

# Nonlinear preferences in group decision-making. Extreme values amplifications and extreme values reductions

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## Abstract

Consensus Reaching Processes (CRPs) deal with those group decision-making situations in which conflicts among experts' opinions make difficult the reaching of an agreed solution. This situation, worsens in large-scale group decision situations, in which opinions tend to be more polarized, because in problems with extreme opinions it is harder to reach an agreement. Several studies have shown that experts' preferences may not always follow a linear scale, as it has commonly been assumed in previous CRP. Therefore, the main aim of this paper is to study the effect of modeling this nonlinear behavior of experts' preferences (expressed by fuzzy preference relations) in CRPs. To do that, the experts' preferences will be remapped by using nonlinear deformations which amplify or reduce the distance between the extreme values. We introduce such automorphisms to remap the preferences as Extreme Values Amplifications (EVAs) and Extreme Values Reductions (EVRs), study their main properties and propose several families of these EVA and EVR functions. An analysis about the behavior of EVAs and EVRs when are implemented in a generic consensus

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model is then developed. Finally, an illustrative experiment to study the performance of different families of EVAs in CRPs is provided.

#### KEYWORDS

consensus reaching process, extreme values amplification, extreme values reductions, group decision-making, nonlinear preferences

## 1 | INTRODUCTION

Group decision-making (GDM) problems are those situations in which several individuals or experts have to choose a solution for a given problem which consists of two or more possible solutions or alternatives.<sup>1</sup>

Butler and Rothstein<sup>2</sup> proposed several rules to guide the decision process in real-world problems, like, the majority rule, the minority rule, or unanimity. The main issue around these general rules is the fact that some individuals or experts may not agree with the *solution* chosen by the group because they could consider that their opinions have not been sufficiently taken into account for achieving the solution.

Even though classic GDM problems have been proposed by considering a few number of experts, current technological advances, such as e-democracy<sup>3</sup> or social networks,<sup>4</sup> have led to situations in which many experts can be required. Large-scale GDM (LSGDM) problems are defined as those GDM problems in which there are more than 20 experts involved in the problem.<sup>5</sup> This increment in the number of experts implies more conflicts and polarized opinions,<sup>6</sup> and thus to obtain agreed solutions that become much more complex.

Consensus Reaching Processes (CRPs)<sup>2,7</sup> emerge for those GDM situations which demand an agreement among experts about the chosen solution.

According to Reference [8] the most widely used approach to deal with CRPs may be the idea of *soft consensus* proposed by Kacprzyk.<sup>1</sup> Soft consensus that is based on the notion of fuzzy linguistic majority provides a measure to compute the consensus among experts.<sup>9</sup>

Several CRPs based on the notion of soft consensus have been developed,<sup>10–13</sup> all of them assume the use of linear scales for the preferences elicited from experts. However, recent studies have shown that, in certain situations, better decisions can be obtained by using nonlinear scales for representing users' preferences.<sup>14,15</sup> Masthoff<sup>14</sup> studied people's behavior when they rate their opinions on a numerical scale and concluded that the ratings did not follow a linear scale because the same differences between two values at different levels of the scale represent different differences in people's minds. Therefore, it was concluded that a quadratic scale was a better measure than a linear one. Meanwhile, Delic et al.<sup>15</sup> pointed out that by using polynomial remappings of individual preferences (under both ranking and rating conditions in group decision schemes) the results of the decision-making processes are improved.

Paying attention to previous results, this paper aims at studying the effect of modeling nonlinear behaviors in CRPs for LSGDM. Therefore, we raise the following research questions that stem from our goal:

- RQ1: How are nonlinear scales modeled in CRPs for managing polarization in (LS)GDM?

- RQ2: Does the nonlinear approach improve the CRPs in comparison with linear approach?

Without loss of generality,<sup>16</sup> we assume that the preferences will be elicited by fuzzy preference relations (FPRs) and nonlinear deformations will be applied to each value of the preference relation, to *adjust* the initial experts' preferences to a more realistic nonlinear scale when the (LS)GDM problem faces a situation in which such scales are needed. The impact of the nonlinear remapping procedure in CRPs will be evaluated by comparing its convergence and degree of consensus achieved.

Consequently, we will introduce Extreme Values Amplifications (EVAs) as those functions that increase in a nonlinear way the distance between extreme values of the FPRs. Additionally, Extreme Values Reductions (EVRs) will be defined as those nonlinear deformations which reduce the distance between the extreme values of the FPRs.

Several families of these EVAs and EVRs are proposed. EVAs will be then applied in different CRPs to LSGDM problems to show the effectiveness of this nonlinear preference modeling by using the software AFRYCA.<sup>10</sup>

Such EVAs (resp., EVRs) will act as:

1. They remap the original linear-scaled FPRs into nonlinear-scaled FPRs.
2. They amplify (resp., reduce) the distance between the extreme values, and reduce (resp., amplify) the distance between the intermediate ones.
3. They have a concrete geometrical pattern.
4. The amplification (resp., reduction) of distances is greater when preferences are close to the extremes.

Finally, we analyze the performance of EVAs and EVRs in CRPs for LSGDM problems evaluating if EVA/EVR approaches outperform the classic linear approach in CRPs.

The remaining of this paper is set up as follows: In Section 2, a brief review of GDM problems and CRPs is presented. Section 3 introduces an exhaustive study of the properties of those automorphisms on the interval  $[0, 1]$  which remaps linear-scaled FPRs into nonlinear-scaled FPRs by increasing or reducing distances between extreme preferences. Section 4 defines the main concepts of this contribution, namely, EVA and EVR, and presents its fundamental characteristics. Section 5 proposes a general method to construct EVAs and introduces several families of EVAs and EVRs. In Section 6 we will discuss the performance of EVAs and EVRs when applied in a generic consensus model. In Section 7, we simulate the performance of EVAs when they are applied in CRPs for LSGDM problems. Finally, Section 8 will conclude the contribution.

## 2 | PRELIMINARIES

This section revises some essential concepts about GDM and CRP to easily understand the proposal.

### 2.1 | Group decision-making

A GDM problem<sup>1,2</sup> is a situation in which two or more individuals have to choose a collective solution for a certain problem. Formally, the main elements in a GDM problem are:

- A set  $X = \{X_1, X_2, \dots, X_n\}$ ,  $2 \leq n \in \mathbb{N}$ , of alternatives or possible solutions to the given problem.
- A set  $E = \{e_1, e_2, \dots, e_m\}$ , where  $2 \leq m \in \mathbb{N}$ , of experts who express their opinions about the alternatives in  $X$  throughout certain preference structure.

In this study, without loss of generality,<sup>16</sup> we will assume that experts elicit their preferences by using an FPR, which has been proved to be effective in managing the uncertainty.<sup>16,17</sup> To obtain these FPRs, each expert  $e_k$ ,  $k = 1, \dots, m$  will elicit the degree to which she/he prefers the alternative  $X_i$  over the alternative  $X_j$ , which will be denoted by  $p_{ij}^k$ . The FPR associated with the expert  $e_k$  will be the matrix  $P_k \in \mathcal{M}_{n \times n}([0, 1])$  whose items are the values  $p_{ij}^k \in [0, 1]$  which must satisfy the symmetry condition  $p_{ij}^k + p_{ji}^k = 1 \quad \forall i, j \in \{1, 2, \dots, n\}, k \in \{1, 2, \dots, m\}$ .

Nowadays, technological developments have led to GDM situations<sup>3,4</sup> which demand a large number of experts. LSGDM problems are defined as those decision situations in which 20 or more experts are required to solve the GDM problems.<sup>5</sup>

## 2.2 | Consensus reaching processes

GDM solving processes may fail when using classical GDM rules, like, the majority rule, since experts may feel unsatisfied with the solution and think that their opinions have not been sufficiently considered.<sup>10,18</sup> To avoid such disagreements, it is necessary to include in the GDM solving process a CRP to obtain agreed solutions that reflect the opinion of all the experts involved in the GDM problem.<sup>11,19</sup>

A CRP is an iterative discussion process<sup>7</sup> usually coordinated by a moderator whose main responsibilities are to evaluate the level of agreement achieved in each round of discussion (and if it is enough), identify those experts' opinions that are far away from the collective opinion and provide some feedback/recommendations to such experts to increase the consensus degree in the next round.<sup>8</sup>

A general scheme of a CRP (see Figure 1) is briefly summarized as follows:

- *Gathering preferences*: Each expert elicits her/his preferences through a certain preference structure.<sup>16</sup>
- *Determining the level of consensus*: The moderator computes the level of agreement throughout a certain consensus measure.<sup>10</sup>
- *Consensus control*: The consensus level is compared with a threshold level previously established as acceptable. If either this consensus threshold is reached or the maximum number of rounds is surpassed, the process finishes. Otherwise the consensus progress keeps going.
- *Consensus progress*: To increase the level of consensus, experts should change their preferences according to the moderator's recommendations.

When large-scale contexts are considered, CRPs become more complex<sup>10,18</sup> because there are usually more conflicts and the opinions are more polarized.<sup>6</sup> Additionally, new challenges emerge to deal with a large number of experts in CRPs and several proposals have been presented to cope with them<sup>19–22</sup> in recent years.

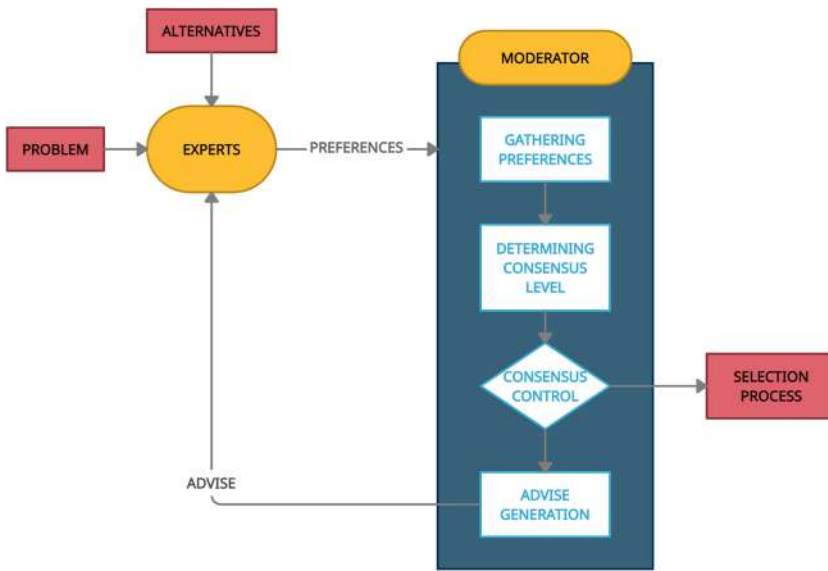


FIGURE 1 General scheme of consensus reaching process [Color figure can be viewed at [wileyonlinelibrary.com](http://wileyonlinelibrary.com)]

### 3 | AMPLIFYING DISTANCES BETWEEN EXTREME PREFERENCES

Our main aim is to study the performance of CRPs when the experts' preferences are modeled by a nonlinear scale. To do this, the preferences elicited from experts with FPRs will be transformed by a nonlinear deformation to obtain more realistic FPRs in which extreme values are deformed so that the distances between them are increased or decreased.

To remap the experts' preferences by using a function  $D: [0, 1] \rightarrow [0, 1]$  it is necessary to consider two different factors:

- (i) The function  $D: [0, 1] \rightarrow [0, 1]$  must remap FPRs into FPRs. This implies not only that the codomain of  $D$  must be the interval  $[0, 1]$ , but also that the nonlinear preferences keep the symmetry condition of an FPR  $P \in \mathcal{M}_{n \times n}([0, 1])$ , that is,  $p_{ij} + p_{ji} = 1$ ,
- (ii) The function  $D: [0, 1] \rightarrow [0, 1]$  must transform the unit interval  $[0, 1]$  such that the distance between the extreme values increases (or decreases) with respect to their original distance. In this section we will focus on those functions which deform the preferences by increasing the distance between extreme values and decreasing it between the intermediate ones, but similar arguments could be developed to describe those functions which increase the distance between the intermediate values and decrease it between the more extreme values.

For the sake of clarity, the description of these nonlinear deformations will be developed heuristically, by progressively adding requirements to a function  $D: [0, 1] \rightarrow [0, 1]$ . Therefore, some mathematical properties will be imposed to the function  $D$  due to their practical application and, in other cases, the mathematical properties will lead to useful features of these functions. In the following section all of these properties will be then compiled in the main definitions of our proposal, namely, EVAs and EVRs.

### 3.1 | Regularity

To obtain a proper deformation of the interval  $[0, 1]$  the function  $D: [0, 1] \rightarrow [0, 1]$ , it must be a bijection. Otherwise, different values of the preferences would be mapped into the same value, which is not reasonable if we want to compare how different the preferences are.

In this context, both the strictly increasing character of  $D$  and the values  $D(0) = 0$  and  $D(1) = 1$  are mandatory.

**Property 1.**  $D: [0, 1] \rightarrow [0, 1]$  is a strictly increasing bijection which satisfies the boundary conditions  $D(0) = 0$  and  $D(1) = 1$ ; that is,  $D$  is an automorphism on the interval  $[0, 1]$ .

The following well-known result will assure that a function satisfying this property is also a continuous function.

**Proposition 1.** Let  $f: [a, b] \rightarrow [c, d]$  be a bijection. Then  $f$  is strictly monotonous if and only if  $f$  is continuous.

*Proof.* Let us prove first the sufficiency. Suppose  $f$  is strictly increasing (the decreasing case is similar) and pick  $x_0 \in [a, b]$  and  $\epsilon > 0$ . Since  $f$  is a bijection we can find  $A \subset [a, b]$  such that  $f(A) = [f(x_0) - \epsilon, f(x_0) + \epsilon] \cap [c, d]$ . Now, if we pick  $x, y \in A$  such that  $x < y$ , because of the monotonicity of  $f$ , we obtain

$$\max\{f(x_0) - \epsilon, c\} < f(x) < f(z) < f(y) < \min\{f(x_0) + \epsilon, d\}$$

for every  $z \in ]x, y[$ . In such a case  $A$  is an interval containing  $x_0$  and we can find  $\delta > 0$  such that  $|f(z) - f(x_0)| < \epsilon \quad \forall z \in [a, b]: |z - x_0| < \delta$ , that is,  $f$  is continuous in  $x_0$ .

To prove the necessary condition pick  $x, y, z \in [0, 1]$  such that  $x < y < z$ . Suppose  $f(x) < f(y)$  and  $f(y) > f(z)$ . In that case, by using the Intermediate Value Theorem, we obtain  $f(]x, y]) \cap f(]y, z]) \neq \emptyset$ , which is impossible because of the bijectivity of  $f$ . We can apply the same reasoning to the remaining case (i.e.,  $f(x) > f(y)$  and  $f(y) < f(z)$ ) and conclude it must be either  $f(x) < f(y) < f(z)$  or  $f(x) > f(y) > f(z)$ .  $\square$

Since a function satisfying Property 1 is continuous, small changes on the original preferences are mapped into small changes of the deformed values.

As we will see, we will need some extra regularity on  $D$  to characterize the amplification of the distances between extreme values according to the value of  $D'$ , so we impose now some additional *smoothness*:

**Property 2.**  $D: [0, 1] \rightarrow [0, 1]$  is a differentiable function whose derivative  $D': [0, 1] \rightarrow [0, 1]$  is continuous, that is,  $D$  is a  $C^1$  function.

Note that if  $D: [0, 1] \rightarrow [0, 1]$  is a function satisfying Properties 1 and 2, then  $D$  also satisfies  $D'(x) \geq 0 \quad \forall x \in [0, 1]$ .

### 3.2 | Symmetry

When using an FPR to represent expert's preferences, it is usual to make the calculations only in the superior triangle due to that triangle and the inferior one are related by the standard negation  $N: [0, 1] \rightarrow [0, 1]$  defined by  $N(x) = 1 - x \quad \forall x \in [0, 1]$ .

We have to translate this symmetry into an equivalent property for our function  $D$ , that is, the modified distance from 0.8 to 0.85 should be the same that the modified distance from 0.2 to 0.15. This kind of symmetry around the value  $x = \frac{1}{2}$  will be imposed by the following property.

**Property 3.**  $D: [0, 1] \rightarrow [0, 1]$  must be a symmetric function in the sense that  $D(x) = 1 - D(1 - x) \quad \forall x \in [0, 1]$ . In particular,  $D(\frac{1}{2}) = \frac{1}{2}$ .

In other words, this property guarantees that  $D$  remaps FPRs into FPRs. Furthermore, this property has a clear practical purpose since it allows us to construct these nonlinear deformations by only focusing on one half of the interval  $[0, 1]$ : if we manage to obtain  $D_1: [\frac{1}{2}, 1] \rightarrow [\frac{1}{2}, 1]$  we can define  $D_2: [0, \frac{1}{2}] \rightarrow [0, \frac{1}{2}]$  by  $D_2(x) := 1 - D_1(1 - x) \quad \forall x \in [0, \frac{1}{2}]$  and construct  $D: [0, 1] \rightarrow [0, 1]$  as a piecewise function. Note that, when  $D_1$  is a differentiable function,  $D_2$  is also differentiable and its derivative satisfies  $D_2'(x) = D_1'(1 - x) \quad \forall x \in [0, \frac{1}{2}]$ , so  $D_2'(\frac{1}{2}) = D_1'(\frac{1}{2})$  and  $D$  will be a differentiable function such that  $D'(0) = D'(1)$ . Furthermore, if  $D_1'$  is continuous,  $D'$  will be also continuous.

It should be highlighted that a function  $D: [0, 1] \rightarrow [0, 1]$  satisfying these three properties induces the restricted dissimilarity  $d_D: [0, 1] \times [0, 1] \rightarrow [0, 1]$ <sup>23</sup> given by

$$d_D(x, y) = |D(x) - D(y)| \quad \forall x, y \in [0, 1],$$

and the Restricted Equivalence Function<sup>23</sup>  $S_D: [0, 1] \times [0, 1] \rightarrow [0, 1]$  defined by

$$S_D(x, y) = 1 - |D(x) - D(y)| \quad \forall x, y \in [0, 1].$$

These tools allow one to compare *how similar* are the preferences taking into account the nonlinear approach.

### 3.3 | Distance amplification and derivatives

Here it is studied the relation between the first derivative of an arbitrary automorphism defined in  $[0, 1]$  and the modification of the distances between elements that it will produce.

First, a theorem that characterizes those functions which amplify the distance between the elements of their domain is proposed.

**Theorem 1.** *Let  $f: [a, b] \rightarrow \mathbb{R}$  be a  $C^1$  function defined on the interval  $[a, b] \subset \mathbb{R}$ . Then the following statements are equivalent:*

1.  $f$  is an increasing function satisfying  $|f(y) - f(x)| \geq |y - x| \quad \forall x, y \in [a, b]$ ,
2.  $f'(z) \geq 1 \quad \forall z \in [a, b]$ .

*Proof.* (1)  $\rightarrow$  (2) Suppose first that  $|f(y) - f(x)| \geq |y - x| \quad \forall x, y \in [a, b]$  and pick  $z \in [a, b[$ . Let us choose  $h > 0$  such that  $z + h < b$ . In that case the auxiliary function  $g: ]0, h[ \rightarrow \mathbb{R}$  defined by  $g(t) = \frac{f(z+t) - f(z)}{t} \quad \forall t \in ]0, h[$  is a continuous function such that  $g(t) \geq 1 \quad \forall t \in ]0, h[$  and therefore  $f'(z) = \lim_{t \rightarrow 0} g(t) \geq 1$ . On the other hand, when  $z = b$ , we can consider  $h > 0$  such that  $b - a > h$  and use the analogue reasoning for the function  $g: ]0, h[ \rightarrow \mathbb{R}$  defined by  $g(t) = \frac{f(b) - f(b-t)}{t} \quad \forall t \in ]0, h[$ .

(2)  $\rightarrow$  (1) Since  $f'(z) \geq 1 \quad \forall z \in [a, b]$ ,  $f$  is increasing. To show the inequality pick  $x, y \in [a, b]$  such that  $x < y$ . We can use the Mean Value Theorem to obtain  $\xi \in ]x, y[$  such that  $f(y) - f(x) = f'(\xi)(y - x)$  and therefore it must be  $|f(y) - f(x)| \geq |y - x|$ .  $\square$

*Remark 1.* Note that if the derivative of  $D$  is between 0 and 1 in some subinterval, as it may occur on the intermediate values of  $[0, 1]$ , then the distance between the deformations of those elements will be lower than the distance between the original ones.

In the following proposition we will use the idea of Theorem 1 to describe the amplification of distances between values close to 1 when we are deforming the interval  $[0, 1]$ . The key is to ask for the derivative in  $x = 1$  to be higher than 1 and use the continuity of the derivative to obtain a neighborhood of 1 wherein the derivative is greater or equal than 1.

**Proposition 2.** *Let  $f: [1 - r, 1] \rightarrow [0, 1]$  be a  $C^1$  function defined on  $[1 - r, 1]$  for some  $r \in ]0, 1[$ . Then,*

- *If  $f'(1) > 1$ , there exists  $r' \in ]0, r[$  such that  $|f(y) - f(x)| > |y - x| \quad \forall x, y \in [1 - r', 1]; x \neq y$ ,*
- *$f$  is an increasing function satisfying  $|f(y) - f(x)| \geq |y - x| \quad \forall x, y \in [1 - r, 1]$ , if and only if  $f'(z) \geq 1 \quad \forall z \in [1 - r, 1]$ .*

*Proof.* To prove the first assumption let us use the continuity of  $f'$  around 1 to get  $r' \in ]0, r[$  such that  $f'(z) > 1 \quad \forall z \in [1 - r', 1]$ . Then, for every  $x < y$ ,  $x, y \in [1 - r', 1]$ , we can use the Mean Value Theorem to obtain  $\xi \in ]x, y[$  such that  $f(y) - f(x) = f'(\xi)(y - x)$ . Since  $f'(\xi) > 1$  and, in particular,  $f$  is strictly increasing we have  $|f(y) - f(x)| = f(y) - f(x) = f'(\xi)(y - x) > |y - x|$ . To show the second statement we just have to use Theorem 1.  $\square$

Using a similar argument we can show the analogous result for distance amplification in values close to 0:

**Proposition 3.** *Let  $f: [0, r] \rightarrow [0, 1]$  be a  $C^1$  function defined on  $[0, r]$  for some  $r \in ]0, 1[$ . Then:*

- *If  $f'(0) > 1$ , there exists  $r' \in ]0, r[$  such that  $|f(y) - f(x)| > |y - x| \quad \forall x, y \in [0, r']; x \neq y$ ,*
- *$f$  is an increasing function satisfying  $|f(y) - f(x)| \geq |y - x| \quad \forall x, y \in [0, r]$ , if and only if  $f'(z) \geq 1 \quad \forall z \in [0, r]$ .*



The practical interpretation of these results is simple: to amplify distances close to the extremes we need that the derivative of  $D$  is greater than 1 in the extremes. Since  $D$  is asked to be a  $C^1$  function, the continuity of  $D'$  will give us two neighborhoods (one around 0 and the other around 1) where  $D'$  is always greater than 1 and thus the distance between two values which are inside one of these neighborhoods will increase when we apply  $D$ .

**Property 4.**  $D$  must be a  $C^1$  function satisfying  $D'(0) > 1$  and  $D'(1) > 1$ .

### 3.4 | Distance amplification and convexity

Finally, we study the relation between the convexity and the distance amplification on extreme values. First, we compare the deformations  $D$  with the identity function on the interval  $[0, 1]$ .

**Proposition 4.** Let  $f: [0, 1] \rightarrow [0, 1]$  be a  $C^1$  increasing function such that  $f(0) = 0, f(1) = 1, f'(0) > 1$ , and  $f'(1) > 1$ . Then we can find  $r \in ]0, \frac{1}{2}[$  such that  $f(x) > x \quad \forall x \in ]0, r[$  and  $f(x) < x \quad \forall x \in ]1 - r, 1[$ .

*Proof.* Due to  $f'(0) > 1$  and  $f'(1) > 1$ , the function  $g: [0, 1] \rightarrow [0, 1]$  given by  $g(x) = f(x) - x \quad \forall x \in [0, 1]$  satisfies  $g'(0) > 0$  and  $g'(1) > 0$  and we can find  $r \in ]0, \frac{1}{2}[$  such that  $g$  is strictly increasing in both  $[0, r]$  and  $[1 - r, 1]$ .

In that case  $g(0) < g(x) \quad \forall x \in ]0, r[$  and therefore  $f(0) - 0 < f(x) - x \quad \forall x \in ]0, r[ \Leftrightarrow x < f(x) \quad \forall x \in ]0, r[$  and  $g(x) < g(1) \quad \forall x \in ]1 - r, 1[ \Leftrightarrow f(x) - x < f(1) - 1$  which means  $f(x) < x \quad \forall x \in ]1 - r, 1[$  □

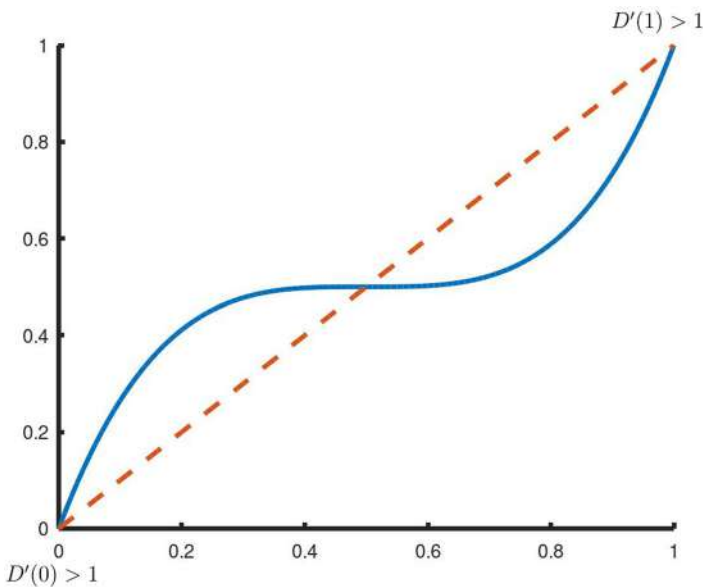


FIGURE 2 A function  $D$  satisfying Proposition 4 ( $r = \frac{1}{2}$ ) [Color figure can be viewed at [wileyonlinelibrary.com](http://wileyonlinelibrary.com)]

This result provides a clear geometrical interpretation: the graph of  $D$  is over the diagonal of the square  $[0, 1] \times [0, 1]$  for values close enough to 0 and it is under the same diagonal for those values close enough to 1 (see Figure 2).

We have already shown that the derivative of  $D$  must be greater than 1 close to the extremes. The following proposition, which is an immediate consequence of the previous one, will prove that  $D'$  must also be under 1 in some subinterval of  $[0, 1]$  and thus the distances decrease between the values in such subinterval. Additionally, we will obtain that  $D$  cannot be convex nor concave on its full domain.

**Proposition 5.** *Let  $f: [0, 1] \rightarrow [0, 1]$  be a  $C^1$  increasing function such that  $f(0) = 0, f(1) = 1, f'(0) > 1$ , and  $f'(1) > 1$ . Then*

- *There exists an interval  $I \subset [0, 1]$  such that  $0 \leq f'(x) < 1 \quad \forall x \in I$ ,*
- *$f'$  cannot be increasing nor decreasing on the full domain  $[0, 1]$ .*

*Proof.* Let us define  $g: [0, 1] \rightarrow \mathbb{R}$  by  $g(x) = f(x) - x \quad \forall x \in [0, 1]$  as before. On the one hand, suppose that  $f'(x) \geq 1 \quad \forall x \in [0, 1]$ . In that case,  $g' \geq 0$  and  $g$  is strictly increasing, which is a contradiction due to  $g(0) = g(1)$ . That means we can find some  $x \in [0, 1]$  where  $f'(x) < 1$  and the continuity of  $f'$  will give us the interval we are looking for.

On the other hand, if  $f'$  is increasing then  $g'$  is also increasing and due to  $g'(0) > 0$  we obtain  $g'(x) > 0 \quad \forall x \in [0, 1]$ . Then  $g$  is strictly increasing and  $g(0) < g(1)$ , which is impossible.  $\square$

Propositions 6 and 7 provide a characterization of those functions which amplify the difference between nearby elements when we approach extreme values (0 or 1).

**Proposition 6.** *Let  $f: [a, b] \rightarrow \mathbb{R}$  be a  $C^1$  increasing function. The following statements are equivalent:*

1.  *$f$  is a convex function,*
2. *For each  $x < y, \quad x, y \in ]a, b]$  the inequality*

$$|f(x) - f(x - t)| \leq |f(y) - f(y - t)|$$

*holds for any  $t \in [0, h]$ , where  $h = \min\{x - a, y - x\}$ .*

*Proof.* (1)  $\rightarrow$  (2) If  $f$  is convex,  $f'$  is an increasing function in  $[a, b]$ . On the other hand, the Mean Value Theorem gives us  $\xi_1 \in ]x - t, x[$  and  $\xi_2 \in ]y - t, y[$  such that

$$\begin{aligned} |f(x) - f(x - t)| &= f(x) - f(x - t) = f'(\xi_1)t \leq f'(\xi_2)t = f(y) - f(y - t) \\ &= |f(y) - f(y - t)|, \end{aligned}$$

where we have used that  $f'$  is increasing and therefore  $f'(\xi_1) \leq f'(\xi_2)$ .

(2)  $\rightarrow$  (1) Let us fix  $x < y, \quad x, y \in ]a, b]$  and define  $g: [0, h] \rightarrow \mathbb{R}$  by

$$g(t) = \frac{f(y) - f(y - t) - (f(x) - f(x - t))}{t} \quad \forall t \in [0, h].$$

Since  $f$  is increasing  $g \geq 0$ . In addition

$$\lim_{t \rightarrow 0} g(t) = f'(y) - f'(x)$$

and the continuity of  $g$  leads to  $f'(y) \geq f'(x)$ , which is the convexity of  $f$ . □

Using a similar proof, we obtain:

**Proposition 7.** *Let  $f: [a, b] \rightarrow \mathbb{R}$  be a  $C^1$  increasing function. The following statements are then equivalent:*

1.  $f$  is a concave function,
2. For each  $x < y, \quad x, y \in ]a, b]$  the inequality

$$|f(x) - f(x - t)| \geq |f(y) - f(y - t)|$$

holds for any  $t \in [0, h]$ , where  $h = \min\{x - a, y - x\}$ .

These propositions show that the convexity is related to an increment of the distances between consecutive values of the preferences when we approach the extremes of the interval  $[0, 1]$ . We summarize this in Property 5:

**Property 5.**  $D: [0, 1] \rightarrow [0, 1]$  should be concave in a neighborhood of 0 and convex in a neighborhood of 1.

## 4 | EXTREME VALUES AMPLIFICATIONS AND EXTREME VALUES REDUCTIONS

This section introduces the concept of EVA and its dual concept EVR. First, we present the definition of EVAs as those functions satisfying the properties stated in Section 3.

**Definition 1** (Extreme Values Amplification). Let  $D: [0, 1] \rightarrow [0, 1]$  be a function satisfying:

1.  $D$  is an automorphism on the interval  $[0, 1]$ ,
2.  $D$  is a  $C^1$  function,
3.  $D$  satisfies  $D(x) = 1 - D(1 - x) \quad \forall x \in [0, 1]$ ,
4.  $D'(0) > 1$  and  $D'(1) > 1$ ,
5.  $D$  is concave in a neighborhood of 0 and convex in a neighborhood of 1.  $D$  will be called then an EVA on the interval  $[0, 1]$ .

This notation reminds that the main purpose of  $D$  is to remap FPRs of a GDM problem in a nonlinear way by amplifying the distance between the extreme values. So the new preferences show a larger distance between extreme elements and a smaller distance between elements close to  $\frac{1}{2}$ .

The following theorem, which compiles the main properties of the EVAs, is obtained by using the results discussed in Section 3:

**Theorem 2.** *Let  $D: [0, 1] \rightarrow [0, 1]$  be an EVA on  $[0, 1]$ . Then,*

1. *The function  $d_D: [0, 1] \times [0, 1] \rightarrow [0, 1]$  given by*

$$d_D(x, y) = |D(x) - D(y)| \quad \forall x, y \in [0, 1]$$

*is a restricted dissimilarity<sup>23</sup> and the function  $S_D: [0, 1] \times [0, 1] \rightarrow [0, 1]$  defined by*

$$S_D(x, y) = 1 - |D(x) - D(y)| \quad \forall x, y \in [0, 1].$$

*is a Restricted Equivalence Function.<sup>23</sup>*

2. *We can find three intervals  $I_1, I_2, I_3 \subset [0, 1]$  such that  $0 \in I_1, 1 \in I_3$ , and  $I_1 < I_2 < I_3$  satisfying that*

$$\begin{aligned} |D(y) - D(x)| &> |y - x| && \forall x, y \in I_1: x \neq y, \\ |D(y) - D(x)| &< |y - x| && \forall x, y \in I_2: x \neq y, \\ |D(y) - D(x)| &> |y - x| && \forall x, y \in I_3: x \neq y. \end{aligned}$$

3. *The graph of  $D$  is over the diagonal of the square  $[0, 1] \times [0, 1]$  for values close enough to 0 and it is under the same diagonal for those values close enough to 1,*
4. *There exist a neighborhood  $U_0$  containing 0 and a neighborhood  $U_1$  containing 1 such that for every  $x, y \in U_0^\circ, x < y$ , there exists  $h_0 > 0$  such that the inequality  $|D(x) - D(x - t)| \geq |D(y) - D(y - t)|$  holds for any  $t \in [0, h_0]$  and for every  $x, y \in U_1^\circ, x < y$ , there exists  $h_1 > 0$  such that the inequality  $|D(x - t) - D(x)| \leq |D(y - t) - D(y)|$  holds for any  $t \in [0, h_1]$ .*

Therefore, EVAs behave exactly as we aimed at:

1. They remap the original linear-scaled FPRs into nonlinear-scaled FPRs.
2. They amplify the distance between the extreme values, and reduce the distance between the intermediate ones.
3. They have a concrete geometrical pattern (see Figure 3A).
4. The amplification of distances is greater close to the extremes.

We can also consider the analogous notion of EVR:

**Definition 2** (Extreme Values Reduction). *Let  $D: [0, 1] \rightarrow [0, 1]$  be a function satisfying:*

1.  *$D$  is an automorphism on the interval  $[0, 1]$ ,*
2.  *$D$  is a  $C^1$  function,*
3.  *$D$  satisfies  $D(x) = 1 - D(1 - x) \quad \forall x \in [0, 1]$ ,*
4.  *$D'(0) < 1$  and  $D'(1) < 1$ ,*
5.  *$D$  is convex in a neighborhood 0 and concave in a neighborhood of 1.*

*Then  $D$  will be called an EVR on the interval  $[0, 1]$ .*

In this case, we obtain:

**Theorem 3.** Let  $D: [0, 1] \rightarrow [0, 1]$  be an EVR on  $[0, 1]$ . Then:

1. The function  $d_D: [0, 1] \times [0, 1] \rightarrow [0, 1]$  given by

$$d_D(x, y) = |D(x) - D(y)| \quad \forall x, y \in [0, 1]$$

is a restricted dissimilarity<sup>23</sup> and the function  $S_D: [0, 1] \times [0, 1] \rightarrow [0, 1]$  defined by

$$S_D(x, y) = 1 - |D(x) - D(y)| \quad \forall x, y \in [0, 1]$$

is a Restricted Equivalence Function.<sup>23</sup>

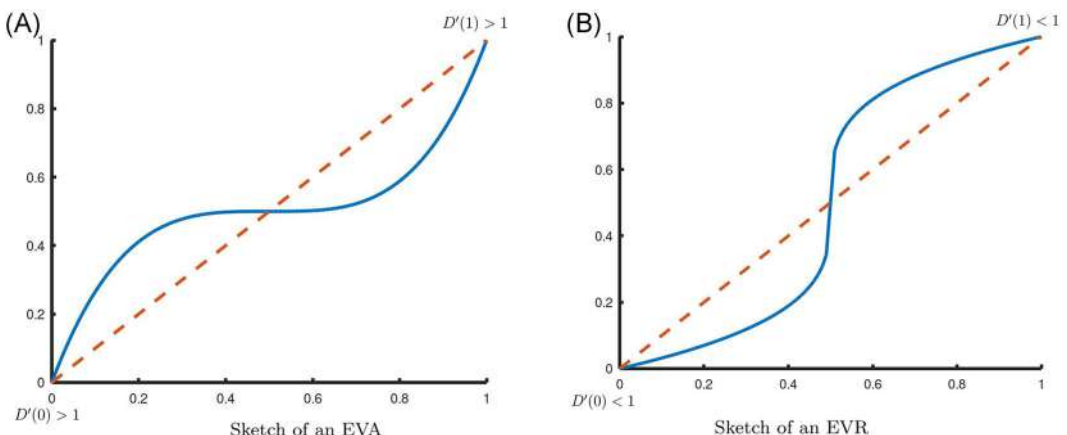
2. We can find three intervals  $I_1, I_2, I_3 \subset [0, 1]$  such that  $0 \in I_1, 1 \in I_3$ , and  $I_1 < I_2 < I_3$  satisfying that

$$\begin{aligned} |D(y) - D(x)| < |y - x| & \quad \forall x, y \in I_1; x \neq y, \\ |D(y) - D(x)| > |y - x| & \quad \forall x, y \in I_2; x \neq y, \\ |D(y) - D(x)| < |y - x| & \quad \forall x, y \in I_3; x \neq y. \end{aligned}$$

- The graph of  $D$  is under the diagonal of the square  $[0, 1] \times [0, 1]$  for values close enough to 0 and it is over the same diagonal for those values close enough to 1,
- There exist a neighborhood  $U_0$  containing 0 and a neighborhood  $U_1$  containing 1 such that for every  $x, y \in U_0, x < y$ , there exists  $h_0 > 0$  such that the inequality  $|D(x) - D(x - t)| \leq |D(y) - D(y - t)|$  holds for any  $t \in [0, h_0]$  and for every  $x, y \in U_1, x < y$ , there exists  $h_1 > 0$  such that the inequality  $|D(x - t) - D(x)| \geq |D(y - t) - D(y)|$  holds for any  $t \in [0, h_1]$ .

The behavior of EVRs is dual to the EVAs as can be seen below:

- They remap the original linear-scaled FPRs into nonlinear-scaled FPRs.
- They reduce the distance between the extreme values, and amplify the distance between the intermediate ones.



**FIGURE 3** Comparing the shapes of EVAs and EVRs. EVA, Extreme Values Amplification; EVR, Extreme Values Reduction [Color figure can be viewed at [wileyonlinelibrary.com](http://wileyonlinelibrary.com)]

3. They have a concrete geometrical pattern (see Figure 3B).
4. The reduction of distances is greater close to the extremes.

Both EVAs and EVRs are valid approaches to model nonlinear preferences, which gives an answer to the first research question.

## 5 | GENERATING EVAS AND EVRS

In this section several examples of EVAs and EVRs are provided. First, a generic method to construct EVAs is developed.

Let  $h: [\alpha, \beta] \rightarrow [a, b]$  be the standard affine transformation given by

$$h(x) = \left( \frac{b-a}{\beta-\alpha} \right) \cdot (x - \alpha) + a \quad \forall x \in [\alpha, \beta]$$

and let us consider the special cases  $h_1: \left[ \frac{1}{2}, 1 \right] \rightarrow [0, 1]$  and  $h_2: [0, 1] \rightarrow \left[ \frac{1}{2}, 1 \right]$ .

**Proposition 8.** *Let  $f: [0, 1] \rightarrow [0, 1]$  be a  $C^1$  convex automorphism on the interval  $[0, 1]$  such that  $f'(0) < 1$  and  $f'(1) > 1$ . Then the mapping  $D: [0, 1] \rightarrow [0, 1]$  given by*

$$D(x) = \begin{cases} 1 - h_2 \circ f \circ h_1(1 - x), & 0 \leq x < \frac{1}{2}, \\ h_2 \circ f \circ h_1(x), & \frac{1}{2} \leq x \leq 1 \end{cases}$$

is an EVA.

*Remark 2.* The previous result could be adapted for EVRs by requiring concavity instead of convexity and changing the direction of the inequalities, that is,  $f'(0) > 1$  and  $f'(1) < 1$ .

In the following we will show several families of EVAs and EVRs and study their properties.

### 5.1 | Sin-based EVAs and EVRs

Let  $\alpha \in \left[ 0, \frac{1}{2\pi} \right]$ . Then the  $C^\infty$  function  $s_\alpha: [0, 1] \rightarrow [0, 1]$  given by

$$s_\alpha(x) = x - \alpha \cdot \sin(2\pi x - \pi) \quad \forall x \in [0, 1] \tag{1}$$

is an EVA (see Figure 4). Note that

$$\begin{aligned} s'_\alpha(x) &= 1 - \alpha 2\pi \cdot \cos(2\pi x - \pi) \quad \forall x \in [0, 1], \\ s''_\alpha(x) &= \alpha (2\pi)^2 \cdot \sin(2\pi x - \pi) \quad \forall x \in [0, 1], \end{aligned}$$

so  $s_\alpha$  is strictly increasing, concave in  $\left[ 0, \frac{1}{2} \right]$  and convex in  $\left[ \frac{1}{2}, 1 \right]$ . In addition  $s'_\alpha(1) = s'_\alpha(0) > 1$  and  $s'_\alpha\left(\frac{1}{2}\right) < 1$ .

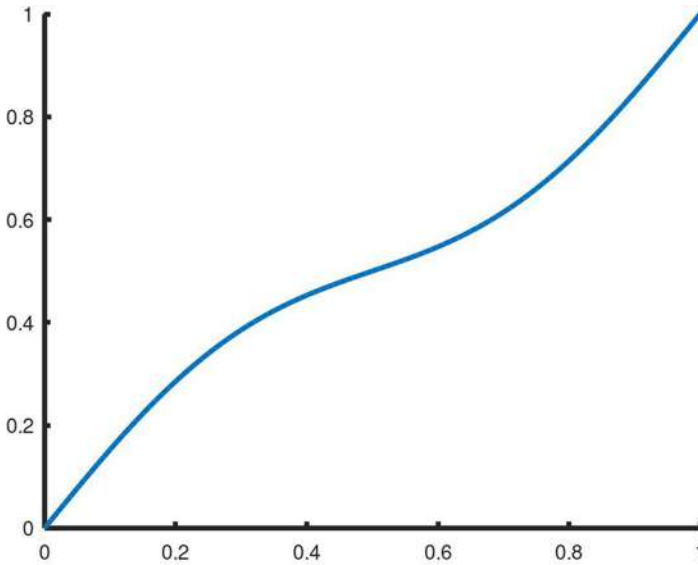


FIGURE 4 Extreme Values Amplification  $s_{0,09}$  [Color figure can be viewed at [wileyonlinelibrary.com](http://wileyonlinelibrary.com)]

Some interesting values are those where  $s'_\alpha(x) = 1$ . The solutions to this trigonometric equation are  $x_1 = \frac{1}{4}$  and  $x_2 = \frac{3}{4}$  (they do not depend on  $\alpha$ ), which are a kind of threshold for the distance amplification/reduction in the sense of

$$\begin{aligned}
 |s_\alpha(x) - s_\alpha(y)| &\geq |x - y| & \forall x, y \in \left[0, \frac{1}{4}\right], \\
 |s_\alpha(x) - s_\alpha(y)| &\leq |x - y| & \forall x, y \in \left[\frac{1}{4}, \frac{3}{4}\right], \\
 |s_\alpha(x) - s_\alpha(y)| &\geq |x - y| & \forall x, y \in \left[\frac{3}{4}, 1\right].
 \end{aligned}$$

Note that by defining  $\hat{s}_\alpha: [0, 1] \rightarrow [0, 1]$  by

$$\hat{s}_\alpha(x) = x + \alpha \cdot \sin(2\pi x - \pi) \quad \forall x \in [0, 1]$$

for  $\alpha \in \left[0, \frac{1}{2\pi}\right]$  we obtain a family of EVRs.

### 5.2 | Polynomial EVAs and EVRs

By applying Proposition 8 to the automorphism  $f_\alpha: [0, 1] \rightarrow [0, 1]$  given by

$$f_\alpha(x) = x^\alpha \quad \forall x \in [0, 1],$$

where  $\alpha > 1$ , we obtain

$$m_\alpha(x) = \begin{cases} \frac{1}{2} - \frac{1}{2}(1 - 2x)^\alpha, & 0 \leq x < \frac{1}{2}, \\ \frac{1}{2} + \frac{1}{2}(2x - 1)^\alpha, & \frac{1}{2} \leq x \leq 1 \end{cases} \tag{2}$$

(see Figure 5), whose derivatives are

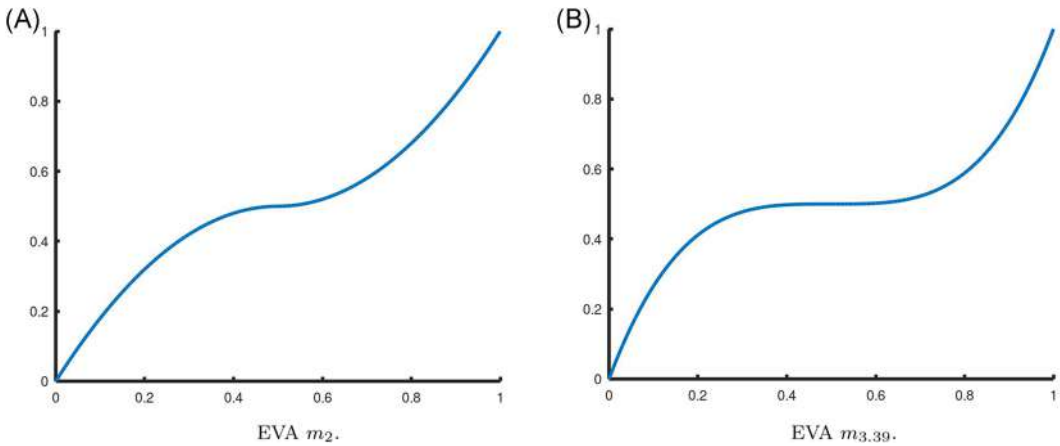


FIGURE 5 Graphs of the EVAs  $m_\alpha$ : (A) EVA  $m_2$  and (B) EVA  $m_{3.39}$ . EVA, Extreme Values Amplification; EVR, Extreme Values Reduction [Color figure can be viewed at [wileyonlinelibrary.com](http://wileyonlinelibrary.com)]

$$m'_\alpha(x) = \begin{cases} \alpha(1 - 2x)^{\alpha-1}, & 0 \leq x < \frac{1}{2}, \\ \alpha(2x - 1)^{\alpha-1}, & \frac{1}{2} \leq x \leq 1, \end{cases}$$

$$m''_\alpha(x) = \begin{cases} -2\alpha(\alpha - 1)(1 - 2x)^{\alpha-2}, & 0 \leq x < \frac{1}{2}, \\ 2\alpha(\alpha - 1)(2x - 1)^{\alpha-2}, & \frac{1}{2} \leq x \leq 1. \end{cases}$$

So  $m_\alpha$  is strictly increasing, concave in  $[0, \frac{1}{2}]$  and convex in  $[\frac{1}{2}, 1]$ . In addition  $m'_\alpha(1) = m'_\alpha(0) = \alpha > 1$  and  $m'_\alpha(\frac{1}{2}) = 0$ .

In this case the calculation for the amplification/reduction threshold values is not as easy as before, but we can compute a numeric approximation. For  $\alpha = 2$ , we obtain that  $m'_2(\frac{3}{4}) = 1$ , and for  $\alpha = 3.39$ ,  $m'_{3.39}(0.8) \approx 1$ .

Note that for  $0 < \alpha < 1$ , the functions  $\hat{m}_\alpha: [0, 1] \rightarrow [0, 1]$  given by

$$\hat{m}_\alpha(x) = \begin{cases} \frac{1}{2} - \frac{1}{2}(1 - 2x)^\alpha, & 0 \leq x < \frac{1}{2}, \\ \frac{1}{2} + \frac{1}{2}(2x - 1)^\alpha, & \frac{1}{2} \leq x \leq 1 \end{cases}$$

behave like an EVR. Although they are not differentiable functions in  $x = \frac{1}{2}$ , this family satisfies  $\lim_{x \rightarrow \frac{1}{2}^-} f'(x) = \lim_{x \rightarrow \frac{1}{2}^+} f'(x) = +\infty$ . Therefore these functions can be used in almost any situation in which a proper EVR can be applied.

### 5.3 | Piecewise polynomial-based EVAs

Let us consider again the automorphism  $f_\alpha: [0, 1] \rightarrow [0, 1]$  given by

$$f_\alpha(x) = x^\alpha \quad \forall x \in [0, 1],$$

where  $\alpha > 1$ .



For  $r, s \in \left[\frac{1}{2}, 1\right]$  such that  $r > s$  and  $\epsilon \in ]0, 1[$ , the following standard affine transformations are considered:

$$\begin{aligned} h_a: \left[\frac{1}{2}, r\right] &\rightarrow \left[\frac{1}{2}, s\right], \\ h_b: [r, 1] &\rightarrow [\epsilon, 1], \\ h_c: [\epsilon^\alpha, 1] &\rightarrow [s, 1]. \end{aligned}$$

We aim to use these affine transformations to construct a parametric EVA from the function  $b_1: \left[\frac{1}{2}, 1\right] \rightarrow \left[\frac{1}{2}, 1\right]$  given by

$$b_1(x) = \begin{cases} h_a(x), & \frac{1}{2} \leq x \leq r, \\ h_c \circ f_\alpha \circ h_b(x), & r < x \leq 1 \end{cases}$$

by defining  $b_\alpha^{r,s}: [0, 1] \rightarrow [0, 1]$  by

$$b_\alpha^{r,s}(x) = \begin{cases} 1 - b_1(1 - x), & 0 \leq x < \frac{1}{2}, \\ b_1(x), & \frac{1}{2} \leq x \leq 1. \end{cases}$$

Since

$$h'(x) = \frac{s - \frac{1}{2}}{r - \frac{1}{2}},$$

the parameters  $r$  and  $s$  allow us to control where and how much the differences between intermediate values are decreased. For instance, if we want the derivative of the EVA around  $[1 - r, r]$  be equals to some  $\lambda \in ]0, 1[$  we just need to fix  $s := \frac{1}{2} + \lambda(r - \frac{1}{2})$ . The parameter  $\alpha > 1$  controls *how faster* the distances between extreme values are amplified.

It is clear that the function  $b_1$  is not  $C^1$  for any combination of the parameters  $\alpha, r, s$ . The parameter  $\epsilon$  has been introduced to solve this issue. Note that

$$(h_c \circ f_\alpha \circ h_b)'(x) = \frac{1 - s}{1 - r} \frac{1 - \epsilon}{1 - \epsilon^\alpha} \alpha h_b^{\alpha-1}(x) = \frac{1 - s}{1 - r} \frac{1 - \epsilon}{1 - \epsilon^\alpha} \alpha \left(\frac{x - r}{1 - r}(1 - \epsilon) + \epsilon\right)^{\alpha-1}.$$

Therefore, for fixed  $1 > r > s > \frac{1}{2}$  and  $\alpha > 1$  we need to find  $\epsilon \in ]0, 1[$  such that the equality  $(h_c \circ f_\alpha \circ h_b)'(r) = h'(r)$  holds, that is,

$$\frac{s - \frac{1}{2}}{r - \frac{1}{2}} = \frac{1 - s}{1 - r} \frac{1 - \epsilon}{1 - \epsilon^\alpha} \alpha \epsilon^{\alpha-1}.$$

**Proposition 9.** Consider  $r, s \in \left[\frac{1}{2}, 1\right]$  such that  $r > s$  and  $\epsilon \in ]0, 1[$  and the standard affine transformations:

$$\begin{aligned} h_a: \left[\frac{1}{2}, r\right] &\rightarrow \left[\frac{1}{2}, s\right], \\ h_b: [r, 1] &\rightarrow [\epsilon, 1], \\ h_c: [\epsilon^\alpha, 1] &\rightarrow [s, 1]. \end{aligned}$$

Then the function  $b_\alpha^{r,s}: [0, 1] \rightarrow [0, 1]$  given by

$$b_{\alpha}^{r,s}(x) = \begin{cases} 1 - b_1(1 - x), & 0 \leq x < \frac{1}{2}, \\ b_1(x), & \frac{1}{2} \leq x \leq 1, \end{cases} \tag{3}$$

where  $b_1: [\frac{1}{2}, 1] \rightarrow [\frac{1}{2}, 1]$  is defined by

$$b_1(x) = \begin{cases} h_a(x), & \frac{1}{2} \leq x \leq r, \\ h_c \circ f_{\alpha} \circ h_b(x), & r < x \leq 1 \end{cases}$$

is an EVA if and only if the following equality holds

$$\lambda = \frac{s - \frac{1}{2}}{r - \frac{1}{2}} = \frac{1 - s}{1 - r} \frac{1 - \epsilon}{1 - \epsilon^{\alpha}} \alpha \epsilon^{\alpha - 1},$$

where  $\lambda$  is the derivative of  $h_a$  and indicates how much the intermediate values become closer.

Some useful combinations of these parameters are shown in Table 1 and the respective graphs are included in Figure 6.

Two limit cases are considered below.

### 5.3.1 | Special case $r = s > 1/2$

Let us first consider the limit case  $r = s > \frac{1}{2}$ . We would need

TABLE 1 Useful combinations of parameters for the EVA  $b$

$r$	$s$	$\lambda$	$\alpha$	$\epsilon$
0.6	0.55	0.5	2	0.28571
0.6	0.533	0.333	3	0.38

Abbreviation: EVA, Extreme Values Amplification.

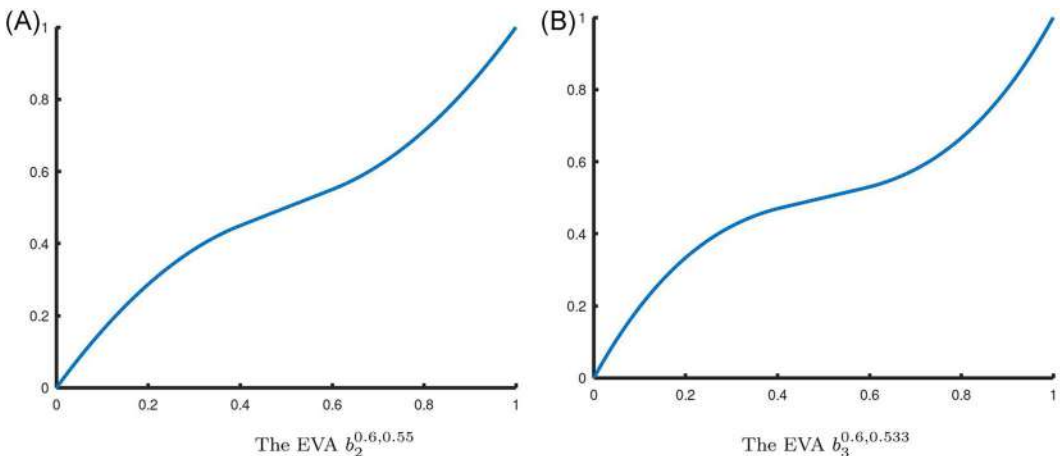


FIGURE 6 Graphs of the EVAs  $b_{\alpha}^{r,s}$ : (A) The EVA  $b_2^{0.6,0.55}$  and (B) The EVA  $b_3^{0.6,0.533}$ . EVA, Extreme Values Amplification [Color figure can be viewed at [wileyonlinelibrary.com](http://wileyonlinelibrary.com)]

$$1 = \frac{1 - \epsilon}{1 - \epsilon^\alpha} \alpha \epsilon^{\alpha-1} \Leftrightarrow 1 = \frac{\epsilon^{\alpha-1} - \epsilon^\alpha}{1 - \epsilon^\alpha} \alpha.$$

Consider the function  $g: ]0, 1[ \rightarrow \mathbb{R}$  defined by

$$g(\epsilon) = \alpha(\epsilon^{\alpha-1} - \epsilon^\alpha) - 1 + \epsilon^\alpha \quad \forall \epsilon \in ]0, 1[.$$

Note that

$$\begin{aligned} g'(\epsilon) &= \alpha((\alpha - 1)\epsilon^{\alpha-2} - \alpha\epsilon^{\alpha-1}) + \alpha\epsilon^{\alpha-1} \\ &= \alpha(\alpha - 1)\epsilon^{\alpha-2} + \alpha\epsilon^{\alpha-1}(1 - \alpha) \\ &= \alpha(\alpha - 1)(\epsilon^{\alpha-2} - \epsilon^{\alpha-1}) \\ &= \alpha(\alpha - 1)\epsilon^{\alpha-2}(1 - \epsilon) > 0 \quad \forall \epsilon \in ]0, 1[. \end{aligned}$$

Since  $\lim_{\epsilon \rightarrow 0} g(\epsilon) = -\infty$ , and  $g' > 0$  the equation  $g(\epsilon) = 0$  has the unique solution  $\epsilon = 1$ , which is not admissible in this problem.

### 5.3.2 | Special case $r = s = 1/2$

Note that this assumption implies that the affine transformation  $h_a$  disappears and  $b_1(x) = h_c \circ f_\alpha \circ h_b(x) \quad \forall x \in [\frac{1}{2}, 1]$ .

Define  $g: ]0, 1[ \rightarrow \mathbb{R}$  by

$$g(\epsilon) = \alpha \frac{\epsilon^{\alpha-1} - \epsilon^\alpha}{1 - \epsilon^\alpha} \quad \forall \epsilon \in ]0, 1[.$$

Note that  $\lim_{\epsilon \rightarrow 0} g(\epsilon) = 0$  and

$$\lim_{\epsilon \rightarrow 1} g(\epsilon) = \lim_{\epsilon \rightarrow 1} \alpha \epsilon^{\alpha-1} \frac{1 - \epsilon}{1 - \epsilon^\alpha} = 1.$$

In addition

$$\begin{aligned} g'(\epsilon) &= \frac{\alpha}{(1 - \epsilon^\alpha)^2} \left( ((\alpha - 1)\epsilon^{\alpha-2} - \alpha\epsilon^{\alpha-1})(1 - \epsilon^\alpha) \right. \\ &\quad \left. + (\epsilon^{\alpha-1} - \epsilon^\alpha)\alpha\epsilon^{\alpha-1} \right) \\ &= \frac{\alpha\epsilon^{\alpha-2}}{(1 - \epsilon^\alpha)^2} \left( (\alpha - 1 - \alpha\epsilon)(1 - \epsilon^\alpha) + \alpha(\epsilon^\alpha - \epsilon^{\alpha+1}) \right) \\ &= \frac{\alpha\epsilon^{\alpha-2}}{(1 - \epsilon^\alpha)^2} \left( \alpha(1 - \epsilon)(1 - \epsilon^\alpha) - (1 - \epsilon^\alpha) + \alpha\epsilon^\alpha(1 - \epsilon) \right) \\ &= \frac{\alpha\epsilon^{\alpha-2}}{(1 - \epsilon^\alpha)^2} \left( \alpha(1 - \epsilon) - (1 - \epsilon^\alpha) \right) \\ &\quad \forall \epsilon \in ]0, 1[. \end{aligned}$$

To study the sign of  $g'$  let us consider  $h: [0, 1] \rightarrow \mathbb{R}$  defined by

$$h(\epsilon) = \alpha(1 - \epsilon) - (1 - \epsilon^\alpha) \quad \forall \epsilon \in ]0, 1[.$$

Since  $h'(\epsilon) = \alpha(\epsilon^{\alpha-1} - 1) < 0 \quad \forall \epsilon \in ]0, 1[$  and  $\lim_{\epsilon \rightarrow 0} h(\epsilon) = \alpha - 1, \lim_{\epsilon \rightarrow 1} h(\epsilon) = 0$  we can conclude that  $h > 0$  in its domain and therefore  $g'(\epsilon) > 0 \quad \forall \epsilon \in ]0, 1[$ . In that case  $g$  is increasing and such that  $\lim_{\epsilon \rightarrow 0} g(\epsilon) = 0, \lim_{\epsilon \rightarrow 1} g(\epsilon) = 1$ . This fact allows one to state the following result.

**Proposition 10.** *Let  $\epsilon \in ]0, 1[$  and consider the standard affine transformations  $h_b: [\frac{1}{2}, 1] \rightarrow [\epsilon, 1]$  and  $h_c: [\epsilon, 1] \rightarrow [\frac{1}{2}, 1]$  and the function  $\hat{b}_1: [\frac{1}{2}, 1] \rightarrow [\frac{1}{2}, 1]$  defined by  $\hat{b}_1(x) = h_c \circ f_\alpha \circ h_b(x) \quad \forall x \in [\frac{1}{2}, 1]$ .*

*Then, for every  $\lambda \in ]0, 1[$  we can find  $\epsilon \in ]0, 1[$  (i.e., the unique one which satisfies  $\lambda = \alpha \frac{\epsilon^{\alpha-1} - \epsilon^\alpha}{1 - \epsilon^\alpha}$ ) such that the function  $\hat{b}: [0, 1] \rightarrow [0, 1]$  defined by*

$$\hat{b}(x) = \begin{cases} \hat{b}_1(x), & \frac{1}{2} \leq x \leq 1, \\ 1 - \hat{b}_1(1 - x), & 0 \leq x \leq \frac{1}{2} \end{cases}$$

*is an EVA such that  $\hat{b}'(\frac{1}{2}) = \lambda$ .*

Note that by taking  $\epsilon = 0$  we would obtain the  $m_\alpha$  family of EVAs.

### 5.4 | Comparing sin-based EVAs, polynomial EVAs, and piecewise polynomial EVAs

The main difference between these families of EVAs is the value of their derivative in  $\frac{1}{2}$ . Note that in all cases the derivative function reaches its minimum value at this point.

For *sin*-based EVAs, the derivative at  $x = \frac{1}{2}$  is  $s'_\alpha(\frac{1}{2}) = 1 - \alpha 2\pi$  and the derivative of the polynomial-based EVAs is zero. For piecewise polynomial EVAs, this derivative is a value  $\lambda \in [0, 1]$

In the first case, by choosing a proper  $\alpha \in ]0, \frac{1}{2\pi}[$ , we can adjust *how much* the intermediate values will move closer. For example, for the intermediate parameters  $\alpha = 0.08$  and  $0.09$  one obtains  $s'_{0.08}(\frac{1}{2}) = 1 - 0.08 \cdot 2\pi \approx 0.49735$  and  $s'_{0.09}(\frac{1}{2}) = 1 - 0.09 \cdot 2\pi \approx 0.43451$ .

On the other hand, the main advantage of *polynomial*-based EVAs is the fact that we can decide where the values are going to start to move away by fixing  $x_0 > \frac{1}{2}$  and solving (numerically) the equation  $m'_\alpha(x_0) = 1$  for the variable  $\alpha > 1$ .

The *piecewise polynomial* EVA allows both to choose how much the intermediate values will become closer, by adjusting  $\lambda$ , and how much the extreme values will become distant, by choosing  $\alpha$ . However, this family of EVAs is more complex and loses the regularity of the other two families.

## 6 | A THEORETIC DISCUSSION ABOUT THE PERFORMANCE OF EVAS AND EVRS IN CRPS FOR LSGDM

In Section 4 the EVA approach and the EVR approach have been introduced as models for nonlinear preferences. However, the strategy to deal with polarized opinions is totally different in EVA and EVR. According to References [24,25], the less extreme values have a more cohesive effect and greater success to reach an agreement. Therefore, this section is devoted to

provide a sustained proof about why EVRs are not a good strategy to remap FPRs in CRPs meanwhile EVAs tend to improve the performance of the consensus models.

### 6.1 | EVAs and order of alternatives in CRPs

First a study of *how much* EVAs deform the original preferences is provided.

**Proposition 11.** *Let  $D: [0, 1] \rightarrow [0, 1]$  be an EVA whose first derivative is strictly increasing in  $[\frac{1}{2}, 1]$ . Then, the equation*

$$D'(x) = 1$$

*has an unique solution  $x_0 \in ]\frac{1}{2}, 1[$  which satisfies*

$$|x - D(x)| \leq |x_0 - D(x_0)|$$

*for every  $x \in [0, 1]$ .*

*Proof.* The existence and the uniqueness of  $x_0$  are given by the bijectivity of  $D'$ .

Now consider the function  $g: [\frac{1}{2}, 1] \rightarrow \mathbb{R}$  given by

$$g(x) = x - D(x) \quad \forall x \in [\frac{1}{2}, 1].$$

Since  $g'(x) = 1 - D'(x) \quad \forall x \in [\frac{1}{2}, 1]$  the candidates to be relative extremes for the function  $g$  are  $(\frac{1}{2}, 0)$ ,  $(1, 0)$  and  $(x_0, g(x_0))$ . Due to the continuity of  $D'$ ,  $g'$  is a continuous function which also satisfies that  $g'(\frac{1}{2}) > 0$  and  $g'(1) < 0$ . In that case  $(x_0, g(x_0))$  is a relative maximum for  $g$  and both  $(\frac{1}{2}, 0)$  and  $(1, 0)$  are minimums, which proves the inequality of the proposition. □

Note that we do not need that  $D'$  is strictly increasing in  $[\frac{1}{2}, 1]$ . It suffices to consider an EVA  $D$  whose first derivative is strictly increasing in a neighborhood of 1 which contains a value  $r > \frac{1}{2}$  such that  $D'(r) < 1$ . The general version, whose proof is analogous, is stated:

**Proposition 12.** *Let  $D: [0, 1] \rightarrow [0, 1]$  be an EVA whose first derivative is strictly increasing in  $[r, 1]$  for some  $r > \frac{1}{2}$  such that  $D'(r) < 1$ . If  $D'$  is monotonous in  $[\frac{1}{2}, 1]$  then, the equation*

$$D'(x) = 1$$

*has an unique solution  $x_0 \in ]r, 1[$  which satisfies*

$$|x - D(x)| \leq |x_0 - D(x_0)|$$

*for every  $x \in [0, 1]$ .*

*Remark 3.* Note that the symmetry of  $D$  around  $\frac{1}{2}$  would provide another  $\hat{x}_0 \in [0, \frac{1}{2}]$  satisfying the same inequality.

*Remark 4.* The three families of EVAs introduced in this contribution satisfy the hypotheses of this proposition.

Suppose now that we have obtained the FPR  $P = (p_{ij}) \in \mathcal{M}_{n \times n}([0, 1])$  from a certain expert. We want to analyze how different the order of the alternatives chosen by that expert will be after applying an EVA.

To compute the order of the alternatives we just assign each alternative a score depending on the value of the preferences:

$$sc(x_i) = \frac{1}{n - 1} \sum_{j=1, j \neq i}^n p_{ij}, \quad i \in \{1, 2, \dots, n\}.$$

Then we order the alternatives according to the score they have received.

We cannot prove that the order of the alternatives will not change after applying any EVA, but we can show that there exists a threshold which enables us to control the distance between the score obtained for the deformed preferences and the original score.

**Proposition 13.** *Let  $D: [0, 1] \rightarrow [0, 1]$  be an EVA whose first derivative is monotonous in  $[\frac{1}{2}, 1]$  and strictly increasing in  $[r, 1]$  for some  $r > \frac{1}{2}$  such that  $D'(r) < 1$ . Consider an FPR  $P = (p_{ij}) \in \mathcal{M}_{n \times n}([0, 1])$  given for the alternatives  $x_1, x_2, \dots, x_n$ . Then*

$$|sc(x_i) - sc(D(x_i))| \leq |x_0 - D(x_0)| \quad \forall i \in \{1, 2, \dots, n\},$$

where  $x_0 \in ]\frac{1}{2}, 1[$  is the unique solution for the equation  $D'(x) = 1$ .

*Proof.* Note that for any alternative  $x_i$  we obtain

$$\begin{aligned} |sc(x_i) - sc(D(x_i))| &= \left| \frac{1}{n-1} \sum_{j=1, j \neq i}^n (p_{ij} - D(p_{ij})) \right| \\ &\leq \frac{1}{n-1} \sum_{j=1, j \neq i}^n |p_{ij} - D(p_{ij})| \leq \frac{1}{n-1} \sum_{j=1, j \neq i}^n |x_0 - D(x_0)| \\ &\leq \frac{|x_0 - D(x_0)|}{n-1} \sum_{j=1, j \neq i}^n 1 = |x_0 - D(x_0)|. \end{aligned}$$

□

*Remark 5.* Note that we would have obtained the same result if we had used any other aggregation operator based on weights to compute the score.

It should be highlighted that  $x_0 \in ]\frac{1}{2}, 1[$  such that  $D'(x_0) = 1$  is not only a threshold for the amplification of the distances, but also provides a bound to study how similar is the order alternatives after applying the EVA with respect to the original order. Therefore, the value  $x_0$  will receive the name of *amplification threshold*, and the value  $R_0 := |x_0 - D(x_0)|$  will be called *maximum deformation*.

Let us study these quantities for different families of EVAs.

### 6.1.1 | Sin-based EVAs

Let  $\alpha \in ]0, \frac{1}{2\pi}]$  and consider the  $C^\infty$  function  $s_\alpha: [0, 1] \rightarrow [0, 1]$  given by

$$s_\alpha(x) = x - \alpha \cdot \sin(2\pi x - \pi) \quad \forall x \in [0, 1].$$

Note that  $s'(x) = 1, x > \frac{1}{2}$  satisfies if and only if

$$\alpha 2\pi \cdot \cos(2\pi x - \pi) = 0$$

and therefore  $x_0 = \frac{3}{4}$  and

$$R_0 = g(x_0) = \frac{3}{4} - s_\alpha\left(\frac{3}{4}\right) = \alpha \cdot \sin\left(2\pi \frac{3}{4} - \pi\right) = \alpha.$$

### 6.1.2 | Polynomial-based EVAs

Consider the automorphism  $f_\alpha: [0, 1] \rightarrow [0, 1]$  given by

$$f_\alpha(x) = x^\alpha \quad \forall x \in [0, 1],$$

where  $\alpha > 1$ . We obtain the EVA

$$m_\alpha(x) = \begin{cases} \frac{1}{2} - \frac{1}{2}(1 - 2x)^\alpha, & 0 \leq x < \frac{1}{2}, \\ \frac{1}{2} + \frac{1}{2}(2x - 1)^\alpha, & \frac{1}{2} \leq x \leq 1. \end{cases}$$

In this case  $m'(x) = 1, x > \frac{1}{2}$  satisfies if and only if

$$\alpha(2x - 1)^{\alpha-1} = 1,$$

thus the amplification threshold is  $x_0 = \frac{1}{2}\left(\alpha^{-\frac{1}{\alpha-1}} + 1\right)$  and the maximum deformation is

$$\begin{aligned} R_0 &= x_0 - m_\alpha(x_0) \\ &= \frac{1}{2}\left(\alpha^{-\frac{1}{\alpha-1}} + 1\right) - \left(\frac{1}{2} + \frac{1}{2}\left(2\left(\frac{1}{2}\left(\alpha^{-\frac{1}{\alpha-1}} + 1\right)\right) - 1\right)^\alpha\right) \\ &= \frac{1}{2}\left(\alpha^{-\frac{1}{\alpha-1}} + 1\right) - \left(\frac{1}{2} + \frac{1}{2}\left(\alpha^{-\frac{1}{\alpha-1}}\right)^\alpha\right) \\ &= \frac{1}{2}\left(\alpha^{-\frac{1}{\alpha-1}} - \alpha^{-\frac{\alpha}{\alpha-1}}\right) = \frac{1}{2}\left(\alpha^{-\frac{1}{\alpha-1}} - \alpha^{-\frac{\alpha}{\alpha-1}}\right) \\ &= \frac{1}{2}\left(\alpha^{-\frac{1}{\alpha-1}} - \alpha^{-\frac{\alpha+1-1}{\alpha-1}}\right) = \frac{1}{2}\left(\alpha^{-\frac{1}{\alpha-1}} - \alpha^{-\frac{1}{\alpha-1}\alpha^{-1}}\right) \\ &= \frac{1}{2^{\alpha-\frac{1}{\alpha}}}\left(1 - \frac{1}{\alpha}\right). \end{aligned}$$

For  $\alpha = 2$  we obtain  $R_0 = \frac{1}{8}$  and for  $\alpha = 3$  it is  $R_0 \approx 0.2$ .

### 6.1.3 | Piecewise polynomial-based EVA

Consider  $r, s \in \left[\frac{1}{2}, 1\right]$  such that  $r > s$  and  $\epsilon \in [0, 1]$ . Consider the standard affine transformations:

TABLE 2 Useful combinations of parameters for  $b_\alpha^{r,s}$

$r$	$s$	$\lambda$	$\alpha$	$\epsilon$	$R_0$
0.6	0.55	0.5	2	0.28571	0.09
0.6	0.533	0.333	3	0.38	0.13

$$\begin{aligned} h_a: \left[\frac{1}{2}, r\right] &\rightarrow \left[\frac{1}{2}, s\right], \\ h_b: [r, 1] &\rightarrow [\epsilon, 1], \\ h_c: [\epsilon^\alpha, 1] &\rightarrow [s, 1], \end{aligned}$$

and the function  $b_1: \left[\frac{1}{2}, 1\right] \rightarrow \left[\frac{1}{2}, 1\right]$

$$b_1(x) = \begin{cases} h_a(x), & \frac{1}{2} \leq x \leq r, \\ h_c \circ f_\alpha \circ h_b(x), & r < x \leq 1. \end{cases}$$

Then the EVA  $b_\alpha^{r,s}: [0, 1] \rightarrow [0, 1]$  defined, for some proper  $\epsilon$  such that  $b$  is  $\mathcal{C}^1$ , by

$$b_\alpha^{r,s}(x) = \begin{cases} b_1(x), & \frac{1}{2} \leq x \leq 1, \\ 1 - b_1(1 - x), & 0 \leq x \leq \frac{1}{2} \end{cases}$$

has its amplification threshold at

$$x_0 = \left( \alpha^{-1} \sqrt{\frac{(1-r)(1-\epsilon^\alpha)}{(1-s)(1-\epsilon)\alpha}} - \epsilon \right) \frac{1-r}{1-\epsilon} + r,$$

since this value satisfies  $(h_c \circ f_\alpha \circ h_b)'(x_0) = 1$ , that is,

$$\frac{1-s}{1-r} \frac{1-\epsilon}{1-\epsilon^\alpha} (h_b(x_0))^{\alpha-1} \alpha = 1.$$

In this case, the explicit formula for the amplification radius is complex and offers no advantage, so we can obtain that value by computing the numeric value of  $x_0$  and then considering  $R_0 = x_0 - b_\alpha^{r,s}(x_0)$ . We show some values in Table 2.

## 6.2 | Why EVAs do work for improving CRPs for LSGDM and why EVRs do not

In Section 6.1 it has been proved that the impact of EVAs in the order of the alternatives is pretty low.

On the other hand, it has been shown in References [24,25] that the less extreme values have a more cohesive effect and make easier the reaching of an agreement whereas the more



extreme values of the preferences tend to polarize situations, so any worthy CRP model should prioritize intermediate values of the preferences in its aggregations.

Let us consider all of these together: when the EVA approach is used on a consensus model which aggregates the preferences by prioritizing intermediate values, the model will ignore the extreme values and the intermediate ones will become closer because of the properties of the EVA function. In this case, the model will need a lower amount of rounds to reach the consensus, since the EVA has done part of the work by making the intermediate values closer. Since the order of the alternatives has not been changed *too much*, a consensus model which uses the EVA approach will choose a similar alternative faster than the original one.

This also explains why we are not using EVRs to model the nonlinear approach in CRPs. Although EVRs also modify the preferences in a nonlinear way, in this case the distances between extreme preferences are reduced and the distance between intermediate preferences is amplified. When EVRs are implemented in a consensus model, which usually prioritizes intermediate preferences to facilitate the consensus, the model will probably need more rounds to reach the consensus, since intermediate elements are less similar.

Both EVAs and EVRs are valid approaches to model nonlinear scales in CRPs (RQ1), but to improve classic models the EVA approach outperforms both the linear approach and the EVR approach (RQ2).

## 7 | EXTREME VALUES AMPLIFICATIONS IN GDM

This section aims at verifying, validating, and showing the better performance of EVA functions in LSGDM problems with respect to lineal preference modeling. Therefore, an illustrative LSGDM problem is solved by using the specialized CRP software AFRYCA<sup>10</sup> and comparing the performance of two widely used consensus models (see Remark 7) when they use linear and nonlinear preference scales.

Therefore, we have implemented the EVA families  $s_\alpha$  (Equation 1) and  $m_\alpha$  (Equation 2) into the CRPs introduced in References [16,26] (both included in AFRYCA) and then carried out several simulations to compare their performance by using EVAs and linear preferences.

*Remark 6.* In Section 6 it was pointed out that extreme values make more difficult the achievement of agreements.<sup>24,25</sup> Therefore, due to the fact that EVRs amplify the distances between less extreme values, we will only consider the study in further detail of EVAs for CRPs in LSGDM because EVRs are not suitable for smoothing the achievement of agreements.

*Remark 7.* The consensus model proposed by Herrera-Viedma et al.<sup>16</sup> has been selected since it has been widely used in the literature and several consensus models are based on its performance.<sup>18</sup> Quesada et al.'s proposal<sup>26</sup> has been chosen since it deals with GDM problems with a large number of experts and considers important aspects, such as the experts' behavior.

Unlike, most of the proposals about LSGDM in the specialized literature whose examples are based on 20 experts, we assume an LSGDM problem in which there are 100 experts who elicit their FPRs over four alternatives  $X = \{X_1, X_2, X_3, X_4\}$ . To accomplish the CRPs proposed in our example, several simulations have been run by using the default values of the parameters

TABLE 3 Average results on Herrera-Viedma et al.<sup>16</sup> (500 simulations)

EVA	Average rounds	Average consensus
Classical	5.158	0.8807
$s_{0.08}$	4.482	0.8835
$s_{0.09}$	4.384	0.8816
$m_2$	4.08	0.8805
$m_{3,39}$	3.606	0.8851

Abbreviation: EVA, Extreme Values Amplification.

TABLE 4 Average results on Quesada et al.<sup>26</sup> (500 simulations)

EVA	Average rounds	Average consensus
Classical	9.256	0.8579
$s_{0.08}$	7.532	0.8575
$s_{0.09}$	7.188	0.8592
$m_2$	6.074	0.8581
$m_{3,39}$	2.842	0.8598

Abbreviation: EVA, Extreme Values Amplification.

established in AFRYCA for such consensus models, by setting the consensus level at 0.85 and the maximum number of rounds at 15.

This section is divided into two subsections. In both, the performance of the classical models Herrera-Viedma et al.<sup>16</sup> and Quesada et al.<sup>26</sup> is compared with the EVA-modified models. To do so, for each consensus model five different scenarios are considered: the classical model (no EVA is used), the EVAs  $s_{0.08}$  and  $s_{0.09}$  (defined by Equation 1), and the EVAs  $m_2$  and  $m_{3,39}$  (defined by Equation 2).

In Section 7.1, 500 simulations are developed for each one of these scenarios. In all of them, 100 randomly defined FPRs are used to model experts' preferences. For these 500 simulations, both the average number of rounds required to obtain the consensus and the average degree of consensus are computed. In Section 7.2, the simulations are developed by using concrete values for the experts' FPRs<sup>27</sup> to be able to compare graphically the evolution of the experts' opinions through the different rounds of the CRPs.

## 7.1 | Average performance of EVAs

To validate the EVA approach, the average performance of EVA-modified models has been compared with the average performance of the classic models. To do so, 500 simulations with 100 randomly defined FPRs have been developed for each EVA in both Herrera-Viedma et al.<sup>16</sup> and Quesada et al.<sup>26</sup> models.

The obtained results, which are summarized in Tables 3 and 4, show that the EVA approach always outperforms the classic approach in terms of convergence, by keeping a similar average

**TABLE 5** Results on Herrera-Viedma et al.<sup>16</sup> with and without EVA

EVA	Order of alternatives	Rounds	Consensus
Classical	$x_1 > x_2 > x_4 > x_3$	6	0.87
$s_{0.08}$	$x_1 > x_2 > x_4 > x_3$	6	0.89
$s_{0.09}$	$x_1 > x_2 > x_4 > x_3$	6	0.92
$m_2$	$x_1 > x_2 > x_4 > x_3$	5	0.86
$m_{3,39}$	$x_1 > x_2 > x_4 > x_3$	5	0.91

Abbreviation: EVA, Extreme Values Amplification.

**TABLE 6** Results on Quesada et al.<sup>26</sup> with and without EVA

EVA	Order of alternatives	Rounds	Consensus
Classical	$x_4 > x_1 \sim x_2 \sim x_3$	10	0.85
$s_{0.08}$	$x_4 > x_1 > x_2 \sim x_3$	7	0.86
$s_{0.09}$	$x_4 > x_1 > x_2 \sim x_3$	7	0.87
$m_2$	$x_4 > x_1 > x_2 \sim x_3$	7	0.86
$m_{3,39}$	$x_4 > x_1 > x_2 \sim x_3$	5	0.87

Abbreviation: EVA, Extreme Values Amplification.

consensus. This fact provides a clear answer to the second research question: the classic consensus models improve on average when the nonlinear approach is modeled by an EVA.

## 7.2 | Performance of EVAs in a concrete example

To clarify the performance of EVAs in CRPs for LSGDM, we have chosen an individual simulation with the FPRs values provided in Reference [27] and then show the results obtained in Tables 4 and 5 together a graphical evolution of the consensus progress. The results obtained are summarized in Tables 5 and 6.

To facilitate the understanding of the simulation results, AFRYCA provides a visualization of the different CRP simulations based on the multidimensional scaling technique<sup>28</sup> (see Figures 7 and 8). This representation shows the collective opinion of the experts' group in the center of the plot. Around the collective opinion, the experts' preferences are represented. The closer the experts' preferences to the collective opinion, the greater the consensus reached. In this way, we can appreciate the state of the experts' preferences for each round and the evolution of the CRPs in the simulations. Furthermore, we have also shown the results obtained from AFRYCA in Tables 5 and 6.

The classical model<sup>16</sup> reached a consensus level of 0.87 in six rounds. For this model the EVAs  $s_{0.08}$  and  $s_{0.09}$  have not reduced the number of rounds required to reach the consensus, but have improved the consensus level reached. The latter can be appreciated in Figure 7, since in the last round (round 6) the experts are closer each other than with linear preferences. In addition, the polynomial-based EVA  $m_2$  has reduced the amount of rounds required to reach a similar consensus level, whereas the EVA  $m_{3,39}$  has improved both aspects by needing just five

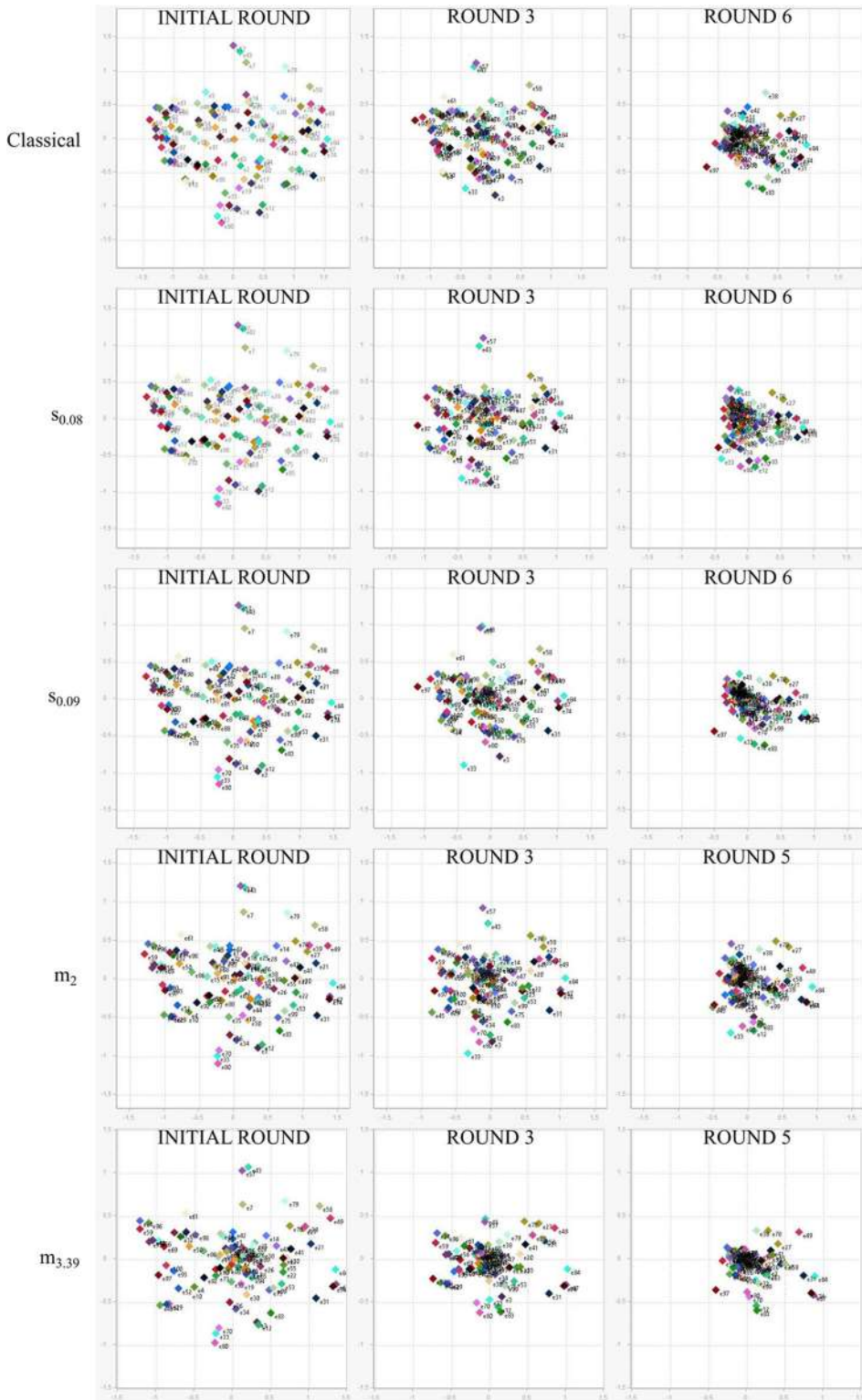


FIGURE 7 Herrera Viedma et al.<sup>16</sup> simulations [Color figure can be viewed at [wileyonlinelibrary.com](http://wileyonlinelibrary.com)]



FIGURE 8 Quesada et al.<sup>26</sup> simulations [Color figure can be viewed at [wileyonlinelibrary.com](http://wileyonlinelibrary.com)]

rounds to reach a consensus level greater than the obtained in the original model. Again, the latter can be visualized in Figure 7.

On the other hand, the classical model<sup>26</sup> obtained a consensus level of 0.85 in 10 rounds (see Table 6). In this case, both families of EVAs have obtained significantly better performance than the original model (see Figure 8). The EVAs  $s_{0.08}$ ,  $s_{0.09}$ , and  $m_2$  have slightly improved the consensus level in just seven rounds. In this case, the EVA  $m_{3.39}$  has performed surprisingly well by increasing the level of consensus reached in only five rounds.

The simulation has shown that the implemented EVAs improve the performance of both models. By keeping the same order for the alternatives, after using the EVAs either the number of rounds has been reduced or the consensus level is increased. These simulations clarify and reinforce the positive answer to the second research question when the nonlinear approach used is an EVA.

## 8 | CONCLUSIONS

Nowadays, CRPs are a prominent line of research in GDM. Several models have been proposed in the literature, but usually these models assume linear scales for experts' preferences.<sup>16,26</sup> This contribution has studied and proposed the use of nonlinear scales to obtain more realistic preference modeling from the original experts' preferences, even in large-scale contexts.

We have exhaustively studied the analytical properties of these nonlinear scales, obtaining the main mathematical characteristics of those functions which are good candidates to become a proper nonlinear deformation for the original preferences. These particular deformations of the preferences have received the name of EVAs. These EVA functions remap linear-scaled FPRs into nonlinear-scaled FPRs and deform the preferences in the way that the distances between extreme values are increased and the distances between intermediate values are decreased. In addition, we have stated the dual definition of EVRs, that is, those functions that reduce the distance between extreme values by amplifying the distance between the intermediate ones.

After introducing a general method to construct EVAs and EVRs, we have proposed several families of EVAs:  $s_\alpha$  (Equation 1),  $m_\alpha$  (Equation 2), and  $b_\alpha^{r,s}$  (Equation 3). The first one is based on the sin function, the second one is constructed from a polynomial and the last one is obtained from a piecewise polynomial function. Finally, we have simulated the performance of some of these EVAs in two classical consensus models by using the software AFRYCA.<sup>10</sup>

The use of the nonlinear scales provided by the EVAs improves the performance of both classical models used in this study. The simulations with random FPRs showed that the EVA approach reduces the average number of rounds required to reach the consensus in both models. In addition, when using the same FPRs for the comparisons, the novel EVA-modified models either reach the consensus in a faster way or increase the level of consensus when we use the proper EVA.

Further studies should focus on either suggesting new EVAs or optimizing the parameters of the existing EVAs for concrete CRPs. In addition, a deeper study of the performance of EVAs in different consensus models should be developed. Additionally, since every EVA (resp., EVR) induces a similarity measure, it is also interesting to study the effects of using these proximity measures when comparing FPRs by moving closer (resp., bringing near) extreme values and bringing near (resp., moving closer) the intermediate ones. Another possible research work would be finding a concrete GDM problem adequate to the properties of EVRs. Furthermore, future works could be related to the application of the proposed framework to real-world problems, such as the high-speed rail passenger satisfaction and bid evaluation with LSGDM.<sup>29,30</sup>

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