation, which may lead to inaccurate decision results. Hence, TD2L model based on the 2-tuple linguistic model was proposed by Zhu et al. [6], in which an additional linguistic value is added to assess the reliability of the subjective preference assessments. For TD2L the reliability assessment is a subjective one directly provided by experts [7], but this idea is not realistic in real-world decision problems, especially for guiding experts to reach an agreement for obtaining the solution [8-11] by modifying their initial preferences. There are different consensus models in which the modification of initial preferences is carried out by the DMs but others are automatic CRPs that adjust the initial preferences without DMs' feedback. Therefore, after an automatic CRP, the reliability of the adjusted preferences is different from the original ones, and it must be considered to know how valid and robust is the solution based on the adjusted preferences. Thus, an objective detection on the reliability of the adjusted preferences is necessary.

Classically many CRPs consider that the minimum distance between original preferences and the adjusted preferences is the key rule for achieving the agreement, but in classical minimum adjustment consensus model (MACM) [12] the number of adjusted preferences should be also considered. Zhang et al. [13] proposed an MACM with these two consensus rules, however, they are separately used in the consensus mechanism, which complicates consensus process. With the aim of simplifying such consensus process, an Algorithm is designed to ensure the minimum number of adjusted preferences with the minimum distance between original and adjusted preferences. In order to balance these two consensus rules, a DM tolerance degree that defines how much is willing the DM to change his original opinion will play an important role, which means DMs only accept the adjusted preferences within tolerance interval.

For solving previous problems, this paper designs a novel CRP with minimum adjustment based on the DMs' tolerance degree and develops a measurement of reliability of adjusted preferences. The final assessments of alternatives are modelled by TD2L expressions. Such a CRP-GDM method based on TD2Ls consists of the following aspects:

(1) Considering the DMs tolerance degree, a novel CRP with minimum adjustment based on two consensus rules is proposed, where the two consensus rules are: minimize the distance between the original and adjusted preferences, and minimize the number of the adjusted preferences.

(2) A measurement of reliability degree of adjusted preferences is discussed and then applied to GDM by using TD2L information.

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(3) A new comparison way for TD2Ls is proposed, then the alternatives ranking is derived from the comparison results.

The rest of this paper is arranged as follows. In Section Ⅱ, we present the basic knowledge regarding the 2-tuple linguistic model, TD2L expression and the classical MACM. In Section Ⅲ, a CRP with the consideration of tolerance degree of DMs is designed. In Section Ⅳ, the measurement of the reliability degree of the adjusted preferences, and a novel comparison rule for TD2L expressions are proposed. In Section Ⅴ, an illustrative example is provided to certificate the effectiveness of the proposed method. Section Ⅵ concludes this paper with final remarks.

II. PRELIMINARIES

This section provides a brief revision about necessary concepts in our proposal such as, the 2-tuple linguistic model, TD2L expressions and MACM.

A. The 2-tuple Linguistic Representation Model

Herrera and Martínez [2] proposed the 2-tuple linguistic model, where the linguistic information is presented by a 2-tuple.

Definition 1 [2]: Let $S=\{s_0,s_1,...,s_g\}$ be a linguistic term set and $\beta\in[0,g]$ be a value representing the results of a symbolic aggregation. The 2-tuple linguistic value (s_i,α) is equivalent to β through the function Δ as follows:

$$\Delta:[0,g]\rightarrow S\times[-0.5,0.5) \tag{1}$$

$$\Delta(\beta)=(s_i,\alpha),\text{ with }\begin{cases}s_i, i=round(\beta)\\ \alpha=\beta-i, \alpha\in[-0.5,0.5)\end{cases} \tag{2}$$

where $round(\cdot)$ is the usual round operation.

Obviously, Δ is a one to one mapping function. There exist a function Δ^{-1} from β to (s_i,α) as follows.

$$\Delta^{-1}:S\times[-0.5,0.5)\rightarrow[0,g] \tag{3}$$

$$\Delta^{-1}(s_i,\alpha)=i+\alpha=\beta \tag{4}$$

B. Two-Dimensional 2-tuple Linguistic Expression

The two-dimensional linguistic representation model was introduced by Zhu and Zhou [14], then Zhu et al. [6] extended it to TD2L expression:

Definition 2 [6]: Let $S=\{s_0,s_1,...,s_g\}$ and $\dot{S}=\{\dot{s}_0,\dot{s}_1,...,\dot{s}_h\}$ be two linguistic term sets, $\alpha,\dot{\alpha}\in[-0.5,0.5)$ be two numerical numbers. Then $\hat{S}=((s_u,\alpha),(\dot{s}_v,\dot{\alpha}))$ is a TD2L expression, $s_u\in S$, $\dot{s}_v\in\dot{S}$.

(s_u,α), the first dimension information of $\hat{S}$, represents the assessment about the decision object. $(\dot{s}_v,\dot{\alpha})$, the second dimension information of $\hat{S}$, expresses the subjective assessment on the reliability of the first dimension information.

C. Minimum Adjustment Consensus Model

Suppose there exist n DMs participating in GDM, o_i and $\overline{o}_i$ are the DMs', $e_i(i=1,2,...,n)$, original and adjusted opinion respectively, o' and $\overline{o}'$ are the collective original and adjusted opinion reached by group. Dong et al. [12] proposed the classical MACM as follows:

$$\min\sum_{i=1}^{n}|o_i-\overline{o}_i|$$
$$s.t.\begin{cases}|\overline{o}_i-\overline{o}'|\le\varepsilon, i=1,2,...,n\\ \overline{o}'=F(\overline{o}_1,\overline{o}_2,...,\overline{o}_n)\end{cases} \tag{5}$$

where $|o_i-\overline{o}_i|$ is the distance between o_i and $\overline{o}_i$. $\varepsilon\ge0$ is the consensus threshold. $F(\overline{o}_1,\overline{o}_2,...,\overline{o}_n)=\sum_{i=1}^{n}w_i\overline{o}_i$, w_i is the importance weight of DM e_i.

Obviously, the less the total adjustment the better, denote the optimal solution of model (5) as $\overline{o}_i^*(i=1,2,...,n)$, in which $\overline{o}_i^*$ is the optimal adjusted opinion of DM e_i.

III. A CRP WITH MINIMUM ADJUSTMENT WITH THE CONSIDERATION OF TOLERANCE DEGREE OF DMS

This section proposes a novel CRP that considers the DMs' tolerance degree and two consensus rules: minimum distance between original and adjusted preferences, and minimum the number of adjusted preferences. The DMs' tolerance degree is an importance factor to balance the two consensus rules. This CRP is an iterative process that evaluates the consensus level among experts if it is good enough the GDM problem is solved if not an adjustment process based on the tolerance degree is carried out to improve the agreement, both steps are further detailed below.

A. Consensus Level

Suppose that m DMs $E=\{e_k|k=1,2,...,m\}$ are invited to join in a GDM problem, the alternative set is $X=\{x_i|i=1,2,...,n\}$. The linguistic term set is $S=\{s_0,s_1,...,s_g\}$. The preferences are provided as preference relation matrices $R_k=(r_{ij}^k)_{n\times n}$ from DM e_k, where $r_{ij}^k\in S$.

First, we check if the overall consensus level (CL) is satisfied based on the following three levels [15].

(1) CL for each pair of alternatives. It is based on the similarities among preferences.

$$CL_{ij}=sim_{ij} \tag{6}$$

where the similarity between alternatives x_i and x_j is as

$$sim_{ij}=1-\sum_{k_1=1}^{m-1}\sum_{\substack{k_2=k_1+1\\k_2\ne k_1}}^{m}\frac{\omega_{k_1}+\omega_{k_2}}{m-1}\left|\Delta^{-1}(r_{ij}^{k_1})-\Delta^{-1}(r_{ij}^{k_2})\right|/g \tag{7}$$

(2) CL for each alternative. It can be obtained the results of the average of each row of consensus matrix $(CL_{ij})_{n\times n}$ as

$$CL_{x_i}=\frac{1}{n-1}\sum_{\substack{j=1\\j\ne i}}^{n}CL_{ij} \tag{8}$$

where $i=1,2,...,n$ and consensus matrix $(CL_{ij})_{n\times n}$ is based on the results of Eq. (6).

(3) Overall CL for all preferences. It is obtained as

$$CL=\frac{1}{n}\sum_{i=1}^{n}CL_{x_i} \tag{9}$$

After the check of the overall CL, compare the overall CL and the consensus threshold ε given in advance. If $CL \geq \varepsilon$, then the CRP is finished. If not, then the CRP keeps going.

B. A Consensus Model with Tolerance Degree of DMs

The adjustment for DMs' preferences is necessary if $CL < \varepsilon$. Hence, by taking into account a DM's tolerance degree that is the maximum change that the DM is willing to accept the adjusted opinions, the proposed CRP considers the following two consensus rules: (1) minimize the distance between the original and adjusted preferences. (2) minimize the number of adjusted preferences.

Let θ_k be the DM's, e_k, tolerance degree then the adjusted preferences to be accepted must satisfy

$$\left|\Delta^{-1}(r_{ij}^k) - \Delta^{-1}(\overline{r}_{ij}^k)\right| / g \leq \theta_k \qquad (10)$$

where $i, j = 1, 2, ..., n$. $\theta_k \in [0,1]$. If $\theta_k = 0$, then DM e_k does not accept any change of the original preferences, where he/she is a stubborn DM. If $\theta_k = 1$, then DM e_k could accept any change of the original preferences, where he/she is a benevolent DM. $[\Delta(\Delta^{-1}(r_{ij}^k) - \theta_k g), \Delta(\Delta^{-1}(r_{ij}^k) + \theta_k g)]$ is denoted as tolerance interval. DMs accept the adjusted preferences if $\overline{r}_{ij}^k \in [\Delta(\Delta^{-1}(r_{ij}^k) - \theta_k g), \Delta(\Delta^{-1}(r_{ij}^k) + \theta_k g)]$.

In fact, the consideration of tolerance degree of DMs is a strict view for minimizing the number of the adjusted preferences. If the minimum number of adjusted preferences is the only condition to be considered, then the distance between the original and adjusted preferences may out of the tolerance interval of DMs. In such situation, the minimum number of adjusted preferences is meaningless.

Thus, it is important to consider both DMs' tolerance degree of DMs and the minimum number of adjusted preferences. To simplify the CRP, the following consensus mechanism is designed:

- To minimize the number of the adjusted preferences. A priority-adjustment rule is proposed. Owing to the preferences information are presented as symmetric matrix, only the upper triangular elements in matrix are considered. Thus, there are $n(n-1)/2$ CLs for each pair of alternatives need to be considered. Sort the value of CL for each pair of alternatives CL_{ij} with $i = 1, 2, ..., n-1$ and $j = 2, 3, ..., n$ in ascending order as $CL_{i_1 j_1} \leq CL_{i_2 j_2} \leq \cdots \leq CL_{i_L j_L}$, where $L = n(n-1)/2$, $i_l = 1, 2, ..., n-1$, $j_l = 2, 3, ..., n$, $l = 1, 2, ..., L$. If $CL < \varepsilon$, then at least there exist one or more $CL_{i_l j_l}$ satisfy $CL_{i_l j_l} < \varepsilon$. Then original preferences $r_{i_1 j_1}^k (k = 1, 2, ..., m)$ which satisfies $r_{i_1 j_1}^k = \min\{r_{i_l j_l}^k \,|\, l = 1, 2, ..., L\}$ need to be adjusted preferentially. Let $T^{l^*} = \{(i_l, j_l) | i = 1, 2, ..., l^*, CL_{i_{l^*}} \leq \varepsilon, CL_{i_{l^*+1}} > \varepsilon\}$ be a set, then the original preferences $r_{i_l j_l}^k (l = 1, 2, ..., L)$ need to be considered to adjusted preferentially if it satisfies $(i_l, j_l) \in T^{l^*}$. First, set $l = 1$ to adjust the corresponding original preference, if CL is not satisfied, then set $l = 2$, if not, set $l = 3$ and repeat until there is a l^* make adjusted preference $\overline{r}_{i_1 j_1}^k, ..., \overline{r}_{i_{l^*} j_{l^*}}^k$ and original preference $\overline{r}_{i_{l^*+1} j_{l^*+1}}^k, ..., \overline{r}_{i_L j_L}^k$ satisfy $CL > \varepsilon$, $l^* = 1, 2, ..., L$. Obviously, the smaller the value of l^* the better.
- To minimize the number of adjusted preferences with the consideration of DMs' tolerance degree. The novel consensus model is set up as follows.

$$\min \sum_{(i_l, j_l) \in T^{l^*}} \sum_{k=1}^{m} \left|\Delta^{-1}(r_{i_l j_l}^k) - \Delta^{-1}(\overline{r}_{i_l j_l}^k)\right|$$
$$s.t. \begin{cases} \left|\Delta^{-1}(r_{i_l j_l}^k) - \Delta^{-1}(\overline{r}_{i_l j_l}^k)\right| / g \leq \theta_k \\ CL' \geq \varepsilon \\ r_{i_l j_l}^k = \overline{r}_{i_l j_l}^k \; if \; (i_l, j_l) \notin T^{l^*} \end{cases} \qquad (11)$$

where CL' is the overall CL of adjusted preferences.

For better understanding, the Algorithm Ⅰ shows how to obtain the optimal adjusted preference with minimum number of adjusted preferences and within the tolerance interval of DMs.

Algorithm I

Input: The preference matrix provided by DM e_k is presented as $R_k = (r_{ij}^k)_{n \times n}$, $k = 1, 2, ..., m$, $i, j = 1, 2, ..., n$; The tolerance degree of DMs $\theta_k (k = 1, 2, ..., m)$; The consensus threshed ε.

Output: The final adjusted preference $\overline{r}_{ij}^k ((i, j) \in T^{l^*})$, $\overline{r}_{ij}^k = r_{ij}^k ((i, j) \notin T^{l^*})$.

Step 1: Check the overall CL of DMs' preferences based on Eqs. (6)-(9), if $CL \geq \varepsilon$, then the CRP is done. If $CL < \varepsilon$, then continue to the next step.

Step 2: Set up consensus model with $l = 1$ based on model (11), if it can be solved by software LINGO 11 and obtain the optimal preference relations as $\overline{r}_{ij}^k ((i, j) \in T^1)$. Then output $\overline{r}_{ij}^k ((i, j) \in T^1)$ as the adjusted preference relations and $\overline{r}_{ij}^k = r_{ij}^k ((i, j) \notin T^1)$. If the model is unsolved, then go to the next step.

Step 3: Set up consensus model with $l = 2$ based on model (11) and repeat the process as described in Step 2, if it can be solved and the optimal adjusted preference relations as $\overline{r}_{ij}^k ((i, j) \in T^2)$. Then output $\overline{r}_{ij}^k ((i, j) \in T^2)$ as the adjusted preference relations and $\overline{r}_{ij}^k = r_{ij}^k ((i, j) \notin T^2)$. If the model is unsolved, then repeat the above steps with $l = 3, 4, ...$ until the consensus model can be solved with $l = l^* (l^* = 1, 2, ...)$.

According to the relating operation research theory, model (11) must have an optimal solution without the consideration of DMs' tolerance degree. Only when the DM's tolerance degree is small enough, model (11) has no solution. In this paper, we suppose that DMs are not demanding, which with appropriate tolerance degree.

IV. The GDM Based on the Measurement of Reliability Degree of the Adjusted Agreed Preferences

Generally, the reliability degree of original preferences given by DMs is considered by existing research, however, the reliability degree of the adjusted preferences has been ignored.

Here we give a reliability model to compute the reliability degree of the adjusted preferences based on the proposed consensus model, where the reliability degree comes from the concept of stability degree of the original preferences. In this section, we introduce a concept: stability degree of original preferences. Then, a comparison measure for TD2L is provided in order to facilitate the selection of the best alternative of the GDM problem.

A. The Reliability Degree of the Adjusted Preferences

The reliability degree of the adjusted preferences derives from the stability degree of original preferences, which describes the similarity between the original and adjusted preferences after CRP. The more similar the original preference to adjusted preference is, the higher the stability degree of adjusted preferences is.

Let r_{ij}^k be as before described, then the stability degree of preference r_{ij}^k is as

$$SD(r_{ij}^k)=1-\frac{\left|\Delta^{-1}(r_{ij}^k)-\Delta^{-1}(\overline{r}_{ij}^k)\right|}{\min\{g,\Delta(\Delta^{-1}(r_{ij}^k)+\theta_k g)\}-\max\{0,\Delta(\Delta^{-1}(r_{ij}^k)-\theta_k g)\}} \quad (12)$$

where $SD(r_{ij}^k)\in[0,1]$. The larger the value of $SD(r_{ij}^k)$ is, the more stable the original preference r_{ij}^k is, then the reliability degree of adjusted preference $\overline{r}_{ij}^k$ is more likely higher.

The reliability degree of the adjusted preference $\overline{r}_{ij}^k$ derives from the stability degree of the original preference r_{ij}^k, which expressed as:

$$RD(\overline{r}_{ij}^k)=\Delta(SD(r_{ij}^k)\cdot h) \quad (13)$$

where $h+1$ is the cardinality of set $\dot{S}$, used for expressing the reliability degree. $RD(\overline{r}_{ij}^k)$ is a 2-tuple linguistic represents the reliability degree of the adjusted preference $\overline{r}_{ij}^k$.

Obviously, the stability degree of the original preferences is related to the tolerance degree of DMs. The higher the value of tolerance degree of DMs is, the higher the value of $SD(r_{ij}^k)$ is and the higher the reliability degree of $\overline{r}_{ij}^k$ is. Because if the value of θ_k is higher, then $\min\{g,\Delta(\Delta^{-1}(r_{ij}^k)+\theta_k g)\}-\max\{0,\Delta(\Delta^{-1}(r_{ij}^k)-\theta_k g)\}$ is smaller, thus the value of $SD(r_{ij}^k)$ is higher based on Eq. (12), therefore the reliability degree of $\overline{r}_{ij}^k$ is higher based on Eq. (13).

B. The comparison of TD2Ls

After the Algorithm Ⅰ and the obtaining of reliability degree of the adjusted preferences, the final overall assessment of each alternative is presented as TD2Ls, DMs will select the alternative with higher assessment value. Thus the comparison of TD2Ls is important for choosing the optimal alternative.

Suppose that the adjusted preference of DM e_k is $\overline{o}_k$, which is presented as a 2-tuple linguistic $(\overline{s}_{u_k},\overline{\alpha}_k)$ and the reliability degree of $\overline{o}_k$ is expressed as $(\dot{s}_{v_k},\dot{\alpha}_k)$. Then the TD2L expression of DM e_k for the final assessment can be presented as $\hat{S}_k=((\overline{s}_{u_k},\overline{\alpha}_k),(\dot{s}_{v_k},\dot{\alpha}_k))$. A comparison way for distinguishing any two TD2Ls is proposed.

Definition 3 Let $\hat{S}_k=((\overline{s}_{u_k},\overline{\alpha}_k),(\dot{s}_{v_k},\dot{\alpha}_k))$ be a TD2L, then the score function of $\hat{S}$ is as

$$SC(\hat{S}_k)=2\eta a_k+2(1-\eta)b_k+4\eta(1-\eta)a_k b_k \quad (14)$$

where η represents the relative importance of the first dimension of TD2Ls provided by DMs, $\eta\in[0,1]$. (a_k,b_k) is the corresponding 2-tuple array of $\hat{S}_k=((\overline{s}_{u_k},\overline{\alpha}_k),(\dot{s}_{v_k},\dot{\alpha}_k))$, $a_k=\Delta^{-1}(\overline{s}_{u_k},\overline{\alpha}_k)/g$, $b_k=\Delta^{-1}(\dot{s}_{v_k},\dot{\alpha}_k)/h$, $a_k,b_k\in[0,1]$.

In special, if $\eta=0.5$, then the score function of $\hat{S}_k$ is as

$$SC(\hat{S}_k)=a_k+b_k+a_k b_k \quad (15)$$

Obviously, $2\eta a_k\in[0,2]$, $2(1-\eta)b_k\in[0,2]$, $4\eta(1-\eta)a_k b_k\in[0,4]$, then $SC(\hat{S}_k)\in[0,8]$. If $\eta\in(0.5,1]$, then it means the DMs pay more attention on the first dimension information of TD2Ls. If $\eta\in[0,0.5)$, then DMs pay more attention on the second dimension information of TD2Ls. In special, if $\eta=0.5$, then the two dimensions have equal importance weight.

Suppose that $\hat{S}_1=((\overline{s}_{u_1},\overline{\alpha}_1),(\dot{s}_{v_1},\dot{\alpha}_1))$ and $\hat{S}_2=((\overline{s}_{u_2},\overline{\alpha}_2),(\dot{s}_{v_2},\dot{\alpha}_2))$ be two TD2Ls, then score functions of $\hat{S}_1$ and $\hat{S}_2$ are $SC(\hat{S}_1)$ and $SC(\hat{S}_2)$, respectively. Then the comparison rules are as follows.

If $SC(\hat{S}_1)>SC(\hat{S}_2)$, then $\hat{S}_1\succ\hat{S}_2$;

If $SC(\hat{S}_1)<SC(\hat{S}_2)$, then $\hat{S}_1\prec\hat{S}_2$;

If $SC(\hat{S}_1)=SC(\hat{S}_2)$, then $\hat{S}_1=\hat{S}_2$.

V. An Illustrative Example

This section provides a numerical example to illustrate the use of the novel consensus model for GDM based on reliability degree detection with the final assessment presented as TD2Ls information.

Suppose that there are four alternatives $X=\{x_1,x_2,x_3,x_4\}$ evaluated by four DMs $E=\{e_1,e_2,e_3,e_4\}$. The DMs' weights are $W=(0.2,0.3,0.25,0.25)$. The original preferences are expressed as 2-tuple linguistic as shown in Table Ⅰ. $S=\{s_0=extremely\ poor, s_1=very\ poor, s_2=poor, s_3=medium, s_4=good, s_5=very\ good\ s_6=extremely\ good\}$ is the assessment linguistic term set. The linguistic term set for expressing reliability degree is $\dot{S}=\{\dot{s}_0=very\ low,\ \dot{s}_1=low,\ \dot{s}_2=medium,\ \dot{s}_3=high,\ \dot{s}_4=very\ high\}$. The tolerance degrees of DMs are $\theta_1=0.15$, $\theta_2=0.2$, $\theta_3=0.3$, $\theta_4=0.2$. The predefined consensus threshold is $\varepsilon=0.85$.

Alternative x_3 are both the optimal no matter we consider the reliability degree of the adjusted preferences, however, $x_1 > x_4$ if the reliability degree of the adjusted preferences is considered and $x_4 > x_1$ if we only consider the overall assessment of alternatives.

In our proposal, the reliability degree of the adjusted preferences can be as important information for decision making. Especially for the large value of adjusted preference but with low reliability degree, the reliability degree detection is necessary to ensure the accuracy of the decision result.

To deal with the influence of parameter η on decision results, the overall assessments with the different η is shown in Table III as follows.

TABLE III. THE RANKING OF ALTERNATIVES WITH DIFFERENT η

η	**Alternatives ranking**	**The optimal alternative**
0	$x_1 \sim x_3 > x_2 \sim x_4$	x_1, x_3
0.1	$x_3 > x_4 > x_1 > x_2$	x_3
0.2	$x_3 > x_1 > x_4 > x_2$	x_3
0.3	$x_3 > x_1 > x_4 > x_2$	x_3
0.4	$x_3 > x_1 > x_4 > x_2$	x_3
0.5	$x_3 > x_1 > x_4 > x_2$	x_3
0.6	$x_3 > x_1 > x_4 > x_2$	x_3
0.7	$x_3 > x_1 > x_4 > x_2$	x_3
0.8	$x_3 > x_1 > x_4 > x_2$	x_3
0.9	$x_3 > x_1 > x_4 > x_2$	x_3
1	$x_3 > x_4 > x_1 > x_2$	x_3

From Table III, we conclude that the parameter η has a very limited influence on the adjusted preferences, which shows the stability of the alternatives ranking with the proposed method.

VI. CONCLUSION

In this paper, we propose a consensus model based on the consideration of tolerance degree of DMs, and two consensus rules are considered: minimum distance between the original and adjusted preferences, and minimum number of adjusted preferences. Furthermore, the reliability degree detection of adjusted preference is presented.

The main contributions in this paper are: First, a consensus model based on the consideration of tolerance degree of DMs is proposed, following two consensus rules: minimum adjustment between original and adjusted preferences, and minimum number of adjusted preferences. Second, we provide a measurement of reliability degree of adjusted preferences, then the TD2L expressions used for describing the overall assessment are presented. Third, a comparison rule for TD2Ls is proposed. Finally, an illustrative example is shown to certificate the effectiveness of the proposed method.

In the future, we plan to study the CRP with the consideration of the reliability degree of adjusted preferences in the large scale decision making problems [16, 17].

ACKNOWLEDGMENT

The work was supported by the grant (No. 61773123) from National Nature Science Foundation of China, by the Spanish National research project PGC2018-099402-B-I00 and by the Postdoctoral fellow Ramón y Cajal (RYC-2017-21978).

REFERENCES

[1] L.A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning—III, Inf. Sci. 9(1) (1975) 43-80.
[2] F. Herrera, L. Martínez, A 2-tuple fuzzy linguistic representation model for computing with words, IEEE Trans. Fuzzy Syst. 8(6) (2000) 746-752.
[3] O.S. A, M.N.A. B, W.H. A, L.M. C, Cloud computing model selection for e-commerce enterprises using a new 2-tuple fuzzy linguistic decision-making method, Comput. Ind. Eng. 132 (2019) 47-58.
[4] L. Wang, Y. Wang, W. Pedrycz, Hesitant 2-tuple linguistic Bonferroni operators and their utilization in group decision making, Appl. Soft Comput. 77 (2019) 653-664.
[5] L. Zhang, Y. Wang, X. Zhao, A new emergency decision support methodology based on multi-source knowledge in 2-tuple linguistic model, Knowl.-Based Syst. 144 (2018) 77-87.
[6] H. Zhu, J.B. Zhao, Y. Xu, 2-dimension linguistic computational model with 2-tuples for multi-attribute group decision making, Knowl.-Based Syst. 103 (2016) 132-142.
[7] X. Liu, Y.J. Xu, R. Montes, R.-X. Ding, F. Herrera, Alternative ranking-based clustering and reliability index-based consensus reaching process for hesitant fuzzy large scale group decision making, IEEE Trans. Fuzzy Syst. 27(1) (2019) 159-171.
[8] F.J. Cabrerizo, F. Chiclana, R. Al-Hmouz, A. Morfeq, A.S. Balamash, E. Herrera-Viedma, Fuzzy decision making and consensus: challenges, J. Intell. Fuzzy Syst. 29(3) (2015) 1109-1118.
[9] E. Herrera-Viedma, F. Javier Cabrerizo, J. Kacprzyk, W. Pedrycz, A review of soft consensus models in a fuzzy environment, Inf. Fusion. 17 (2014) 4-13.
[10] S. Zadrożny, An approach to the consensus reaching support in fuzzy environment, Consensus under fuzziness, Springer1997, pp. 83-109.
[11] I. Palomares, F.J. Estrella, L. Martínez, F. Herrera, Consensus under a fuzzy context: Taxonomy, analysis framework AFRYCA and experimental case of study, Inf. Fusion. 20 (2014) 252-271.
[12] Y. Dong, Y. Xu, H. Li, B. Feng, The OWA-based consensus operator under linguistic representation models using position indexes, Eur. J. Oper. Res. 203(2) (2010) 455-463.
[13] B. Zhang, Y. Dong, Y. Xu, Multiple attribute consensus rules with minimum adjustments to support consensus reaching, Knowl.-Based Syst. 67 (2014) 35-48.
[14] W.D. Zhu, G.Z. Zhou, S.L. Yang, An approach to group decision making based on 2-dimension linguistic assessment information, Syst Eng 27(2) (2009) 113-118.
[15] Z.B. Wu, J.P. Xu, A consensus model for large-scale group decision making with hesitant fuzzy information and changeable clusters, Inf. Fusion. 41 (2018) 217-231.
[16] R.M. Rodríguez, Á. Labella, G. De Tré, L. Martínez, A large scale consensus reaching process managing group hesitation, Knowl.-Based Syst. 159 (2018) 86-97.
[17] Á. Labella, Y.Y. Liu, R.M. Rodríguez, L. Martínez, Analyzing the performance of classical consensus models in large scale group decision making: A comparative study, Appl. Soft Comput. 67 (2018) 677-690.

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Chapter 5

Conclusions and Future Works

Chapter 5 concludes our research memory by revising the conclusions about the main proposals and results obtained, and pointing out possible future works.

5.1 Conclusions

GDM is widely applied in real life to solve important and complicated problems in a range of domains, such as emergency decision making [51, 71, 180], medical service assessment [150, 179], the selection of supplier [16, 17, 48] etc. Since the importance of GDM in selecting and evaluating the management and economic issues, many models and approaches for GDM problems have been proposed [81, 103, 122, 152].

MAGDM involves that DMs provide evaluations regarding the performance of the alternatives under multiple attributes [75]. With the increasing complexity of the decision making environment and the limited DMs' expressiveness, the MAGDM based on linguistic assessment has attracted more attention [129, 132, 143, 196]. Considering the complexity and uniqueness of the linguistic expression, the general MAGDM methods are not always suitable for linguistic MAGDM. Even though, the existing research has achieved numerous and successful achievements [27, 101, 188], but there are still many methods and theoretical systems need to be improved. Besides, in some situations, the use of only one dimension to represent information by linguistic information is not enough to ensure the accuracy of such information, the reliability of the assessment is also an important factor to be considered. Therefore, the study of TD2L is necessary and meaningful. Furthermore, several new methods need to be developed aiming at solving MAGDM based on TD2L information.

Across our research we have obtained novel, remarkable and relevant results regarding those challenges that not only fulfill the objectives indicated in Section 1.2, but also provide new views in the solving processes of MAGDM based on TD2L labels and new research opportunities for the future.

Consequently, we should conclude from our research the following results:

1. In spite of the successful application of the two-dimension linguistic information to deal with the representation and computation of two-dimension linguistic labels, the analysis of the uncertainty of assessments according to the second dimension information has not been explored. Thus, a new representation model of TD2L from stochastic perspective has been proposed. A corresponding rule from TD2L label to a stochastic variable and its inverse have been presented, which is more suitable for the computation of large scale GDM. And the comparison and similarity measurement between two TD2L labels have been developed with the consideration of the relative importance degree of the two dimensions of information from the stochastic perspective, which has made that the decision analysis provides more useful information.

2. The reliability of the initial assessment provided by DMs is usually presented as two dimension linguistic information for MAGDM problems, however, during the CRP, especially for automatic CRP, the reliability of the adjusted preferences is often neglected. Based on this observation, a two-stage minimum adjustment consensus model for large scale GDM problems based on TD2L expressions with the consideration of reliability degree of the adjusted preferences has been proposed, which not only considers the minimum adjustment, but also minimizes the number of adjusted preferences. And the relations between the total preference adjustment and the reliability degree of the adjusted preferences have been discussed. The proposed method has completed the two dimension 2-tuple linguistic approach for large scale GDM.

3. The measurement of the reliability of the adjusted opinions can improve the accuracy of the decision making with automatic CRPs. However, the reliability is based on the reasonable tolerance of DMs on changing their opinions. Therefore, the tolerance degree of the DMs is an important factor to consider in advance during the CRP. The proposal of the new consensus model based on the consideration of tolerance degree of DMs has improved the MAGDM method with TD2L information based on reliability measurement.

5.2 Future Works

Even though several methods, tools and approaches have been proposed in this research, there are still challenges within GDM based on linguistic assessment and the TD2L approach that should be further studied. In near future, we will focus on the extension of the proposals presented and the development of solutions for new problems:

1. Usually, it is considered that DMs are completely rational in most existing researches, however, in real situation, DMs are bounded rational, and they may feel uncomfortable when they are suggested to adjust their opinions within a minor adjustment range, thus non-cooperative behavior could appear in GDM. Therefore, the psychological behavior of DMs would be considered in the future work.

2. For CRP in GDM, the minimum adjustment is often considered to make the DMs' opinions changed, however, make DMs to change their opinions have different levels of difficulty, the unit adjustment cost is almost defined under the assumption that they are non-directional, in fact, the unit adjustment cost is not always equal in upward and download adjustment directions [70]. Therefore, the determination of the unit adjustment cost is also needed to be considered, especially for the symmetric unit cost, which is a puzzle for minimum cost consensus model and is the next step worth thing about it.

3. As the linguistic term set for expressing the initial assessment is sometimes unbalanced, the linguistic term for expressing the reliability of the adjusted

opinions could be also unbalanced, how to design the representation and computation model with the unbalanced reliability information will be studied in the future.

5.3 Additional Publications

Regarding the diffusion of our scientific results, besides the publications included in this memory, we highlight the following contributions:

- International Journals

  - Z. L. Wang, Y. M. Wang. Prospect theory-based group decision-making with stochastic uncertainty and 2-tuple aspirations under linguistic assessments. *Information Fusion*, vol. 56, issue 1, pp. 81-92, 2020.

  - Z. L. Wang, Y. M. Wang, L. Wang. Tri-level multi-attribute group decision making based on regret theory in multi-granular linguistic contexts. *Journal of Intelligent & Fuzzy Systems*, vol 35, issue 3, pp. 793-806, 2018.

- International Conferences

  - Z. L. Wang, R. M. Rodríguez, Y. M. Wang, L. Martínez. A Novel Method for Group Decision Making based on Two-dimensional 2-tuple Linguistic from a Stochastic Perspective.
    International Virtual Workshop on Business Analytics Eureka 2021 held in Ciudad Juárez (México), 2-4 June 2021.

Appendix A

Resumen escrito en Español

Título de la tesis: *Enfoque lingüístico de dos dimensiones y 2 tupla para toma de decisión en grupo con múltiples atributos bajo incertidumbre*

Este apéndice incluye el título, índice, introducción, resumen y conclusiones escritas en español, como parte de los requisitos necesarios para obtener el doctorado según el artículo 23.2 del Reglamento de Estudios de Doctorado de la Universidad de Jaén.

En primer lugar, se presenta el índice de la memoria. A continuación, se introduce de forma breve la investigación llevada cabo, indicando motivación, objetivos y la estructura de los capítulos que la componen. Seguidamente, se presenta un resumen de la misma, para finalmente concluir con el apartado de conclusiones obtenidas y trabajos futuros.

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A.1 Motivación

La toma de decisiones en grupo (TDG) es una rama de la teoría de decisiones que se ha aplicado ampliamente en escenarios del mundo real para resolver problemas de decisión importantes y complicados en una variedad de dominios, como salud pública [5], proyectos de ingeniería civil [127] y política exterior [8]. En los problemas de TDG, los decisores suelen evaluar alternativas basadas en múltiples atributos, lo que se conoce como un problema de toma de decisiones en grupo con múltiples atributos (TDGMA) [82]. Sin embargo, debido a la complejidad de proporcionar las opiniones y la racionalidad limitada de los seres humanos, el uso de términos lingüísticos es más intuitivo, flexible y cercano al lenguaje utilizado por los seres humanos para evaluar los criterios en TDGMA que el uso de valores numéricos. El concepto de variable lingüística fue introducido por Zadeh [206], para modelar la incertidumbre de la información. Una variable lingüística es una variable cuyos valores no son números sino palabras u oraciones en lenguaje natural o artificial. Es una herramienta muy utilizada para resolver problemas de TDGMA con información cualitativa. Por tanto, existen muchos enfoques de TDGMA que utilizan variables lingüísticas para modelar la incertidumbre [54, 109, 111, 117, 177].

Para resolver un problema de TDGMA con información lingüística, es necesario realizar procesos de computación con palabras (CWW) [121, 208, 210] (ver Figure A.1), que es una de las metodologías más utilizadas en toma de decisión lingüística.

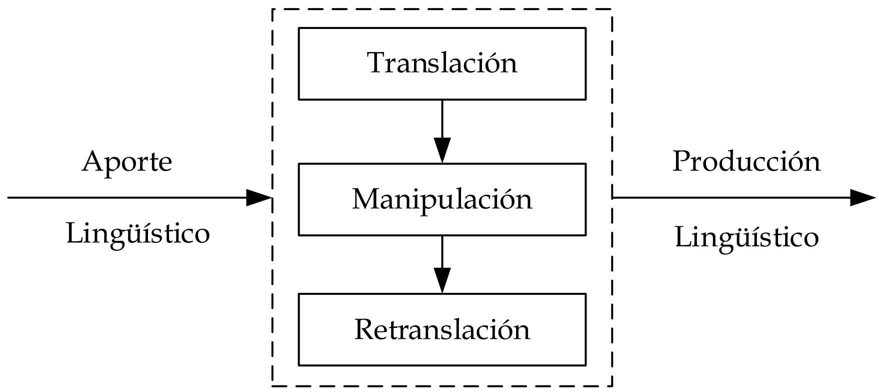

Figure A.1: Proceso de computación con palabras

En los procesos de CWW en TDGMA, los resultados lingüísticos se obtienen a partir de entradas lingüísticas, que son fácilmente comprensibles y se representan adecuadamente. En consecuencia, se han desarrollado varios modelos computacionales lingüísticos para llevar a cabo los procesos de CWW [3, 60, 61, 118, 172, 197]. Estos modelos siguen el esquema de computación descrito por Yager [198, 199] que señala la importancia de los procesos de translación y retranslación en

CWW. Sin embargo, existen algunas limitaciones cuando se realizan procesos de fusión sobre variables lingüísticas, según estos modelos originales. En estos enfoques, los resultados no suelen coincidir exactamente con ninguno de los términos lingüísticos iniciales, por lo que se debe desarrollar un proceso de aproximación para expresar el resultado en el dominio de la expresión inicial [63]. Esto produce la consiguiente pérdida de información y por ende la falta de precisión.

Para evitar tal inexactitud en el paso de retranslación, se propuso el modelo lingüístico 2-tupla [60]. Una representación lingüística de 2-tupla está compuesta por un término lingüístico y un valor numérico llamado translación simbólica que representa el desplazamiento del término lingüístico. Por tanto, evita la pérdida de información y obtiene resultados más precisos e interpretables. Por ello, el modelo lingüístico 2-tupla destaca como uno de los más utilizados en toma de decisiones [119, 142].

Además, se han propuesto varias extensiones del modelo lingüístico 2-tupla para resolver problemas de TDGMA, como el modelo semántico de 2-tupla [1, 163, 164], el modelo lingüístico multigranular de 2 tupla [38, 62, 197], el modelo lingüístico proporcional de 2-tupla [172, 173], modelo numérico escalar [34, 36, 37], etc. Teniendo en cuenta la extensa y exitosa investigación de modelos lingüísticos basados en 2-tupla, Martínez y Herrera [120] realizaron una revisión del estado del arte de estos modelos. Los modelos lingüísticos de 2-tupla se han utilizado con éxito para obtener resultados precisos e interpretables, pero la fiabilidad de las evaluaciones también es un tema importante para los decisores. Los modelos de toma de decisiones existentes basados en información lingüística 2-tupla asumen que todas las evaluaciones tienen el mismo nivel de confianza [112], lo que no es realista en la práctica. Por tanto, Zhu et al. [225] propusieron el concepto de información lingüística bidimensional, que incluye la información de fiabilidad de las evaluaciones subjetivas. Posteriormente, se propuso el concepto de información lingüística bidimensional de 2-tupla (LB2T) [224] combinando la expresión lingüística de dos dimensiones y la información lingüística 2-tupla.

Evidentemente, la información expresada como LB2T es más precisa y razonable, porque la valoración y la fiabilidad de la valoración se proporcionan al mismo tiempo. Debido a las ventajas de utilizar evaluaciones LB2T, se han desarrollado diferentes enfoquespara resolver problemas de TDGMA con evaluación lingüística bidimensional [98, 99, 220], tales como:

– *Modelo de representación de etiquetas LB2T.* Generalmente, las etiquetas LB2T se representan como un término lingüístico binario [223]. Las dos clases de información lingüística proceden de dos conjuntos de términos lingüísticos diferentes. El primer conjunto de términos lingüísticos representa las evaluaciones proporcionadas por los decisores y el segundo conjunto de términos lingüísticos representa la fiabilidad de la evaluación anterior, que también es información subjetiva proporcionada por los decisores [202].

- *Operaciones y comparación de etiquetas LB2T.* Se han desarrollado diferentes operadores para diferentes tipos de expresión lingüística bidimensional, como los operadores de agregación de información lingüística incierta bidimensional [106, 110] utilizados para agregar las etiquetas lingüísticas bidimensionales bajo incertidumbre, operadores de agregación generalizados de potencia lingüística trapezoidal de dos dimensiones. Los operadores de agregación [99] se utilizan para agregar las etiquetas LB2T. Además, las operaciones de comparación entre LB2T se han desarrollado sobre la base de las operaciones de comparación tradicionales del modelo lingüístico 2-tupla [60], como el álgebra de implicación lingüística bidimensional [224] que se utiliza para expresar y comparar las LB2Ts, la notación de expectativa de las LB2Ts [110] propuesta para comparar variables lingüísticas inciertas de dos dimensiones, etc.

- *Métodos TDG basados en expresión LB2T.* Dado que LB2T tiene ventajas importantes en la expresión de información, su investigación y aplicación combinadas con estos métodos clásicos de TDG han atraído la atención de los investigadores y se han extendido a varios métodos de TDG en el entorno LB2T, como PROMETHEE [220], TODIM extendido [105], VIKOR-QUALIFLEX extendido [98], modo de fallo y análisis de efectos [104], teoría prospectiva extendida-VIKOR [33] , etc.

- *Aplicación de métodos TDG basados en etiquetas LB2T en la vida real.* En algunas situaciones reales, los términos lingüísticos se han considerado el modelo más adecuado para evaluar atributos, como la toma de decisiones de emergencia [32, 33], la evaluación de la calidad [97], la selección del lugar para la construcción de una central eléctrica [185], la evaluación de riesgos [186], etc.

La investigación en TDG muestra que es necesario un proceso de alcance de consenso (PAC) para asegurar el acuerdo sobre los resultados de las decisiones en los problemas de TDG basados en información lingüística. Sin embargo, los PAC generalmente exigen que las preferencias u opiniones iniciales se modifiquen si no se satisface el nivel de consenso esperado durante el PAC. En esta situación, vale la pena pensar en la fiabilidad de las preferencias u opiniones modificadas. Obviamente, los decisores podrían dar por adelantado la fiabilidad de sus preferencias; sin embargo, esta fiabilidad debería obtenerse de una forma de medición objetiva.

A pesar de que existen múltiples modelos y enfoques que tratan los problemas de TDGMA y la información LB2T de manera conjunta, tanto en la teoría como en la práctica, estos modelos y enfoques no son lo suficientemente buenos cuando se aplican a problemas de TDGMA del mundo real en los que se aplican PAC. Así, los nuevos desafíos que se describen a continuación son las principales motivaciones de esta memoria de investigación:

- *La agregación de los LB2T en TDGMA:* Agregar las opiniones modeladas mediante LB2T de los decisores para clasificar u ordenar las alternativas, y seleccionar la mejor opción, es un proceso necesario. En los problemas de TDGMA basados en etiquetas LB2T, las preferencias de los decisores individuales deben agregarse de forma colectiva y bien estructurada para tomar la decisión final. La agregación de los LB2T es de gran importancia en TDGMA porque diferentes operadores de agregación pueden conducir a resultados diferentes. Sin embargo, interpretar y analizar las preferencias de estos decisores es una tarea compleja. Y en los métodos existentes, independientemente del operador de agregación utilizado, la información bidimensional de las etiquetas LB2T se toma por separado para su cálculo [99, 107, 110, 167, 200]. De hecho, cuando las evaluaciones no son completamente fiables, se vuelven aleatorias, lo que significa que el valor de la preferencia u opinión es muy incierto. Por lo tanto, un operador de agregación para agregar las etiquetas LB2T de una perspectiva estocástica parece adecuado.

- *Medición de la fiabilidad de la evaluación LB2T modificada:* Las etiquetas LB2T expresan la valoración y su fiabilidad y se han aplicado a muchos problemas de TDGMA [32, 185, 186]. En un PAC, las etiquetas LB2T iniciales se modifican y es necesario volver a calcular la fiabilidad de la evaluación modificada. La fiabilidad de la evaluación inicial es subjetiva, sin embargo, es necesaria una medición objetiva para mejorar el uso de las etiquetas LB2T en TDGMA.

- *Cálculo de los pesos de los decisores en problemas TDGMA:* El cálculo de la importancia de los decisores se puede dividir en métodos subjetivos, métodos objetivos y métodos que combinan los enfoques objetivo y subjetivo [42, 178]. Los métodos que obtienen el peso subjetivo, como el proceso de jerarquía analítica (analytic hierarchy process, AHP) [146] y los métodos Delphi [73], asignan pesos a los decisores en función de características subjetivas como su formación, nivel profesional y experiencia con los problemas de toma de decisiones. Los métodos de cálculo del peso objetivo [85], como el peso obtenido de la entropía [46], la técnica de orden de preferencia por similitud a una solución ideal (the order preference technique for similarity to an ideal solution, TOPSIS) [68] y los métodos de proyección [204], etc., son algunos de los más usados. Los métodos mixtos (subjetivos y objetivos) para calcular los pesos de los decisores combinan los pesos subjetivos y objetivos para obtener los pesos de los decisores [116, 147, 176]. Cuando los pesos de los decisores no se dan por adelantado, el método objetivo de cálculo de las ponderaciones es importante. Por tanto, es un desafío encontrar una forma más eficaz y adecuada de determinar los pesos de los decisores para los problemas de TDGMA con evaluaciones lingüísticas.

- *Clustering para manejar grandes grupos:* Los métodos de clustering pueden simplificar eficazmente el PAC cuando hay una gran cantidad de decisores involucrados en el problema. Por tanto, es importante aplicar clustering para resolver problemas de TDGMA. Muchos investigadores se han centrado en el método de clustering utilizado, como el algoritmo de clustering k-means [187], fuzzy c-means [151], clustering jerárquico [21], etc. Usando un método de clustering, los decisores se pueden dividir en varios grupos pequeños, por lo que las preferencias de los decisores tienen mayor consistencia y menor grado de conflicto para cada grupo. Sin embargo, los métodos de clustering existentes son complejos de aplicare ignoran el grado de soporte en cada alternativa de diferentes decisores.

Por lo tanto, es necesario desarrollar un nuevo método de clustering basado en el grado de soporte de cada alternativa de los decisores para obtener más información durante el PAC.

- *La consistencia y el consenso de las opiniones de los decisores:* La consistencia y el consenso son otros retos dignos de mención en el proceso TDGMA. La consistencia está directamente relacionada con la credibilidad de los resultados. El consenso, por otro lado, significa el acuerdo de los decisores para aceptar los resultados del proceso. Durante el PAC, algunos decisores no modifican sus opiniones, lo que podría suceder cuando no hay tiempo suficiente para persuadirlos o cuando mantienen su grado de tolerancia. Los decisores podrían aceptar modificar sus preferencias como máximo un valor que esté dentro de su grado de tolerancia. Por lo tanto, es un reto coordinar las preferencias de los decisores que no quieren modificar sus preferencias y el proceso de feedback automático con el grado de aceptación y tolerancia de la opinión modificada de los decisores.

En los problemas de TDGMA del mundo real, los retos encontrados si son superados pueden hacer que los enfoques de TDGMA satisfagan mejor las situaciones y necesidades en la toma de decisiones. Esta memoria de investigación se centra en estudiar en profundidad dichos retos y como superarlos.

A.2 Objetivos

Según los retos señalados anteriormente en los enfoques de TDGMA basados en etiquetas LB2T, esta memoria de investigación se centra en proponer nuevos modelos que permitan hacer frente a los retos indicados.

En base a tal propósito, se consideran los siguientes tres objetivos de investigación:

1. Desarrollar un modelo computacional LB2T que considere la información de dos dimensiones con etiquetas LB2T desde una perspectiva estocástica y permita comparar los modelos computacionales mediante un caso de estudio. Además, se introducirán algunos nuevos operadores de agregación y reglas de comparación para mejorar los estudios anteriores.

2. Considerar el grado de fiabilidad de las preferencias modificadas durante el PAC. En general, las preferencias iniciales las proporcionan los decisores mediante términos lingüísticos, y las preferencias modificadas se presentan mediante un término lingüístico o la extensión de un término lingüístico, como el valor lingüístico 2-tupla. En cualquier caso, falta la información de fiabilidad de las preferencias modificadas. Por tanto, es necesario otra dimensión de información lingüística para representar la fiabilidad de las preferencias modificadas. Considerando el ajuste mínimo durante el PAC, se propone un modelo de consenso de ajuste mínimo de dos etapas basado en información lingüística para mostrar la preferencia modificada obtenida y su fiabilidad. Además, se discuten las relaciones entre la fiabilidad de las preferencias modificadas y la distancia entre la preferencia original y la modificada.

3. Definir un modelo de TDGMA en el que se manejen grandes grupos de decisores y se considere el grado de tolerancia de los decisores al cambiar de opinión. Además, se desarrollará un método de clustering basado en el grado de soporte para clasificar a los decisores en varios subgrupos haciendo más manejable situaciones con gran cantidad de decisores. Se considerará el grado de tolerancia de los decisores para mejorar la fiabilidad de las opiniones modificadas, y se presentará un modelo de consenso de ajuste mínimo con dos reglas de consenso para mejorar gradualmente el nivel de consenso. Eventualmente, las preferencias modificadas se modelarán con etiquetas LB2T. Usando la forma de comparación propuesta entre LB2T, se puede obtener un ranking de alternativas.

A.3 Estructura

Para alcanzar los objetivos presentados en el apartado 1.2, y teniendo en cuenta el artículo 23, punto 3, de la normativa vigente de Estudios de Doctorado en la Universidad de Jaén, de acuerdo con el programa establecido en el RD 99/2011, esta memoria de investigación se presenta como un compendio de artículos publicados por la estudiante de doctorado durante su período de doctorado.

Se han publicado dos artículos en revistas internacionales indexadas por la base de datos JCR, producida por ISI y una contribución en el congreso IEEE CIS

International Conference on Fuzzy Systems 2020 (Clasificación en la lista de conferencias Core Ranking 2020 como CORE A). En resumen, el informe se compone de un total de dos artículos que han sido publicados en revistas internacionales de alta calidad (Q1) y una contribución a la conferencia CORE A.

La estructura de esta memoria de investigación se describe brevemente a continuación:

- Capítulo 2: Se revisan algunos conceptos básicos que se utilizan en la memoria de investigación para lograr nuestros objetivos, tales como conceptos relacionados con la toma de decisiones, TDG, TDGMA, TDGMA bajo incertidumbre, TDGMA basado en información lingüística. Y se introducen brevemente los métodos y modelos que se utilizan en nuestras propuestas, tales como, el enfoque lingüístico difuso, el modelo lingüístico 2-tupla, la etiqueta lingüística 2-tupla de dos dimensiones, proceso de consenso, el modelo de coste de ajuste mínimo etc.

- Capítulo 3: Se introducen brevemente las propuestas publicadas que componen la memoria de investigación, además, se presenta una breve discusión de los resultados obtenidos para esclarecer los logros alcanzados en nuestra investigación.

- Capítulo 4: Este capítulo es el núcleo de esta tesis doctoral, ya que recoge las publicaciones obtenidas como resultado de la investigación. Para cada publicación se indica la revista en la que se ha publicado, así como su factor de impacto y cuartil.

- Capítulo 5: Se señalan las conclusiones finales sobre esta investigación y posibles trabajos futuros.

A.4 Resumen

Los términos lingüísticos son más intuitivos y cercanos al lenguaje usado por los seres humanos para representar las preferencias de los decisores que participan en

los problemas de TDGMA. Por tanto, se han investigado ampliamente los enfoques de TDGMA que utilizan información lingüística. Las metodologías y modelos existentes que manejan información lingüística no hubieran sido posibles sin metodologías para llevar a cabo los procesos de computación con palabras [87, 209]. El modelo lingüístico 2-tupla [60] fue introducido para evitar la pérdida de información y obtener resultados más interpretables y precisos durante los procesos de computación con palabras. Como consecuencia es uno de los modelos computacionales lingüísticos más utilizados en TDGMA. Por lo tanto, una revisión profunda en la literatura especializada muestra el rápido crecimiento y aplicabilidad del modelo de representación lingüística de 2-tupla, que se ha aplicado a distintos problemas del mundo real. Sin embargo, con el aumento de la complejidad de los problemas de toma de decisiones, la información lingüística de una dimensión no siempre es suficiente para que los decisores tomen decisiones con precisión. Por lo tanto, se proponen etiquetas LB2T, que son una extensión del modelo lingüístico 2-tupla [60]. Consecuentemente, en los últimos años se han estudiado los enfoques correspondientes para resolver problemas TDGMA basados en información LB2T. Sin embargo, todavía existen algunas limitaciones en los estudios existentes, tales como, la precisión del modelo computacional y la agregación de las etiquetas LB2T, el PAC durante la TDGMA teniendo en cuenta la fiabilidad de las preferencias modificadas, el grado de tolerancia de los decisores cuando se sugiere modificar las preferencias originales, etc. Para superar estos retos, esta investigación ha realizado las siguientes propuestas.

1. Se han presentado unas funciones para transformar una etiqueta LB2T en una variable estocástica y su inversa. Además, para la comparación y la medición de la similitud entre dos etiquetas LB2T se han desarrollado operadores teniendo en cuenta el grado de importancia relativa de las dos dimensiones de la información. Es evidente que el nuevo modelo de representación y computación de LB2T puede reflejar la influencia de la información de la segunda dimensión en los resultados de la decisión final, y también se han discutido el impacto de los diferentes métodos en los resultados de la decisión final.

2. Se ha propuesto un modelo de consenso de ajuste mínimo en dos etapas, que no solo considera el ajuste mínimo, sino que también minimiza el número de preferencias modificadas. La primera etapa es maximizar la mejora del nivel de consenso para cada par de alternativas con un ajuste mínimo. La segunda etapa es obtener las preferencias ajustadas con un

determinado nivel de consenso en la primera etapa dentro de un ajuste mínimo. La información de la segunda dimensión se puede obtener a través de un modelo matemático como la fiabilidad de las preferencias modificadas, que evitan la subjetividad de la información lingüística. Se construyen las relaciones entre el ajuste de preferencia total y el grado de fiabilidad de las preferencias ajustadas, lo que mejora la precisión del ranking de alternativas.

3. Se ha propuesto un modelo de consenso basado en la consideración del grado de tolerancia de los decisores, siguiendo dos reglas de consenso: ajuste mínimo entre preferencias originales y modificadas, y el número mínimo de preferencias modificadas. Mediante el uso de la función de comparación para LB2Ts, las expresiones LB2T utilizadas para describir la evaluación general brindan más información para la toma de decisiones y mejoran la fiabilidad del ranking de alternativas.

A.5 Conclusiones y Trabajos Futuros

El capítulo 5 concluye nuestra memoria de investigación revisando las conclusiones sobre las principales propuestas y resultados obtenidos, y señalando posibles trabajos futuros.

A.5.1 Conclusiones

La TDG se utiliza ampliamente en la vida real para resolver problemas importantes y complicados en una variedad de dominios, como la toma de decisiones de emergencia [51, 71, 180], la evaluación de servicios médicos [150, 179], la selección de proveedores [16, 17, 48], etc. Dada la importancia de TDG en la selección y evaluación de la gestión y los problemas económicos, se han propuesto muchos modelos y enfoques para los problemas de TDG [81, 103, 122, 152].

La TDGMA implica que los decisores proporcionan evaluaciones con respecto al desempeño de las alternativas bajo múltiples criterios [75]. Con el aumento de la complejidad de los problemas de toma de decisiones y la limitación de la expresión de los decisores, la TDGMA basada en la evaluación lingüística ha atraído más atención [129, 132, 143, 196]. Teniendo en cuenta la complejidad y singularidad de la expresión lingüística, los métodos TDGMA generales no siempre son adecuados para resolver problemas de TDGMA con información lingüística. A pesar de que la investigación existente ha obtenido numerosos logros [27, 101, 188], todavía hay muchos métodos que deben mejorarse. Además, en algunas situaciones, el uso de solo una dimensión de la información lingüística no es suficiente para garantizar la precisión de la información inicial, la fiabilidad de la evaluación también es un factor importante a considerar. Por tanto, el estudio de LB2T es necesario y significativo. Además, es necesario desarrollar varios métodos nuevos con el objetivo de resolver TDGMA basados en información LB2T.

A lo largo de nuestra memoria de investigación hemos obtenido resultados novedosos, destacables y relevantes respecto a aquellos retos que no solo cumplen con los objetivos señalados en el apartado 1.2, sino que también aportan nuevas visiones en los procesos de resolución de TDGMA basados en etiquetas LB2T y nuevas oportunidades de investigación para el futuro.

En consecuencia, debemos concluir de los resultados de nuestra investigación que:

1. A pesar de la aplicación exitosa de la información lingüística bidimensional mediante el modelo de representación y computación de etiquetas lingüísticas bidimensionales, no se había explorado el análisis de la incertidumbre de las evaluaciones según la información de la segunda dimensión. Así, se ha propuesto un nuevo modelo de representación de LB2T desde una perspectiva estocástica. Se han presentado funciones de transformación entreuna etiqueta LB2T y una variable estocástica y su inversa, que es más adecuada para el cálculo de TDG a gran escala. La comparación y medida de similitud entre dos etiquetas LB2T se ha desarrollado teniendo en cuenta el grado de importancia relativa de las dos dimensiones de información desde la perspectiva estocástica, lo que ha hecho que el análisis de decisiones brinde información más útil.

2. La fiabilidad de la preferencia inicial proporcionada por los decisores generalmente se presenta como información lingüística de dos dimensiones para problemas TDGMA basados en información lingüística, sin embargo, durante el PAC, especialmente para los PAC automáticos, la fiabilidad de las preferencias modificadas a menudo se obvia. Con base a

esta observación, se ha propuesto un modelo de consenso de ajuste mínimo de dos etapas para problemas de TDG a gran escala basado en expresiones LB2T con la consideración del grado de fiabilidad de las preferencias ajustadas, que no solo considera el ajuste mínimo, sino que también minimiza el número de preferencias ajustadas. Y se han discutido las relaciones entre el ajuste de preferencia total y el grado de fiabilidad de las preferencias modificadas. El método propuesto ha completado el enfoque lingüístico de dos dimensiones y 2-tupla para la TDG a gran escala.

3. La medición de la fiabilidad de las opiniones modificadas puede mejorar la precisión de la toma de decisiones con los PAC automáticos. Sin embargo, la fiabilidad se basa en la tolerancia de los decisores al cambiar sus opiniones. Por tanto, el grado de tolerancia de los decisores es un factor importante a considerar de antemano durante el PAC. La propuesta del nuevo modelo de consenso basado en el grado de tolerancia de los decisores ha mejorado el método de TDGMA con información LB2T basada en la medición de fiabilidad.

A.5.2 Trabajos Futuros

Aunque se han propuesto varios métodos, herramientas y enfoques en esta investigación, todavía existen retos dentro de la TDG basados en la evaluación lingüística y el enfoque LB2T que deben estudiarse más a fondo. En un futuro próximo, nos centraremos en la extensión de las propuestas presentadas y el desarrollo de soluciones para nuevos problemas:

1. En la mayoría de los estudios existentes, los decisores se consideran totalmente racionales, pero en la realidad, los no lo son ya que presentan ciertas limitaciones respecto a su racionalidad a la hora de tomar decisiones. Debido a esta racionalidad limitada algunos decisores pueden no estar cómodos cuando se les sugiere que ajusten sus opiniones en un rango menor y limitado, por lo que se pueden producir comportamientos no cooperativos en la TDG. . Esto nos lleva a la necesidad de estudiar y considerar el comportamiento psicológico de los decisores en nuestros trabajos futuros.

2. Para PAC en TDG, a menudo se considera el ajuste mínimo para hacer que las opiniones de los decisores cambien, sin embargo, hacer que los decisores cambien sus opiniones tiene diferentes niveles de dificultad, el coste de ajuste unitario está definido bajo el supuesto de que no esdireccional, de hecho, el coste de ajuste unitario no siempre es igual en las direcciones de ajuste ascendente y descendente [71]. Por lo tanto, también se debe considerar la determinación del coste de ajuste unitario, especialmente para el coste unitario simétrico, que es un reto a considerar para el modelo de consenso de coste mínimo.

3. Como el conjunto de términos lingüísticos establecido para expresar la evaluación inicial a veces no está balanceado, el conjunto de términos lingüísticos para expresar la fiabilidad de las opiniones modificadas también podría ser no balanceado, por tanto, cómo diseñar el modelo de representación y computación con la información de fiabilidad no balanceada se estudiará en el futuro.

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Bibliography

[1] M. A. Abchir and I. Truck, Towards an extension of the 2-tuple linguistic model to deal with unbalanced linguistic term sets, *Kybernetika,* 49(1):164-180, 2013.

[2] O. Alagoz, H. Hsu, A. J. Schaefer, and M. S. Roberts, Markov decision processes: a tool for sequential decision making under uncertainty, *Medical Decision Making,* 30(4):474-483, 2010.

[3] K. Anagnostopoulos, H. Doukas, and J. Psarras, A linguistic multicriteria analysis system combining fuzzy sets theory, ideal and anti-ideal points for location site selection, *Expert Systems with Applications,* 35(4):2041-2048, 2008.

[4] N. O. Attoh-Okine and B. M. Ayyub, *Applied research in uncertainty modeling and analysis*. Springer, 2005.

[5] J. Balagna, C. R. Williams, J. Wang, S. Burch, E. Dalton, J. Kirchick, and P. Sosa, Consensus-driven approach for decision-making in diverse groups, *American Public Health Association,* 110(1):5-5, 2020.

[6] N. Banaeian, H. Mobli, B. Fahimnia, I. E. Nielsen, and M. Omid, Green supplier selection using fuzzy group decision making methods: A case study from the agri-food industry, *Computers & Operations Research,* 89:337-347, 2018.

[7] T. Bao, X. Xie, P. Long, and Z. Wei, MADM method based on prospect theory and evidential reasoning approach with unknown attribute weights under intuitionistic fuzzy environment, *Expert Systems with Applications,* 88:305-317, 2017.

[8] K. Barr and A. Mintz, Public Policy Perspective on Group Decision - Making Dynamics in Foreign Policy, *Policy Studies Journal,* 46:69-90, 2018.

[9] D. Ben-Arieh and T. Easton, Multi-criteria group consensus under linear cost opinion elasticity, *Decision Support Systems,* 43(3):713-721, 2007.

[10] D. Ben-Arieh, T. Easton, and B. Evans, Minimum cost consensus with quadratic cost functions, *IEEE Transactions on Systems, Man, and Cybernetics - Part A: Systems and Humans,* 39(1):210-217, 2009.

[11] J. C. Bezdek, R. Ehrlich, and W. Full, FCM: The fuzzy c-means clustering

algorithm, *Computers & geosciences,* 10(2-3):191-203, 1984.

[12] E. H. Bowman, Production scheduling by the transportation method of linear programming, *Operations Research,* 4(1):100-103, 1956.

[13] P. Byrne, *Risk, uncertainty and decision-making in property*. Routledge, 2002.

[14] N. Capuano, F. Chiclana, H. Fujita, E. Herrera-Viedma, and V. Loia, Fuzzy group decision making with incomplete information guided by social influence, *IEEE Transactions on Fuzzy Systems,* 26(3):1704-1718, 2017.

[15] E. Celik, M. Gul, M. Yucesan, and S. Mete, Stochastic multi-criteria decision-making: an overview to methods and applications, *Beni-Suef University Journal of Basic and Applied Sciences,* 8(1):1-11, 2019.

[16] J. Chai and E. W. Ngai, Decision-making techniques in supplier selection: Recent accomplishments and what lies ahead, *Expert Systems with Applications,* 140:112903, 2020.

[17] J. Chai, J. N. Liu, and E. W. Ngai, Application of decision-making techniques in supplier selection: A systematic review of literature, *Expert systems with applications,* 40(10):3872-3885, 2013.

[18] X. Chao, G. Kou, Y. Peng, and E. H. Viedma, Large-Scale Group Decision-Making with Non-cooperative Behaviors and Heterogeneous Preferences: An Application in Financial Inclusion, *European Journal of Operational Research*, 2020.

[19] X. Chao, G. Kou, Y. Peng, and E. H. Viedma, Large-scale group decision-making with non-cooperative behaviors and heterogeneous preferences: an application in financial inclusion, *European Journal of Operational Research,* 288(1):271-293, 2021.

[20] X. Chao, G. Kou, Y. Peng, E. Herrera-Viedma, and F. Herrera, An efficient consensus reaching framework for large-scale social network group decision making and its application in urban resettlement, *Information Sciences,* 575:499-527, 2021.

[21] Z.-S. Chen, L. Martínez, K.-S. Chin, and K.-L. Tsui, Two-stage aggregation paradigm for HFLTS possibility distributions: A hierarchical clustering perspective, *Expert Systems with applications,* 104:43-66, 2018.

[22] D. Cheng, F. Cheng, Z. Zhou, and Y. Wu, Reaching a minimum adjustment consensus in social network group decision-making, *Information Fusion,*

59:30-43, 2020.

[23] F. Chiclana, F. Herrera, E. Herrera-Viedma, and M. Poyatos, A classification method of alternatives for multiple preference ordering criteria based on fuzzy majority, *Journal of Fuzzy Mathematics,* 4:801-814, 1996.

[24] F. Chiclana, J. T. García, M. J. del Moral, and E. Herrera-Viedma, A statistical comparative study of different similarity measures of consensus in group decision making, *Information Sciences,* 221:110-123, 2013.

[25] K.-S. Chin, C. Fu, and Y. Wang, A method of determining attribute weights in evidential reasoning approach based on incompatibility among attributes, *Computers & Industrial Engineering,* 87:150-162, 2015.

[26] A. J. Chorin and O. H. Hald, *Stochastic tools in mathematics and science*. Springer, 2009.

[27] A. Cid-López, M. J. Hornos, R. A. Carrasco, and E. Herrera-Viedma, Applying a linguistic multi-criteria decision-making model to the analysis of ICT suppliers' offers, *Expert Systems with Applications,* 57:127-138, 2016.

[28] A. J. Conejo, M. Carrión, and J. M. Morales, *Decision making under uncertainty in electricity markets*. Springer, 2010.

[29] J. H. Dahooie, S. H. R. Hajiagha, S. Farazmehr, E. K. Zavadskas, and J. Antucheviciene, A novel dynamic credit risk evaluation method using data envelopment analysis with common weights and combination of multi-attribute decision-making methods, *Computers & Operations Research,* 129:105223, 2021.

[30] D. Dalalah, M. Hayajneh, and F. Batieha, A fuzzy multi-criteria decision making model for supplier selection, *Expert systems with applications,* 38(7):8384-8391, 2011.

[31] G. DeSanctis and R. B. Gallupe, A foundation for the study of group decision support systems, *Management science,* 33(5):589-609, 1987.

[32] X. F. Ding and H. C. Liu, A 2-dimension uncertain linguistic DEMATEL method for identifying critical success factors in emergency management, *Applied Soft Computing,* 71:386-395, 2018.

[33] X. F. Ding and H. C. Liu, An extended prospect theory–VIKOR approach for emergency decision making with 2-dimension uncertain linguistic information, *Soft Computing,* 23(22):12139-12150, 2019.

[34] Y. C. Dong, Y. F. Xu, and S. Yu, Computing the numerical scale of the linguistic term set for the 2-tuple fuzzy linguistic representation model, *IEEE Transactions on Fuzzy Systems,* 17(6):1366-1378, 2009.

[35] Y. C. Dong, Y. F. Xu, H. Y. Li, and B. Feng, The OWA-based consensus operator under linguistic representation models using position indexes, *European Journal of Operational Research,* 203(2):455-463, 2010.

[36] Y. C. Dong, G. Q. Zhang, W. C. Hong, and S. Yu, Linguistic computational model based on 2-tuples and intervals, *IEEE Transactions on Fuzzy Systems,* 21(6):1006-1018, 2013.

[37] Y. C. Dong, C. C. Li, Y. F. Xu, and X. Gu, Consensus-based group decision making under multi-granular unbalanced 2-tuple linguistic preference relations, *Group Decision and Negotiation,* 24(2):217-242, 2015.

[38] Y. C. Dong, Y. Z. Wu, H. J. Zhang, and G. Q. Zhang, Multi-granular unbalanced linguistic distribution assessments with interval symbolic proportions, *Knowledge-Based Systems,* 82:139-151, 2015.

[39] R. Durrett, *Probability: theory and examples*. Cambridge university press, 2019.

[40] T. Erdem and M. P. Keane, Decision-making under uncertainty: Capturing dynamic brand choice processes in turbulent consumer goods markets, *Marketing science,* 15(1):1-20, 1996.

[41] Z.-P. Fan, Y. Liu, and B. Feng, A method for stochastic multiple criteria decision making based on pairwise comparisons of alternatives with random evaluations, *European Journal of Operational Research,* 207(2):906-915, 2010.

[42] C. Fu and Y. Wang, An interval difference based evidential reasoning approach with unknown attribute weights and utilities of assessment grades, *Computers & Industrial Engineering,* 81:109-117, 2015.

[43] J. Fülöp, Introduction to decision making methods, in *BDEI-3 workshop, Washington*, pages 1-15, 2005.

[44] S. I. Gass, *Linear programming: methods and applications*. Courier Corporation, 2003.

[45] X. Gou and Z. Xu, Managing noncooperative behaviors in large-scale group decision-making with linguistic preference orderings: The application in Internet Venture Capital, *Information Fusion,* 69:142-155, 2021.

[46] S. Guiaşu, Weighted entropy, *Reports on Mathematical Physics,* 2(3):165-179, 1971.

[47] E. Güney, An efficient linear programming based method for the influence maximization problem in social networks, *Information Sciences,* 503:589-605, 2019.

[48] S. Gupta, U. Soni, and G. Kumar, Green supplier selection using multi-criterion decision making under fuzzy environment: A case study in automotive industry, *Computers & Industrial Engineering,* 136:663-680, 2019.

[49] F. Haag, J. Lienert, N. Schuwirth, and P. Reichert, Identifying non-additive multi-attribute value functions based on uncertain indifference statements, *Omega,* 85:49-67, 2019.

[50] Y. B. Haim, "Info-gap decision theory," ed: California: Academic Press, 2006.

[51] Z. Hao, Z. Xu, H. Zhao, and H. Fujita, A dynamic weight determination approach based on the intuitionistic fuzzy Bayesian network and its application to emergency decision making, *IEEE Transactions on Fuzzy Systems,* 26(4):1893-1907, 2017.

[52] P. T. Harker, Alternative modes of questioning in the analytic hierarchy process, *Mathematical modelling,* 9(3-5):353-360, 1987.

[53] Y. He, H. Guo, M. Jin, and P. Ren, A linguistic entropy weight method and its application in linguistic multi-attribute group decision making, *Nonlinear Dynamics,* 84(1):399-404, 2016.

[54] E. Herrera-Viedma, F. Herrera, and F. Chiclana, A consensus model for multiperson decision making with different preference structures, *IEEE Transactions on Systems, Man, and Cybernetics-Part A: Systems and Humans,* 32(3):394-402, 2002.

[55] E. Herrera-Viedma, F. Herrera, F. Chiclana, and M. Luque, Some issues on consistency of fuzzy preference relations, *European journal of operational research,* 154(1):98-109, 2004.

[56] E. Herrera-Viedma, L. Martínez, F. Mata, and F. Chiclana, A consensus support system model for group decision-making problems with multigranular linguistic preference relations, *IEEE Transactions on Fuzzy Systems,* 13(5):644-658, 2005.

[57] E. Herrera-Viedma, L. Martínez, F. Mata, and F. Chiclana, A consensus

support system model for group decision-making problems with multigranular linguistic preference relations, *IEEE Transactions on fuzzy Systems,* 13(5):644-658, 2005.

[58] E. Herrera-Viedma, S. Alonso, F. Chiclana, and F. Herrera, A consensus model for group decision making with incomplete fuzzy preference relations, *IEEE Transactions on fuzzy Systems,* 15(5):863-877, 2007.

[59] E. Herrera-Viedma, F. Javier Cabrerizo, J. Kacprzyk, and W. Pedrycz, A review of soft consensus models in a fuzzy environment, *Information Fusion,* 17:4-13, 2014.

[60] F. Herrera and L. Martínez, A 2-tuple fuzzy linguistic representation model for computing with words, *IEEE Transactions on Fuzzy Systems,* 8(6):746-752, 2000.

[61] F. Herrera and E. Herrera-Viedma, Linguistic decision analysis: steps for solving decision problems under linguistic information, *Fuzzy Sets and systems,* 115(1):67-82, 2000.

[62] F. Herrera and L. Martínez, A model based on linguistic 2-tuples for dealing with multigranular hierarchical linguistic contexts in multi-expert decision-making, *IEEE Transactions on Systems, Man, and Cybernetics, Part B* 31(2):227-234, 2001.

[63] F. Herrera and L. Martinez, The 2-tuple linguistic computational model: Advantages of its linguistic description, accuracy and consistency, *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems,* 9(supp01):33-48, 2001.

[64] F. Herrera, E. Herrera-Viedma, and J. L. Verdegay, Direct approach processes in group decision making using linguistic OWA operators, *Fuzzy Sets and Systems,* 79(2):175-190, 1996.

[65] F. Herrera, E. Herrera-Viedma, and F. Chiclana, Multiperson decision-making based on multiplicative preference relations, *European journal of operational research,* 129(2):372-385, 2001.

[66] C. A. Holloway and C. E. Holloway, *Decision making under uncertainty: models and choices*. Prentice Hall, 1979.

[67] V.-N. Huynh and Y. Nakamori, A satisfactory-oriented approach to multiexpert decision-making with linguistic assessments, *IEEE Transactions on Systems, Man, and Cybernetics, Part B,* 35(2):184-196, 2005.

[68] C.-L. Hwang and K. Yoon, "Methods for multiple attribute decision making," in *Multiple attribute decision making*: Springer, 1981, pp. 58-191.

[69] G. R. Jahanshahloo, F. H. Lotfi, and A. Davoodi, Extension of TOPSIS for decision-making problems with interval data: Interval efficiency, *Mathematical and Computer Modelling,* 49(5-6):1137-1142, 2009.

[70] Y. Ji, H. H. Li, and H. J. Zhang, Risk-averse two-stage stochastic minimum cost consensus models with asymmetric adjustment cost, *Group Decision and Negotiation,* https://doi.org/10.1007/s10726-021-09752-z, 2021.

[71] G.-J. Jiang, H.-X. Chen, H.-H. Sun, M. Yazdi, A. Nedjati, and K. A. Adesina, An improved multi-criteria emergency decision-making method in environmental disasters, *Soft Computing*:1-29, 2021.

[72] Y. Ju, Y. Liang, M. Luis, E. D. S. Gonzalez, M. Giannakis, P. Dong, and A. Wang, A new framework for health-care waste disposal alternative selection under multi-granular linguistic distribution assessment environment, *Computers & Industrial Engineering,* 145:106489, 2020.

[73] R. C. Judd, Use of Delphi methods in higher education, *Technological forecasting and social change,* 4(2):173-186, 1972.

[74] D. Julong, Introduction to grey system theory, *The Journal of grey system,* 1(1):1-24, 1989.

[75] Ö. Kabak and B. Ervural, Multiple attribute group decision making: A generic conceptual framework and a classification scheme, *Knowledge-Based Systems,* 123:13-30, 2017.

[76] J. Kacprzyk, M. Fedrizzi, and H. Nurmi, Group decision making and consensus under fuzzy preferences and fuzzy majority, *Fuzzy Sets and Systems,* 49(1):21-31, 1992.

[77] J. Kacprzyk, S. Zadrozny, and Z. W. Ras, How to support consensus reaching using action rules: a novel approach, *International Journal of Uncertainty Fuzziness and Knowledge-Based Systems,* 18(4):451-470, 2010.

[78] N. H. Kamis, F. Chiclana, and J. Levesley, Preference similarity network structural equivalence clustering based consensus group decision making model, *Applied Soft Computing,* 67:706-720, 2018.

[79] A. Kandel, *Fuzzy mathematical techniques with applications*. Addison-Wesley Longman Publishing Co., Inc., 1986.

[80] G. A. Keskin, Using integrated fuzzy DEMATEL and fuzzy C: means algorithm for supplier evaluation and selection, *International Journal of Production Research,* 53(12):3586-3602, 2015.

[81] S. H. Kim and B. S. Ahn, Interactive group decision making procedure under incomplete information, *European Journal of Operational Research,* 116(3):498-507, 1999.

[82] S. H. Kim, S. H. Choi, and J. K. Kim, An interactive procedure for multiple attribute group decision making with incomplete information: Range-based approach, *European Journal of Operational Research,* 118(1):139-152, 1999.

[83] C. W. Kirkwood, Strategic decision making, *Duxbury Press,* 149, 1997.

[84] M. J. Kochenderfer, *Decision making under uncertainty: theory and application*. MIT press, 2015.

[85] E. Koksalmis and Ö. Kabak, Deriving decision makers' weights in group decision making: An overview of objective methods, *Information Fusion,* 49:146-160, 2019.

[86] A. Kolios, V. Mytilinou, E. Lozano-Minguez, and K. Salonitis, A comparative study of multiple-criteria decision-making methods under stochastic inputs, *Energies,* 9(7):566, 2016.

[87] J. K. L. Zadeh, *Computing with words in Information/Intelligent systems 1 – Part 1: Foundations; Part 2: Applications*. Heidelberg, Germany: Physica-Verlag, 1999.

[88] Á. Labella, H. B. Liu, R. M. Rodríguez, and L. Martínez, A cost consensus metric for consensus reaching processes based on a comprehensive minimum cost model, *European Journal of Operational Research,* 281(2):316-331, 2020.

[89] R. Lahdelma and P. Salminen, SMAA-2: Stochastic multicriteria acceptability analysis for group decision making, *Operations research,* 49(3):444-454, 2001.

[90] P. Lattimore and A. Witte, Models of decision making under uncertainty: The criminal choice, *The reasoning criminal*:129-155, 2017.

[91] H. Levy and M. Robinson, *Stochastic dominance: Investment decision making under uncertainty*. Springer, 2006.

[92] C.-C. Li, Y. Dong, and F. Herrera, A consensus model for large-scale linguistic group decision making with a feedback recommendation based on clustered personalized individual semantics and opposing consensus groups, *IEEE Transactions on Fuzzy Systems,* 27(2):221-233, 2018.

[93] C. C. Li, Y. C. Dong, and F. Herrera, A consensus model for large-scale linguistic group decision making with a feedback recommendation based on clustered personalized individual semantics and opposing consensus groups, *IEEE Transactions on Fuzzy Systems,* 27(2):221-233, 2018.

[94] D.-F. Li and J.-B. Yang, Fuzzy linear programming technique for multiattribute group decision making in fuzzy environments, *Information Sciences,* 158:263-275, 2004.

[95] G. Li, G. Kou, and Y. Peng, Heterogeneous large-scale group decision making using fuzzy cluster analysis and its application to emergency response plan selection, *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, 2021.

[96] H. Li and V. C. Yen, *Fuzzy sets and fuzzy decision-making*. CRC press, 1995.

[97] M. Li, Y. Li, Q. Peng, and J. Wang, A hybrid MCDM model combining DANP with TODIM to evaluate the information quality of community question answering in a two - dimensional linguistic environment, *Expert Systems,* 38(2):e12619, 2021.

[98] Y. Li and Y. Liu, Extended VIKOR-QUALIFLEX method based on trapezoidal fuzzy two-dimensional linguistic information for multiple attribute decision-making with unknown attribute weight, *Mathematics,* 9(1):37, 2021.

[99] Y. Li, Y. Wang, and P. Liu, Multiple attribute group decision-making methods based on trapezoidal fuzzy two-dimension linguistic power generalized aggregation operators, *Soft Computing,* 20(7):2689-2704, 2016.

[100] Y. Li, X. Chen, Y. Dong, and F. Herrera, Linguistic group decision making: Axiomatic distance and minimum cost consensus, *Information Sciences,* 541:242-258, 2020.

[101] M. Lin, Z. Xu, Y. Zhai, and Z. Yao, Multi-attribute group decision-making under probabilistic uncertain linguistic environment, *Journal of the Operational Research Society*:1-15, 2017.

[102] Y.-H. Lin, P.-C. Lee, and H.-I. Ting, Dynamic multi-attribute decision

making model with grey number evaluations, *Expert Systems with Applications,* 35(4):1638-1644, 2008.

[103] B. Liu, Q. Zhou, R.-X. Ding, I. Palomares, and F. Herrera, Large-scale group decision making model based on social network analysis: Trust relationship-based conflict detection and elimination, *European Journal of Operational Research,* 275(2):737-754, 2019.

[104] H.-C. Liu, Y.-P. Hu, J.-J. Wang, and M. Sun, Failure mode and effects analysis using two-dimensional uncertain linguistic variables and alternative queuing method, *IEEE Transactions on Reliability,* 68(2):554-565, 2018.

[105] P. Liu and F. Teng, An extended TODIM method for multiple attribute group decision - making based on 2 - dimension uncertain linguistic variable, *Complexity,* 21(5):20-30, 2016.

[106] P. Liu and F. Teng, Multiple attribute decision-making method based on 2-dimension uncertain linguistic density generalized hybrid weighted averaging operator, *Soft Computing,* 22(3):797-810, 2018.

[107] P. Liu and Y. Li, Multiattribute decision method for comprehensive logistics distribution center location selection based on 2-dimensional linguistic information, *Information Sciences,* 538:209-244, 2020.

[108] P. Liu, H. Diao, L. Zou, and A. Deng, Uncertain multi-attribute group decision making based on linguistic-valued intuitionistic fuzzy preference relations, *Information Sciences,* 508:293-308, 2020.

[109] P. Liu, F. Jin, X. Zhang, Y. Su, and M. Wang, Research on the multi-attribute decision-making under risk with interval probability based on prospect theory and the uncertain linguistic variables, *Knowledge-based systems,* 24(4):554-561, 2011.

[110] P. D. Liu and X. C. Yu, 2-Dimension uncertain linguistic power generalized weighted aggregation operator and its application in multiple attribute group decision making, *Knowledge-Based Systems,* 57:69-80, 2014.

[111] S. Liu, F. T. Chan, and W. Ran, Multi-attribute group decision-making with multi-granularity linguistic assessment information: An improved approach based on deviation and TOPSIS, *Applied Mathematical Modelling,* 37(24):10129-10140, 2013.

[112] W. Q. Liu, Y. C. Dong, F. Chiclana, F. J. Cabrerizo, and E. Herrera-Viedma,

Group decision-making based on heterogeneous preference relations with self-confidence, *Fuzzy Optimization and Decision Making,* 16(4):429-447, 2017.

[113] J. Lu and D. Ruan, *Multi-objective group decision making: methods, software and applications with fuzzy set techniques*. Imperial College Press, 2007.

[114] Y. L. Lu, Y. J. Xu, E. Herrera-Viedma, and Y. F. Han, Consensus of large-scale group decision making in social network: the minimum cost model based on robust optimization, 547:910-930, Information Sciences, 2021.

[115] F. C. Lunenburg, THE DECISION MAKING PROCESS, in *National Forum of Educational Administration & Supervision Journal*, vol. 27, no. 4, 2010.

[116] J. Ma, Z.-P. Fan, and L.-H. Huang, A subjective and objective integrated approach to determine attribute weights, *European journal of operational research,* 112(2):397-404, 1999.

[117] X.-B. Mao, M. Wu, J.-Y. Dong, S.-P. Wan, and Z. Jin, A new method for probabilistic linguistic multi-attribute group decision making: Application to the selection of financial technologies, *Applied Soft Computing,* 77:155-175, 2019.

[118] L. Martinez, J. Liu, and J.-B. Yang, A fuzzy model for design evaluation based on multiple criteria analysis in engineering systems, *International Journal of Uncertainty Fuzziness and Knowledge-Based Systems,* 14(3):317-336, 2006.

[119] L. Martinez, R. M. Rodríguez, and F. Herrera, *The 2-tuple Linguistic Model Computing with Words in Decision Making*, 1 ed. Springer International Publishing, 2015.

[120] L. Martínez and F. Herrera, An overview on the 2-tuple linguistic model for computing with words in decision making: Extensions, applications and challenges, *Information Sciences,* 207:1-18, 2012.

[121] J. M. Mendel, L. A. Zadeh, E. Trillas, R. Yager, J. Lawry, H. Hagras, and S. Guadarrama, What computing with words means to me [discussion forum], *IEEE computational intelligence magazine,* 5(1):20-26, 2010.

[122] M. Mohammadi and J. Rezaei, Bayesian best-worst method: A probabilistic group decision making model, *Omega,* 96:102075, 2020.

[123] R. E. Moore, *Methods and applications of interval analysis*. SIAM, 1979.

[124] R. E. Moore, R. B. Kearfott, and M. J. Cloud, *Introduction to interval analysis*. SIAM, 2009.

[125] S. M. Mousavi, F. Jolai, and R. Tavakkoli-Moghaddam, A fuzzy stochastic multi-attribute group decision-making approach for selection problems, *Group Decision and Negotiation,* 22(2):207-233, 2013.

[126] N. Mukherjee, L. V. Dicks, G. E. Shackelford, B. Vira, and W. J. Sutherland, Comparing groups versus individuals in decision making: a systematic review protocol, *Environmental Evidence,* 5(1):1-9, 2016.

[127] A. Noori, H. Bonakdari, A. H. Salimi, and B. Gharabaghi, A group Multi-Criteria Decision-Making method for water supply choice optimization, *Socio-Economic Planning Sciences,* 77:101006, 2021.

[128] I. Palomares and L. Martínez, Low-dimensional visualization of experts' preferences in urgent group decision making under uncertainty, *Procedia Computer Science,* 29:2090-2101, 2014.

[129] J. Pang and J. Liang, Evaluation of the results of multi-attribute group decision-making with linguistic information, *Omega,* 40(3):294-301, 2012.

[130] Z. Pawlak, Rough set theory and its applications to data analysis, *Cybernetics & Systems,* 29(7):661-688, 1998.

[131] M. Pearson, Goodwill hunting and profit sharing: Decision-making in a newspaper chain, *European journal of operational research,* 181(3):1593-1606, 2007.

[132] I. J. Pérez, F. J. Cabrerizo, and E. Herrera-Viedma, Group decision making problems in a linguistic and dynamic context, *Expert Systems with Applications,* 38(3):1675-1688, 2011.

[133] I. J. Pérez, F. J. Cabrerizo, S. Alonso, and E. Herrera-Viedma, A new consensus model for group decision making problems with non-homogeneous experts, *IEEE Transactions on Systems, Man, and Cybernetics: Systems,* 44(4):494-498, 2013.

[134] K. Qi, Q. Wang, Q. Duan, L. Gong, J. Sun, K.-M. Liew, and L. Jiang, A multi criteria comprehensive evaluation approach for emergency response capacity with interval 2-tuple linguistic information, *Applied Soft Computing,* 72:419-441, 2018.

[135] X. Qi, C. Liang, and J. Zhang, Generalized cross-entropy based group

decision making with unknown expert and attribute weights under interval-valued intuitionistic fuzzy environment, *Computers & Industrial Engineering,* 79:52-64, 2015.

[136] F. J. Quesada, I. Palomares, and L. Martínez, "Using computing with words for managing non-cooperative behaviors in large scale group decision making," in *Granular Computing and Decision-Making*: Springer, 2015, pp. 97-121.

[137] J. R. Quinlan, Decision trees and decision-making, *IEEE Transactions on Systems, Man, and Cybernetics,* 20(2):339-346, 1990.

[138] M. Rabiee, B. Aslani, and J. Rezaei, A decision support system for detecting and handling biased decision-makers in multi criteria group decision-making problems, *Expert Systems with Applications,* 171:114597, 2021.

[139] R. Ramanathan, Stochastic decision making using multiplicative AHP, *European Journal of Operational Research,* 97(3):543-549, 1997.

[140] J. Reason, *Human error*. Cambridge university press, 1990.

[141] R. M. Rodriguez, L. Martínez, and F. Herrera, Hesitant fuzzy linguistic term sets for decision making, *IEEE Transactions on Fuzzy Systems,* 20(1):109-119, 2012.

[142] R. M. Rodríguez and L. Martínez, An analysis of symbolic linguistic computing models in decision making, *International Journal of General Systems,* 42(1):121-136, 2013.

[143] R. M. Rodríguez, L. Martínez, and F. Herrera, A group decision making model dealing with comparative linguistic expressions based on hesitant fuzzy linguistic term sets, *Information Sciences,* 241:28-42, 2013.

[144] R. M. Rodríguez, L. Martínez, and F. Herrera, A group decision making model dealing with comparative linguistic expressions based on hesitant fuzzy linguistic term sets, *Information Sciences,* 241:28-42, 2013.

[145] R. M. Rodríguez, Á. Labella, G. De Tré, and L. Martínez, A large scale consensus reaching process managing group hesitation, *Knowledge-Based Systems,* 159:86-97, 2018.

[146] T. L. Saaty, "What is the analytic hierarchy process?," in *Mathematical models for decision support*: Springer, 1988, pp. 109-121.

[147] M. Sahoo, S. Sahoo, A. Dhar, and B. Pradhan, Effectiveness evaluation of objective and subjective weighting methods for aquifer vulnerability assessment in urban context, *Journal of Hydrology,* 541:1303-1315, 2016.

[148] M. K. Sayadi, M. Heydari, and K. Shahanaghi, Extension of VIKOR method for decision making problem with interval numbers, *Applied Mathematical Modelling,* 33(5):2257-2262, 2009.

[149] F. Seo and M. Sakawa, Fuzzy multiattribute utility analysis for collective choice, *IEEE Transactions on Systems, Man, and Cybernetics,* (1):45-53, 1985.

[150] M. Shafii, S. Rafiei, F. Abooee, M. A. Bahrami, M. Nouhi, F. Lotfi, and K. Khanjankhani, Assessment of service quality in teaching hospitals of Yazd University of Medical Sciences: using multi-criteria decision making techniques, *Osong public health and research perspectives,* 7(4):239-247, 2016.

[151] Z. Shi, X. Wang, I. Palomares, S. Guo, and R.-X. Ding, A novel consensus model for multi-attribute large-scale group decision making based on comprehensive behavior classification and adaptive weight updating, *Knowledge-Based Systems,* 158:196-208, 2018.

[152] H.-S. Shih, H.-J. Shyur, and E. S. Lee, An extension of TOPSIS for group decision making, *Mathematical and computer modelling,* 45(7-8):801-813, 2007.

[153] S. Siddharth, S. Saurav, J. Kan, W. Bimlesh, and D. J. Song, Model Driven Inputs to aid Athlete's Decision Making, in *2020 27th Asia-Pacific Software Engineering Conference (APSEC)*, pages 485-489. IEEE, 2020.

[154] R. Słowiński, A multicriteria fuzzy linear programming method for water supply system development planning, *Fuzzy sets and systems,* 19(3):217-237, 1986.

[155] A. Soroudi and T. Amraee, Decision making under uncertainty in energy systems: State of the art, *Renewable and Sustainable Energy Reviews,* 28:376-384, 2013.

[156] A. L. Soyster, Convex programming with set-inclusive constraints and applications to inexact linear programming, *Operations research,* 21(5):1154-1157, 1973.

[157] P. Strandmark, Y. Qu, and T. Curtois, First-order linear programming in a column generation-based heuristic approach to the nurse rostering problem, *Computers & Operations Research,* 120:104945, 2020.

[158] E. Szmidt and J. Kacprzyk, A consensus‐reaching process under intuitionistic fuzzy preference relations, *International Journal of Intelligent Systems,* 18(7):837-852, 2003.

[159] X. Tan, J. Zhu, and Y. Zhang, A consensus reaching process with quantum subjective adjustment in linguistic group decision making, *Information Sciences,* 533:150-168, 2020.

[160] T. Tanino, "On group decision making under fuzzy preferences," in *Multiperson decision making models using fuzzy sets and possibility theory*: Springer, 1990, pp. 172-185.

[161] T. Tervonen, J. R. Figueira, R. Lahdelma, J. A. Dias, and P. Salminen, A stochastic method for robustness analysis in sorting problems, *European Journal of Operational Research,* 192(1):236-242, 2009.

[162] V. Torra, Hesitant fuzzy sets, *International Journal of Intelligent Systems,* 25(6):529-539, 2010.

[163] I. Truck, Comparison and links between two 2-tuple linguistic models for decision making, *Knowledge-Based Systems,* 87:61-68, 2015.

[164] I. Truck and H. Akdag, Manipulation of qualitative degrees to handle uncertainty: formal models and applications, *Knowledge and Information Systems,* 9(4):385-411, 2006.

[165] M. Valipour, S. Yousefi, M. J. Rezaee, and M. Saberi, A clustering-based approach for prioritizing health, safety and environment risks integrating fuzzy C-means and hybrid decision-making methods, *Stochastic Environmental Research and Risk Assessment*:1-20, 2021.

[166] R. Van den Honert, Stochastic group preference modelling in the multiplicative AHP: A model of group consensus, *European Journal of Operational Research,* 110(1):99-111, 1998.

[167] R. Verma and J. M. Merigó, Multiple attribute group decision making based on 2-dimension linguistic intuitionistic fuzzy aggregation operators, *Soft Computing,* 24(22):17377-17400, 2020.

[168] H. Wang and Z. Xu, Some consistency measures of extended hesitant fuzzy linguistic preference relations, *Information Sciences,* 297:316-331, 2015.

[169] J.-Q. Wang, D.-D. Wang, H.-Y. Zhang, and X.-H. Chen, Multi-criteria group decision making method based on interval 2-tuple linguistic information

and Choquet integral aggregation operators, *Soft Computing,* 19(2):389-405, 2015.

[170] J.-W. Wang, Multi-criteria decision-making method based on a weighted 2-tuple fuzzy linguistic representation model, *International Journal of Information Technology & Decision Making,* 20(02):619-634, 2021.

[171] J. Wang, J.-Q. Wang, H.-Y. Zhang, and X.-H. Chen, Multi-criteria group decision-making approach based on 2-tuple linguistic aggregation operators with multi-hesitant fuzzy linguistic information, *International Journal of Fuzzy Systems,* 18(1):81-97, 2016.

[172] J. H. Wang and J. Y. Hao, A new version of 2-tuple fuzzy linguistic representation model for computing with words, *IEEE Transactions on Fuzzy Systems,* 14(3):435-445, 2006.

[173] J. H. Wang and J. Y. Hao, An approach to computing with words based on canonical characteristic values of linguistic labels, *IEEE Transactions on Fuzzy Systems,* 15(4):593-604, 2007.

[174] L. Wang, Y. Wang, and W. Pedrycz, Hesitant 2-tuple linguistic Bonferroni operators and their utilization in group decision making, *Applied Soft Computing,* 77:653-664, 2019.

[175] P. Wang, X. Xu, J. Wang, and C. Cai, Interval-valued intuitionistic linguistic multi-criteria group decision-making method based on the interval 2-tuple linguistic information, *Journal of Intelligent & Fuzzy Systems,* 33(2):985-994, 2017.

[176] T.-C. Wang and H.-D. Lee, Developing a fuzzy TOPSIS approach based on subjective weights and objective weights, *Expert systems with applications,* 36(5):8980-8985, 2009.

[177] X. Wang, Z. Xu, and X. Gou, Nested probabilistic-numerical linguistic term sets in two-stage multi-attribute group decision making, *Applied Intelligence,* 49(7):2582-2602, 2019.

[178] Y.-M. Wang and Y. Luo, Integration of correlations with standard deviations for determining attribute weights in multiple attribute decision making, *Mathematical and Computer Modelling,* 51(1-2):1-12, 2010.

[179] B. S. Wilfond, J. Zabrowski, and L.-M. Johnson, A pragmatic trial of suicide risk assessment and ambulance transport decision making among emergency medical services providers: implications for patient consent, *The*

American Journal of Bioethics, 19(10):97-98, 2019.

[180] B. Wu, C. Zhao, T. L. Yip, and D. Jiang, A novel emergency decision-making model for collision accidents in the Yangtze River, *Ocean Engineering,* 223:108622, 2021.

[181] J. Wu, L. Dai, F. Chiclana, H. Fujita, and E. Herrera-Viedma, A minimum adjustment cost feedback mechanism based consensus model for group decision making under social network with distributed linguistic trust, *Information Fusion,* 41:232-242, 2018.

[182] X. Wu, H. Liao, Z. Xu, A. Hafezalkotob, and F. Herrera, Probabilistic linguistic MULTIMOORA: A multicriteria decision making method based on the probabilistic linguistic expectation function and the improved Borda rule, *IEEE Transactions on Fuzzy Systems,* 26(6):3688-3702, 2018.

[183] Y. Wu, H. Xu, C. Xu, and K. Chen, Uncertain multi-attributes decision making method based on interval number with probability distribution weighted operators and stochastic dominance degree, *Knowledge-Based Systems,* 113:199-209, 2016.

[184] Y. Wu, Y. Dong, J. Qin, and W. Pedrycz, Flexible linguistic expressions and consensus reaching with accurate constraints in group decision-making, *IEEE transactions on cybernetics,* 50(6):2488-2501, 2019.

[185] Y. N. Wu, X. K. Sun, Z. M. Lu, J. L. Zhou, and C. B. Xu, Optimal site selection of straw biomass power plant under 2-dimension uncertain linguistic environment, *Journal of Cleaner Production,* 212:1179-1192, 2019.

[186] Y. N. Wu, C. B. Xu, L. W. Li, Y. Wang, K. F. Chen, and R. H. Xu, A risk assessment framework of PPP waste-to-energy incineration projects in China under 2-dimension linguistic environment, *Journal of Cleaner Production,* 183:602-617, 2018.

[187] Z. Wu and J. Xu, A consensus model for large-scale group decision making with hesitant fuzzy information and changeable clusters, *Information Fusion,* 41:217-231, 2018.

[188] Z. B. Wu and Y. H. Chen, The maximizing deviation method for group multiple attribute decision making under linguistic environment, *Fuzzy Sets and Systems,* 158(14):1608-1617, 2007.

[189] Z. B. Wu and J. P. Xu, A consensus model for large-scale group decision making with hesitant fuzzy information and changeable clusters,

Information Fusion, 41:217-231, 2018.

[190] J. Xu and Z. Wu, A discrete consensus support model for multiple attribute group decision making, *Knowledge-Based Systems,* 24(8):1196-1202, 2011.

[191] X.-h. Xu, Z.-j. Du, and X.-h. Chen, Consensus model for multi-criteria large-group emergency decision making considering non-cooperative behaviors and minority opinions, *Decision Support Systems,* 79:150-160, 2015.

[192] X. H. Xu, Z. J. Du, and X. H. Chen, Consensus model for multi-criteria large-group emergency decision making considering non-cooperative behaviors and minority opinions, *Decision Support Systems,* 79:150-160, 2015.

[193] Y. Xu, X. Wen, and W. Zhang, A two-stage consensus method for large-scale multi-attribute group decision making with an application to earthquake shelter selection, *Computers & Industrial Engineering,* 116:113-129, 2018.

[194] Y. J. Xu, X. W. Wen, and W. C. Zhang, A two-stage consensus method for large-scale multi-attribute group decision making with an application to earthquake shelter selection, *Computers & Industrial Engineering,* 116:113-129, 2018.

[195] Z. Xu, On method for uncertain multiple attribute decision making problems with uncertain multiplicative preference information on alternatives, *Fuzzy Optimization and Decision Making,* 4(2):131-139, 2005.

[196] Z. Xu, Group decision making based on multiple types of linguistic preference relations, *Information Sciences,* 178(2):452-467, 2008.

[197] Z. S. Xu, A method based on linguistic aggregation operators for group decision making with linguistic preference relations, *Information sciences,* 166(1-4):19-30, 2004.

[198] R. Yager, "Computing with words and information/intelligent systems 2: applications, chapter Approximate reasoning as a basis for computing with words," ed: Physica Verlag Heidelberg, 1999.

[199] R. R. Yager, On the retranslation process in Zadeh's paradigm of computing with words, *IEEE Transactions on Systems, Man, and Cybernetics, Part B (Cybernetics),* 34(2):1184-1195, 2004.

[200] M. Yan, J. Wang, Y. Dai, and H. Han, A method of multiple-attribute group decision making problem for 2-dimension uncertain linguistic variables

based on cloud model, *Optimization and Engineering*:1-25, 2021.

[201] T. Yang and P. Chou, Solving a multiresponse simulation-optimization problem with discrete variables using a multiple-attribute decision-making method, *Mathematics and Computers in simulation,* 68(1):9-21, 2005.

[202] X. H. Yu, Z. S. Xu, S. S. Liu, and Q. Chen, Multicriteria decision making with 2-dimension linguistic aggregation techniques, *International Journal of Intelligent Systems,* 27(6):539-562, 2012.

[203] Z. Yue, A method for group decision-making based on determining weights of decision makers using TOPSIS, *Applied Mathematical Modelling,* 35(4):1926-1936, 2011.

[204] Z. Yue, Approach to group decision making based on determining the weights of experts by using projection method, *Applied Mathematical Modelling,* 36(7):2900-2910, 2012.

[205] L. A. Zadeh, Fuzzy sets, *Information and Control,* 8(3):338-353, 1965.

[206] L. A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning - I, *Information Sciences,* 8(3):199-249, 1975.

[207] L. A. Zadeh, "Fuzzy sets," in *Fuzzy sets, fuzzy logic, and fuzzy systems: selected papers by Lotfi A Zadeh*: World Scientific, 1996, pp. 394-432.

[208] L. A. Zadeh, "Fuzzy logic= computing with words," in *Computing with Words in Information/Intelligent Systems 1*: Springer, 1999, pp. 3-23.

[209] L. A. Zadeh, *Computing with words: Principal concepts and ideas*. Springer, 2012.

[210] L. A. Zadeh, *Computing with words in Information/Intelligent systems 1: Foundations*. Physica, 2013.

[211] B. Zhang, Y. Dong, and Y. Xu, Multiple attribute consensus rules with minimum adjustments to support consensus reaching, *Knowledge-Based Systems,* 67:35-48, 2014.

[212] B. Zhang, H. Liang, Y. Gao, and G. Zhang, The optimization-based aggregation and consensus with minimum-cost in group decision making under incomplete linguistic distribution context, *Knowledge-Based Systems,* 162:92-102, 2018.

[213] G. Q. Zhang, Y. C. Dong, Y. F. Xu, and H. Y. Li, Minimum-cost consensus models under aggregation operators, *IEEE Transactions on Systems, Man, and Cybernetics-Part A: Systems and Humans,* 41(6):1253-1261, 2011.

[214] H. Zhang, Y. Dong, and X. Chen, The 2-rank consensus reaching model in the multigranular linguistic multiple-attribute group decision-making, *IEEE Transactions on Systems, Man, and Cybernetics: Systems,* 48(12):2080-2094, 2017.

[215] H. Zhang, G. Kou, and Y. Peng, Soft consensus cost models for group decision making and economic interpretations, *European Journal of Operational Research,* 277(3):964-980, 2019.

[216] H. Zhang, Y. Dong, I. Palomares-Carrascosa, and H. Zhou, Failure mode and effect analysis in a linguistic context: A consensus-based multiattribute group decision-making approach, *IEEE Transactions on Reliability,* 68(2):566-582, 2018.

[217] H. J. Zhang, S. H. Zhao, G. Kou, C.-C. Li, Y. C. Dong, and F. Herrera, An overview on feedback mechanisms with minimum adjustment or cost in consensus reaching in group decision making: Research paradigms and challenges, *Information Fusion,* 60:65-79, 2020.

[218] Z. Zhang and C. Guo, Consistency and consensus models for group decision-making with uncertain 2-tuple linguistic preference relations, *International journal of systems Science,* 47(11):2572-2587, 2016.

[219] Z. Zhang, W. Yu, L. Martínez, and Y. Gao, Managing multigranular unbalanced hesitant fuzzy linguistic information in multiattribute large-scale group decision making: A linguistic distribution-based approach, *IEEE Transactions on Fuzzy Systems,* 28(11):2875-2889, 2019.

[220] J. Zhao, H. Zhu, and H. Li, 2-Dimension linguistic PROMETHEE methods for multiple attribute decision making, *Expert Systems with Applications,* 127:97-108, 2019.

[221] X. Zhong, X. Xu, and B. Pan, A non-threshold consensus model based on the minimum cost and maximum consensus-increasing for multi-attribute large group decision-making, *Information Fusion*, 2021.

[222] B. Zhu, Z. Xu, and J. Xu, Deriving a ranking from hesitant fuzzy preference relations under group decision making, *IEEE transactions on cybernetics,* 44(8):1328-1337, 2013.

[223] H. Zhu, J. B. Zhao, and Y. Xu, 2-dimension linguistic computational model

with 2-tuples for multi-attribute group decision making, *Knowledge-Based Systems*, 103:132-142, 2016.

[224] H. Zhu, J. Zhao, and Y. Xu, 2-dimension linguistic computational model with 2-tuples for multi-attribute group decision making, *Knowledge-Based Systems,* 103:132-142, 2016.

[225] W. D. Zhu, G. Z. Zhou, and S. L. Yang, An approach to group decision making based on 2-dimension linguistic assessment information, *System Engineering*, 27(2):113-118, 2009.

[226] H.-J. Zimmermann, *Fuzzy set theory – and its applications*. Springer Science & Business Media, 2011.