UNIVERSIDAD DE JAÉN

Departamento de Informática

Modelado Complejo de Información Lingüística en Problemas de Toma de Decisión en Grupo Bajo Incertidumbre

Memoria de Tesis Presentada por

Álvaro Labella Romero

UNIVERSIDAD DE JAÉN

Escuela Politécnica Superior de Jaén
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Álvaro Labella Romero

para optar al grado de Doctor en Informática

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Una vez escuché a Enrique Bunbury decir una frase que me llamó la atención. *La meta es el camino y cuanto más largo, mejor*. A lo largo de esta tesis doctoral he podido comprender el significado detrás de esta frase. En mi camino han aparecido muchas personas que me han ayudado, y que sin ellas, hubiera sido imposible llegar hasta aquí. Será difícil expresar todo el agradecimiento que merecen, pero lo intentaré.

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Capítulo 1

Introducción

En este primer capítulo de la memoria, se presenta una introducción para contextualizar esta tesis doctoral. En primer lugar, abordaremos el área de investigación sobre la que se centra la tesis y las motivaciones que nos han empujado a llevarla a cabo. Seguidamente, expondremos los objetivos a alcanzar y, por último, la estructura de la misma.

1.1. Motivación

En nuestra vida diaria estamos acostumbrados a hacer frente a múltiples situaciones de *Toma de Decisión* (TD), tan cotidianas como elegir la ropa que nos vamos a poner o lo que vamos a desayunar. Formalmente, la TD se define como un proceso cognitivo en el que, a través de diferentes procesos mentales y de razonamiento, un experto selecciona entre múltiples alternativas o posibles soluciones la mejor [73]. En ciertos ámbitos de la sociedad, es muy común que la resolución de un problema de TD no se lleve a cabo solamente por una persona, sino por un conjunto de expertos con diferentes puntos de vista y conocimiento, dando lugar a lo que se conoce como *Toma de Decisión en Grupo* (TDG) [34, 62, 66, 76, 80].

La participación de varios expertos en la resolución de problemas de TDG implica inevitablemente la aparición de conflictos y desacuerdos entre los expertos a la hora de escoger la solución del problema [16, 17]. Los esquemas de resolución clásicos para problemas de TDG [22] no tenían en cuenta este aspecto, por lo que era posible obtener una solución donde no todos los expertos estuvieran de acuerdo, sintiéndose ignorados y fuera del proceso de decisión [7, 61]. Por esta razón, se incluye previamente al proceso de selección de la mejor alternativa un *Proceso de Alcance de Consenso* (PAC), donde los expertos discuten y modifican sus opiniones iniciales con el objetivo de alcanzar una solución que satisfaga al mayor número de expertos posible. Aunque los PAC son clave para obtener soluciones consensuadas en problemas de TDG, existe tal cantidad de modelos de consenso propuestos en la literatura [34, 35, 56, 83], que a menudo resulta realmente complejo determinar cuál modelo se ajusta mejor a un problema de decisión dado [49]. La falta de métricas que permitan evaluar el desempeño de estos modelos sobre un problema de TDG, se presenta como una de las principales limitaciones dentro de este ámbito de la TD.

Por otro lado, la mayoría de problemas de TDG del mundo real y sus correspondientes PAC están definidos en contextos cambiantes, lo que genera falta de información y la aparición de incertidumbre. Por lo tanto, no todos los problemas de TD son tan simples y cotidianos como los mencionados anteriormente, muchos presentan incertidumbre cuya naturaleza no es probabilística y que se denominan problemas de *TDG bajo incertidumbre* [33]. En este tipo de problemas, a los expertos les resulta complejo expresar apropiadamente su conocimiento, por lo que prefieren usar expresiones lingüísticas más cercanas a su forma de pensar. Bajo estas circunstancias, la *lógica difusa* [88], el *enfoque lingüístico difuso* [89] y otras herramientas de *soft computing*, se han utilizado con gran éxito a la hora de modelar la incertidumbre en problemas de TDG mediante variables lingüísticas, dando lugar a la *Toma de Decisión Lingüística* (TDL).

El uso de expresiones lingüísticas para modelar la opinión de los expertos en problemas de TDL, implica la necesidad de llevar a cabo operaciones con información de este tipo. Existen numerosas metodologías para realizar estas operaciones, pero dentro del ámbito de la lógica difusa, destaca la metodología de *Computación con Palabras* (Computing with Words, CW) [20, 41, 84, 90]. A través de esta metodología, se realizan cálculos sobre palabras o frases dadas en un lenguaje natural o artificial en lugar de valores numéricos, imitando el proceso de razonamiento que tienen los seres humanos. Una premisa fundamental de esta metodología es que la información de entrada deber de ser de tipo lingüístico y, una vez manipulada, los resultados deben de expresarse de igual forma para garantizar su comprensión (ver Fig. 1.1).

Figura 1.1: Esquema general de un proceso de Computación con Palabras.

Hoy en día, existen muchos modelos computacionales aplicados a problemas de TDL que siguen la metodología de CW y que permiten modelar la opinión de los expertos mediante información lingüística [59]. Uno de los más destacados es el *modelo lingüístico 2-tupla* [39] que, gracias al uso de la translación simbólica, permite llevar a cabo operaciones en un dominio continuo con total precisión. Sin embargo, un valor lingüístico 2-tupla, al igual que la mayoría de las propuestas para el modelado de información en TDL, está compuesto únicamente por un término lingüístico, que puede ser insuficiente en problemas con alta complejidad donde los expertos dudan y no son capaces de decantarse por un único término lingüístico. Para superar esta limitación, otros enfoques han definido procesos para elaborar expresiones lingüísticas más complejas que permiten modelar la duda de los expertos, como son los *conjuntos de términos lingüísticos difusos dudosos* (CTLDD) [57], las *expresiones lingüísticas comparativas*

(ELC) [58], etc. Con todo, estas nuevas propuestas todavía presentan varias limitaciones en términos de expresividad y/o precisión que son resumidas a continuación:

1. *Modelado de la incertidumbre*: algunas propuestas [39, 78] no son capaces de representar la duda de los expertos en la resolución de problemas de TDL y limitan la representación lingüística a un solo término, insuficiente si tenemos en cuenta que estos problemas son cada vez más complejos y la duda en las opiniones de los expertos cada vez más común.

2. *Modelado de las expresiones lingüísticas*: aunque algunas propuestas modelan la opinión de los expertos mediante expresiones lingüísticas más complejas que un único término lingüístico [19, 71], a menudo estas expresiones están tan alejadas de la forma de expresarse que tienen los seres humanos, que las hace difíciles de entender y prácticamente inutilizables.

3. *Precisión e interpretabilidad*: es habitual en muchos enfoques [45, 75] transformar la información lingüística de entrada en valores numéricos, lo que implica pérdida de información y precisión en los resultados. Además, dichos resultados se representan mediante estructuras no lingüísticas difíciles de interpretar por parte de los expertos, violando la principal característica de la metodología de CW [42].

Como hemos mencionado anteriormente, existe una gran cantidad de modelos de decisión y PAC propuestos en la literatura, cada uno con sus características, ventajas y desventajas. Sin embargo, estos modelos a menudo no resultan sencillos de comprender, la mayoría se presentan como algoritmos compuestos por múltiples pasos o basados en modelos matemáticos como la programación lineal [69]. Teniendo en cuenta la alta complejidad a la que se enfrentan los expertos a la hora de resolver un problema de TDG y más aún bajo condiciones de incertidumbre donde la información es vaga e imprecisa, resulta impensable que además tengan que invertir su tiempo en comprender, analizar y aplicar manualmente estos modelos de decisión y PAC, incrementando aún más si cabe dicha complejidad. Sin perder de vista que, a menudo, ciertas situaciones de decisión son tomadas bajo presión y requieren de una solución rápida. Por lo tanto, el desarrollo de *Sistemas de Soporte a la Decisión* (SSD) que faciliten la labor de los expertos en la resolución de problemas de TD en cualquier contexto, adquiere gran importancia. A pesar de ello, existe una importante falta de herramientas software enfocadas a este objetivo y las existentes, presentan limitaciones como la imposibilidad de tratar con problemas de TD bajo incertidumbre [26], una batería insuficiente de modelos de decisión disponibles [18] o la incapacidad de resolver los problemas aplicando la metodología de CW [25].

Las principales limitaciones en los actuales modelos lingüísticos para la resolución de problemas de TDG bajo incertidumbre y sus PAC y la falta de herramientas software para el tratamiento de dichos problemas, nos condujo al inicio de esta investigación a formular las siguientes hipótesis:

1. *La definición de un nuevo y mejor marco metodológico a partir de modelos, metodologías y herramientas basadas en soft computing para el modelado difuso de la incertidumbre que, mediante modelos lingüísticos complejos para procesos de TDG bajo incertidumbre y PAC, permitirá superar distintos retos impuestos por las nuevas circunstancias y tendencias en las que han de desarrollarse dichos problemas y que actualmente no pueden resolverse.*

2. *La definición de una métrica para PAC facilitará una mejor evaluación del desempeño de los distintos PAC actuales o de nuevas propuestas.*

3. *La aplicación de un nuevo marco metodológico en nuevos modelos de PAC y TDG. Además de su integración en un sistema software que producirá un importante avance en los PAC y TDG del mundo real al facilitar la resolución de problemas de forma automática y dar soporte a los decisores con herramientas comprensibles y adecuadas a los problemas.*

1.2. Objetivos

Teniendo en cuenta las limitaciones previamente expuestas en los actuales modelos lingüísticos de TD y las hipótesis de partida, nuestra meta en esta tesis doctoral se centra en la investigación y definición de modelos de TD y PAC lingüísticos complejos, que permitan superar dichas limitaciones. En base a ésto, nos proponemos los siguientes objetivos:

1. *Establecimiento de un marco metodológico para el modelado y tratamiento de incertidumbre en TDG y sus PAC empleando expresiones lingüísticas complejas*, que permita modelar de forma apropiada las opiniones de los expertos y obtener resultados fácilmente interpretables y precisos.

2. *Definición de nuevos modelos de consenso lingüísticos complejos para problemas de TDG bajo incertidumbre* que superen las limitaciones de las propuestas existentes en la literatura especializada, mejorando la detección del disenso en el grupo y las recomendaciones de cambio sobre las opiniones de los expertos y así incrementar el consenso entre expertos en el menor tiempo posible.

3. *Elaborar métricas para procesos de consenso que establezcan referencias de funcionamiento en el alcance de consenso* y de esta forma analizar y seleccionar el mejor PAC a aplicar en cada problema de TDG.

4. *Investigar distintos problemas de TDG y PAC en el mundo real*, identificando sus principales características y los retos que nos platean para así poder analizar y seleccionar el enfoque de resolución que proporciona la mejor solución posible.

5. *Soporte a la TDG de los problemas anteriores* mediante el desarrollo de SSD que ayuden a los expertos a manejar la creciente complejidad inherente en los problemas de TDG.

1.3. Estructura

Esta tesis doctoral, de acuerdo a lo establecido en el artículo 25, punto 2, de la normativa vigente de los Estudios de Doctorado en la Universidad de Jaén (RD. 99/2011), se compondrá de una serie de artículos publicados por el doctorando, cuya finalidad se basa en alcanzar los objetivos marcados en la sección anterior. Concretamente, esta memoria está compuesta de diez artículos, nueve de ellos publicados en revistas internacionales indexadas en la base de datos Journal Citation Reports (JCR) y otro publicado en una revista indexada en Scopus.

La memoria se divide en los siguientes capítulos:

1. **Capítulo 2**: se revisan conceptos básicos relacionados con la temática de la tesis doctoral. Introduciremos la teoría de la lógica difusa y el enfoque lingüístico difuso. Posteriormente, nos centraremos en conceptos relacionados con la TDG, TDL y la metodología de CW. Analizaremos las ventajas y limitaciones de los modelos de decisión lingüísticos existentes, centrándonos principalmente en el modelo lingüístico 2-tupla, los CTLDD y las ELC. Finalmente, expondremos la necesidad de los PAC para alcanzar soluciones consensuadas.

2. **Capítulo 3**: resumirá las principales propuestas que componen esta memoria, poniendo de manifiesto los resultados obtenidos y las conclusiones extraídas en cada una de ellas.

3. **Capítulo 4**: los diez artículos anteriormente mencionados dan forma a esta sección.

4. **Capítulo 5**: para finalizar, se extraen las principales conclusiones obtenidas a lo largo del desarrollo de la tesis doctoral y se esbozan posibles trabajos futuros de investigación.

Adicionalmente, se incluye un Apéndice (Apéndice A) donde se realiza un resumen en inglés de la investigación llevada a cabo, con el objetivo de alcanzar la Mención Internacional de Doctorado. Finalmente, esta memoria concluye con una recopilación bibliográfica de los artículos más relevantes relacionados con esta tesis doctoral.

Capítulo 2

Conceptos Teóricos y Antecedentes

En este capítulo haremos un breve resumen de los conceptos teóricos y antecedentes relacionados con la investigación presentada en esta memoria. Inicialmente, introduciremos conceptos básicos sobre lógica difusa y el enfoque lingüístico difuso. A continuación, profundizaremos en la definición de toma de decisión bajo incertidumbre y analizaremos algunas de las propuestas más importantes que permiten modelar dicha incertidumbre mediante expresiones lingüísticas. Por último, describiremos los procesos de alcance de consenso en toma de decisión.

2.1. Lógica Difusa y Enfoque Lingüístico Difuso

L. Zadeh introdujo la *Teoría de la Lógica Difusa* [88] con el propósito de modelar la incertidumbre o imprecisión. Para ello, extendió la definición de conjunto clásico a la de *conjunto difuso*, donde los límites del conjunto no están estrictamente definidos. Por una parte, un conjunto clásico está marcado por una estricta restricción de dicotomía, es decir, un objeto puede pertenecer o no a un conjunto. Esta clasificación binaria se puede definir mediante la siguiente función característica:

Definición 1 *Sea A un conjunto en el universo X, la función característica asociada a A, $A(x), x \in X$, se define como:*

$$A(x) = \begin{cases} 1, & si\ x \in A \\ 0, & si\ x \notin A. \end{cases}$$

De acuerdo a la Definición 1, la pertenencia o no de un objeto x al conjunto A se define mediante una función $A : X \to \{0, 1\}$ cuyo valor es 1 cuando el objeto pertenece al conjunto y 0 en caso contrario. La definición de conjunto difuso suaviza la restricción de la función característica de un conjunto clásico, permitiendo obtener valores intermedios. En un conjunto difuso, la función característica pasa a denominarse *función de pertenencia*:

Definición 2 *[88]. Un conjunto difuso $\tilde{A}$ sobre X está definido por una función de pertenencia que transforma los elementos del universo del discurso X en el intervalo* $[0, 1]$.

$$\mu_{\tilde{A}} : X \longrightarrow [0, 1]$$

Por lo tanto, un conjunto difuso $\tilde{A}$ en X puede representarse como un conjunto de pares ordenados de un elemento $x \in X$ y su grado de pertenencia $\mu_{\tilde{A}}(x)$:

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x))/\ x \in X, \mu_{\tilde{A}}(x) \in [0, 1]\}$$

La función de pertenencia de un conjunto difuso resulta más compleja que la función característica de un conjunto clásico, ya que permite obtener un valor de pertenencia entre 0 y 1. Cuanto más cercano a 1, mayor es el grado de pertenencia. Por lo tanto, es necesario definir funciones que describan la pertenencia a un conjunto difuso. Aunque los conjuntos difusos pueden representarse mediante muchos tipos de funciones, siempre que cumplan la condición $\mu_{\tilde{A}} \in [0, 1]$, las funciones paramétricas son las más utilizadas. Dentro de esta familia de funciones, la más comunes son las de tipo triangular y trapezoidal (ver Fig. 2.1), cuyas funciones de pertenencia se definen a continuación:

- *Función de pertenencia triangular*:

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & \text{si} \quad x \leq a \\ \dfrac{x-a}{b-a}, & \text{si} \quad x \in (a, b] \\ \dfrac{c-x}{c-b}, & \text{si} \quad x \in (b, c) \\ 0, & \text{si} \quad x \geq c \end{cases}$$

- *Función de pertenencia trapezoidal*:

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & \text{si} \quad x \leq a \\ \dfrac{x-a}{b-a}, & \text{si} \quad x \in (a, b] \\ 1 & \text{si} \quad x \in (b, c] \\ \dfrac{d-x}{d-c}, & \text{si} \quad x \in (c, d) \\ 0, & \text{si} \quad x \geq d \end{cases}$$

La lógica difusa desempeña un papel fundamental a la hora de enfrentarse a la mayoría de problemas del mundo real, que se definen habitualmente bajo un contexto de incertidumbre y falta de información. La cuestión clave reside en cómo modelar dicha incertidumbre de una manera sencilla e interpretable por los seres humanos. La respuesta ya ha sido abordada con gran éxito a través del *modelado lingüístico* [38]. El modelado lingüístico de la incertidumbre nos permite utilizar palabras del lenguaje natural como *alto*, *sencillo* o *cómodo* para evaluar

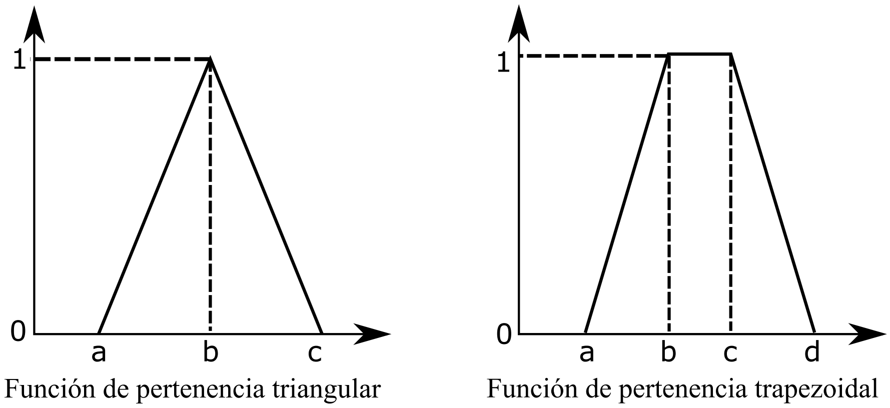

Figura 2.1: Funciones paramétricas.

aspectos cualitativos de un problema que tienen que ver con percepciones o sensaciones. Existen múltiples enfoques para el modelado de información lingüística [42, 43, 89], pero en el desarrollo de esta tesis doctoral se ha empleado el *enfoque lingüístico difuso*.

El enfoque lingüístico difuso sustenta sus bases en la Teoría de la Lógica Difusa y permite modelar la información lingüística mediante el concepto de *variable lingüística* definido por L. Zadeh [89]. En palabras de L. Zadeh, una variable lingüística es *"una variable cuyos valores no son números sino palabras o frases en un lenguaje natural o artificial"*. La definición formal de una variable lingüística se presenta a continuación:

Definición 3 *[89]. Una variable lingüística se compone de una quíntupla (H, T(H), U, G, M), donde H representa el nombre de la variable, T(H) un conjunto de términos lingüísticos de H, donde cada valor es una variable difusa notada como X y que varía a lo largo del universo de discurso U, G es una regla sintáctica para generar los nombres de los valores de H y M es una regla semántica que asocia significado M(X) a cada elemento de H, el cual es un conjunto difuso de U.*

En resumen, una variable lingüística está principalmente formada por un valor sintáctico o etiqueta (una palabra perteneciente a un conjunto de términos lingüísticos) y un valor semántico representado por un conjunto difuso dado en un universo de discurso.

En la Fig. 2.2 podemos ver un ejemplo de conjunto de términos lingüísticos. A partir de este conjunto de términos una persona podría expresar su conocimiento empleando cualquiera de los descriptores lingüísticos que componen el conjunto, en este caso, *Horrible*, *Muy malo*, *Malo*, *Medio*, *Bueno*, *Muy bueno* o *Excelente*. También podemos apreciar que la semántica de las variables son representadas por funciones de pertenencia triangulares, aunque podrían usarse otro tipo de funciones como las trapezoidales, mencionadas anteriormente.

En un conjunto de términos lingüístico, el número de términos que lo componen (también denominado *cardinalidad*) es un aspecto importante a tener en cuenta. Esta decisión dependerá del grado de conocimiento que se pretenda expresar. Un conjunto con pocos términos implica falta de conocimiento y, a su vez, pérdida de expresividad. Por el contrario, un conjunto con una cardinalidad alta presenta una mayor granularidad de la incertidumbre, que es adecuada cuando el grado de conocimiento es alto. Los valores más comunes de cardinalidad suelen ser valores impares como 5, 7 o 9 [44], donde el término medio representa un valor aproximado de 0.5 y el resto de términos son distribuidos alrededor de éste [6].

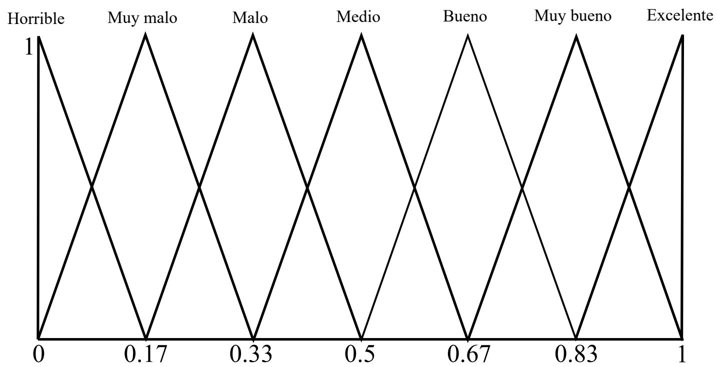

Figura 2.2: Conjunto de términos lingüísticos.

Por otro lado, el modelado difuso de información lingüística no se ha limitado exclusivamente al uso de términos lingüísticos simples. La necesidad de expresiones lingüísticas más complejas y flexibles para representar de forma apropiada el conocimiento de las personas, ha dado lugar a numerosas propuestas basadas en el enfoque lingüístico difuso. En esta tesis doctoral se presenta, en la Sección 4.1, una publicación con una revisión bibliográfica de algunas de estas extensiones.

2.2. Toma de Decisión Lingüística bajo Incertidumbre

La *Toma de Decisión* (TD) es una actividad cotidiana en la vida de los seres humanos que implica seleccionar, entre un conjunto de posibles alternativas, la mejor como solución a un problema de decisión. Algunos problemas de TD son tan sencillos y cotidianos que pueden ser resueltos en un breve espacio de tiempo y por una sola persona. Sin embargo, otros problemas de TD resultan mucho más complejos y requieren de la participación de varios expertos con diferentes puntos de vista y conocimiento [28, 46, 52, 77, 82], dando lugar a la *Toma de Decisión en Grupo* (TDG). Formalmente, un problema de TDG se compone de un conjunto finito de expertos $E = \{e_1, e_2, \ldots, e_m\}$ cuya principal tarea consiste en seleccionar una o varias alternativas dentro de un conjunto finito de posibles opciones $X = \{x_1, x_2, \ldots, x_n\}$ como solución/es al problema de decisión. En múltiples problemas, las alternativas son evaluadas a partir de un conjunto finito de atributos o criterios $C = \{c_1, c_2, \ldots, c_s\}$, dando lugar a la *Toma de Decisión Multi-criterio* [27, 60, 87].

El proceso clásico de resolución de un problema de TDG está compuesto de dos fases (ver Fig. 2.3):

1. *Agregación*: las opiniones individuales de los expertos sobre cada alternativa y criterio son agregadas empleando un operador de agregación adecuado. De esta forma, se obtiene un valor colectivo para cada una de las alternativas del problema.

2. *Explotación*: los valores colectivos obtenidos en la fase anterior son ordenados mediante funciones de selección que permiten seleccionar la/s mejor/es alternativa/s como solución del problema.

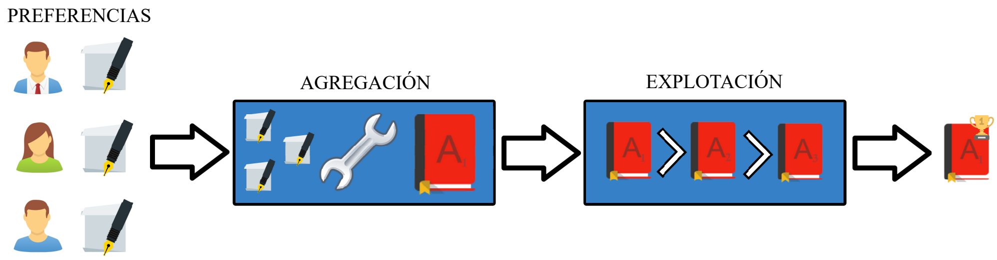

Figura 2.3: Esquema general de resolución de un problema de TDG.

En el mundo real, los seres humanos se enfrentan a problemas de TD condicionados por la falta de información y la inevitable aparición de incertidumbre, ya que es casi imposible conocer todos los estados de la naturaleza del problema. El modelado de dicha incertidumbre mediante información lingüística ha ofrecido excelentes resultados [21, 40], dando lugar a los problemas de *Toma de Decisión Lingüísticos* (TDL). En este tipo de problemas, el enfoque lingüístico difuso se presenta como uno de los enfoques más utilizados a la hora de modelar las preferencias de los expertos mediante expresiones lingüísticas (ver Sección 2.1).

El esquema de resolución de un problema de TDL varía ligeramente con respecto al de un problema de TD clásico, incorporando dos fases adicionales [21] (ver Fig. 2.4):

1. *Selección del conjunto de términos lingüísticos y su semántica*: se define el conjunto de términos lingüísticos que los expertos usarán para expresar apropiadamente su conocimiento sobre el conjunto de alternativas.

2. *Selección de un operador de agregación para información lingüística*: las opiniones proporcionadas por los expertos mediante expresiones lingüísticas son agregadas mediante un operador lingüístico, obteniendo un valor colectivo para cada una de las alternativas.

El esquema presentado en la Fig. 2.4 evidencia la necesidad de llevar a cabo operaciones con información lingüística para encontrar la solución a un problema de TDL. En este sentido, la metodología de *Computación con Palabras* (Computing with Words, CW) [42, 90] imita el proceso de razonamiento del ser humano, generando resultados lingüísticos a partir de premisas también lingüísticas. Según la definición proporcionada por L. Zadeh, la CW es *"una metodología en la que se usan palabras en lugar de números para calcular, razonar y tomar decisiones"*. La metodología de CW ha sido aplicada con éxito para llevar a cabo procesos

Figura 2.4: Esquema general de resolución de un problema de TD lingüístico.

computacionales en problemas de diferentes ámbitos como la TDL [12, 20], aprendizaje [48] o base de datos [86] entre otros.

En esta memoria nos centraremos en los procesos computacionales llevados a cabo a través de la CW en problemas de TDL, donde esta metodología se ha aplicado de forma especialmente intensa [12, 20, 39, 40] y que ha dado lugar a diferentes esquemas de CW [67, 84, 85]. Sin embargo, todos ellos enfatizan la necesidad de obtener resultados lingüísticos precisos y fáciles de interpretar. R.R. Yager introdujo [84, 85] un esquema de CW formado por dos procesos principales, *transformación* y *retransformación* (representado en Fig. 2.5). El primero consiste en transformar la información lingüística de entrada a un formato basado en la lógica difusa y que pueda ser manipulado por una máquina. El segundo se encarga de transformar la información manipulada de nuevo a un formato lingüístico que sea fácil de interpretar por los seres humanos.

Figura 2.5: Esquema de CW propuesto por R. R. Yager.

2.3. Modelos Lingüísticos Computacionales

De acuerdo a lo mencionado en la sección anterior, el modelado de incertidumbre mediante información lingüística implica llevar a cabo procesos de CW. En base a esto, se han propuesto una gran cantidad de modelos de representación lingüísticos que llevan a cabo operaciones con información lingüística. En esta sección revisaremos brevemente los modelos más relevantes relacionados con la investigación desarrollada en esta memoria. Estos mismos modelos, además de otros, son revisados junto a sus procesos computacionales en mayor profundidad en el artículo incluido en la Sección 4.1.

2.3.1. Modelo de Representación Lingüístico 2-tupla

El *modelo de representación lingüístico 2-tupla* [39], basado en el enfoque lingüístico difuso, es uno de los modelos de representación lingüísticos más usados en TDL. Este modelo presenta como principales características la alta interpretabilidad y precisión de los resultados. La primera de ellas, se consigue gracias al desarrollo de procesos de CW que permiten obtener resultados representados de forma lingüística. La segunda viene determinada por la representación en un espacio continuo de los valores lingüísticos, que permite obtener resultados precisos sin pérdida de información.

Uno de los conceptos más importantes presentados en el modelo lingüístico 2-tupla [23, 39] es el de *translación simbólica*, un valor numérico que representa el desplazamiento de la función de pertenencia de una etiqueta lingüística. Formalmente, la información lingüística en el modelo lingüístico 2-tupla se representa a partir de un par de valores (s_i, α) donde s_i es una etiqueta lingüística que pertenece a un conjunto de términos lingüísticos $S = \{s_0, s_1, \ldots, s_g\}$ y el valor de translación simbólica $\alpha \in [-0{,}5, 0{,}5)$ que representa el desplazamiento de la función de pertenencia del término s_i (ver Fig 2.6). El valor de α se define como:

$$\alpha = \begin{cases} [-0{,}5, 0{,}5) & si\ s_i \in \{s_1, s_2, \ldots, s_{g-1}\} \\ [0, 0{,}5) & si\ s_i = s_0 \\ [-0{,}5, 0] & si\ s_i = s_g \end{cases}$$

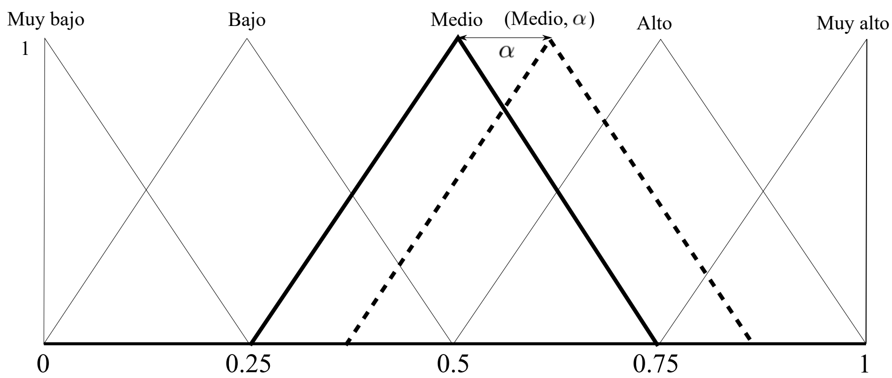

Figura 2.6: Translación simbólica.

Un valor lingüístico 2-tupla $(s_i, \alpha) \in \overline{S}$, siendo $\overline{S}$ el conjunto de valores lingüísticos 2-tuplas asociado a S definido como $\overline{S} = S \times [-0{,}5, 0{,}5)$, también puede representarse mediante un valor numérico equivalente $\beta \in [0, g]$:

Proposición 1 *[39] Dado $S = \{s_0, \ldots, s_g\}$ un conjunto de términos lingüísticos y $(s_i, \alpha) \in \overline{S}$ un valor lingüístico 2-tupla. Existe una función, Δ^{-1} tal que:*

$$\Delta^{-1} : \overline{S} \to [0, g]$$
$$\Delta_S^{-1}(s_i, \alpha) = \alpha + i = \beta$$

A su vez, un valor numérico $\beta \in [0, g]$ puede ser transformado a su correspondiente valor lingüístico 2-tupla de la siguiente forma:

Definición 4 *[39] Dado $S = \{s_0, \ldots, s_g\}$ un conjunto de términos lingüísticos y $\overline{S}$ el conjunto de 2-tuplas asociado a S definido como $\overline{S} = S \times [-0{,}5, 0{,}5)$. La función $\Delta_S : [0, g] \to \overline{S}$ es dada por:*

$$\Delta_S(\beta) = (s_i, \alpha), \ \ con \ \begin{cases} i = round(\beta) \\ \alpha = \beta - i \end{cases}$$

donde $round(\cdot)$ es una función que asigna el entero más cercano $i \in \{0, \ldots, g\}$ a β.

El modelo de representación lingüístico 2-tupla fue definido junto a un modelo computacional que puede ser consultado en detalle en las referencias [23, 39].

2.3.2. Conjunto de Términos Lingüísticos Difusos Dudosos

El modelo de representación lingüístico 2-tupla presenta ventajas muy destacables en términos de precisión e interpretabilidad. Sin embargo, los valores lingüísticos 2-tupla están representados por un único término lingüístico, que puede ser insuficiente en situaciones donde los expertos duden entre varios términos lingüísticos al expresar sus opiniones. Con el objetivo de superar esta limitación y modelar la duda de los expertos, se definieron los *Conjuntos de Términos Lingüísticos Difusos Dudosos* (CTLDD) [57].

Definición 5 *[57] Sea S un conjunto de términos lingüísticos, $S = \{s_0, \ldots, s_g\}$, un CTLDD H_S se define como un subconjunto finito ordenado de términos lingüísticos consecutivos pertenecientes a S.*

$$H_S = \{s_i, s_{i+1}, \ldots, s_j\}$$

Para clarificar este concepto, veamos un ejemplo:

Ejemplo 1 *Suponiendo un conjunto de términos lingüísticos S = {Muy inseguro, Inseguro, Medio, Seguro, Muy seguro}, algunos posibles CTLDD serían:*

$$H_S^1 = \{Inseguro, Medio\}$$
$$H_S^2 = \{Medio, Seguro, Muy\ seguro\}$$
$$H_S^3 = \{Seguro, Muy\ seguro\}$$

2.3.3. Expresiones Lingüísticas Comparativas

Los CTLDD permiten a los expertos expresarse mediante varios términos lingüísticos en situaciones de duda y no tienen claro cuál de ellos escoger. Sin embargo, éstos están bastante lejos de la forma de expresarse que tienen los seres humanos. Por lo tanto, es evidente la necesidad de crear expresiones lingüísticas más complejas que permitan modelar la duda de los expertos con una estructura similar a las expresiones que usan los seres humanos para expresar su conocimiento. Con este objetivo, Rodríguez et al. [58] definió un nuevo tipo de expresiones lingüísticas más expresivas y complejas denominadas *Expresiones Lingüísticas Comparativas* (ELC).

Las ELC se basan en los CTLDD, pero son generadas mediante una *gramática libre de contexto* que permite modelar expresiones más cercanas al lenguaje que usan los expertos en problemas de TDL. Rodríguez et al. introdujo la siguiente gramática libre de contexto para generar ELC [58]:

Definición 6 *[58] Dado G_H una gramática libre de contexto y $S = \{s_0, \ldots, s_g\}$ un conjunto de términos lingüísticos. Los elementos de $G_H = (V_N, V_T, I, P)$ se definen como:*

$V_N = \{$*⟨término primario⟩, ⟨término compuesto⟩,*
⟨relación unaria⟩, ⟨relación binaria⟩, ⟨conjunción⟩$\}$
$V_T = \{$*al menos, como mucho, menor que, mayor que, entre, y,* $s_0,\ s_1, \ldots, s_g\}$
$I \in V_N$

La reglas de producción se definen a partir de la forma extendida de Backus-Naur:

$P = \{I$ ::= *⟨término primario⟩*|*⟨término compuesto⟩*
⟨término compuesto⟩ ::= *⟨relación unaria⟩⟨término primario⟩*|
⟨relación binaria⟩⟨término primario⟩⟨conjunción⟩⟨término primario⟩
⟨término primario⟩ ::= $s_0|s_1|\ldots|s_g$
⟨relación unaria⟩ ::= *al menos*|*como mucho*|*menor que*|*mayor que*
⟨relación binaria⟩ ::= *entre*
⟨conjunción⟩ ::= *y*$\}$

Ejemplo 2 *Suponiendo un conjunto de términos lingüísticos S = {Muy incómodo, Incómodo, Normal, Cómodo, Muy cómodo} y la gramática libre de contexto G_H mostrada en la Definición 6, algunos ejemplos de ELC serían:*

$$ELC_1 = \textit{Al menos cómodo}$$
$$ELC_2 = \textit{Como mucho Normal}$$
$$ELC_3 = \textit{Menor que cómodo}$$
$$ELC_4 = \textit{Mayor que Normal}$$
$$ELC_5 = \textit{Entre Cómodo y Muy cómodo}$$
$$ELC_6 = \textit{Normal}$$

2.4. Procesos de Alcance de Consenso

En la sección 2.2 hemos visto diferentes esquemas de resolución para problemas de TDG y TDL (ver Figs. 2.2 y 2.3). En ambos esquemas se aprecia que la opiniones de los expertos son agregadas directamente, ignorando los posibles desacuerdos que pueden existir entre ellos. La principal consecuencia de esta omisión es que algunos expertos podrían no estar de acuerdo con la solución obtenida, sintiéndose ignorados y poniendo en entredicho la confianza en el proceso de decisión. Hoy en día, las decisiones consensuadas son realmente valoradas en diferentes ámbitos de la sociedad [16, 49, 82] por lo que parece evidente la necesidad de añadir un *Proceso de Alcance de Consenso* (PAC) en el esquema de resolución de un problema de TDG antes de la selección de la mejor alternativa.

Antes de definir en detalle qué es un PAC, explicaremos qué se entiende por *consenso*. El concepto de consenso puede generar cierta controversia, ya que existen múltiples puntos de vista sobre su significado. Algunos enfoques más estrictos definen el consenso como la *unanimidad* o acuerdo total, que difícilmente puede ser alcanzado en la práctica [32]. Otros enfoques son más flexibles. Como la visión de Kacprzyk, que propuso el concepto de *soft consensus* [29, 30], un enfoque basado en la *mayoría difusa* que establece el consenso en un grupo cuando *"la mayoría de expertos más relevantes están de acuerdo en casi todas las opciones relevantes"*. En esta memoria de investigación, tomaremos la visión de Kacprzyk de soft consensus.

Un PAC es un proceso iterativo y dinámico en el que los expertos discuten, revisan y modifican sus opiniones iniciales con el objetivo de acercar posturas y llegar a una solución consensuada en un número determinado de rondas de debate. Este proceso suele estar guiado por un *moderador*, una persona encargada de identificar aquellos expertos más alejados de la opinión general del grupo y de sugerir los cambios necesarios en las opiniones de los mismos para evitar estancamientos en el proceso de decisión. Generalmente, un PAC está formado por las siguientes fases (representadas en Fig. 2.7):

1. *Recopilación de preferencias*: se recopilan las preferencias que los expertos han proporcionado sobre las alternativas.

2. *Calcular nivel de consenso*: se calcula el nivel de consenso actual en el grupo de expertos a través de una medida de consenso. Existen dos tipos básicos de medidas de consenso [49]:

   - *Medida de consenso basada en la distancia a la opinión colectiva del grupo*: se calcula la opinión colectiva del grupo mediante la agregación de las opiniones individuales de los expertos. Posteriormente, se calcula la distancia entre la opinión colectiva y la individual de cada experto.

   - *Medida de consenso basada en la distancia entre las opiniones de los expertos*: se calcula la similitud de opiniones para cada par de expertos. Posteriormente, los valores de similitud son agregados para obtener el valor de consenso en el grupo.

3. *Control del consenso*: se comprueba si el consenso actual del grupo ha alcanzado un mínimo nivel de consenso requerido y predefinido antes de iniciar el proceso de consenso. Si se ha alcanzado, el PAC termina y daría comienzo el proceso de selección de la mejor

alternativa. En caso contrario, es necesaria otra ronda de debate. El número máximo de rondas de debate también se establece a priori, con el fin de evitar procesos interminables. Si se alcanza el máximo número de rondas pero no el mínimo consenso requerido, el proceso terminará sin llegar a un acuerdo.

4. *Generación de recomendaciones*: en el caso de no alcanzar un acuerdo en la ronda actual, el moderador identifica los mayores puntos de disenso en el grupo y aconseja a los expertos que cambien ciertas opiniones con el objetivo de elevar el nivel de consenso. Existen modelos de consenso que eliminan el papel del moderador y llevan a cabo los cambios en las opiniones de forma automática sin necesidad de involucrar a los expertos. Estos modelos suelen usarse como herramienta de apoyo a PAC del mundo real [76, 91].

Figura 2.7: Esquema general de un proceso de consenso.

Existe una inmensa cantidad de modelos de consenso propuestos en la literatura [35, 56, 83]. Algunos de los más relevantes fueron revisados en el desarrollo de esta tesis doctoral en el artículo incluido en la Sección 4.4. Este hecho llevó a Palomares et al. [49] a introducir una taxonomía de modelos de consenso para problemas de TDG, que clasifica los modelos en base a dos aspectos básicos (ver Fig. 2.8):

- *Con o sin generación de recomendaciones*: los modelos son clasificados dependiendo de si incorporan o no un mecanismo de generación de recomendaciones.

- *Medida de consenso*: los modelos son clasificados dependiendo de la medida que empleen para calcular el consenso, ya sea basada en la distancia a la opinión colectiva o basada en la distancia entre las opiniones individuales de los expertos.

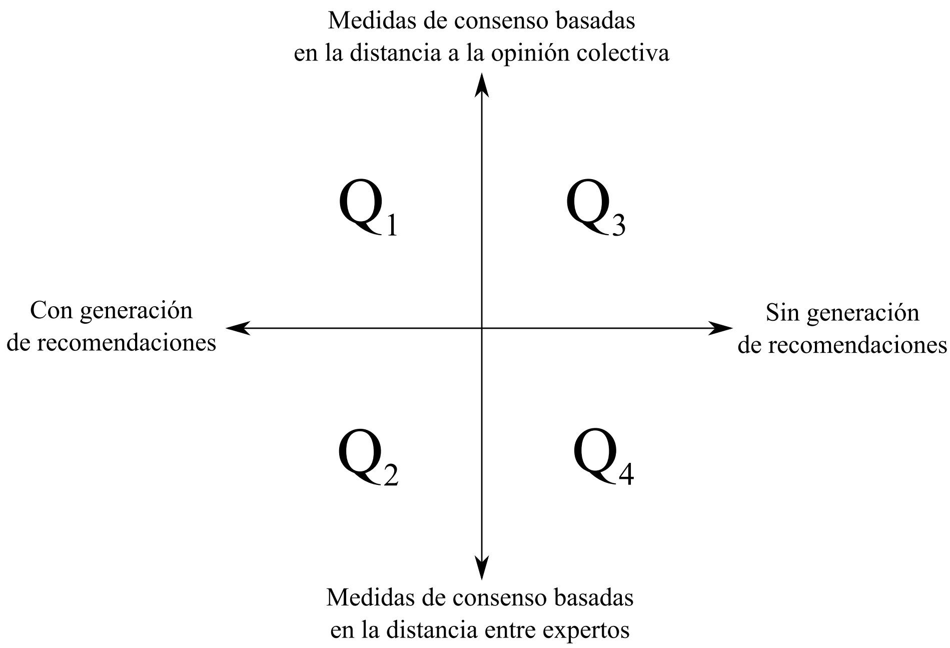

Figura 2.8: Taxonomía de modelos de consenso.

Aunque la taxonomía propuesta por Palomares et al. nos permite categorizar de forma clara los modelos de consenso en base a sus principales características, la gran cantidad de propuestas dificulta en gran medida la selección del modelo de consenso más adecuado para un problema de TDG dado. Este problema es abordado en esta memoria desde diferentes puntos de vista. Por un lado, en la Sección 4.8 del capítulo 4, se incluye un artículo donde se presenta una métrica de consenso que permite evaluar el desempeño de un modelo de consenso aplicado a un problema de TDG. Por otro lado, el desarrollo de un sistema de soporte a la decisión nos permitiría llevar a cabo simulaciones de diferentes modelos de consenso y determinar cuál es el que mejor se ajusta a las necesidades del problema. También en el capítulo 4, Sección 4.9, se presenta un artículo donde se introduce un software enfocado al soporte de PAC en TDG.

Capítulo 3

Discusión de los Resultados

En este capítulo se resumirán las propuestas que dan forma a esta memoria de investigación junto con los resultados y conclusiones obtenidas a partir de las mismas. Este capítulo se estructura en tres propuestas principales que se dividen a su vez en diferentes partes:

1. *Modelado y Tratamiento de Información Lingüística mediante Expresiones Lingüísticas Complejas.* Esta propuesta se divide en tres propuestas específicas:

   - *Visión General de las Propuestas Existentes sobre Modelado de Preferencias mediante Expresiones Lingüísticas en la Toma de Decisión.*
   - *Definición de Expresiones Lingüísticas Comparativas con Translación Simbólica en Toma de Decisión.*
   - *Operadores de Agregación para Expresiones Lingüísticas Comparativas con Translación Simbólica.*

2. *Procesos de Alcance de Consenso en Toma de Decisión en Grupo.* Esta propuesta se divide en cinco propuestas específicas:

   - *Estudio Comparativo de Modelos de Consenso Clásicos en Problemas de Toma de Decisión en Grupo a Gran Escala.*
   - *Proceso de Alcance de Consenso a Gran Escala para la Gestión de Dudas en Grupos de Expertos.*
   - *Proceso de Alcance de Consenso con Expresiones Lingüísticas Comparativas.*
   - *Proceso de Alcance de Consenso con Expresiones Lingüísticas Comparativas con Translación Simbólica.*
   - *Métrica basada en Modelos Integrales de Mínimo Coste para Procesos de Alcance de Consenso.*

3. *Soporte a Problemas de Toma de Decisión en Grupo y Procesos de Alcance de Consenso.* Esta propuesta se divide en dos propuestas específicas:

- *Software para el Análisis de Procesos de Alcance de Consenso: AFRYCA 2.0.*
- *Software para el Soporte de Problemas de Decisión basados en Política Climática: APOLLO*

3.1. Modelado y Tratamiento de Información Lingüística mediante Expresiones Lingüísticas Complejas

Esta parte comienza analizando ventajas e inconvenientes de las principales propuestas que existen en la literatura basadas en el modelado lingüístico de preferencias en problemas de TDL. Posteriormente, se propone un nuevo modelo de representación lingüístico basado en *Expresiones Lingüísticas Comparativas con Translación Simbólica* (ELICIT) que supera las limitaciones de los modelos existentes. Finalmente, cualquier modelo de representación lingüístico debe tener asociado un modelo computacional que permita llevar a cabo las operaciones con información lingüística. Para ello, el proceso de agregación de la información es clave, por lo que proponemos diferentes operadores de agregación que trabajen con información ELICIT.

3.1.1. Visión General de las Propuestas Existentes sobre Modelado de Preferencias mediante Expresiones Lingüísticas en la Toma de Decisión

En esta parte, se revisan las principales propuestas basadas en el enfoque lingüístico difuso para el modelado de preferencias mediante expresiones lingüísticas complejas en problemas de TDL [11, 37, 57, 58, 64, 70, 71]. Del análisis de cada una de estas propuestas, se extraen sus principales ventajas e inconvenientes y una visión clara de cuáles son los principales aspectos a mejorar en el modelado de preferencias mediante expresiones lingüísticas. Algunos de estos aspectos se resumen a continuación:

- A pesar de que algunas propuestas son bastante flexibles a la hora de generar expresiones lingüísticas [11, 37], no definen un proceso formal para su generación o están lejos del lenguaje natural del ser humano. Por otro lado, aquellas expresiones más cercanas al lenguaje común de los seres humanos son menos flexibles [58]. Por lo tanto, es clave elaborar expresiones lingüísticas cercanas al pensamiento del ser humano y que a su vez sean más flexibles.

- El modelado de incertidumbre en problemas de TD generalmente se establece aplicando una única técnica. Sin embargo, ésto podría no ser realista, teniendo en cuenta los múltiples enfoques que se pueden aplicar para resolver un problema. Por lo tanto, sería digno de estudio el modelado de la incertidumbre mediante la combinación de varios enfoques de forma simultánea, aprovechando las bondades de cada uno de ellos.

- Las propuestas analizadas proporcionan un significado único a las expresiones lingüísticas que generan. Sin embargo, una expresión lingüística puede tener diferentes significados dependiendo de la persona, por lo que sería interesante elaborar mecanismos de representación para expresiones lingüísticas que consideren este aspecto.

El articulo asociado a esta revisión se encuentra en la Sección 4.1. Cabe destacar, que este articulo es, según los InCites Essential Science Indicators publicados por Clarivate Analytics, altamente citado (Highly Cited Paper).

3.1.2. Definición de Expresiones Lingüísticas Comparativas con Translación Simbólica en Toma de Decisión

Este trabajo toma como punto de partida las conclusiones extraídas de la revisión presentada en la sección anterior, que ponen de manifiesto la necesidad de definir un nuevo modelo de representación lingüístico que permita superar las limitaciones existentes en otros modelos publicados en la literatura. Estas limitaciones se engloban principalmente en dos aspectos básicos, la precisión en los procesos llevados a cabo con las expresiones lingüísticas y la interpretabilidad de las mismas. La revisión de modelos lingüísticos permitió analizar dos propuestas que presentan buenas características en relación a estos dos aspectos, aunque por separado. Por un lado, el modelo de representación lingüístico 2-tupla (ver Sección 2.3.1), lleva a cabo procesos de CW precisos gracias al uso del concepto de translación simbólica. Sin embargo, estas expresiones están formadas por un único término lingüístico, insuficiente en situaciones donde los expertos dudan entre varios términos lingüísticos. Esta limitación es superada por las ELC basadas en CTLDD (ver Sección 2.3.3), que permiten modelar la duda de los expertos además de ofrecer una representación lingüística rica y cercana a la forma de pensar del ser humano. A pesar de que múltiples modelos han empleado las ELC [9, 45, 54], todos presentan inconvenientes desde diferentes puntos de vista. La mayoría de estas propuestas transforman las ELC para llevar a cabo los procesos computacionales, perdiendo información en dicho proceso y, en consecuencia, también la principal característica de estas expresiones, su interpretabilidad.

Lo mencionado anteriormente evidencia las limitaciones tanto del modelo de representación lingüístico 2-tupla como de las ELC, pero también sus bondades, lo que nos llevó a pensar que un uso combinado de ambas propuestas podría ofrecer excelentes resultados en el modelado de información lingüística. Otras propuestas existentes en la literatura especializada ya combinaron en menor o mayor medida conceptos relacionados con las ELC, CTLDD y el modelo lingüístico 2-tupla [1, 63, 74, 95], aunque ninguna de ellas de una forma plenamente satisfactoria. Por esta razón, proponemos un nuevo modelo de representación lingüístico que combina la expresividad de las ELC y la precisión del modelo lingüístico 2-tupla. Este nuevo modelo de representación lingüístico modela la información lingüística a partir de información ELICIT, ELC extendidas a un dominio continuo mediante el uso de la translación simbólica. Las expresiones propuestas se generan a través de una gramática libre de contexto cuyos términos están formados por valores lingüísticos 2-tupla en lugar de términos lingüísticos simples.

Junto al modelo de representación lingüístico ELICIT se propone un enfoque de CW para información ELICIT que permite, partiendo de premisas lingüísticas representadas por ELC e información ELICIT, llevar a cabo procesos de CW precisos basados en operaciones difusas [53] y obtener resultados fáciles de interpretar representados mediante información ELICIT. Para llevar a cabo los procesos de CW con información ELICIT, se define un modelo computacional con operaciones básicas como la negación, comparación entre ELICIT u operadores de agregación.

Por último, las bondades del nuevo modelo lingüístico se ponen de manifiesto mediante la resolución de un problema de TDL y la comparación con otros modelos lingüísticos previos. Los resultados obtenidos muestran que el modelo de representación ELICIT proporciona una solución más precisa, interpretable y fiable que otros enfoques.

El artículo asociado a esta parte se encuentra en la Sección 4.2.

3.1.3. Operadores de Agregación para Expresiones Lingüísticas Comparativas con Translación Simbólica

La anterior propuesta nos permite modelar preferencias lingüísticas mediante información ELICIT y llevar a cabo procesos de CW para la resolución de problemas de TDL. Una etapa clave en la resolución de un problema de TDL es la fase de agregación de información mediante operadores de agregación lingüísticos, donde se combinan las opiniones individuales de los expertos sobre las alternativas en base a diferentes atributos (ver Sección 2.2). En ocasiones, dichos atributos están relacionados entre sí, siendo necesario modelar dicha interacción para llevar a cabo correctamente el proceso de agregación y obtener una solución fiable del problema. Sin embargo, en el anterior trabajo no se propuso ningún operador de agregación que considerara la interrelación entre las ELICIT agregadas ni tampoco que tuviera en cuenta la importancia individual de los atributos en el proceso de agregación, clave en muchos procesos de decisión.

Teniendo en cuenta la falta de propuestas, en este trabajo nos marcamos como objetivo definir nuevos operadores de agregación lingüísticos para información ELICIT que consideren diferentes patrones de relación entre los atributos y la importancia de los mismos en el proceso de agregación. Dichos operadores están basados en la media de Bonferroni y sus variantes [5, 13, 14], capaces de capturar diferentes tipos de relación entre los atributos agregados. En total, se proponen tres nuevos operadores de agregación, el primero enfocado a expresiones ELICIT cuya interrelación es homogénea o, en otras palabras, en donde cada expresión de entrada tiene relación con el resto. El segundo enfoque se basa en un operador de agregación que trata con expresiones ELICIT con una interrelación heterogénea, es decir, donde ciertas expresiones pueden tener o no relación con el resto. Por último, el tercer operador de agregación trata la interrelación partida de las expresiones ELICIT, en la cual, las expresiones de entrada son divididas en conjuntos formados por expresiones ELICIT con una interrelación entre ellas, pero no así entre expresiones de diferentes conjuntos.

Por último, los operadores ELICIT propuestos se aplican en la resolución de un problema de TDL comparando su funcionamiento con operadores similares pero que no consideran la interrelación entre los atributos del problema. Como conclusión, observamos que el ranking de las alternativas obtenido a través de los operadores propuestos es totalmente diferente al que ofrecen los operadores que no consideran la relación entre atributos, demostrando la necesidad de considerar la relación subyacente entre los atributos en un problema de TDL.

El artículo asociado a esta parte se encuentra en la Sección 4.3.

3.2. Procesos de Alcance de Consenso en Toma de Decisión en Grupo

En primer lugar, esta propuesta estudia los principales retos que existen hoy en día en los PAC en problemas de TDG, teniendo en cuenta aspectos como el incremento del número de expertos involucrados en el proceso de decisión y analiza si algunos de lo modelos de consenso más usados en la literatura pueden afrontar estos nuevos retos. Posteriormente, se presentan diferentes propuestas de PAC con diferentes características capaces de tratar con problemas de TDG del mundo real. Por último, se propone una métrica para PAC que permite evaluar el desempeño de los mismos y determinar que PAC es más adecuado para un problema de TDG a resolver.

3.2.1. Estudio Comparativo de Modelos de Consenso Clásicos en Problemas de Toma de Decisión en Grupo a Gran Escala

En esta parte, se estudian y analizan los nuevos retos que plantean los problemas de TDG y sus PAC debido a la expansión de paradigmas tecnológicos en nuestra sociedad, como por ejemplo, las redes sociales o el Big Data y que han dado paso a nuevos problemas de TDG donde el número de personas involucradas pueden ser de cientos o miles. En este tipo de problemas los PAC son aún más necesarios si cabe, ya que un gran número de decisores implica una polarización de las opiniones y, a su vez, la aparición inevitable de un gran número de conflictos, la necesidad de tratar con múltiples puntos de vista y comportamientos frente al proceso de decisión, la detección de coalición entre grupos, etc.

Los modelos de PAC clásicos presentados en la literatura especializada, tratan con problemas de TDG donde el número de expertos se asume como pequeño, lo que lleva a formularnos una pregunta obvia, ¿son los modelos de PAC enfocados a problemas de TDG con pocos expertos adecuados para hacer frente a problemas donde el número de expertos es mucho más elevado? Para dar respuesta a esta pregunta, en este trabajo se lleva a cabo una revisión de los modelos de consenso más influyentes de la literatura orientados a pequeños grupos de expertos y, debido al gran número de propuestas existentes, se procede a realizar una selección de los mismos. Con el objetivo de que dicha selección sea lo más representativa posible, hacemos uso de la taxonomía propuesta por Palomares et al. [49] (ver Sección 2.4), escogiendo modelos de consenso representativos para cada una de las categorías que se definen en dicha taxonomía [10, 24, 79, 81, 92].

Una vez escogidos los modelos de consenso, el siguiente paso consiste en analizar su funcionamiento empleando para ello un problema de TDG con un número significativo de expertos bajo diferentes escenarios de decisión. En concreto, definimos tres escenarios en base al comportamiento de los expertos: (i) todos los expertos aceptan las recomendaciones proporcionadas por el modelo, (ii) el 80 % de los expertos acepta las recomendaciones y el 20 % restante las rechaza y, por último, (iii) el 70 % acepta las recomendaciones, el 20 % las rechaza y el 10 % restante presenta un actitud defensiva que pretende sabotear el consenso en el grupo. La simulación de los modelos de consenso sobre los diferentes escenarios se lleva a cabo mediante AFRYCA 2.0, un sistema de soporte a PAC desarrollado en el transcurso de esta tesis doctoral y que será introducido en la Sección 3.3.1.

Las simulaciones permiten extraer valiosas conclusiones. Los modelos de consenso clásicos que no emplean un mecanismo de generación de recomendaciones no se ven afectados por el comportamiento de los expertos, ya que no se requiere de su participación en el PAC y lo que, aparentemente, los hace adecuados en problemas con grandes grupos. Sin embargo, esta característica junto con su naturaleza matemática podrían ser sus principales limitaciones, ya que, por un lado los expertos podrían no confiar en la solución obtenida al ser apartados del PAC y, por otro lado, el modelo matemático podría no encontrar una solución factible. Debido a esto, los modelos clásicos que incorporan un mecanismo de generación de recomendaciones se podrían considerar los más adecuados para resolver este tipo de problemas. Sin embargo, los modelos clásicos basan su funcionamiento en un comportamiento colaborativo de los expertos, si este comportamiento no se produce, se podrían producir bloqueos y no alcanzarse nunca el consenso deseado. Por lo tanto, es evidente que los modelos de consenso clásicos no pueden hacer frente a problemas de TDG con grandes grupos de expertos, por lo que es necesario desarrollar nuevas propuestas que permitan afrontar los retos que este tipo de problemas proponen.

El artículo asociado a esta parte se encuentra en la Sección 4.4.

3.2.2. Proceso de Alcance de Consenso a Gran Escala para la Gestión de Dudas en Grupos de Expertos

En el anterior trabajo se puso de manifiesto la necesidad de desarrollar nuevos modelos de consenso que sean capaces de hacer frente a los nuevos retos que presentan los actuales problemas de TDG. Uno de estos retos es la escalabilidad, que aparece en problemas de TDG que involucran a una gran cantidad de expertos y, en consecuencia, el tratamiento de forma simultánea de una gran cantidad de información. Por otra parte, resulta lógico pensar que los problemas con grandes grupos de expertos tienen de forma implícita asociados una alta complejidad y por lo tanto, incertidumbre y falta de información que pueden hacer que los expertos duden en el momento de expresar sus preferencias. En base a estas premisas, este trabajo presenta un nuevo modelo de consenso enfocado a problemas con grandes grupos de expertos que permite superar los problemas de escalabilidad y modelar la duda de los expertos.

Para mitigar el problema de la escalabilidad, el modelo de consenso propuesto aplica un *proceso de agrupamiento* o *clustering* basado en el algoritmo Fuzzy C-means [4] que clasifica a los expertos en diferentes subgrupos en base a la similitud entre sus opiniones. Por lo tanto, aquellos expertos cuyas opiniones son similares formaran parte de un mismo subgrupo. De esta forma, el procesamiento de la información no se aplica sobre todo el conjunto de expertos, sino sobre los diferentes subgrupos. Un aspecto clave en cualquier técnica de agrupamiento, es la asignación de pesos a los subgrupos. Estos pesos determinaran la influencia del subgrupo en el PAC, cuanto mayor sea el peso, mayor será su influencia sobre el PAC y sobre la solución del problema. Habitualmente, la ponderación de los subgrupos se basa exclusivamente en su tamaño, cuanto mayor sea el número de miembros que tiene el grupo, mayor peso asociado. Sin embargo, el que un subgrupo esté formado por expertos con opiniones aparentemente similares, no garantiza que no existan desacuerdos dentro de él. Por esta razón, este propuesta incluye un mecanismo para calcular la importancia de los subgrupos en base a dos aspectos: el tamaño del subgrupo y su cohesión. De esta forma, dos subgrupos con igual número de miembros pero con diferente cohesión, serán ponderados de manera diferente, dando prioridad a aquellos con mayor cohesión.

El modelo de consenso también incluye un proceso de generación de recomendaciones adaptativo. Dependiendo del nivel de consenso global que exista en ese momento, las recomendaciones se aplican de forma grupal o individual. Esta distinción se establece a partir de un valor umbral de consenso preestablecido. Si el consenso actual se encuentra por debajo del umbral, se considera que el grupo de expertos está aún lejos de alcanzar el consenso deseado y que es necesario un cambio significativo en las preferencias de los expertos, por lo que se procede a detectar los subgrupos formados por expertos cuyas opiniones están más alejadas del resto y se recomienda a todos los expertos que forman el subgrupo cambiar sus preferencias. Si por el contrario el consenso actual está por encima del umbral fijado, significa que el grupo está cerca de alcanzar el consenso deseado y que no es necesario realizar muchos cambios en las preferencias, por lo que se procede a detectar individualmente a los expertos cuyas opiniones están más alejadas de la mayoría y son exclusivamente esas opiniones las que se recomienda modificar.

Cabe destacar que las preferencias de los expertos son modeladas mediante *conjuntos difusos dudosos* (CDD) [68]. Estos conjuntos, son una extensión de los conjuntos difusos que permiten asignar varios grados de pertenencia de un elemento a un conjunto difuso, modelando así la duda de los expertos y conservando la mayor cantidad de información posible en los procesos computacionales llevados a cabo en el PAC.

Por último, la nueva propuesta se aplica a la resolución de un problema de TDG a gran escala y se lleva a cabo un análisis comparativo con diferentes modelos de consenso publicados en la literatura [10, 31]. Los resultados obtenidos a partir del software AFRYCA 2.0 (ver Sección 3.3.1) muestran que el modelo de consenso es capaz de resolver problemas de TDG con grandes grupos, alcanzando el consenso deseado en pocas rondas de debate. Además, el análisis comparativo muestra que el consenso alcanzado por la propuesta es mayor que el alcanzado por otros modelos de consenso y necesita menos rondas para alcanzarlo.

El artículo asociado a esta parte se encuentra en la Sección 4.5.

3.2.3. Proceso de Alcance de Consenso con Expresiones Lingüísticas Comparativas

Hoy en día, los problemas de TDG y sus PAC son cada vez más complejos y difíciles de resolver, por lo que es bastante común la aparición de incertidumbre y duda en las opiniones de los expertos. La mayoría de modelos de consenso presentados en la literatura [8, 72, 94], modelan dicha incertidumbre mediante términos lingüísticos simples, que no son lo suficientemente expresivos como para modelar la duda de los expertos. Con el objetivo de cubrir esta falta de propuestas, este trabajo presenta un modelo de consenso que modela las preferencias de los expertos mediante ELC, expresiones lingüísticas flexibles y complejas que permiten representar la duda en las opiniones de los expertos (ver Sección 2.3.3).

Esta propuesta emplea la representación difusa de las ELC haciendo uso del concepto de *envolvente difusa* [36], una función que permite transformar el CTLDD asociado a una ELC en un número difuso. De esta forma, es posible llevar a cabo los cálculos del PAC de una forma precisa y sin pérdida de información.

El modelo de consenso también propone un mecanismo de generación de recomendaciones. Este mecanismo se basa en el cálculo de la proximidad entre las opiniones individuales de los expertos y la opinión colectiva del grupo. Si el consenso colectivo sobre algunas de las alternativas está por debajo del umbral de consenso deseado, se recomienda modificar las opiniones sobre dichas alternativas. Los expertos que deben de llevar a cabo esas modificaciones, serán aquellos cuyas opiniones sobre estas alternativas están más alejadas de la opinión del grupo. Una vez que se han identificado los expertos y las alternativas en disenso, es necesario definir cómo se llevarán a cabo las recomendaciones. Al contrario que otras propuestas, este trabajo aplica los cambios directamente sobre las ELC que han proporcionado los expertos inicialmente, facilitando la interpretabilidad de los resultados.

El buen funcionamiento del modelo de consenso propuesto se pone a prueba mediante la resolución de un problema de TDG. El uso de ELC y su representación difusa, junto con la formalización de un proceso de generación de recomendaciones aplicado directamente sobre las ELC proporcionadas inicialmente por los expertos, permite resolver el problema planteado en muy pocas rondas de debate. Estas características hacen que la propuesta sea superior a otros modelos de consenso presentados en la literatura, como se demuestra en el análisis comparativo llevado a cabo. De nuevo cabe destacar que la resolución del problema de TDG y el análisis comparativo con otros modelos de consenso se lleva a cabo mediante el software AFRYCA 2.0 presentado en la Sección 3.3.1.

Sin embargo, este trabajo también presenta una importante limitación, ya que los expertos expresan sus opiniones a partir de un dominio de expresión discreto formado por el conjunto finito de términos lingüísticos que los expertos pueden usar para expresar sus opiniones. Por lo tanto, los cambios sobre las preferencias están limitados a la granularidad del conjunto de términos lingüísticos, lo que podría perjudicar el PAC.

El artículo asociado a esta parte se encuentra en la Sección 4.6.

3.2.4. Proceso de Alcance de Consenso con Expresiones Lingüísticas Comparativas con Translación Simbólica

El anterior trabajo evidenció la falta de modelos de consenso que fueran capaces de modelar la incertidumbre y duda de los expertos en los problemas de TDG y sus PAC. Por ello, se propuso un modelo de consenso que modelaba las preferencias de los expertos mediante ELC y llevaba a cabo un mecanismo de generación de recomendaciones que se aplicaba directamente sobre estas expresiones. Sin embargo, la propuesta presentaba una importante limitación, ya que estas recomendaciones estaban limitadas por el dominio de expresión discreto que los expertos usan para expresar sus opiniones. Esta limitación podría suponer un obstáculo a la hora de alcanzar el consenso deseado. Este trabajo tiene como objetivo superar dicha limitación.

El nuevo modelo de consenso sustituye las ELC por la información ELICIT (ver Sección 3.1.2), lo que permite mantener la interpretabilidad de las ELC y emplear expresiones lingüísticas definidas bajo un dominio continuo de expresión y que por lo tanto no están limitadas exclusivamente al conjunto finito de términos que forma dicho dominio. Los procesos computacionales llevados a cabo en el modelo de consenso se realizan de una forma precisa y sin pérdida de información, gracias al uso de la representación difusa de las expresiones ELICIT.

Esta propuesta también incluye un mecanismo de generación de recomendaciones. En este caso, se identifican las alternativas donde existe mayor disenso dentro del grupo. Si el consenso colectivo sobre una alternativa está por debajo del umbral de consenso deseado, es necesario recomendar a ciertos expertos que modifiquen sus opiniones sobre dicha alternativa. Los expertos que deberían modificar sus preferencias, son aquellos cuyas opiniones sobre las alternativas en disenso están más alejadas de la opinión del grupo. Una vez identificados los expertos y las alternativas, es necesario definir la recomendación de cambio sobre la preferencia. La propuesta incluye un proceso adaptativo que identifica si el cambio a aplicar debe ser más o menos drástico, un aspecto clave de nuestra contribución ya que, al contrario de otras propuestas existentes, la información ELICIT permite modificar las preferencias de los expertos en un dominio continuo. Mientras que otros modelos de consenso aplican el cambio en las preferencias de los expertos únicamente sobre los términos lingüísticos pertenecientes a un conjunto de términos lingüísticos predefinido que conforman un dominio discreto, nuestra propuesta puede utilizar el concepto de translación simbólica de la información ELICIT para aplicar los cambios en los valores continuos que existen entre los términos lingüísticos. Esto ayuda a generar recomendaciones más precisas, evitando modificaciones excesivas en las preferencias que pueden provocar una desviación de los resultados y un bloqueo en el proceso de consenso.

Para evaluar el funcionamiento de la propuesta, se procede a la resolución de un problema de TDG y a un posterior análisis comparativo con la propuesta presentada en la sección anterior, debido a la similitud entre ambas. La simulación llevada a cabo mediante el software AFRYCA 2.0 nos muestra excelentes resultados. Por una parte, el modelo de consenso es capaz de resolver el problema planteado en pocas rondas de debate y alcanzando un alto nivel de consenso. Ésto se consigue gracias al uso de información ELICIT, que permite llevar a cabo operaciones difusas que evitan la pérdida de información en el proceso de resolución y generar recomendaciones en su justa medida, evitando cambios excesivos en las preferencias que influyen negativamente en el acuerdo del grupo. Además, los cambios son aplicados sobre las expresiones ELICIT, lo que facilita su interpretación por parte de los expertos. El análisis comparativo también muestra un mejor funcionamiento con respecto a la propuesta anterior, ya que ésta última no es capaz de alcanzar el consenso deseado en el máximo número de rondas de debate predefinido.

El artículo asociado a esta parte se encuentra en la Sección 4.7.

3.2.5. Métrica basada en Modelos Integrales de Mínimo Coste para Procesos de Alcance de Consenso

Como hemos visto en trabajos anteriores, los PAC tienen gran importancia dentro de la TDG ya que, en muchas ocasiones, ciertos problemas de decisión requieren de una solución consensuada. Por esta razón, se han propuesto una gran cantidad de modelos de consenso en la literatura. El número de modelos de consenso es tal, que resulta realmente complejo decidir cual usar en la resolución de un problema de TDG. Además, no existe ningún tipo de medida objetiva que permita evaluar el buen o mal funcionamiento de un PAC y que facilite dicha decisión. Este trabajo pretende superar esta limitación mediante la definición de una métrica que permita evaluar el desempeño de modelos de consenso.

La métrica propuesta inicialmente está basada en *modelos de consenso de mínimo coste* (MMC) [2, 3, 93]. Estos modelos definen el consenso como la mínima distancia entre las opiniones individuales de los expertos y la opinión colectiva y buscan minimizar el coste de modificar dichas opiniones mediante una función lineal. Por lo tanto, son capaces de obtener una solución en consenso modificando lo mínimo posible las opiniones iniciales de los expertos en base a un valor umbral predefinido de distancia máxima entre las opiniones de los expertos y la colectiva. Cuanto más pequeño es el valor de ese umbral, menor distancia entre las opiniones de los expertos y la colectiva y, consecuentemente, mayor será el nivel de acuerdo alcanzado en el grupo. Sin embargo, pequeñas distancias entre las opiniones individuales de los expertos y la opinión colectiva no siempre garantizan alcanzar el nivel de acuerdo deseado dentro del grupo.

Para resolver la anterior problemática, en este trabajo se proponen nuevos MMC que incluyen una restricción adicional relacionada con el cálculo del consenso dentro del grupo de expertos y que denominaremos *modelos de consenso integrales de mínimo coste* (MIMC). De esta forma, se garantiza que, en caso de encontrar una solución factible, ésta cumplirá con las necesidades de consenso que requiera el problema. En total se proponen cuatros MIMC en base a dos aspectos. El primer aspecto está relacionado con la medida de consenso que se utiliza para calcular el consenso dentro del grupo, que puede estar basada en la similitud entre la opinión de los expertos y la opinión colectiva o basada en la similitud entre expertos. El segundo aspecto está relacionado con el tipo de estructura de preferencia que los expertos usan para expresar sus opiniones. En este caso, consideramos dos posibles estructuras, las formadas por *vectores de utilidad* [65] o por *relaciones de preferencia difusas* [47].

El siguiente paso es definir una métrica para modelos de consenso. La métrica, denominada *métrica de coste de consenso*, podría emplear cualquiera de los cuatro modelos anteriormente descritos, la selección dependerá de las características del modelo de consenso a analizar. Una vez seleccionado el MIMC, éste proporcionará, si existe, la solución óptima del problema, que es aquella de menor coste o que requiere del menor cambio en las preferencias de los expertos en base a las condiciones de consenso y distancia fijadas para el problema de decisión. Posteriormente, la métrica compara esta solución óptima con la solución proporcionada por el modelo de consenso analizado, calculando la distancia entre ambas soluciones y devolviendo un valor entre -1 y 1. Si el valor resultante es 0, el modelo de consenso analizado proporciona la misma solución que el MIMC, es decir, la solución óptima. Si el resultado es 1, significa que el modelo de consenso proporciona una solución donde las preferencias de los expertos no han sido en ningún momento modificadas. Para valores intermedios, cuanto más cercano a uno, peor es la solución propuesta por el modelo de consenso. En el caso de los valores negativos, -1 representa la peor solución posible por exceso de coste, es decir, que las preferencias de los expertos han sido modificadas en exceso. Para valores intermedios, cuanto más negativo el valor, peor solución.

Finalmente, para mostrar la utilidad de la métrica, se seleccionan una serie de modelos de consenso representativos con el objetivo de evaluar su funcionamiento sobre un problema de TDG. Las simulaciones llevadas a cabo por AFRYCA 2.0 demuestran que la nueva métrica puede utilizarse eficazmente para realizar comparaciones entre los PAC, ya que permite detectar situaciones anómalas en su rendimiento que no pueden detectarse con otros criterios.

El artículo asociado a esta parte se encuentra en la Sección 4.8.

3.3. Soporte a Problemas de Toma de Decisión en Grupo y Procesos de Alcance de Consenso

Los problemas de TD del mundo real son cada vez más complejos debido al continuo desarrollo de la sociedad. Los expertos a menudo tienen que hacer frente a problemas de decisión envueltos de incertidumbre y falta de información y que, en ocasiones, demandan soluciones en un breve periodo de tiempo. En estas condiciones, los expertos se encuentran expuestos a situaciones de alta presión que pueden afectar directamente a su comportamiento e influir negativamente en el proceso de decisión. Los sistemas de soporte a la decisión son creados con el objetivo de apoyar a los expertos y facilitar su labor en la toma de decisión.

En esta parte, mostraremos dos sistemas de soporte a la decisión que fueron desarrollados en el transcurso de esta tesis doctoral. Primeramente, introduciremos *AFRYCA 2.0*, una versión mejorada del software orientado al análisis de procesos de alcance consenso propuesto por Palomares et al. [49]. Esta nueva versión del software incluye nuevos modelos de consenso y características que permiten el tratamiento de un mayor número de problemas de TDG entre otras ventajas que serán desarrolladas de forma detallada en la siguiente sección. También presentaremos el sistema de soporte a la decisión APOLLO, acrónimo de "A grouP decisiOn fuzzy TOoL in support of cLimate change pOlicy making", que permite resolver problemas de TDG relativos a políticas sobre el cambio climático.

3.3.1. Software para el Análisis de Procesos de Alcance de Consenso: AFRYCA 2.0

Los PAC son fundamentales cuando se requieren soluciones consensuadas en problemas de TDG. Existen múltiples modelos de consenso propuestos en la literatura que simulan estos PAC y sirven como herramienta de soporte a los expertos en la resolución del problema. Sin embargo, la mayoría de estos modelos tienen una estructura algorítmica compleja compuesta por diferentes pasos, por lo que los expertos no pueden dedicar el ya escaso tiempo que tienen en determinar qué modelo de consenso usar y llevar a cabo manualmente todas los cálculos relativos al funcionamiento del mismo. Con esta idea, se diseñó inicialmente AFRYCA [49], acrónimo de "A FRamework for the analYsis of Consensus Approaches", un sistema de soporte a la decisión que incluye diferentes modelos de consenso que permiten simular un PAC real. Sus principales objetivos son (i) descubrir las ventajas y desventajas de los modelos de consenso, (ii) determinar el modelo más adecuado para un problema de TDG específico y (iii) llevar a cabo comparaciones entre los diferentes modelos.

El uso de AFRYCA en múltiples situaciones de TDG evidenció ciertas carencias en el software, como una tecnología obsoleta, una estructura compleja que dificulta la inclusión de nuevos modelos de consenso y sus parámetros, la imposibilidad de modificar por parte del usuario varios parámetros relevantes de la simulación, falta de información sobre los resultados de dicha simulación y la incapacidad de analizar los modelos de forma más detallada. Con todas estas limitaciones en mente, en este trabajo se presenta una versión mejorada del software, AFRYCA 2.0.

La versión 2.0 de AFRYCA presenta las siguientes ventajas con respecto a su predecesora:

- *Migración e independencia*: AFRYCA 2.0 se desarrolla bajo la nueva rama 4.0 de Eclipse RCP [15] que incluye varias novedades a nivel tecnológico como la inyección de dependencias, servicios declarativos, aplicación del modelo, etc. Además, la primera versión de AFRYCA hacía uso de librerías externas para ciertas funcionalidades como el uso del lenguaje estadístico de programación R [55], lo que dificultaba su migración a otras plataformas. En la versión 2.0, el lenguaje se incorpora de forma nativa, por lo que el entorno estadístico puede seleccionarse en tiempo de ejecución.

- *Inclusión de nuevos modelos*: AFRYCA 2.0 incorpora un nuevo mecanismo más simple para añadir nuevos modelos de consenso al software. Ahora es posible definir todos los parámetros asociados al modelo y aplicarles una serie de restricciones y relaciones entre ellos, evitando que los usuarios tengan que comprobar manualmente si todos los valores son correctos. Además, se han incluido dos nuevos modelos de consenso en el software [50, 51].

- *Configuración de comportamientos*: AFRYCA 2.0 otorga mayor flexibilidad en la configuración de la simulación del comportamiento de expertos, siendo posible modelar la distribución de probabilidad asociada a ellos. También se ha facilitado el mecanismo para incluir nuevos comportamientos y se ha incluido uno nuevo denominado *adverso*, que permite simular expertos reticentes a aceptar las recomendaciones.

- *Evolución de los PAC*: la primera versión de AFRYCA visualizaba el estado de las preferencias de los expertos al final del PAC. Sin embargo, AFRYCA 2.0 muestra dicha visualización para cada una de las rondas de debate necesarias en el transcurso del PAC.

- *Métricas*: en AFRYCA 2.0 se incluyen varias métricas que permiten estudiar diferentes aspectos de los modelos de consenso y analizar su funcionamiento.

Este trabajo también incluye un estudio experimental donde se llevan a cabo varias simulaciones de PAC en diferentes problemas de TDG con el objetivo de mostrar las nuevas características y ventajas de AFRYCA 2.0.

Cabe destacar que, en el transcurso de esta tesis, se ha ido mejorando de forma constante el software, incluyendo nuevas características como una visualización tridimensional de las preferencias de los expertos, inclusión de nuevos modelos de consenso, soporte para nuevos tipos de estructuras de preferencia, etc. Actualmente, el software está en proceso de registro para el reconocimiento de su autoría.

El artículo asociado a esta parte se encuentra en la Sección 4.9.

3.3.2. Software para el Soporte de Problemas de Decisión basados en Política Climática: APOLLO

Hoy en día, muchos de los problemas más importantes de TD están relacionados con cuestiones de sostenibilidad. Los efectos del cambio climático son cada vez más evidentes y sus repercusión en nuestra sociedad, economía y medio ambiente a día de hoy y en el futuro es una de nuestras principales preocupaciones. Este reto se ha abordado mediante diferentes políticas climáticas. Sin embargo, su enorme complejidad hace que los expertos deban evaluar los riesgos de aplicar diferentes políticas en una determinada zona geográfica, dejándose llevar por una serie de suposiciones que no reflejan las limitaciones del mundo real. Este trabajo se centra en reducir dicha complejidad mediante el desarrollo de un sistema de soporte a la decisión enfocado a políticas climáticas denominado APOLLO.

El principal objetivo de APOLLO es facilitar el proceso de consenso de un grupo de expertos para alcanzar la mejor solución posible en un problema de TDG relacionado con cuestiones climáticas. Para ello, APOLLO presenta un esquema de resolución dividido en varios pasos:

1. *Definición del problema*: en este paso se define el problema de TDG y todos los elementos relacionados con el mismo, las alternativas, los criterios para evaluarlas, expertos o dominios de expresión. Concretamente, APOLLO se centra en el modelado de preferencias mediante información lingüística con el objetivo de facilitar la labor de los expertos.

2. *Asignación de dominios de expresión*: en esta fase los dominios de expresión creados en la primera etapa son asignados a los diferentes expertos. De esta forma, los expertos pueden usar el dominio de expresión con el que se sientan más cómodos a la hora de expresar su conocimiento.

3. *Consenso*: las políticas climáticas afectan al conjunto de la sociedad, por lo que las soluciones consensuadas son mucho más valoradas. APOLLO mide el nivel de consenso en el grupo de expertos, llevando a cabo un PAC si éste no alcanza el valor deseado y

con el objetivo de que los expertos modifiquen sus preferencias iniciales e incrementen el acuerdo entre ellos.

4. *Valoración*: finalmente, en esta etapa APOLLO lleva a cabo la resolución del problema de TDG mediante el método lingüístico TOPSIS 2-tupla [62] proporcionando un ranking de las alternativas en base a las opiniones consensuadas de los expertos.

El funcionamiento de APOLLO se pone a prueba mediante la resolución de un caso de estudio real relacionado con la descarbonización de la producción de hierro y acero en Austria. En el caso de estudio se pretende facilitar el camino de la transición de la industria siderúrgica austriaca, priorizando los riesgos asociados a esta transición mediante la participación de las partes interesadas en un proceso que proporcionará información sobre lo que más temen los actores clave del sistema. Se consideran un total de veinticinco riesgos posibles que son evaluados en base a cuatro criterios diferentes. APOLLO permite detectar los desacuerdos que existen en el grupo de expertos, simular un PAC que sirva de apoyo a estos últimos para modificar sus preferencias y alcanzar un mayor nivel de acuerdo y, por último, ofrecer un ranking de los diferentes riesgos. Todo ello sin perder de vista que el software hace uso de información lingüística en todo momento, lo que facilita la comprensión de los resultados por parte de los expertos.

El artículo asociado a esta parte es se encuentra en la Sección 4.10.

Capítulo 4

Publicaciones

En virtud de lo establecido en el artículo 25, punto 2, de la normativa vigente para los Estudios de Doctorado de la Universidad de Jaén, correspondiente al programa RD. 99/2011, en este capítulo se presentan las publicaciones que componen el núcleo de la presente tesis doctoral.

Dichas publicaciones se corresponden con nueve artículos científicos publicados en Revistas Internacionales indexadas por la base de datos JCR (Journal Citation Reports), producida por Clarivate Analytics y un artículo publicado en una revista internacional indexada en Scopus.

4.1. Perspectiva General del Modelado Difuso de las Preferencias Lingüísticas Complejas en la Toma de Decisión

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- Factor de impacto (JCR 2016): 1.140
    - Cuartiles por Área de Conocimiento:
        - Cuartil 3 en *Computer Science, Artificial Intelligence*. Ranking 92/133.
        - Cuartil 3 en *Computer Science, Interdisciplinary Applications*. Ranking 77/105.

An Overview on Fuzzy Modelling of Complex Linguistic Preferences in Decision Making

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Abstract

Decision makers involved in complex decision making problems usually provide information about their preferences by eliciting their knowledge with different assessments. Usually, the complexity of these decision problems implies uncertainty that in many occasions has been successfully modelled by means of linguistic information, mainly based on fuzzy based linguistic approaches. However, classically these approaches just allow the elicitation of simple assessments composed by either one label or a modifier with a label. Nevertheless, the necessity of more complex linguistic expressions for eliciting decision makers' knowledge has led to some extensions of classical approaches that allow the construction of expressions and elicitation of preferences in a closer way to human beings cognitive process. This paper provides an overview of the broadest fuzzy linguistic approaches for modelling complex linguistic preferences together some challenges that future proposals should achieve to improve complex linguistic modelling in decision making.

Keywords: Fuzzy Linguistic Approach, Fuzzy Logic, Computing with Words, Decision Making, Preference Modelling

1. Introduction

In spite of decision making processes have been an object of research during many years, new requirements and challenges within the topic arise often, because of new problems and new necessities of decision makers. Nowadays the complexity of decision making problems is not only due to the existence of multiple and conflicting goals and the necessity of dealing with huge amounts of information and alternatives, but also because of time pressure, lack of knowledge and so on. It implies that these problems are *ill-structured* whose definition framework often involves uncertainty, vagueness and incomplete information that cannot be properly modelled by probabilistic models. In such decision situations with non-probabilistic uncertainty the use of linguistic information has provided successful results in different fields [10,24,30,31]. To model and cope with the inherent uncertainty and vagueness of linguistic descriptors, it has been extensively used the fuzzy linguistic approach [2,47] based on the fuzzy sets theory [17]. Hence, decision making problems could use the fuzzy linguistic approach in its solving process whenever its fuzzy representation would be adequate for the decision situations.

The usefulness of using a fuzzy representation to model linguistically preferences in decision making comes from the interpretation of the semantics of a fuzzy set as a *degree of preference* [8], such that the fuzzy semantics represents the values of a decision variable more o less preferred. Therefore, by using the interpretation of degree of preference for semantics of fuzzy sets, the use of fuzzy linguistic labels to express the intensity of preference for a given alternative in a decision-making problem seems natural.

The use of linguistic information in decision making implies to carry out computing with words (CW) processes. CW is defined as a methodology for reasoning, computing and making decisions using information described in natural language [29]. Therefore, it emulates human cognitive processes to improve solving processes of problems dealing with uncertainty. Thus, CW has been applied as computational basis to decision making problems that deal with linguistic information [22,26], because it provides tools close to human beings reasoning processes related to decision making, enhances the reliability and flexibility of classical decision models and improves the resolution of decision making under uncertainty with linguistic information. Consequently, different linguistic computational models have been developed to manage linguistic decision making [14,20,37,41,43].

Across specialized literature different fuzzy linguistic based approaches for modelling preferences in decision making and computational models for CW processes can be found [18,22,26,28,32], however these approaches provide just either simple terms or labels that hardly can express in many complex decision situations the decision makers' knowledge in a proper and adequate way according to decision makers' aims. Hence, recently different researchers have proposed different attempts to facilitate the elicitation of linguistic preferences by expressions to some extent more elaborated than simple labels [20,33,38,42,49]. Such extensions have used different fuzzy tools to model and compute with such linguistic expressions in a closer way to decision makers' needs. This paper aims at providing an overview of the fuzzy approaches that model complex linguistic expressions together with their computational models. Eventually several challenges related to the modelling of complex linguistic expressions within decision making are also pointed out.

This paper is structured as follows: Section 2 provides a brief review of the use of fuzzy linguistic information in decision making. Section 3 presents an overview of different fuzzy based approaches for modelling complex linguistic expressions paying attention to their computational models. Section 4 points out different challenges that must be achieved for improving this linguistic modelling in decision making problems, and finally Section 5 concludes this paper.

2. Decision Making and Linguistic Information

The introductory section pointed out that complex real world decision making problems are often *ill-structured* problems that cannot be solved straightforwardly because of the uncertainty, vagueness and incomplete information involved. In such a type of decision making problems, the use of linguistic descriptors by decision makers is a straightforward and natural tool to elicit their preferences on the alternatives. The fuzzy linguistic approach [2,47] which is based on the fuzzy sets theory [17], has been widely used to model and manage the vagueness and inherent uncertainty of the linguistic descriptors by linguistic variables.

Therefore, before providing an overview about different fuzzy based approaches to model complex linguistic preferences, this section reviews in short necessary concepts to understand such approaches. First, a brief revision of fuzzy linguistic approach is provided. Afterwards, the decision making solving scheme used when linguistic information takes part in the decision process is reviewed and eventually classical fuzzy linguistic computational models are shown.

2.1. Fuzzy Linguistic Approach

The fuzzy linguistic approach [47] based on the fuzzy set theory is a common approach for modelling the linguistic information by using the concept of linguistic variable [47], *"a variable whose values are not numbers, but words or sentences in a natural or artificial language"*. A linguistic value is less precise than a number, but it is closer to human cognitive processes used to solve successfully problems dealing with uncertainty. Formally a linguistic variable is defined as follows:

Definition 1. [48]: A linguistic variable is characterized by a quintuple (*V,T(V),U,G,M*) in which *V* is the name of the variable; *T(V)* (or simply *T*) denotes the term set of *V*, i.e., the set of names of linguistic values of *V*, with each value being a fuzzy variable denoted generically by *X* and ranging across a universe of discourse *U* which is associated with the base variable *u*; *G* is a syntactic rule (*which usually takes the form of a grammar*) for generating the names of values of *H*; and *M* is a semantic rule for associating its meaning with each *V*, $M(X)$, which is a fuzzy subset of *U*.

The use of linguistic variables needs the selection of appropriate linguistic descriptors for the term set, including the analysis of their *granularity of uncertainty*, and their syntax and semantics. The former commonly noted as, $g+1$, determines the level of discrimination among different counts of uncertainty modeled by the linguistic descriptors in the linguistic term set, $S = \{s_0, \ldots, s_g\}$. A fine granule means a high level of discrimination, however a coarse granule means a low discrimination level. The selection of the syntax and suitable semantics are crucial to determine the validity of the fuzzy linguistic approach, and exist different approaches to choose the linguistic descriptors and different ways to define their linguistic semantics [21,44,47]. The semantics of the terms is represented by fuzzy numbers, described by membership functions. The linguistic assessments given by users are just approximate ones. A way to characterize a fuzzy number is to use a representation based on parameters of its membership function [3]. Figure 1 shows an example of a linguistic term set with the syntax and semantics defined.

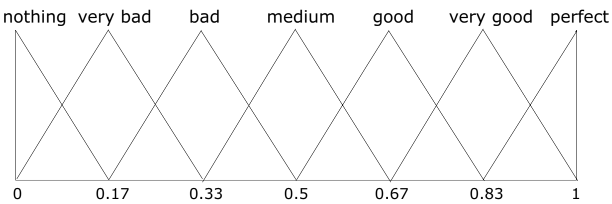

Fig. 1. A linguistic term set of 7 labels

2.2. Linguistic decision making solving scheme

A classical decision making solving scheme consists of two main steps [35]:

1. An *aggregation phase* that aggregates the values provided by the decision makers to obtain a collective assessment for the alternatives.
2. An *exploitation phase* of the collective assessments to rank, sort or choose the best one/s among the alternatives.

The use of linguistic information in decision making modifies the previous solving scheme adding two new steps: (i) selecting the linguistic term set and its semantics and (ii) selecting the aggregation operator for linguistic information. Therefore, the linguistic decision making solving scheme is composed by 4 steps (see Fig. 2).

- *Selecting the linguistic term set and the semantics*: In this step, the linguistic domain in which decision makers provide their assessments about the alternatives is defined according to each specific decision problem.
- *Selecting the aggregation operator for linguistic information*: A proper linguistic aggregation operator is selected to aggregate the linguistic assessments provided by decision makers in accordance to the goal of the problem.
- *Aggregation*: The linguistic assessments are aggregated by using the aggregation operator previously selected to obtain a collective value for each alternative of the decision problem.
- *Exploitation*: The collective values obtained in the previous aggregation step are ranked to select the best alternative(s).

2.3. Linguistic computing models

The linguistic decision making solving scheme depicted in Figure 2 shows the necessity of developing linguistic computing models to operate with linguistic information. Different linguistic computing models have been developed to facilitate such processes. Here a brief revision of the most extended models to deal with linguistic variables are revised.

2.3.1. Classical linguistic computing models

Initially, two linguistic computing models based on the fuzzy linguistic approach [47] were defined to perform CW processes.

1. *Linguistic computing model based on membership functions*: It makes the computations with linguistic terms by operating directly on their membership functions using the Extension Principle [16]. The use

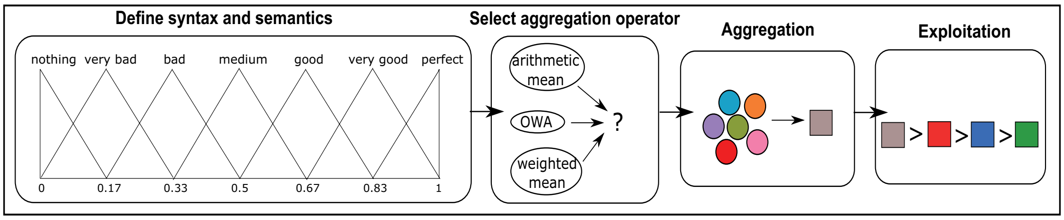

Fig. 2. Linguistic decision making solving scheme

of fuzzy arithmetic based on the Extension Principle increases the vagueness of the results. Therefore, the results obtained are fuzzy numbers that usually do not match with any linguistic term in the initial linguistic term set. Taking into account these results, there are two possible ways:

- If in the decision problem, it is more relevant to obtain precise results than interpretable ones, the results are expressed by fuzzy numbers [1].
- If an interpretable and linguistic result is required, then it is necessary an approximation function, $app_1(\cdot)$, to associate the fuzzy result with a linguistic term in S [23]:

$$S^n \xrightarrow{\widetilde{F}} F(\mathscr{R}) \xrightarrow{app_1(\cdot)} S$$

  where S^n symbolizes the n Cartesian product of S, $\widetilde{F}$ is an aggregation operator based on the Extension Principle and $F(\mathscr{R})$ the set of fuzzy sets over the set of real numbers $\mathscr{R}$.

The approximation process implies a loss of information and lack of accuracy of the results.

A later computational approach based on membership functions for linguistic information is the one based on type-2 fuzzy sets. This computational model makes use of type-2 fuzzy sets to model the linguistic assessments [27,39]. The use of type-2 fuzzy sets has been justified in order to improve the modelling and management of the uncertainty in linguistic information [28,39]. The majority of the contributions dealing with this fuzzy representation use interval type-2 fuzzy sets which maintain the uncertainty modelling properties of general type-2 fuzzy sets, but reducing the computational efforts that are needed to operate with them. Different aggregation operators for type-2 representation were introduced in [7,50]. As the type-1 linguistic based representation, the type-2 fuzzy sets computational based model needs to approximate the resulting type-2 fuzzy set from a linguistic operation by mapping the result into a linguistic assessment producing a loss of information.

2. *Symbolic linguistic computing model*: Symbolic models have been widely used in CW, because they are simple and provide interpretable results. Such models use the ordered structure of the linguistic term set, $S = \{s_0, s_1, \ldots, s_g\}$ where $s_i < s_j$ if $i < j$, to carry out the computations. The intermediate results are numerical values $\gamma \in [0, g]$, that must be approximated by an approximation function $app_2(\cdot)$ to obtain a numerical value.

$$app_2 : [0, g] \rightarrow \{0, \ldots, g\}$$

Yager in [45] introduced the symbolic model based on ordinal scales and max-min operators, it obtains linguistic results easy to understand, but their accuracy is low because they are computed by using the maximum or minimum values ignoring the intermediated ones. Later on, the linguistic symbolic computational model based on convex combinations was introduced by Delgado et al. [5], which directly acts over the label indexes, $\{0, \ldots, g\}$, of the linguistic term set, $S = \{s_0, \ldots, s_g\}$, in a recursive way producing a real value on the granularity interval, $[0, g]$, of the linguistic term set S. It is worthy to note that this model usually assumes that the cardinality of the linguistic term set is odd and that linguistic labels are symmetrically placed around a middle term. The result of a symbolic convex combination aggregation usually does not match with a term of the label set S, therefore it is also necessary to introduce an approximation function $app_2(\cdot)$ for obtaining a solution in the linguistic term set S.

Hence, similarly to the linguistic computing based on membership functions, the approximation process in the symbolic based models produces loss of information.

Therefore, both types of linguistic classical computing models produce loss of information due to the approximation processes and hence a lack of accuracy in the results. This loss of information is produced because the information representation model of the fuzzy linguistic approach is discrete in a continuous domain. In order to overcome these limitations different linguistic computing models have been proposed in the literature [26], the most widely used in decision making with linguistic information is the 2-tuple linguistic model [14,?] that is briefly revised below, because some of the proposals to deal with complex linguistic expressions either extend it or are based on it.

2.3.2. 2-tuple linguistic model

As it was aforementioned, *the 2-tuple linguistic model* [14] was developed to avoid the loss of information and the lack of accuracy that present the classical computing models in the CW processes. Many approaches that deal with complex linguistic expressions either make or can make use of it, thus a short revision about the model it is introduced.

The 2-tuple linguistic model represents the linguistic information by means of a pair of values (s,α), where s is a linguistic term and α is a numerical value that represents the *symbolic translation*.

Definition 2. [14,22] The symbolic translation is a numerical value assessed in $[-0.5,0.5)$ that supports the "difference of information" between a counting of information β assessed in the interval of granularity $[0,g]$ of the linguistic term set $S=\{s_0,\dots,s_g\}$ and the closest value in $\{0,\dots,g\}$ which indicates the index of the closest linguistic term in S.

This model defines a set of functions to facilitate the computations with 2-tuple linguistic values.

Definition 3. [14] Let $S=\{s_0,\dots,s_g\}$ be a set of linguistic terms. The 2-tuple linguistic set associated with S is defined as $\overline{S}=S\times[-0.5,0.5)$. The function $\Delta:[0,g]\longrightarrow\overline{S}$ is given by

$$\Delta(\beta)=(s_i,\alpha),\ \text{with}\ \begin{cases} i=\ \text{round}\,(\beta),\\ \alpha=\beta-i, \end{cases} \tag{1}$$

where *round* assigns to β the closest integer number $i\in\{0,\dots,g\}$ to β.

Remark 1. Δ is a bijective function and $\Delta^{-1}:\overline{S}\longrightarrow[0,g]$ is defined by $\Delta^{-1}(s_i,\alpha)=i+\alpha$.

The 2-tuple linguistic model has defined a symbolic computational model based on the functions Δ and Δ^{-1} and defines a negation operator, several aggregation operators and the comparison between two 2-tuple linguistic values [14].

Example 1. Let us suppose an example where decision makers provide their assessments by using the linguistic term set shown in Figure 1. The assessments provided are $\{low, very\ high, medium\}$. These linguistic terms are aggregated by using the 2-tuple arithmetic mean (see [25] for further detail). The result is $\bar{x}=(medium,0.33)$ which is represented in Figure 3.

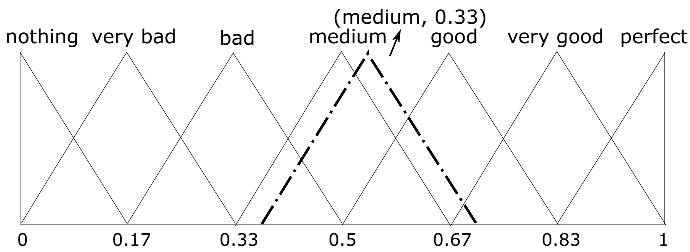

Fig. 3. A 2-tuple linguistic value

3. Modelling Complex Linguistic Preferences

So far, it has been shown that the use of fuzzy linguistic information and its computational models (see Section 2) have been not only broadly used to model and manage the uncertainty in real world decision problems but also to solve such problems in different fields [9,12]. Notwithstanding, some researchers have indicated the necessity of introducing some improvements to model the elicitation of linguistic information in decision making. Because decision makers involved in the problems are limited to express their knowledge by using only a simple linguistic term and often this type of modelling is not enough to reflect the knowledge and preference that they really want to elicit. Additionally, another limitation of current linguistic preference modelling approaches based on the fuzzy linguistic approach consists of the linguistic terms that can be used by decision makers in the decision problem are defined a priory, thus decision makers cannot express their preferences in a more flexible and richer way if it is necessary to elicit the preferences in a more elaborated way.

In order to face these restrictions, different proposals that facilitate the elicitation of elaborated linguistic preferences with complex linguistic expressions have been introduced in the literature [20,33,38,42,49]. Such proposals focus their performance on very different points of view that can be used by decision makers according to their needs in each specific problem. Hence this section provides an overview of the most important proposals to model complex linguistic preferences pointing out the way to construct such expressions and the computing models used by them in order to accomplish the processes of CW in decision making. Additionally, some comments for analysing the main features of each proposal are introduced.

3.1. Proportional 2-tuple linguistic model

The first model that attracts our attention for modelling expressions more elaborated than a single term is the proportional 2-tuple linguistic model introduced by Wang and Hao in [41]. Such a model is a generalization and extension of the 2-tuple linguistic model in which the linguistic modelling is based on the use of proportions of two adjacent linguistic terms represented by two pairs of values.

3.1.1. Representation model

In this model the information is represented by a proportional 2-tuple value which has a linguistic term in each pair that represents the linguistic information and a numerical value that indicates its proportion in the expression.

Definition 4. [41] Let $S = \{s_0, \ldots, s_g\}$ be an ordinal term set, $I = [0,1]$ and

$$IS \equiv I \times S = \{(\alpha, s_i) : \alpha \in [0,1] \text{ and } i = \{0, \ldots, g\} \quad (2)$$

where S is the ordered set of $g+1$ ordinal terms. Given a pair of two consecutive ordinal terms (s_i, s_{i+1}), any two elements (α, s_i), (β, s_{i+1}) of IS is called a symbolic proportion pair, and α, β are called a pair of symbolic proportions of the pair (s_i, s_{i+1}) if $\alpha + \beta = 1$. A symbolic proportion pair (α, s_i), $(1-\alpha, s_{i+1})$ is denoted by $(\alpha s_i, (1-\alpha)s_{i+1})$ and the set of all the symbolic proportion pairs is denoted by $\overline{\overline{S}}$, i.e., $\overline{\overline{S}} = \{(\alpha s_i, (1-\alpha)s_{i+1}) : \alpha \in [0,1] \text{ and } i = \{0, \ldots, g-1\}$.

Remark 2. The ordinal term s_i, $i = \{2, \ldots, g-1\}$, can be represented both $(0s_{i-1}, 1s_i)$ and $(1s_i, 0s_{i+1})$.
$\overline{\overline{S}}$ is called the *ordinal proportional 2-tuple set* generated by S and the members of $\overline{\overline{S}}$, *ordinal proportional 2-tuple values*, that are used to represent the ordinal information.

This model also defines some functions to make easier the operations with this type of information.

Definition 5. [42] Let $S = \{s_0, \ldots, s_g\}$ be an ordinal term set and $\overline{\overline{S}}$ be the ordinal proportional 2-tuple set generated by S. The function $\pi : \overline{\overline{S}} \to [0, g]$ is defined as follows,

$$\pi((\alpha s_i, (1-\alpha)s_{i+1})) = i + (1-\alpha), \quad (3)$$

where $i = \{0, \ldots, g-1\}, \alpha \in [0,1]$ and π is called the position index function of ordinal 2-tuple.

The position index function π is bijective and its inverse $\pi^{-1} : [0, g] \to \overline{\overline{S}}$ is defined as follows,

$$\pi^{-1}(x) = ((1-\beta)s_i, \beta s_{i+1}) \quad (4)$$

where $i = E(x)$, being E the integer part function, $\beta = x - i$.

Example 2. By using the linguistic term set depicted in Figure 1, some assessments represented by proportional 2-tuple values might be,
$(0.66\ \textit{medium}, 0.33\ \textit{good})$
$(0.25\ \textit{good}, 0.75\ \textit{very good})$

3.1.2. Computational model

A computational model based on the functions π and π^{-1} was also defined with the following operations [42].

1. *Comparison of proportional 2-tuple values*

   The comparison of linguistic information represented by proportional 2-tuple value is carried out as follows:

   Let $S = \{s_0, \ldots, s_g\}$ be an ordinal term set and $\overline{\overline{S}}$ be the ordinal proportional 2-tuple set generated by S. For any $(\alpha s_i, (1-\alpha)s_{i+1}), (\beta s_j, (1-\beta)s_{j+1}) \in \overline{\overline{S}}$, defines $(\alpha s_i, (1-\alpha)s_{i+1}) < (\beta s_j, (1-\beta)s_{j+1}) \Leftrightarrow \alpha i + (1-\alpha)(i+1) < \beta j + (1-\beta)(j+1) \Leftrightarrow i + (1-\alpha) < j + (1-\beta)$.
   Therefore, for any two proportional 2-tuple values $(\alpha s_i, (1-\alpha)s_{i+1})$ and $(\beta s_j, (1-\beta)s_{j+1})$:

   - if $i < j$, then

(a) $(\alpha s_i,(1-\alpha)s_{i+1}),(\beta s_j,(1-\beta)s_{j+1})$ represents the same information when $i=j-1$ and $\alpha=0,\beta=1$

(b) $(\alpha s_i,(1-\alpha)s_{i+1})<(\beta s_j,(1-\beta)s_{j+1})$ otherwise

- if $i=j$, then

(a) if $\alpha=\beta$ then $(\alpha s_i,(1-\alpha)s_{i+1}),(\beta s_j,(1-\beta)s_{j+1})$ represents the same information

(b) if $\alpha<\beta$ then $(\alpha s_i,(1-\alpha)s_{i+1})<(\beta s_j,(1-\beta)s_{j+1})$

(c) if $\alpha>\beta$ then $(\alpha s_i,(1-\alpha)s_{i+1})>(\beta s_j,(1-\beta)s_{j+1})$

2. *Negation operator of a proportional 2-tuple value*

The negation of a proportional 2-tuple value is defined as:

$$Neg((\alpha s_i,(1-\alpha)s_{i+1}))=((1-\alpha)s_{g-i-1},\alpha s_{g-i}), \tag{5}$$

where $g+1$ is the cardinality of S, $S=\{s_0,\dots,s_g\}$.

3. *Proportional 2-tuple aggregation operators*

Several aggregation operators were defined by Wang and Hao to accomplish CW processes. The definitions of these aggregation operators are based on canonical characteristic values of linguistic terms. To do so, similar corresponding aggregation operators developed in [14] were defined to aggregate ordinal 2-tuple values by means of their position indexes [42].

In [42] was also introduced a relationship between the proportional 2-tuple linguistic model and the 2-tuple linguistic model [14].

Definition 6. [42] Let $\overline{S}$ be a 2-tuple linguistic set and $\overline{\overline{S}}$ the ordinal proportional 2-tuple set generated by S, the function $h:\overline{\overline{S}}\rightarrow\overline{S}$ is defined as follows,

$$h(\alpha s_i,(1-\alpha)s_{i+1})=\begin{cases}(s_{i+1},-\alpha) & if\ \ 0\leqslant\alpha\leqslant 0.5\\(s_i,1-\alpha) & if\ \ 0.5<\alpha\leqslant 1.\end{cases} \tag{6}$$

h is a bijective function and $\pi=\Delta^{-1}\circ h$. The proof of this relationship can be found in [42].

3.1.3. Analysis of proportional 2-tuple expressions

The expressions represented by this model are still simple and far from common linguistic expressions used by human beings, because from the linguistic point of view decision makers do not provide naturally such expressions but rather they can be computed either from other linguistic representations or after a specific training expert might provide them directly. However, it was an interesting and initial step to provide a way to improve the elicitation of linguistic information.

3.2. Linguistic model based on fuzzy relation

A second step for dealing with the modelling of elaborated linguistic expressions was introduced by Tang and Zheng [38].

3.2.1. Representation model

Tang and Zheng proposed a linguistic model that generates linguistic expressions from a set of linguistic terms $S=\{s_0,\dots,s_g\}$, using logical connectives, such as $(\vee,\wedge,\neg,\longrightarrow)$, whose semantics are represented by fuzzy relations R, that describe the degree of similarity between two linguistic terms s_i and s_j. The set of all linguistic expressions is denoted as LE.

Definition 7. [38] Let LE be the set of linguistic expressions which is defined recursively as follows:

1. $s_i\in LE$ for $i=\{0,\dots,g\}$,
2. if $\theta,\phi\in LE$ then $\neg\theta,\theta\vee\phi,\theta\wedge\phi,\theta\longrightarrow\phi\in LE$.

A formal definition of this set is the following one.

Definition 8. [38] Any linguistic expression $\theta\in LE$ is associated with a set of subsets of S, denoted $\lambda(\theta)$ and defined recursively as follows,

1. $\lambda(s_i)=\{Z\subseteq S|s_i\in Z\}\forall i=\{0,\dots,g\}$,
2. $\lambda(\theta\wedge\phi)=\lambda(\theta)\cap\lambda(\phi)$,
3. $\lambda(\theta\vee\phi)=\lambda(\theta)\cup\lambda(\phi)$,
4. $\lambda(\theta\rightarrow\phi)=\overline{\lambda(\theta)}\cup\lambda(\phi)$,
5. $\lambda(\neg\theta)=\overline{\lambda(\theta)}$.

Example 3. Some examples of linguistic expressions in *LE* generated from the linguistic term set *S* shown in Figure 1 could be the following ones:

$$\neg good \vee very\ good$$

$$medium \wedge good$$

3.2.2. Computational model

A fuzzy relation $R = (r_{ij})_{n\times n}$ is defined on S where the elements $r_{ij} \in [0,1]$ of R represent the degree of similarity between the linguistic terms s_i and s_j. Therefore, r_{ij} is denoted as $r(s_i, s_j)$. A membership function $\mathscr{F}_{s_i}(\cdot) = r(s_i, \cdot)$ on S can be obtained for each s_i.

There is also a correspondence between fuzzy sets and consonant mass assignment functions [11].

Definition 9. [38] Let $\mathscr{F}_{s_i}$ be a membership function that achieves its value in $\{\lambda_1, \ldots, \lambda_z\}$ such as $1 = \lambda_1 > \lambda_2 > \ldots > \lambda_z \geqslant 0$. A consonant mass assignment function m_{s_i} for the membership function $\mathscr{F}_{s_i}$ can be obtained as follows,

$$m_{s_i}(Z_k) = \lambda_k - \lambda_{k+1}, k = \{1, \ldots, z\} \tag{7}$$

where the focal element Z_k is the λ_k-cut set of $\mathscr{F}_{s_i}$.

$$Z_k = \{s_h | \mathscr{F}_{s_i}(s_h) \geqslant \lambda_k\}. \tag{8}$$

And from the consonant mass assignment function m_{s_i}, a membership function $\mathscr{F}_{s_i}$ could be obtained as follows,

$$\mathscr{F}_{s_i}(s_j) = \sum_{s_i \in Z} m_{s_i}(Z) \tag{9}$$

This equation can be rewritten as the following one,

$$r(s_j, s_i) = \sum_{Z \in \lambda(s_j)} m_{s_i}(Z) \tag{10}$$

The fuzzy relation R on S can be generalized to the fuzzy relation R on LE.

Definition 10. [38] Let $\theta, \phi \in LE$ be any two linguistic expressions, the degree similarity between θ and ϕ is defined recursively as follows,

1. r(θ, s_i)=$\sum_{Z\in\lambda(\theta)} m_{s_i}(Z)$, if $\phi = s_i$,
2. r(θ, ϕ)=$\sum_{Z\in\lambda(\theta)} m_{\phi}(Z)$, being the mass assignment function m_ϕ obtained from the membership function $\mathscr{F}_\phi(\cdot) = r(\phi, \cdot)$ on S.

Some properties of this computational linguistic model are defined in [38] to simplify the inference process for the fuzzy relation R on linguistic expressions.

3.2.3. Analysis of fuzzy relation based expressions

The linguistic expressions provided by this approach are more elaborated and flexible than previous one (Section 3.1), but their formalization is still far from common language used by decision makers in decision making, unless for mathematician experts that are familiar with logic expressions. Therefore, it can be very useful in some decision problems in which logic expressions are close to the decision makers and the solving process.

3.3. A fuzzy-set approach to treat determinacy and consistency of linguistic terms

As it has been previously mentioned Ma et al. [20] also pointed out that the use of predefined linguistic terms facilitates the elicitation of linguistic information, but it limits to decision makers to express their preferences freely, because they have to select one linguistic term from the predefined linguistic term set, that might not matching with his/her opinion, and he/she might think in several linguistic terms at the same time. Consequently, a new approach that increases the flexibility of the linguistic expressions allowing to use more than one linguistic term was proposed.

3.3.1. Representation model

This idea consists of decision makers provide their preferences on all the alternatives by using 0 or 1 for each linguistic term. Table 1 shows a general representation of such a model, where $X = \{x_1, \ldots, x_n\}$ is the set of alternatives, $s_i \in S = \{s_0, \ldots, s_g\}$ is the linguistic term set and $e_k \in E = \{e_1, \ldots, e_m\}$ is the set of decision makers. Therefore, $v_{k,i}(x_r) = 1$ means that the decision maker e_k assigns the corresponding linguistic term $s_i \in S$ to the alternative $x_r \in X$, and 0 in otherwise. The selected linguistic terms are then used to generate *synthesized comments*.

Example 4. By using the linguistic term set depicted in Figure 1, a decision maker might provide the synthesized

Table 1: Synthesized comments.

	s_0	s_1	$\cdots$	s_g	synthesized comment
x_1	$v_{k,1}(x_1)$	$v_{k,2}(x_1)$	$\cdots$	$v_{k,g}(x_1)$	$c_{k,1}$
x_2	$v_{k,1}(x_2)$	$v_{k,2}(x_2)$	$\cdots$	$v_{k,g}(x_2)$	$c_{k,2}$
$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$
x_n	$v_{k,1}(x_n)$	$v_{k,2}(x_n)$	$\cdots$	$v_{k,g}(x_n)$	$c_{k,n}$

Table 2: Synthesized comments.

nothing	very bad	bad	medium	good	very good	perfect	Comment
0	0	1	1	0	0	0	*Commonly*
0	0	0	0	0	1	1	*Excellent*

comments shown in Table 2.

3.3.2. *Computational model*

The computational model of this linguistic approach is based on a fuzzy model and two novel concepts namely *determinacy* and *consistency*.

The concept of *determinacy* indicates the understandable degree that the decision maker has on the linguistic terms. For instance, if a decision maker provides his/her preference using only one linguistic term, it means that he/she is sure about the usage of the linguistic terms. However, if the decision maker uses more than one linguistic term, it is because of he/she cannot select one from the set. Formally, it is defined as follows.

Definition 11. [20] The determinacy of a linguistic term $s_i \in S$ presented by a decision maker $k \in E$ is,

$$Det_k(s_i) = 1 - \left(\int_U \mathscr{F}_{s_i} dU\right) \Big/ \int_U dU, \qquad (11)$$

where $\int_U \mathscr{F}_{s_i} dU$ is the fuzzy integral of $\mathscr{F}_{s_i}$ on U.

The *consistency* is related to the rationality of the preferences provided by the decision makers. The linguistic terms obtained by the decision maker should be consistent, otherwise the final result might lead to wrong conclusions in the decision making problem.

Definition 12. [20] Let S be a set of linguistic terms and $\mathscr{F}_{s_i}$, $i = \{0,\ldots,g\}$ be the corresponding fuzzy sets of s_i, the consistency of S is,

$$Con_k(S) = \bigvee\{\alpha : \bigcap_{i=0}^{g} (\mathscr{F}_{s_i})_\alpha \neq \emptyset\}, \qquad (12)$$

being $(\mathscr{F}_{s_i})_\alpha$ the α-cut of $\mathscr{F}_{s_i}$, $i = \{0,\ldots,g\}$.

In order to represent the synthesized comments Ma et al. proposed a strategy similar to the voting strategy in data fusion [46] which uses the definitions of *consistency* and *determinacy*.

Definition 13. Let x_r an alternative, e_k a decision maker, and S the linguistic term set that the decision maker uses to provide his/her opinions, the synthesized comment is,

$$Com_k(x_r) = \{(s_i, Dsync_k(s_i)) : s_i \in S_k^*,$$
$$Dsync_k(s_i) = Det_k(s_i) * Det_k(S^*) * Con_k(S^*)\}$$

where $S^* \subseteq S$ and $S^* = \{s_i \in S : v_{k,i}(x_r) = 1\}$.

The set of synthesized comments $\{Com_k(x_r) : k = 1,\ldots,m\}$ of all decision makers can be aggregated by using any aggregation operator defined in [14,13,46].

3.3.3. *Analysis of expressions based on synthesized comments*

This model is initially quite flexible and suitable to achieve the aim of modelling rich and flexible expressions for eliciting complex linguistic preferences because it allows to build expressions close to natural language used by experts in decision making. However, there is not any formal process or rule defined to fix the syntax of the synthesized comments obtained from multiple linguistic terms that makes this model hard to use in different decision situations with different decision makers chasing comparable results.

3.4. Linguistic distribution

Keeping in mind the proportional 2-tuple linguistic model presented by Wang and Hao [42], Dong et al. developed a generalization of such a model introducing the concept of distribution assessment [6].

3.4.1. Representation model

The representation of this model consists of assigning symbolic proportions to all the terms of the linguistic term set. To do so, the definition of distribution assessment is proposed.

Definition 14. [49] Let $S = \{s_0, \ldots, s_g\}$ be a linguistic term set, a distribution assessment, m, of S is defined as follows, $m = \{(s_i, \beta_i) | i = \{0, \ldots, g\}\}$ where $s_i \in S$, $\beta_i \geqslant 0$, $\sum_{i=0}^{g} \beta_i = 1$ and β_i is the symbolic proportion of s_i.

An example of the representation of this model is the following one.

Example 5. Let's suppose that 10 students has to evaluate to their teacher by using the linguistic term set S, depicted in Figure 1, two of them provide *very good*, five provide *good* and the remaining ones say *bad*. The evaluation could be defined using the following distribution assessment, $\{(nothing, 0), (very\ bad, 0), (bad, 0.3), (medium, 0),$ $(good, 0.5), (very\ good, 0.2), (perfect, 0)\}$

3.4.2. Computational model

A computational model was also proposed to carry out operations with distribution assessments.

1. *A comparison law*

   To compare two distribution assessments, it was necessary to introduce the definition of *Expectation*.

   Definition 15. [49] Let $m = \{(s_i, \beta_i\}, i = \{0, \ldots, g\}$ where $s_i \in S$, $\beta_i \geqslant 0$, $\sum_{i=0}^{g} \beta_i = 1$, be a distribution assessment of S. The expectation of m is,

$$E(m) = \sum_{i=0}^{g} \beta_i s_i \quad (13)$$

   Let m_1 and m_2 be two distribution assessments of S, then,

   - If $E(m_1) < E(m_2)$, then m_1 is smaller than m_2
   - If $E(m_1) = E(m_2)$, then m_1 and m_2 have the same expectation.

2. *A negation operator*

$$Neg(\{s_i, \beta_i\}, s_i \in S) = \{(s_i, \beta_{-i}, i = \{0, \ldots, g\}\} \quad (14)$$

3. *Aggregation operators of distribution assessments*

   Several aggregation operators to aggregate this type of information were defined in [49].

Dong et al. also studied some consistency measures, such as additive and multiplicative consistency for a distribution linguistic preference relation [49], and they proposed a consensus model which identifies those distribution linguistic preference relations that less contribute to achieve the consensus level and modifies them until the consensus level is reached.

3.4.3. Analysis of expressions based on linguistic distributions

The linguistic distributions allow to keep linguistic information in a broad sense taking into account more than a single term in a similar but more complete way than the proportional 2-tuple (see Section 3.1). Hence its interpretability of the linguistic information is still far from common language used by decision makers in decision making problems despite it can be useful in managing computational processes for keeping as much information as possible.

3.5. Complex Linguistic Expressions based on Hesitant Fuzzy Linguistic Term Sets

The linguistic computing models revised previously try to use linguistic expressions richer than single linguistic terms, but some of them provide linguistic expressions far from the common language used by human beings in decision making problems or they do not explain how the linguistic expressions are built formally. Another linguistic model was proposed in [33] to construct complex linguistic expressions, based on the use of Hesitant Fuzzy Linguistic Term Sets (HFLTS) [33] that models decision maker's hesitancy when elicits linguistic preferences. Such complex linguistic expressions not only achieve the improvements pointed out by Ma et al. [20], but also provide decision makers greater flexibility to express their preferences by means of context-free grammars that fix the rules to generate comparative linguistic expressions similar to the natural language used by decision makers in decision making problems.

3.5.1. Representation model

The following context-free grammar G_H, generates comparative linguistic expressions suitable to provide preferences in decision making problems.

Definition 16. [34] Let G_H be a context-free grammar and $S = \{s_0, \ldots, s_g\}$ a linguistic term set. The elements of $G_H = (V_N, V_T, I, P)$ are defined as follows:

$V_N = \{\langle primary\ term\rangle, \langle composite\ term\rangle, \langle unary\ relation\rangle, \langle binary\ relation\rangle, \langle conjunction\rangle\}$

$V_T = \{lower\ than, greater\ than, at\ least, at\ most, between, and, s_0, s_1, \ldots, s_g\}$

$I \in V_N$

$P = \{I ::= \langle primary\ term\rangle | \langle composite\ term\rangle$

$\langle composite\ term\rangle ::= \langle unary\ relation\rangle\langle primary\ term\rangle | \langle binary\ relation\rangle\langle primary\ term\rangle\langle conjunction\rangle \langle primary\ term\rangle$

$\langle primary\ term\rangle ::= s_0|s_1|\ldots|s_g$

$\langle unary\ relation\rangle ::= lower\ than|greater\ than|at\ least|at\ most$

$\langle binary\ relation\rangle ::= between$

$\langle conjunction\rangle ::= and\}$

The comparative linguistic expressions generated by G_H cannot be straightforwardly used to make computations, therefore, they are transformed into HFLTS by means of a transformation function, E_{G_H}.

Definition 17. [33] Let $S = \{s_0, \ldots, s_g\}$ be a linguistic term set, a HFLTS, H_S, is defined as an ordered finite subset of consecutive linguistic terms of S,
$H_S = \{s_i, s_{i+1}, \ldots, s_j\}$ such that, $s_k \in S$, $k \in \{i, \ldots, j\}$

The transformation function E_{G_H}, was defined as follows.

Definition 18. [33] Let E_{G_H} be a function that transforms comparative linguistic expressions, ll, obtained by G_H, into HFLTS, H_S, where S is the linguistic term set used by G_H and S_{ll} is the set of linguistic expressions generated by G_H,

$$E_{G_H} : S_{ll} \longrightarrow H_S$$

The comparative linguistic expressions generated through the context-free grammar G_H, are transformed into HFLTS by using the following transformations:

- $E_{G_H}(s_i) = \{s_i | s_i \in S\}$
- $E_{G_H}(\text{at most } s_i) = \{s_j | s_j \in S\ and\ s_j \leqslant s_i\}$
- $E_{G_H}(\text{lower than } s_i) = \{s_j | s_j \in S\ and\ s_j < s_i\}$
- $E_{G_H}(\text{at least } s_i) = \{s_j | s_j \in S\ and\ s_j \geqslant s_i\}$
- $E_{G_H}(\text{greater than } s_i) = \{s_j | s_j \in S\ and\ s_j > s_i\}$
- $E_{G_H}(\text{between } s_i \text{ and } s_j) = \{s_k | s_k \in S\ and\ s_i \leqslant s_k \leqslant s_j\}$

Example 6. By using the context-free grammar G_H, and the linguistic term set shown in Figure 1, some comparative linguistic expressions might be,
$ll_1 = between\ medium\ and\ very\ good$
$ll_2 = at\ least\ bad$

The transformation of these comparative linguistic expressions into HFLTS are,
$E_{G_H}(between\ medium\ and\ very\ good) = \{medium, good, very\ good\}$
$E_{G_H}(at\ least\ bad) = \{nothing, very\ bad, bad\}$

3.5.2. Computational model

Different computation models can be used to operate with HFLTS depending on its representation, such as an envelope that is an interval value [33] or the fuzzy envelope [19]. Due to the interest in fuzzy based representations of this paper the fuzzy envelope is revised:

Definition 19. [19] Let $H_S = \{s_i, s_{i+1}, \ldots, s_j\}$ be a HFLTS, so that $s_k \in S = \{s_0, \ldots, s_g\}$, $k \in \{i, \ldots, j\}$.

$$env_F(H_S) = T(a, b, c, d), \tag{15}$$

where $T(\cdot)$ is a trapezoidal or triangular fuzzy membership function (see [19] for further details).

The concept of fuzzy envelope $env_F(H_S)$ of an HFLTS facilitates the CW processes with HFLTS [19] because it represents the comparative linguistic expressions by means of a fuzzy membership function obtained of aggregating the linguistic terms that compound the HFLTS and hence the computations can be carried out by the Extension Principle [25] (see Section 2.3.1).

3.5.3. Extension of Hesitant Fuzzy Linguistic Term Sets

Recently, the concept of HFLTS has been extended to use non-consecutive linguistic terms [40]. This generalization is called Extended Hesitant Fuzzy Linguistic Term Set (EHFLTS) and it is defined as follows.

Definition 20. [40] Let S be a linguistic term set, a EHFLTS is an ordered subset of linguistic terms of S, such that,

$$EH_S = \{s_i | s_i \in S\}.$$

This extension was proposed to fuse the preferences provided by different decision makers by using the union operation. The idea consists of combining the HFLTS provided for each decision maker to obtain a EHFLTS that represent the collective preference of the group. Several aggregation operators for EHFLTS have been defined in [40].

Note that this model deals with multiple linguistic terms, but does not provide linguistic expressions similar to the common language.

3.5.4. Analysis of complex linguistic expressions based on HFLTS

It is clear that the comparative linguistic expressions generated by G_H and represented by HFLTS provide an important flexibility to decision makers when eliciting preferences, together a clear formalization of the way of generating expressions that could be close to the expressions used by human beings in decision making depending on the grammar used for such a generation.

4. Challenges and Future in Modelling Complex Linguistic Preferences

The management of uncertain and vague information is always hard and complex, therefore the modelling of information in such an environment presents important difficulties that the fuzzy linguistic modelling has tackled successfully in many decision situations. However, it is clear that the use of simple fuzzy linguistic preferences composed by a single term is not always suitable to represent the real preferences of the decision makers.

Across this paper it has been shown different proposals to model linguistic preferences by means of more elaborated expressions than a single linguistic term. It is easy to observe that each different proposal treats the preference modelling from very different perspectives, all of them quite interesting in specific decision situations. However, despite the different linguistic modelling proposals for complex linguistic preferences introduced in the specialized literature, it seems necessary a further research looking for some aspects that have not been considered yet:

- Some proposals are very flexible to construct linguistic expressions such as in Sections 3.3 and 3.4, but there is not formal processes to build expressions either are far from common language. However, other proposals as comparative linguistic expressions (Section 3.5) are well formalized by means of context-free grammars, but are not so rich as previous ones. Hence, it is important to keep working on proposals able to keep features of the latter and increase its flexibility as the former. Maybe a way to do that, it will be the use of richer grammars than context-free grammars [4,15].
- So far, most of problems dealing with uncertain information have applied a determined technique to model and manage such a uncertainty. However, it is clear that in real world problems the use of only one technique is not realistic, because of multiple perspectives in which a problem can be solved, hence further research on the use of multiple linguistic modelling proposals to model complex linguistic preferences could suit better different real-world decision problems. Therefore, another important challenge to deal with complex linguistic information in uncertain decision making problems, is the development of hybrid modelling and computing proposals to improve the results, such hybridization could include the interoperability among different types of expressions and their computational models.
- Across this overview the proposals revised aim at providing richer and more flexible syntax to decision makers, for eliciting their knowledge, based on a fuzzy semantics. All of them provide a unique meaning for the complex expressions elaborated with each approach, however in CW literature it has been thoroughly discussed that *words means different things for different people* [25,29,36] because of different reasons. Therefore, the current approaches for eliciting linguistic complex expressions should consider this fact and provide mechanisms for representing and managing those dif-

ferent meanings for the linguistic expressions in the problems. Maybe this challenge can be initially tackled by integrating the view of multi-granular linguistic scales and later on by researching on the use of type-2 fuzzy sets. Other approaches and ideas can enrich previous ideas for this challenge.

Even though, there would be other challenges to point out, the previous ones could be the most interesting ones from a decision making and decision analysis point of view.

5. Conclusions

The need to model linguistically preferences in complex decision problems has led to many ways of linguistic modelling and computational approaches in which fuzzy based approaches play a key role. However, most of these approaches provide a priori fixed vocabularies that decision makers are forced to use for eliciting their preferences and usually in a very simple way. To overcome this drawback the ability to generate flexible and complex linguistic expressions to elicit preferences has been recently researched. An overview of the most important fuzzy proposals to deal with this type of preferences has been provided in this paper and pointed out the different points of view used in each proposal to model these complex preferences. Eventually some challenges have been introduced for further research.

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4.2. Computación con Expresiones Lingüísticas Comparativas y Translación Simbólica en la Toma de Decisión: Información ELICIT

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Computing With Comparative Linguistic Expressions and Symbolic Translation for Decision Making: ELICIT Information

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***Abstract*—Many real-world decision making (DM) problems present changing contexts in which uncertainty or vagueness appear. Such uncertainty has been often modeled based on the linguistic information by using single linguistic terms. Dealing with linguistic information in DM demands processes of computing with words whose main characteristic is to emulate human beings reasoning processes to obtain linguistic outputs from linguistic inputs. However, often single linguistic terms are limited or do not express properly the expert's knowledge, being necessary to elaborate richer linguistic expressions easy to understand and able to express greater amount of knowledge, as it is the case of the comparative linguistic expressions based on hesitant fuzzy linguistic terms sets. Nevertheless, current computational models for comparative linguistic expressions present limitations both from understandability and precision points of view. The 2-tuple linguistic representation model stands out in these aspects because of its accuracy and interpretability dealing with linguistic terms, both related to the use of the symbolic translation, although 2-tuple linguistic values are still limited by the use of single linguistic terms. Therefore, the aim of this article is to present a new fuzzy linguistic representation model for comparative linguistic expressions that takes advantage of the goodness of the 2-tuple linguistic representation model and improve the interpretability and accuracy of the results in computing with words processes, resulting the so-called extended comparative linguistic expressions with symbolic translation. Taking into account the proposed model, a new computing with words approach is presented and then applied to a DM case study to show its performance and advantages in a real case by comparing with other linguistic decision approaches.**

***Index Terms*—Computing with words, comparative linguistic expressions, decision making (DM), hesitant fuzzy linguistic term set, symbolic translation.**

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I. Introduction

NOWADAYS, human beings need to deal with problems characterized by multiple alternatives or options and decide which one is the best one/s as solution of the problem, this process is known as *decision making* (DM). DM problems may take place in changing environments, in which the uncertainty and vagueness are common, i.e., DM under uncertainty [1]–[3]. In such conditions, the use of linguistic information based on the fuzzy linguistic approach [4] has given successful results leading to *linguistic decision making* (LDM) [5]. Nevertheless, the use of linguistic information implies to accomplish computations with it. Computing with Words (CW) [6]–[9] is one of the most used methodologies, which carries out on processes where words (in a natural or artificial language) and not numbers are used for computing, emulating in this way, human beings cognitive processes [7], [10]–[13]. CW processes obtain linguistic outcomes from linguistic inputs, obtaining results, which are easily understandable and properly represented.

Consequently, several linguistic computational models have been developed to accomplish such linguistic computations within CW, that can be classified into two groups: 1) *linguistic computational model based on membership functions models* (semantic models) [2], [14]–[16] and 2) *linguistic symbolic computational model based on ordinal scales* (symbolic models) [5], [17]–[19]. However, the symbolic models stand out because their simplicity and high interpretability. Among them, *the 2-tuple linguistic computational model* [11], [20] is a symbolic model that extends the use of indexes, modifying the fuzzy linguistic approach representation by adding a parameter, so-called *symbolic translation*, to the basic linguistic representation in order to improve the accuracy of the linguistic computations, keeping the CW scheme and the interpretability of the results.

Nonetheless, the elicitation of linguistic information by either linguistic terms or 2-tuple linguistic values are still limited because such information must be always expressed by single linguistic terms, defined *a priori*. This limitation is especially important in LDM problems, since experts face uncertain decision situations in which they might hesitate among several linguistic terms and would prefer to employ more complex linguistic expressions to elicit their own knowledge. In order to overcome such a limitation, several proposals aimed at improving the elicitation of the linguistic information [21], but the expressions generated by those proposals were not close to human beings cognitive process or they did not provide a formalization method explaining how to obtain the linguistic expressions. On the other hand, Rodríguez *et al.* [22] proposed a linguistic model to construct linguistic expressions based on the hesitant fuzzy linguistic term sets (HFLTS). Later on, Rodríguez *et al.* [23] proposed the use of HFLTS and context-free grammars to build in a formal way *comparative linguistic expressions* (CLEs) that are more flexible and richer than single linguistic terms and also closer to the language used by human beings in real-world decision problems.

Many DM models have used CLEs and their corresponding HFLTS transformation to model and compute experts' information. Some of them have used a symbolic approach, in which the CLEs are transformed into linguistic terms intervals, losing information during the process [22], [24]–[27]. Others have utilized a symbolic approach in which the *fuzzy envelope* [28] of each CLE is computed [28]–[31]. Following with fuzzy envelope, a collection of proposals including *trapezoidal fuzzy numbers* [32], *discrete fuzzy numbers* [33], *possibility distribution* [34], [35], *proportional hesitant fuzzy linguistic term set* [36], and *probabilistic linguistic term set* [37] have provided successful results to carry out computations with HFLTS. However, although the latter models keep a fuzzy representation or have improved the precision in the results, the interpretability remains their weakness, since the representation of the results is difficult to understand because is far from human beings cognitive process.

CLEs are closer to the way of thinking of human beings but the results obtained in CW with them are still limited both from the point of view of interpretability and precision due to its discrete expression domain. As it was aforementioned, symbolic translation solves this problem for single linguistic terms. Therefore, the combination of CLEs and symbolic translation might lead to an improved CW processes for CLEs. Several proposals that combine both concepts have been introduced in the specialized literature. Some approaches have considered the use of HFLTSs and 2-tuple linguistic model independently [38]. Others proposals define CLEs based on 2-tuple linguistic term set to express experts' hesitancy [39], [40], while others are based on 2-tuple representation whose symbolic translation is formed by several values that represent the experts' hesitancy [41]. However, the abovementioned proposals present limitations and/or drawbacks from CW point of view (see Section II-D).

Taking into account previous drawbacks for computing with CLEs and keeping in mind the CW methodology, this article proposes a new fuzzy linguistic representation for CLEs together with a linguistic computational model that will keep understandability of results and improve their precision. Such a new linguistic model extends the CLEs by using the concept of symbolic translation introduced by the 2-tuple linguistic model [17] resulting the so-called *Extended Comparative LInguistiC Expressions with SymbolIc Translation* (ELICIT) information. These expressions extend the representation of CLEs generated by a context-free grammar into a continuous domain to perform CW processes without any kind of approximation. For sake of clarity, the main novelties of this article are enumerated as follows.

1) A new linguistic model, so-called ELICIT, which represents linguistic information through the generation of ELICIT information, an extension of CLEs in a continuous domain by using the symbolic translation concept related to the 2-tuple linguistic model.
2) A CW approach based on the ELICIT, which takes advantage of the main characteristics of the ELICIT information, interpretability, and precision.
3) A linguistic computational model for ELICIT information composed by several basic operations such as negation and aggregation together with a fuzzy comparison operator.

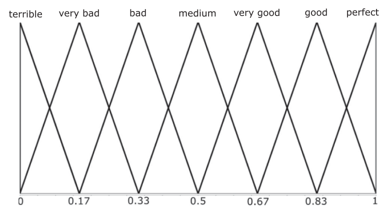

Fig. 1. Seven-term with its semantic.

The new representation and its computational model based on fuzzy arithmetic will be applied in DM research area, in which linguistic expressions have been widely used [5], [11], [42], [43]. Eventually, a real-world case study will be solved by using ELICIT information and a comparison with other previous models will show its advantages.

The rest of article is organized as follows. Section II reviews basic concepts about the DM, CW, and CLEs. Furthermore, a short literature review with the main related approaches to the proposed method is carried out. Section III introduces the new linguistic representation model along with the generation of ELICIT expressions through of a context-free grammar also defined. Furthermore, a new linguistic computational model based on such expressions is also described. Section IV shows the performance of the CW approach based on the ELICIT information in a real-world LDM problem and a comparison with previous models. Finally, Section V concludes this article.

II. Preliminaries

This section reviews some concepts related to fuzzy linguistic approach, DM, CW, and CLEs in order to understand easily this contribution. Moreover, several proposals conceptually related to the proposed model are also revised, to show the originality, novelty of the proposal and for further comparisons in real-world DM problems.

A. Fuzzy Linguistic Approach

Most of real-world problems present incomplete information, vagueness, and uncertainty that often cannot be modeled by probabilistic models. Under these conditions, the use of linguistic information has obtained successful results [44] in different fields [10], [45]–[47]. Several approaches have been presented in the literature to model linguistic information [4], [12], [48]–[51], among them, the *fuzzy linguistic approach* uses fuzzy set theory [52] to manage uncertainty and model linguistic information by using linguistic variables described by Zadeh [4] as *"A variable whose values are not numbers but words or sentences in a natural or artificial language"*. A linguistic variable is characterized by a syntactic value or *label* and a semantic value (see Fig. 1). Whereas the label is a word that belongs to a set of linguistic terms, semantics is provided by a fuzzy set in a discourse universe. Due to words are less precise than numbers, the concept of linguistic variable seems suitable to model complex and uncertain information.

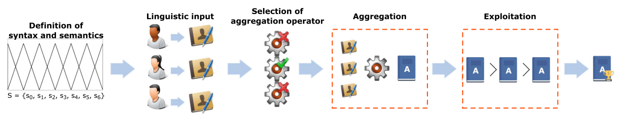

Fig. 2. Linguistic DM scheme.

Fig. 3. CW scheme.

Remark 1: Note that Appendix A related to fuzzy concepts has been introduced for a better understanding of the proposal.

The fuzzy linguistic approach provides the basis to model information linguistically by using a fuzzy representation. Furthermore, one of the main contributions of this approach, is a methodology for CW that merges natural languages and computations with fuzzy variables. Although the fuzzy linguistic approach and the CW methodology have been applied in several fields [10], [45]–[47], this contribution focuses on the DM research area (see Section II-B) due to their convenience and suitability to deal with DM problems under uncertainty [20].

B. Linguistic DM Under Uncertainty

DM is a quotidian process [53]–[58] characterized by a set of alternatives and the need to decide, which one/s is/are the best. In order to solve DM problems, experts provide their knowledge about the set of such alternatives and make decisions by means of reasoning processes [59]–[61]. Formally, a DM problem is characterized by a set of experts $E = \{e_1, \ldots, e_k\}$ who express their assessments over a sequence of alternatives or options $X = \{x_1, \ldots, x_n\}$, defined by a finite set of criteria $C = \{c_1, \ldots, c_m\}$. Due to the fact that, most of the real-world DM problems involve incomplete information, vagueness, and uncertainty that often cannot be modeled by probabilistic models, classical decision theory models cannot be applied. Therefore, under these conditions, the use of linguistic information and its modeling using linguistic variables [4] has obtained successful results. The use of linguistic variables to elicit knowledge and preferences about either alternatives or criteria is often used by decision makers involved in DM problems, raising the concept on LDM. The phases that compose the LDM scheme [5] are *definition of syntax and semantics*, *selection of an aggregation operator*, *aggregation,* and *exploitation*. All of them are graphically shown in Fig. 2.

The linguistic resolution scheme shows the necessity of operating with linguistic information to find a solution in LDM problems. The CW methodology mimics the human beings reasoning, that is, compute and reason by means of words, obtaining linguistic results from linguistic premises. Zadeh [7] defined CW as *"A methodology in which words are used in place of numbers for computing, reasoning, and DM"*.

In recent years, CW methodology has been intensively applied in DM [11], [46], [62] and, thus, multiple CW schemes have been proposed in the literature [13], [63], [64]. These schemes emphasize the need of obtaining accurate linguistic results easy to compute and understand. Fig. 3 shows the CW scheme introduced by Yager in [13], [64], in which the importance of the processes of *translation* and *retranslation* in CW was pointed out. The former translates the linguistic inputs into a machine-manipulate format based on fuzzy tools in which the computations are carried out. The latter consists of converting the computing results into linguistic information again to facilitate the human comprehension.

There are several computational models developed to perform linguistic computations [2], [46] based on the fuzzy linguistic approach [4], [50], [51]. However, the 2-tuple linguistic model proposed by Herrera and Martínez [17] is notable for its interpretability and accuracy [3].

Remark 2: Note that, due to the key role of the 2-tuple linguistic model in our contribution, concepts related to such a linguistic model have been included in Appendix B.

C. Hesitant Fuzzy Linguistic Term Sets and Comparative Linguistic Expressions

In spite of the fuzzy linguistic approach has been successfully applied in DM, the modeling of linguistic information is limited when experts provide their preferences by using just single terms. Experts might face situations in which they hesitate among several linguistic terms at the same time. Therefore, to overcome such a limitation, the concept HFLTS [22] was introduced. HFLTSs are based on the fuzzy linguistic approach that will serve as bases to increase the flexibility of the elicitation of linguistic information.

HFLTS can be directly used by the experts to elicit several linguistic values for a linguistic variable, but they are not close to the expressions used by the human beings. Therefore, it is necessary to define linguistic sentences closer to human beings expressions. Rodríguez *et al.* [21] reviewed the broadest linguistic approaches for modeling complex linguistic preferences. Although there are several proposals that obtain richer linguistic expressions than single linguistic terms, the one presented by Rodríguez *et al.* [22], [23] stands out because it provides a formalization process to generate linguistic expressions close to the common language used by human beings in DM problems. Such expressions, so-called CLEs are based on HFLTSs, that model the decision maker's hesitancy.

CLEs are built by using context-free grammars G_H. In [23] a basic context-free grammar for generating CLEs for eliciting DM preferences was introduced.

Definition 1. (see [23]): Let G_H be a context-free grammar and $S = \{s_0, \ldots, s_g\}$ a linguistic term set. The elements of $G_H = (V_N, V_T, I, P)$ are defined as follows.

$$V_N = \{\langle \text{primary term} \rangle, \langle \text{composite term} \rangle,$$
$$\langle \text{unary relation} \rangle, \langle \text{binary relation} \rangle, \langle \text{conjunction} \rangle\}$$
$$V_T = \{\text{at least, at most, between, and, } s_0, \ s_1, \ldots, s_g\}$$
$$I \in V_N.$$

The production rules defined in an extended Backus–Naur form are

$$P = \{I \ ::= \ \langle \text{primary term} \rangle | \langle \text{composite term} \rangle$$
$$\langle \text{composite term} \rangle \ ::= \ \langle \text{unary relation} \rangle \langle \text{primary term} \rangle |$$
$$\langle \text{binary relation} \rangle \langle \text{primary term} \rangle \langle \text{conjunction} \rangle$$
$$\langle \text{primary term} \rangle$$
$$\langle \text{primary term} \rangle \ ::= \ s_0 | s_1 | \ldots | s_g$$
$$\langle \text{unary relation} \rangle \ ::= \ \text{at least} | \text{at most}$$
$$\langle \text{binary relation} \rangle \ ::= \ \text{between}$$
$$\langle \text{conjunction} \rangle \ ::= \ \text{and}\}.$$

Similar to the simple linguistic terms, the use of CLEs in DM implies processes of CW. To accomplish the computations on linguistic expressions, a transformation function E_G was defined to transform such expressions into HFLTSs.

Definition 2. (see [23]): Let E_{G_H} be a function that transforms CLEs, $ll \in S_{ll}$, obtained by G_H, into HFLTSs, H_S. S is the linguistic term set used by G_H and S_{ll} is the expression domain generated by G_H

$$E_{G_H} : S_{ll} \rightarrow H_S. \tag{1}$$

The CLEs generated by the context-free grammar G_H introduced in Definition 1 are transformed into HFLTSs H_S by means of the following transformations:

$$E_{G_H}(s_i) = \{s_i | s_i \in S\}$$
$$E_{G_H}(\text{ at most } s_i) = \{s_j | s_j \leq s_i \text{ and } s_j \in S\}$$
$$E_{G_H}(\text{ at least } s_i) = \{s_j | s_j \geq s_i \text{ and } s_j \in S\}$$
$$E_{G_H}(\text{ between } s_i \text{ and } s_j) = \{s_k | s_i \leq s_k \leq s_j \text{ and } s_k \in S\}.$$

Once the CLEs are transformed into a HFLTS, different computational models have been proposed [28], [31], [38], mainly based on the fusion of HFLTS by means of an *envelope* that can be obtained in different ways [22], [28]. Nevertheless, the proposal presented by Liu and Rodriguez [28] stands out. This proposal consists of a *fuzzy envelope* for HFLTSs, which represents the semantics of the CLEs by fuzzy membership functions obtained by aggregating the linguistic terms, which belong to the HFLTS, keeping in mind that the CW processes translate the linguistic input into a format based on fuzzy arithmetic to carry out the computations.

Definition 3. (see [28]): The *fuzzy envelope* $\text{env}_F(H_S)$ is defined as a trapezoidal fuzzy membership function as follows:

$$\text{env}_F(H_S) = T(a, b, c, d) \tag{2}$$

where H_S is a HFLTS and $T(a, b, c, d)$ is a fuzzy membership function (see [28] for further detail).

HFLTS fuzzy envelope is obtained by aggregating different fuzzy memberships functions with the OWA operator [65]. One of the most relevant characteristic of such operator is the possibility to set distinct importance to the linguistic terms that compose the HFLTS by means of weights assignment. Such importance will depend on the *optimism degree* related to the weights, which can be measured by the *orness measure*. This measure plays a key role in our contribution, since, thanks to it, we will able to compute fuzzy envelopes, which preserve as much information as possible.

Remark 3: Note that Appendix C related to the orness measure and its influence in our proposal has been included for a better understanding of it.

The CLEs are close to the linguistic structures used by human beings for eliciting preferences, specially in the real-world DM contexts, and improve preferences elicitation regarding single linguistic terms, but still the results obtained in CW with CLEs need improve precision and interpretability because of its current discrete expressions domain. Look at the LDM literature [20], the use of the symbolic translation solved an analogous problem with single linguistic terms. Consequently, it seems reasonable and promising to combine the CLEs and the symbolic translation to improve the CW processes with CLEs. Different attempts have been presented in the literature [38]–[41] that are briefly revised in the following section in which their drawbacks are pointed out in order to clarify the novelty advantages and need of our proposal.

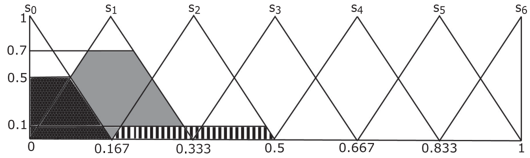

Fig. 4. Tan *et al.* proposal visualization.

D. Related Works

This section reviews several proposals that have combined to some extent the concepts of HFLTSs and the 2-tuple linguistic model to develop new representation models. We analyze such approaches taking into account the way of combining the CLEs and the 2-tuple linguistic model, also from the CW scheme point of view, pointing out their limitations and/or drawbacks.

1) Proposals that consider the use of the HFLTSs and the 2-tuple linguistic model independently.
   a) Zulueta *et al.* [38] presented an *environmental impact significance assessment* approach that allows to manage heterogeneous information, that is, the input information can be provided by the experts through CLEs, crisp numbers, interval-valued or hesitant fuzzy sets, and to carry out the computations with these different types of information, such information is unified into 2-tuple linguistic values. The results are represented by 2-tuple linguistic values. This approach aims at combining information but does not consider the CLEs as output; hence, it loses information in the unification process as well as reduces the expressiveness of assessments to just one term, although the output is linguistic and easy to understand.
2) Proposals define CLEs based on 2-tuple linguistic term sets to express decision makers' hesitancy.
   a) Tang *et al.* [39] introduced a linguistic approach that allows the construction of CLEs based on HFLTSs and the 2-tuple linguistic model, so-called *2-tuple hesitant linguistic fuzzy set* (2-TLFHS). The 2-TLFHS can be defined as follows.

      Definition 4. (see [39]): Let S ={(s_0, α_0), (s_1, α_1), …, (s_g, α_g)} be a 2-tuple linguistic term set. A 2-tuple linguistic hesitant fuzzy set (2-TLHFS), LH, in S can be expressed as follows:

      $$LH = \{((s_{\theta_i}, \alpha_{\theta_i}), lh(s_{\theta_i}, \alpha_{\theta_i}))|(s_{\theta_i}, \alpha_{\theta_i}) \in S\}$$

      where $lh((s_{\theta_i}, \alpha_{\theta_i})) = \{r_1, r_2, r_3, \ldots, r_m\}$ is a set with m_i values in [0,1], denoting the possible membership degrees of the elements $x \in X$ to the set LH.

      Example 1: Supposing that one decision maker evaluates the quality of a wine, by using $S = \{s_0 :$ disgusting, $s_1 :$ very bad, $s_2 :$ bad, $s_3 :$ normal, $s_4 :$ good, $s_5 :$ very good, $s_5 :$ excellent$\}$, and provides the value 0.5 for *disgusting*, the value 0.7 for very bad and the value 0.1 for bad, the 2-TLHFS would be $LH = ((s_0, 0), 0.5), ((s_1, 0), 0.7), ((s_2, 0), 0.1)$ (graphically represented in Fig. 4).

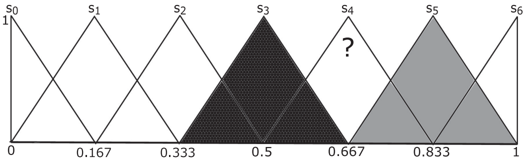

Fig. 5. Wei and Liao proposal visualization.

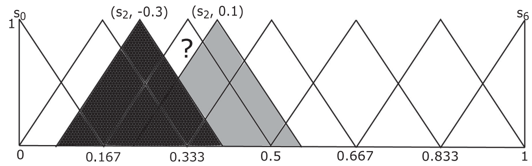

Fig. 6. Beg and Rashid proposal visualization.

These expressions model the decision maker's uncertainty by using hesitant fuzzy sets and the 2-tuple linguistic model. Nevertheless, the expressions present an important lack of interpretability, because they are not close to a human beings cognitive process, and it is really hard that decision makers provide their opinions by means of this type of expressions. Finally, although the computations are carried by using 2-TLHFS to avoid loss of information, the result is transformed into a 2-tuple linguistic value that implies lack of precision and reduces the expressiveness.

b) Wei and Liao [40] presented the concept of hesitant 2-tuple sets together with three kinds of operators to aggregate multigranularity hesitant fuzzy linguistic information without loss of information. The aim is to transform HFLTSs into hesitant 2-tuple sets in order to avoid loss of information. A hesitant 2-tuple set is defined as follows.

Definition 5. (see [40]): Let $S = \{s_0, s_1, \ldots, s_\tau\}$ be a linguistic term set, and (b_i, α_i) be a 2-tuple linguistic value on S, $i = 1, 2, \ldots, n$. If $(b_i, \alpha_i) < (b_j, \alpha_j)$ for any $i < j$, then is $\{(b_1, \alpha_1), (b_2, \alpha_2), \ldots, (b_l, \alpha_l)\}$, a hesitant 2-tuple set on S.

Example 2: $T^2_{S^7} = \{(s^7_3, 0), (s^7_5, 0)\}$ (graphically represented in Fig. 5).

This type of expressions are difficult to understand, because it consists of several 2-tuple linguistic values. Furthermore, there is no a definition that explains formally what happens with the information between two 2-tuple linguistic values whose linguistic terms are not consecutive, thus it is not clear whether such intermediate information is taken into account.

3) Other approaches are based on the 2-tuple representation whose symbolic translation is composed by several values that determine the decision makers' hesitancy.

a) Beg and Rashid [41] proposed a new linguistic representation model based on *hesitant 2-tuple linguistic information* defined as follows.

Definition 6. (see [41]): Let X be an universe of discourse and $S = \{s_1, \ldots, s_t\}$ be a linguistic term set, a hesitant linguistic term set in X is an expression A given by $A = \{(x, h(x)) | x \in X\}$ where $h(x) = (s_i, \beta_{ij}) \, \forall \, x \in X$.

This model represents the hesitant linguistic information by means of 2-tuple linguistic values, (s_i, β_{ij}), where s_i is a linguistic label and β_{ij} is a finite subset of $[-0.5, 0.5)$ that represents the possible symbolic translations of s_i. It is noted that the cardinality of β may be different for each x.

Example 3: $(\, s_2, \, (\, -0.3, \, 0.1 \,)\,)$ (graphically represented in Fig. 6).

As in previous proposals, these expressions are far from structures used by human beings for eliciting preferences. Therefore, they present an important lack of interpretability and it is very difficult for decision makers to express their opinions and preferences through these expressions, which generate more uncertainty, from a semantic point of view, because the multiple values to represent the symbolic translation. Moreover, the results are numeric values, which imply loss of information and interpretability in the output.

III. ELICIT: Comparative Linguistic Expressions With Symbolic Translation

The lack of representation and computational models for CLEs that provide accurate and/or interpretable linguistic results manifests the need of a new approach able to keep the interpretability and accuracy of the results in CW processes dealing with CLEs. Taking into account the previous premises, this section presents the ELICIT linguistic *model*, a new fuzzy linguistic representation model that extends the CLEs by using the concept of symbolic translation used by the 2-tuple linguistic model.

This section is organized as follows. Section III-A introduces the ELICIT linguistic representation model and the ELICIT expressions. Section III-B presents a new CW approach based on the ELICIT linguistic model. Finally, Section III-C presents a computational model to accomplish the processes of CW by using ELICIT information.

A. Representation Model

The ELICIT linguistic model represents the linguistic information through the generation of ELICIT information, an extension of CLEs generated by a context-free grammar into a continuous domain by using the symbolic translation. The ELICIT information takes advantage of the main feature of CLEs that is their interpretability and, when it is necessary, it replaces the linguistic terms of the expressions by 2-tuple linguistic terms. In this way, the CW processes are performed without any kind of approximation, providing accurate results easy to understand.

Definition 7: Let G_H be a context-free grammar and $S = \{s_0, \ldots, s_g\}$ a linguistic term set. The elements of $G_H = (V_N, V_T, I, P)$ are defined as follows:

$$V_N = \{\langle\text{continuous primary term}\rangle, \langle\text{composite term}\rangle$$
$$\langle\text{unary relation}\rangle, \langle\text{binary relation}\rangle, \langle\text{conjunction}\rangle\}$$
$$V_T = \{\text{at least, at most, between, and, } \boldsymbol{(s_0, \alpha)^\gamma}$$
$$\boldsymbol{(s_1, \alpha)^\gamma}, \ldots, \boldsymbol{(s_g, \alpha)^\gamma}\}$$
$$I \in V_N.$$

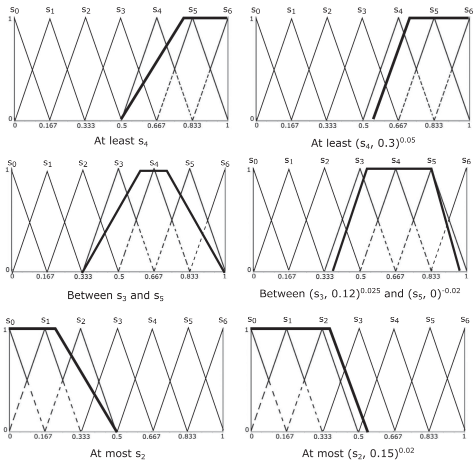

Fig. 7. Fuzzy comparison representation between CLEs and ELICIT.

The new production rules defined in an extended Backus–Naur form are

$$P = \{I \ ::= \ \langle\text{continuous primary term}\rangle|$$
$$\langle\text{composite term}\rangle$$
$$\langle\text{composite term}\rangle \ ::= \ \langle\text{unary relation}\rangle$$
$$\langle\text{continuous primary term}\rangle|$$
$$\langle\text{binary relation}\rangle\langle\text{continuous primary term}\rangle$$
$$\langle\text{conjunction}\rangle\langle\text{continuous primary term}\rangle$$
$$\langle\text{continuous primary term}\rangle \ \boldsymbol{::=} \ \boldsymbol{(s_0, \alpha)^\gamma}|$$
$$\boldsymbol{(s_1, \alpha)^\gamma|\ldots|(s_g, \alpha)^\gamma}$$
$$\langle\text{unary relation}\rangle \ ::= \ \text{at least}|\text{at most}$$
$$\langle\text{binary relation}\rangle \ ::= \ \text{between}$$
$$\langle\text{conjunction}\rangle \ ::= \ \text{and}\}.$$

Thus, the possible ELICIT expressions generated according to the new definition of the context-free grammar are: "at least $(s_i, \alpha)^\gamma$," "at most $(s_i, \alpha)^\gamma$," and "between $(s_i, \alpha_1)^{\gamma_1}$ and $(s_j, \alpha_2)^{\gamma_2}$" (Fig. 7 shows an example of their fuzzy representation). Note that an additional parameter so-called *adjustment* and noted as γ has been included in each continuous primary term in the expression. This parameter provides essential information to carry out accurate computations with the ELICIT information as is further detailed in Section III-B4.

B. CW Approach for ELICIT Information

ELICIT representation model provides a new and flexible way to model linguistic information but it is also necessary to obtain interpretable and precise results in a CW processes by using ELICIT information and overcoming the drawbacks of previous linguistic models pointed out in Section II-D. Consequently, the definition of a new CW approach that takes advantage of these new expressions is logical and necessary. Here, a new CW approach is introduced for ELICIT information (see Fig. 8) based on the fuzzy linguistic approach (see Section II-A) that carries out CW computations in a precise way and provides linguistic results represented by ELICIT information by obtaining interpretable results. This section also describes the different processes carried out in such scheme that fulfil the general CW scheme showed in Fig. 3.

Fig. 8. ELICIT CW approach.

1) Linguistic Input and Output: As it was aforementioned, in a CW approach both inputs and outcomes have to be represented by linguistic information. The outputs will be logically represented by ELICIT expressions that should be generated from linguistic inputs close to the way of thinking of human beings and that facilitate the elicitation of their preferences. Section II-C presented the CLEs as linguistic expressions close to the common language of human beings, especially in DM problems, and that are able to model the hesitancy of the experts. For this reason, for the proposed CW approach, CLEs will be used to represent the linguistic input. Furthermore, ELICIT information previously computed can also be incorporated in the input of the CW approach.

2) Translation: For our proposal, the translation process consists of transforming the initial CLEs and ELICIT expressions into fuzzy numbers. Such transformations are carried out in different ways depending of the type of expression, CLE or ELICIT. CLEs are transformed into fuzzy numbers by computing their *fuzzy envelopes* (See Eq. (2)). Fuzzy envelopes are obtained by means of the aggregation of the linguistic terms that belong to the representative HFLTS of each CLE by using the OWA operator.

Remark 4: Note that the way to compute fuzzy envelopes is a key in our proposal. The OWA operator behavior regarding the importance of the linguistic terms belonging to a HFLTS determines the shape of the fuzzy envelopes and, consequently, the amount of information that fuzzy envelopes preserve. For our proposal, the more information is kept, the more accurate the results are. Such behavior is directly related to the *orness* measure of the OWA operator, thus, the orness measure plays at the same time a pivotal role in our proposal. For this reason, Appendix C introduces several necessary conditions regarding orness measure for preserving as much information as possible in the fuzzy envelopes computation.

On the other hand, the transformation of ELICIT information into fuzzy numbers is carried out by means of a function, noted as ζ^{-1}.

Definition 8: Let x_{el} be an ELICIT expression and $T(a, b, c, d)$ a trapezoidal fuzzy number. The function ζ^{-1} is defined as

$$\zeta^{-1} : x_{\text{el}} \to T(a, b, c, d). \tag{3}$$

Such that, from an ELICIT expression, it returns its equivalent trapezoidal fuzzy number.

The *adjustment* γ of the ELICIT expression, plays a key role in ζ^{-1}, since this parameter is used to obtain the points that define the corresponding fuzzy number, by preserving as much information as possible in the fuzzy representation of the ELICIT information (the *adjustment* computation for each ELICIT expression is introduced in Section III-B4 and further detailed in Appendix D). Depending on the ELICIT expression, the ζ^{-1} function is defined in different ways.

1) *At least expression:* The function ζ^{-1} for an ELICIT expression whose relation is "at least" is defined as follows.

Proposition 1: Let at least $(s_i, \alpha)^\gamma$ be an ELICIT expression and $T_{\text{ELICIT}}(a,' b,' 1, 1)$ the fuzzy envelope of such ELICIT expression. There is a function ζ^{-1}

$$\begin{aligned}\zeta^{-1}(\text{at least } (s_i, \alpha)^\gamma) &= T(a, b, 1, 1)\\ a &= a' + \gamma\\ b &= b'.\end{aligned} \tag{4}$$

2) *At most expression:* The function ζ^{-1} for an ELICIT expression whose relation is "at most" is defined as follows.

Proposition 2: Let at most $(s_i, \alpha)^\gamma$ be an ELICIT expression and $T_{\text{ELICIT}}(0, 0, c,' d')$ the fuzzy envelope of such ELICIT expression. There is a function ζ^{-1}

$$\begin{aligned}\zeta^{-1}(\text{at most } (s_i, \alpha)^\gamma) &= T(0, 0, c, d)\\ c &= c'\\ d &= d' + \gamma.\end{aligned} \tag{5}$$

3) *Between expression:* The function ζ^{-1} for an ELICIT expression whose relation is "between" is defined as follows.

Proposition 3: Let between $(s_i, \alpha_1)^{\gamma_1}$ and $(s_j, \alpha_2)^{\gamma_2}$ be an ELICIT expression and $T_{\text{ELICIT}}(a,' b,' c,' d')$ the fuzzy envelope of such ELICIT expression. There is a function ζ^{-1}

$$\begin{aligned}\zeta^{-1}(\text{between } (s_i, \alpha_1) \text{ and } (s_j, \alpha_2)) &= T(a, b, c, d)\\ a &= a' + \gamma_1\\ b &= b'\\ c &= c'\\ d &= d' + \gamma_2.\end{aligned} \tag{6}$$

3) Manipulation: The manipulation phase consists of carrying out fuzzy arithmetic computations with the fuzzy envelopes previously obtained, by resulting new fuzzy numbers noted as $\tilde{\beta}$. The fuzzy arithmetic operations have to keep the fuzzy representation to guarantee that the resulting fuzzy numbers $\tilde{\beta}$ can be represented in the initial fuzzy linguistic domain used in the input and, subsequently, transformed into ELICIT information.

Rezvani and Molani [66] proved that, by means of the fuzzy numbers shape function and $\alpha -$ cuts (see Appendix A), it is possible to compute different arithmetic operations keeping the fuzzy parametric representation (triangular or trapezoidal). This section presents the addition and subtraction fuzzy operations with ELICIT expressions represented by their envelopes.

Definition 9: Let $T_{\tilde{A}}(a_1, b_1, c_1, d_1)$ and $T_{\tilde{B}}(a_2, b_2, c_2, d_2)$ be two fuzzy envelopes modeled by two trapezoidal fuzzy numbers (TrFNs). Suppose the normal shape functions of $\tilde{A}, \tilde{B}$ as follows:

$$\mu_{\tilde{A}} = \begin{cases} \left(\dfrac{x - a_1}{b_1 - a_1}\right)^n, & \text{when } x \in [a_1, b_1)\\ 1, & \text{when } x \in [b_1, c_1]\\ \left(\dfrac{d_1 - x}{d_1 - c_1}\right)^n, & \text{when } x \in (c_1, d_1]\\ 0, & \text{otherwise}\end{cases}$$

$$\mu_{\tilde{B}} = \begin{cases} \left(\dfrac{x - a_2}{b_2 - a_2}\right)^n, & \text{when } x \in [a_2, b_2)\\ 1, & \text{when } x \in [b_2, c_2]\\ \left(\dfrac{d_2 - x}{d_2 - c_2}\right)^n, & \text{when } x \in (c_2, d_2]\\ 0, & \text{otherwise.}\end{cases}$$

And supposing $\tilde{A}_{\overline{\alpha}}$, $\tilde{B}_{\overline{\alpha}}$ are the $\alpha -$ cuts (see Definition 22) of $\tilde{A}$ and $\tilde{B}$, respectively

$$\begin{aligned}\tilde{A}_{\overline{\alpha}} &= [a_1 + \overline{\alpha}^{1/n}(b_1 - a_1), d_1 - \overline{\alpha}^{1/n}(d_1 - c_1)]\\ \tilde{B}_{\overline{\alpha}} &= [a_2 + \overline{\alpha}^{1/n}(b_2 - a_a), d_2 - \overline{\alpha}^{1/n}(d_2 - c_2)].\end{aligned}$$

It should be noted that, in our proposal, the computational processes deal with normal and complete TrFN, hence $n = 1$ and $\overline{\alpha} = 1$.

Definition 10. (see [66]): The addition of two fuzzy envelopes modeled by two TrFNs $\tilde{A}$, $\tilde{B}$ can be defined with a shape function $\mu_{\tilde{A}+\tilde{B}}$ as

$$\mu_{\tilde{A}+\tilde{B}} = \begin{cases} \dfrac{(x - (a_1 + a_2))^n}{(b_1 + b_2) - (a_1 + a_2)}, & a_1 + a_2 \leq x \leq b_1 + b_2\\ 1, & b_1 + b_2 \leq x \leq c_1 + c_2\\ \dfrac{((d_1 + d_2) - x)^n}{(d_1 + d_2) - (c_1 + c_2)}, & c_1 + c_2 \leq x \leq d_1 + d_2\\ 0, & \text{otherwise.}\end{cases}$$

Definition 11. (see [66]): The subtraction of two fuzzy envelopes modeled by two TrFNs $\tilde{A}$, $\tilde{B}$ can be defined with a shape function $\mu_{\tilde{A}-\tilde{B}}$ as

$$\mu_{\tilde{A}-\tilde{B}} = \begin{cases} \dfrac{(x - (a_1 - d_2))^n}{(b_1 - a_1 + d_2 - c_2)}, & a_1 - d_2 \leq x \leq b_1 - c_2\\ 1, & b_1 - c_2 \leq x \leq c_1 - b_2\\ \dfrac{((d_1 - a_2) - x)^n}{(d_1 - c_1 + b_2 - a_2)}, & c_1 - b_2 \leq x \leq d_1 - a_2\\ 0, & \text{otherwise.}\end{cases}$$

Example 4: Let $T_A(0.2, 0.33, 0.45, 0.5)$ and $T_B(0.15, 0.2, 0.3, 0.4)$ be two fuzzy envelopes modeled by two TrFNs, the normal shape functions $\mu_{\tilde{A}+\tilde{B}}$ and $\mu_{\tilde{A}-\tilde{B}}$ are defined, respectively, as

$$\mu_{\tilde{A}+\tilde{B}} = \begin{cases} \dfrac{(x - 0.35)^n}{0.18}, & 0.35 \leq x \leq 0.53\\ 1, & 0.53 \leq x \leq 0.75\\ \dfrac{(0.9 - x)^n}{0.15}, & 0.75 \leq x \leq 0.9\\ 0, & \text{otherwise}\end{cases}$$

$$\mu_{\tilde{A}-\tilde{B}} = \begin{cases} \dfrac{(x - 0.35)^n}{0.18}, & -0.2 \leq x \leq 0.03\\ 1, & 0.03 \leq x \leq 0.25\\ \dfrac{(0.9 - x)^n}{0.15}, & 0.25 \leq x \leq 0.35\\ 0, & \text{otherwise.}\end{cases}$$

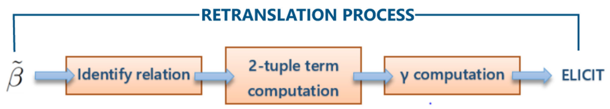

Fig. 9. Steps to build ELICIT expressions.

Remark 5: Note that Appendix A introduces the necessary fuzzy concepts.

4) Retranslation: Last but not least, the resulting fuzzy numbers $\overline{\beta}$ are transformed into ELICIT expressions in the *retranslation* process. This section explains in further detail the necessary steps to build ELICIT expressions from the results obtained by fuzzy computations on CLEs and ELICIT expressions, graphically represented in Fig. 9.

Starting from a fuzzy number $\tilde{\beta}$, the different steps to build an ELICIT expression are:

1) *Identify the relation:* Regarding Definition 7, the possible relations for ELICIT expressions are "at least," "at most," and "between". The relation is determined by the fuzzy number $\tilde{\beta}$ and the ζ function, defined as follows.

Definition 12: Let $S = \{s_0, \ldots, s_g\}$ be a set of linguistic terms and $\tilde{\beta}$ a fuzzy number. The function ζ is given by

$$\zeta(\tilde{\beta}) = x_{\text{el}},$$

$$\text{where} \begin{cases} x_{\text{el}} = \text{at least } (s_i, \alpha)^{\gamma} \text{ if } \tilde{\beta} = T(a, b, 1, 1) \\ x_{\text{el}} = \text{at most } (s_i, \alpha)^{\gamma} \text{ if } \tilde{\beta} = T(0, 0, c, d) \\ x_{\text{el}} = \text{between } (s_i, \alpha_1)^{\gamma_1} \text{and } (s_j, \alpha_2)^{\gamma_2} \\ \text{if } \tilde{\beta} = T(a, b, c, d). \end{cases}$$

Henceforth, for sake of clarity, it is assumed that the ELICIT expression is composed by a "between" relation, others are analogously developed in Appendix D. Thus, according to Definition 12, for a "between" relation the fuzzy number $\overline{\beta}$ is represented by $\tilde{\beta} = T(a, b, c, d)$ and consequently, the ELICIT expression is "between $(s_i, \alpha_1)^{\gamma_1}$ and $(s_j, \alpha_2)^{\gamma_2}$".

2) *2-tuple linguistic terms computation:* The ELICIT expression with the relation "between" is composed by two continuous primary terms $(s_i, \alpha_1)^{\gamma_1}$ and $(s_j, \alpha_2)^{\gamma_2}$. The process of obtaining the terms is divided into different steps.

a) *Compute linguistic terms (see Fig. 10):* To select the linguistic terms s_i and $s_j \in S, i, j \in \{0, \ldots g\}$, whose distance between the coordinates x of their respective centroids [67], $\overline{x}_i$ and $\overline{x}_j$, and the points b and c belonging to $\tilde{\beta}$ is minimal

$$i = \arg\min_h |b - \overline{x}_h|, \; h \in \{0, \ldots, g\}$$

$$j = \arg\min_h |c - \overline{x}_h|, \; h \in \{0, \ldots, g\}. \quad (7)$$

When this process finishes, the ELICIT expression so far is "between $(s_i, ?)^?$ and $(s_j, ?)^?$".

b) *Compute symbolic translations (see Fig. 11):* According to [20], [68], $1/2g$ represents the distance equivalent to a symbolic translation equal to 0.5 in S, where $g+1$ is the cardinality of S

$$\alpha_1 = g \cdot (b - \overline{x}_i) \;\; \alpha_1 \in [-0.5, 0.5)$$

$$\alpha_2 = g \cdot (c - \overline{x}_j) \;\; \alpha_2 \in [-0.5, 0.5). \quad (8)$$

When this process finishes, the ELICIT expression so far is "between $(s_i, \alpha_1)^?$ and $(s_j, \alpha_2)^?$".

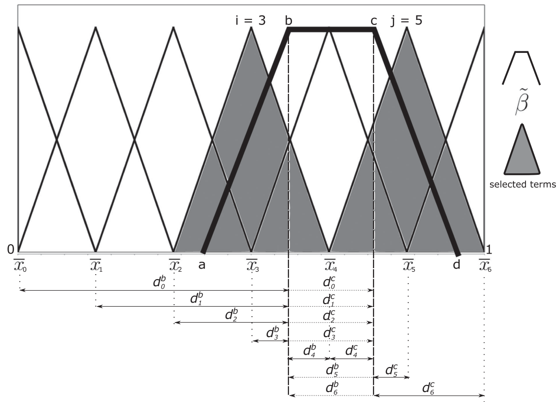

Fig. 10. Select linguistic terms.

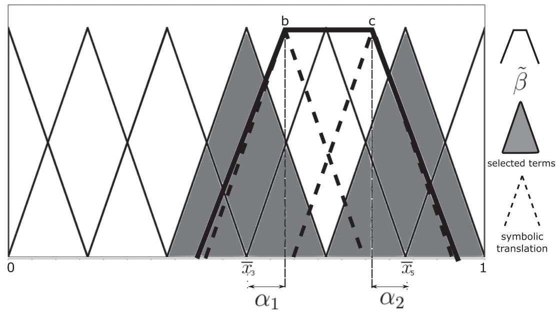

Fig. 11. Symbolic translation.

3) *Compute adjustments:* The *adjustment* is an additional parameter included in the ELICIT expression, which allows to keep information related to the fuzzy number $\tilde{\beta}$. This parameter will be used to obtain the fuzzy number $\tilde{\beta}$ from an ELICIT expression by using its inverse function, ζ^{-1} (see Section III-B2), preserving as much information as possible in the fuzzy representation and facilitating that accurate computations can be carried out in the manipulation phase. The steps to compute the adjustments for the ELICIT expression are as follows.

a) *Compute HFLTS (see Fig. 12):* The HFLTS of an ELICIT expression whose relation is "between" would be composed by

$$E_{\text{ELICIT}}(\text{between } (s_i, \alpha) \;\text{ and } (s_j, \alpha)) = \{s_k | (s_i, \alpha) \;\text{ and } (s_j, \alpha) \;\text{ and } s_i < s_k < s_j \;\text{ where } s_k \in S\}.$$

b) *Compute fuzzy envelope (see Fig. 13):* Applying (2), the fuzzy envelope of the computed HFLTS is noted as $T_{\text{ELICIT}} = T(a,' b,' c,' d')$. To compute b' and c', different weights for the linguistic terms, which compose the previous HFLTS are assigned

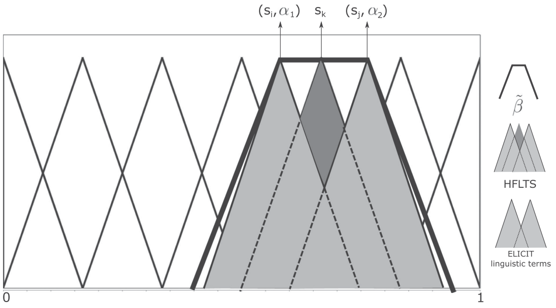

Fig. 12. HFLTS in "between" expression.

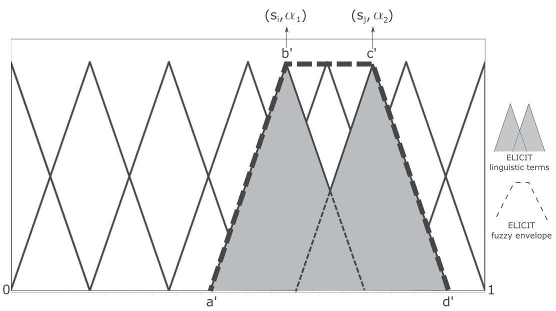

Fig. 13. Fuzzy envelope in "between" expression.

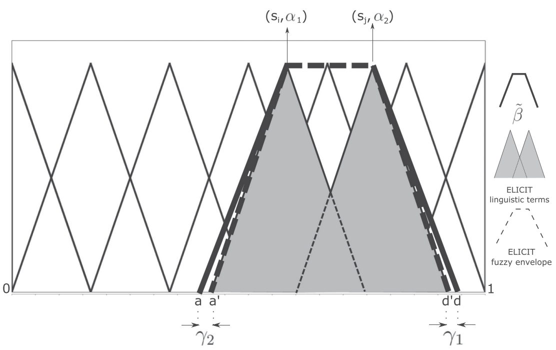

Fig. 14. γ_1 and γ_2 in "between" expression.

by using a parameter noted as ϵ and introduced in Appendix C. Specifically, to compute b', the value of ϵ_1 will be equal to 0 and to compute c', the value of ϵ_2 will be equal to 1. These values are provided to preserve as much information as possible in the ELICIT expressions for CW processes.

c) *Compute adjustments γ_1 and γ_2 (see Fig. 14):* The adjustments γ_1 and γ_2 are determined by the subtraction between the points a and d of $\tilde{\beta} = T(a, b, c, d)$ and the points a' and d' of $T_{\text{ELICIT}}(a,' b,' c,' d')$, so that

$$\begin{aligned} \gamma_1 &= a - a'\gamma_1 \in [0, 1] \\ \gamma_2 &= d - d'\gamma_2 \in [0, 1]. \end{aligned} \tag{9}$$

When this process finishes, the ELICIT expression is completed "between $(s_i, \alpha_1)^{\gamma_1}$ and $(s_j, \alpha_2)^{\gamma_2}$".

C. ELICIT Computational Model

This section introduces basic operations to accomplish the processes of CW with ELICIT information.

1) Comparison Operator: This section introduces a comparison operator for ELICIT information. The comparison among ELICIT expressions is carried out from their respective fuzzy envelopes. Therefore, in order to compare ELICIT expressions, a method for ranking fuzzy numbers is applied. There are several proposals that allow to rank fuzzy numbers [69]–[72] but, the method presented by Abbasbandy and Hajjari in [69] stands out, since it is far simpler and easier than other proposals. According to such a method, the ordering of ELICIT expressions is carried out by raking their respective fuzzy numbers $\tilde{\beta}$ by using the concept of *magnitude*.

Definition 13: Let x_{el1} and x_{el2} be two ELICIT expressions, $\tilde{\beta}_1$ and $\tilde{\beta}_2$ their equivalent fuzzy numbers obtained from $\zeta^{-1}(x_{\text{el1}})$ and $\zeta^{-1}(x_{\text{el2}})$ are as follows:

1) $\text{Mag}(\tilde{\beta}_1) > \text{Mag}(\tilde{\beta}_2) \iff \tilde{\beta}_1 \succ \tilde{\beta}_2$;
2) $\text{Mag}(\tilde{\beta}_1) < \text{Mag}(\tilde{\beta}_2) \iff \tilde{\beta}_1 \prec \tilde{\beta}_2$;
3) $\text{Mag}(\tilde{\beta}_1) = \text{Mag}(\tilde{\beta}_2) \iff \tilde{\beta}_1 \sim \tilde{\beta}_2$;

where $\text{Mag}(\beta)$ is the magnitude of the fuzzy number β.

Remark 6: Note that Appendix E related to ranking of fuzzy numbers and magnitude concept has been introduced for a better understanding of the comparison operator for ELICIT information.

2) Negation Operator: The concept of negation is also essential to deal with linguistic information. Therefore, its definition is relevant for dealing with ELICIT information. Depending on the linguistic expression, several negation operators are defined.

Definition 14: Let "at least $(s_i, \alpha)^\gamma$" be an ELICIT expression and $S = \{s_0, \ldots s_g\}$ a linguistic term set, the negation operator is defined as:

$$\text{Neg}(\text{at least } (s_i, \alpha)^\gamma) = \text{at most } \Delta(g - (\Delta^{-1}(s_i, \alpha)))^\gamma.$$

Definition 15: Let "at most $(s_i, \alpha)^\gamma$" be an ELICIT expression and $S = \{s_0, \ldots s_g\}$ a linguistic term set, the negation operator is defined as

$$\text{Neg}(\text{at most } (s_i, \alpha)^\gamma) = \text{at least } \Delta(g - (\Delta^{-1}(s_i, \alpha)))^\gamma.$$

Definition 16: Let "between $(s_i, \alpha_1)^{\gamma_1}$ and $(s_j, \alpha_2)^{\gamma_2}$" be an ELICIT expression and $S = \{s_0, \ldots s_g\}$ a linguistic term set, the negation operator is defined as

$$\begin{aligned} &\text{Neg}(\text{between } (s_i, \alpha_1)^{\gamma_1} \text{ and } (s_j, \alpha_2)^{\gamma_2}) \\ &= \text{between } \Delta(g - (\Delta^{-1}(s_i, \alpha_1)))^{\gamma_1} \text{ and} \\ &\Delta(g - (\Delta^{-1}(s_j, \alpha_2)))^{\gamma_2}. \end{aligned}$$

3) Aggregation Operators: The aggregation of multiple values into a single one is an essential process for any discipline based on the data processing, for instance DM. The aggregation process for ELICIT expressions is accomplished by using the fuzzy numbers $\tilde{\beta}$ obtained from the function ζ^{-1} of each expression. The operations carried out with the $\tilde{\beta}$ fuzzy numbers are based on the fuzzy arithmetic operations (see Section III-B3) defined by Rezvani and Molani [66]. Such fuzzy operations allow to keep the fuzzy representation and obtain new $\tilde{\beta}$ fuzzy numbers that can be again transformed into ELICIT expressions. A general aggregation operator is defined as follows.

TABLE I
ENVELOPES OF HFLTS

		Criterion		
Expert	Alternative	c_1	c_2	c_3
e_1	a_1	T(0.67, 0.83, 1)	T(0.33, 0.5, 0.67, 0.83)	T(0.83, 1, 1)
	a_2	T(0.33, 0.5, 0.67)	T(0.67, 0.83, 1)	T(0.5, 0.67, 1, 1)
	a_3	T(0.33, 0.5, 0.67)	T(0.17, 0.33, 0.5)	T(0.17, 0.33, 0.5)
	a_4	T(0.67, 0.83, 1)	T(0.67, 0.83, 1, 1)	T(0.83, 1, 1)
e_2	a_1	T(0.33, 0.5, 0.67)	T(0, 0, 0.33, 0.5)	T(0.5, 0.67, 0.83)
	a_2	T(0.67, 0.83, 1)	T(0.83, 1, 1)	T(0.5, 0.67, 1, 1)
	a_3	T(0, 0.17, 0.33)	T(0, 0.17, 0.33)	T(0, 0, 0.33, 0.5)
	a_4	T(0.5, 0.67, 0.83)	T(0.17, 0.33, 0.5)	T(0.33, 0.5, 0.67)
e_3	a_1	T(0.83, 1, 1)	T(0.5, 0.67, 0.83, 1)	T(0.5, 0.67, 1, 1)
	a_2	T(0.17, 0.33, 0.5)	T(0.5, 0.67, 0.83)	T(0.33, 0.5, 0.67)
	a_3	T(0, 0.17, 0.33)	T(0, 0, 0.17, 0.33)	T(0, 0.17, 0.33)
	a_4	T(0.5, 0.67, 0.83)	T(0.17, 0.33, 0.5)	T(0.17, 0.33, 0.5)

Definition 17: Let $\{x_{\text{el}1}, \ldots, x_{elk}\}$ be a set of ELICIT expressions and $\{\tilde{\beta}_1, \ldots, \tilde{\beta}_k\}$ their equivalent fuzzy numbers obtained from $\{\zeta^{-1}(x_{\text{el}1}), \ldots, \zeta^{-1}(x_{elk})\}$, a fuzzy aggregation operator F is defined as

$$F(\tilde{\beta}_1, \ldots, \tilde{\beta}_k) = \tilde{\beta} = T(a, b, c, d) = x_{\text{el}}. \tag{10}$$

IV. CASE STUDY

This section presents a case study to show the usefulness of the proposed fuzzy linguistic representation model. First, a LDM problem is described. Afterwards, the LDM problem is solved by means of the ELICIT CW approach. Finally, the results are compared with another CW approach [20] to show the advantages of the proposal.

A. Definition of DM Problem

Let us suppose a prestigious university that wants to hire a Ph.D. student among four possible candidates $X = \{a_1, a_2, a_3, a_4\}$. The final decision is made for a group of three renowned professors $E = \{e_1, e_2, e_3\}$ who have to evaluate the candidates according to three criteria $C = \{c_1, c_2, c_3\}$, which are, respectively: *communication skills*, *research experience,* and *academic record*.

B. Resolution of DM Problem

In order to solve the DM problem, the ELICIT CW approach is applied. This section is divided into several sections that describes the different processes carried out in such approach (see Fig. 8).

1) Linguistic Input and Output: Due to the DM problem implies uncertainty and imprecision, the set of criteria will be evaluated by means of linguistic information. The experts provide their preferences through a linguistic domain based on such knowledge composed by 7 labels, $S_7 =$ {Horrible (H), Very bad (VB), Bad (B), Medium (M), Good (G), Very good (VG), Excellent (E)} (see Fig. 1), by using CLEs and single linguistic terms. For the sake of space, the preferences have been included as a supplementary material document, which is available online[1].

2) Translation: The experts' preferences are transformed into fuzzy numbers by computing their fuzzy envelopes, which are shown in Table I.

[1][Online]. Available: https://sinbad2.ujaen.es/flintstones/en/study-cases/phdStudentSelection.pdf

TABLE II
RESULTING β FUZZY NUMBERS

Alternative	β
a_1	T(0.5, 0.65, 0.76, 0.88)
a_2	T(0.5, 0.67, 0.75, 0.86)
a_3	T(0.08, 0.21, 0.26, 0.43)
a_4	T(0.45, 0.62, 0.63, 0.76)

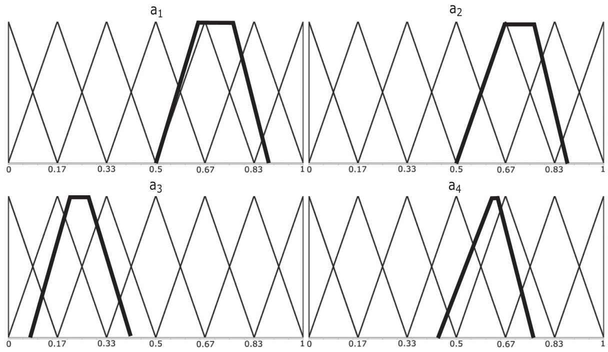

Fig. 15. β fuzzy representation for each alternative.

TABLE III
RESULTING ELICIT EXPRESSIONS

Alternative	ELICIT
a_1	$Between(Good, -0.11)^{0.018}$ and $(\textit{Very good}, -0.44)^{-0.056}$
a_2	$Between(Good, 0)^{0}$ and $(Good, 0.44)^{-0.055}$
a_3	$Between(\textit{Very bad}, 0.22)^{0.037}$ and $(Bad, -0.44)^{-0.001}$
a_4	$Between(Good, -0.33)^{-0.001}$ and $(Good, -0.22)^{-0.037}$

3) Manipulation: The fuzzy envelopes are aggregated by using an aggregation operator to obtain a collective value for each alternative (see Fig. 2). As it was aforementioned in Section III-B3, the fuzzy arithmetic operations carried out have to keep the fuzzy representation to guarantee that the resulting fuzzy numbers $\tilde{\beta}$ can be represented. For sake of simplicity and without losing generality the aggregation operator used in this case study is the *fuzzy arithmetic mean*.

Definition 18: Let $\{\tilde{A}_1, \ldots, \tilde{A}_n\}$ be a set of fuzzy numbers, the fuzzy arithmetic mean $\overline{x}$ is computed as

$$\overline{x}\{\tilde{A}_1, \ldots, \tilde{A}_n\} = \frac{1}{n}(\mu_{\tilde{A}_1 + \tilde{A}_2 + \ldots + \tilde{A}_n}). \tag{11}$$

The results of the aggregation process are showed in Table II and graphically represented in Fig. 15.

4) Retranslation: The resulting β are transformed into ELICIT expressions following the scheme represented in Fig. 9 (see Table III). In this way, the ELICIT model, on the contrary of others representation models (see Section II-D), provides linguistic results easily interpretable represented by ELICIT information, which is closer to the way of thinking of human beings and facilitates the understanding of the results by decision makers. Furthermore, the use of symbolic translation in ELICIT information guarantees that the linguistic computations have been carried out in a continuous domain, which means that the results have been obtained without any kind of approximation. Consequently, the results are more precise and reliable.

To conclude, the ranking of alternatives is obtained as solution of the problem and shown in Table IV. To do so, the approach introduced by Abbasbandy and Hajjari is used (see Appendix E).

TABLE IV
RANKING OF ALTERNATIVES

Alternative	Ranking
a_1	1
a_2	2
a_3	4
a_4	3

TABLE V
RESULTING 2-TUPLE LINGUISTIC VALUES

Alternative	2-tuple	Ranking
a_1	$(Good, 0.14)$	2
a_2	$(Good, 0.16)$	1
a_3	$(Very\ bad, 0.42)$	4
a_4	$(Good, -0.33)$	3

Remark 7: Note that the function $f(r)$ defined to compute the ranking in this proposal is $f(r) = r$.

Thus, the best Ph.D. student to hire among the four possible candidates is a_1.

C. Comparison With Previous Models

Previous section reveals the advantages of the ELICIT linguistic model: fuzzy linguistic representation in a continuous domain, precision in CW processes and improved interpretability in the results. However, it would be convenient to compare such results with another proposed CW scheme to stand out the features that make this model innovative. To do this, we propose a CW approach introduced in [20]. This approach has been selected because it presents a CW scheme similar to our proposal. The experts can provide their preferences by using CLEs and the CW processes are carried out by transforming the initial linguistic preferences into fuzzy envelopes and finally transformed into 2-tuple linguistic values. The results of applying this approach to the case study are shown in Table V.

Remark 8: Note that to obtain the aggregated results for each alternative the arithmetic mean aggregation operator has been used.

Notice that the approaches provide different rankings of alternatives, whereas ELICIT linguistic model chooses a_1 as the best solution of the problem, the 2-tuple based approach [20] selects a_2. However, the former provides results with a greater amount of information that leads to a greater level of discrimination and, hence, greater accuracy, so it can be guaranteed that the solution provided by the ELICIT linguistic model is more precise and robust. To show the latter, a sensitive analysis will be carried out. In this case, one aspect of sensitive analysis is conducted: the analysis about the criteria weight evolution.

The previous results obtained from both approaches have been computed by considering the same weights for all the criteria (0.333) in which the sum of such weights have to be equal to 1. To carry out the sensitive analysis, such weights are modified. Fig. 16 shows the changes that have to take place in the criteria weights (x-axis) for two alternatives to exchange their positions according to the final ranking provided by the CW approach presented in [20] and ELICIT CW approach. Note that, for the former, the pair of alternatives $a_1 - a_2$ exchange their positions in the final ranking with slight changes (-0.067, 0.04, and -0.142) in the weights of the criteria. Concretely, c_2 is the most critical criterion, since with the slightest variation of its weight (0.04), there is a change in the ranking among the alternatives

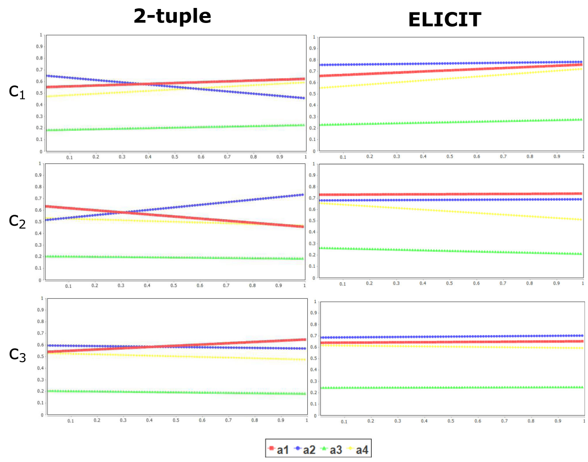

Fig. 16. Sensitive analysis for weights' evolution.

a_1 and a_2, that happens when the weight of c_2 is equal to 0.293 ($0.333-0.04$), being the weights of the remaining two 0.3535. There are also possible changes in the ranking with the pair of alternatives $a_2 - a_4$ for the criteria c_1 and c_2 but, in this case, the changes have to be more significant (-0.552 and 0.281). For the rest of the cases, the sensitive analysis provides non feasible solutions (lines never intersect in Fig. 16). Therefore, according to the sensitive analysis, the results obtained from the CW approach proposed in [20] are not enough robust since, by applying slight changes in the criteria weights, the final ranking is modified.

On the other hand, Fig. 16 shows that the weights evolution of the criteria does not imply any exchange in any pair of alternatives for the ELICIT CW approach, since for all the cases, sensitive analysis provides non feasible solutions. Therefore, the solution provided by the ELICIT CW approach is more robust than the previous one, since the final ranking remains unchanged for any variation of the criteria weights. Such robustness is determined by the ELICIT expressions, that represent more amount of information and allow to obtain more precise results and with a greater level of discrimination.

To conclude, ELICIT CW approach does not only provides more robust and precise results but, in addition, these are represented by ELICIT expressions, which are not limited by single linguistic terms as in the 2-tuple linguistic values and are closer to the linguistic structures used by human beings.

V. CONCLUSION

In this article, we have presented a new fuzzy linguistic representation model, which represents the linguistic information by means of ELICIT information, an extension of CLEs that takes advantage of the concept of symbolic translation used by the 2-tuple linguistic model. Such expressions are generated by a context-free grammar and composed by the relation of continuous primary terms represented by 2-tuple linguistic values. This novel approach has not lost of information when CW processes are applied since information is managed as a continuous range instead of a discrete one and provides linguistic results represented by ELICIT expressions close to the common language used by human beings. This new representation model has been applied in a DM problem and lately compared with another model to show its validity and advantages.

To conclude, as future research, we will study the use of the ELICIT linguistic representation in multigranulariy and combination of linguistic-numerical information contexts and the proposal of new aggregation operators for ELICIT information.

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Aggregating Interrelated Attributes in Multi-Attribute Decision-Making With ELICIT Information Based on Bonferroni Mean and Its Variants

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ABSTRACT

In recent times, to improve the interpretability and accuracy of computing with words processes, a rich linguistic representation model has been developed and referred to as Extended Comparative Linguistic Expressions with Symbolic Translation (ELICIT). This model extends the definition of the comparative linguistic expressions into a continuous domain due to the use of the symbolic translation concept related to the 2-tuple linguistic model. The aggregation of ELICIT information via a suitable rule that reflects the underlying interrelation among the aggregated information in output is the key tool to design decision-making algorithm for solving multi-attribute decision-making problems under linguistic information. In this study, we introduce three aggregation operators for aggregating ELICIT information in aim of capturing three different types of interrelationship patterns among inputs, which we refer to as ELICIT Bonferroni mean, ELICIT extended Bonferroni mean and ELICIT partitioned Bonferroni mean. Further, the key aggregation properties of these proposed operators are investigated with the proposal of weighted forms. Based on the proposed aggregation operators, an approach for solving multi-attribute decision-making problems, in which attributes are interrelated is developed. Finally, a didactic example is presented to illustrate the working of the proposal and demonstrate its feasibility.

1. INTRODUCTION

With the growing complexity of the socio-economic environment, it is quite common to prevail the uncertainty and vagueness in the decision-making process, in particular, the situations, where human judgments/assessments/perceptions are inevitable to reach a final decision over a set of alternatives [1]. The emergence of such scenarios involving human cognition leads us to use linguistic information based on the fuzzy linguistic approach [2] to effectively manage uncertainty in such decision-making processes. The *fuzzy linguistic approach* uses fuzzy set theory [3] to manage uncertainty and model linguistic information by using linguistic variables described by Zadeh [2] as "*A variable whose values are not numbers but words or sentences in a natural or artificial language.*" A linguistic variable is characterized by a syntactic value or *label* and a semantic value. Whereas the label is a word that belongs to a set of linguistic terms, semantics is provided by a fuzzy set in a discourse universe. Over the years, the fuzzy linguistic approach has been applied successfully in solving many practical multi-attribute decision-making (MADM) problems from the different domains [1,4] and many linguistic computational models have been put forwarded to improve and enhance the information modeling and computation process capability of the Zadeh's approach [2]. They can be broadly classified into two distinct categories: symbolic computational models [5–7] and semantic-based computational models [8]. In terms of simplicity and interpretability, symbolic models stand out semantic models. The symbolic models have evolved enormously over the years. The first proposals [4,9,10] made use of single linguistic variables, for instance, *good, horrible, very bad, perfect*, to provide the decision makers' preferences and carried out the linguistic computations. Among these symbolic models, 2-tuple linguistic computational model [4,5], which enhanced the interpretability of the fuzzy linguistic approach by introducing the concept of symbolic translation, has got wide speared acceptance among the community and successfully applied in solving the MADM problems [11,12]. However, in spite of many of these approaches have been applied successfully in decision-making problems, the modeling of linguistic information is limited when experts provide their preferences by using just single terms. To overcome this drawback, several proposals that obtain richer linguistic expressions than single linguistic terms have been proposed [13]. One of the most outstanding proposals is the so-called *Hesitant Fuzzy Linguistic Term Sets* (HFLTSs) [14], which were introduced to model the hesitancy of the experts when they doubt among several linguistic terms at the same time. HFLTSs are also based on the fuzzy linguistic approach that will serve as bases to increase the flexibility of the elicitation of linguistic information. An example of HFLTS might be $\{good, very\, good, excellent\}$. Furthermore,

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several decision-making proposals have been put forwarded in the literature [15]. Although HFLTS can be directly used by the experts to elicit several linguistic values for a linguistic variable, they are not close to the way of expressing opinions used by human beings. For this reason, Rodríguez *et al.* [14,16] proposed a formalization process to generate linguistic expressions close to the common language used by human beings in decision-making problems. Such expressions, so-called *comparative linguistic expressions* (CLEs), are based on HFLTSs and model the decision maker's hesitancy by means of the use of context-free grammars. An example of CLEs might be *between good and very good, at most bad, at least medium*, etc. Several decision-making models in CLEs environment have been proposed by adopting different computational approaches [17–20]. Up to this point, the CLEs are the closest to the way of thinking of the decision makers but the interpretability of the results in the existing computational approaches and information loss in the linguistic computations are the two key concerns that restrict their use as a decision tool under uncertainty. For this reason, in recent times, Labella *et al.* [21] proposed a new fuzzy linguistic representation for CLEs, which they referred as ***E**xtended **C**omparative **LI**nguisti**C** Expressions with Symbol**I**c **T**ranslation* (ELICIT) information. This representation takes advantage of the main characteristic of the CLEs, their interpretability, and improves the precision of the results by extending the representation of CLEs generated by a context-free grammar into a continuous domain to perform computing with words (CW) processes without any kind of approximation. In this way, the proposed ELICIT computational model overcomes the drawbacks of the earlier proposals. Some examples of ELICIT information might be *between* $(good, 0.23)^{0.12}$ *and* $(very\,good, 0.1)^{0.3}$, *at most* $(bad, 0)^{0}$, *at least* $(medium, -0.1)^{0.11}$, etc.

In the same way that representing information in the decision process is key, the aggregation of such information, which comes from different sources via a suitable rule (aggregation operator), plays also a pivotal role in decision-making process by combining several pieces of information into a single information, which represents overall overview [22]. In the context of MADM, aggregation operators are generally used to find overall performance of the alternatives from their performances against the predefined set of criteria. The need of modeling specific interaction among the attributes and computational formalization with different types of linguistic information to conduct decision-making process under specific linguistic environment were the cornerstone behind the development of several classes of aggregation operators in MADM context.

In this vein, to aggregate interrelated linguistic information represented by 2-tuple linguistic information, several 2-tuple linguistic aggregation operators have been proposed in the literature [4,23–27]. On the other hand, to fuse linguistic information, expressed by HFLTSs, many aggregation operators have been developed considering the nature of the interaction (independent/interrelated) among the aggregated HFLTSs [28–33]. Despite many successful uses of the hesitant fuzzy linguistic computational model in decision-making, it has limitations in modeling complex linguistic expressions by HFLTS [34] and can be overcome with the capability of ELICIT expression. The use of ELICIT information in the decision-making makes it necessary to consider the issue of aggregation of ELICIT information. In this view, Labella *et al.* [21] defined an aggregation operator, which we can refer to as ELICIT arithmetic mean, to aggregate ELICIT expressions in the decision-making process. However, the proposed aggregation operator does not consider the interrelationship among the aggregated ELICIT expressions that are connected with the underlying interrelationship structure of associated concepts/objects, like the attributes' interrelationship and the corresponding ratings. Further, considering the importance/weights of the inputs in the aggregation process is vital to take into account in many decision-making processes and that have not been considered by Labella *et al.* [21]. Therefore, in spite of ELICIT information advantages, there is an evident lack of proposals about ELICIT aggregation operators that consider the interrelation among the ELICIT expressions and their importance in the aggregation process. For this reason, this study aims:

- Develop several aggregation operator to aggregate ELICIT information by capturing different interrelationship patterns (homogeneous, heterogeneous and partitioned structure) among the aggregated arguments.
- Capture the homogeneous relationship among ELICIT expressions by developing the ELICIT Bonferroni mean (ELICITBM) operator.
- Reflect the heterogeneous interaction among the aggregated ELICIT expressions by developing the ELICIT extended Bonferroni mean (ELICITEBM) operator
- Capture the partitioned structured interrelationship among aggregated ELICIT expressions by developing the ELICIT partitioned Bonferroni mean (ELICITPBM) operator.
- Study the proposed aggregation operators properties and weighted form to take into account weight information in the aggregation process.
- Based on the proposed aggregation operators, present an approach for solving MADM problems in which attributes follow the different interrelationship patterns.

To this end, the paper is organized as follows. In Section 2, we provide a brief primer of classical aggregation operator that captures interrelationship of among the aggregated arguments along with fuzzy set theory. A brief overview of the ELICIT representation and computational model is also included in Section 2. In Section 3, we develop three aggregation operators to fuse the ELICIT information according to their underlying interrelationship structures, namely, ELICITBM, ELICITEBM and ELICITPBM. The key properties of these operators are also studied along with the weighted forms: ELICITWBM, weighted ELICITEBM (ELICITWEBM) and WELICITPBM. In Section 4, an aggregation operator-based approach to solving the MADM problems, in which attributes are interrelated with different patterns is proposed. A didactic example is presented in Section 5 to illustrate the working of our approach and feasibility. Finally concluding remarks are made in Section 6.

2. PRELIMINARIES

In this section, we overlay the key concepts related to Bonferroni mean (BM), arithmetic operational laws of fuzzy numbers and ELICIT information for easy understanding of our subsequent proposals on aggregation of interrelated ELICIT information and linguistic decision-making process.

2.1. Aggregation Operators for Interrelated Information

In this section, we briefly introduce the BM and its variants, which are capable of capturing different kinds of interrelationship patterns among the aggregated information. We start by recalling the definition of the BM operator.

Definition 1. [35] Let p and $q \geq 0, p+q>0$. For an input vector $\mathbf{a}=(a_1, a_2, ..., a_n) \in [0,1]^n$, the BM can be defined as a mapping $BM\colon [0,1]^n \rightarrow [0,1]$ and given by

$$BM_{p,q}(a_1, a_2, ..., a_n) = \left(\frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} a_i^p a_j^q \right)^{\frac{1}{p+q}} \tag{1}$$

Although, BM was introduced by Bonferroni [35] in 1950, it is analyzed and interpreted in decision-making context by Yager [36]. Specifically, BM captures a homogeneous interrelationship pattern among the inputs that every input $a_i \in \mathbf{a}$ is related to the rest of the inputs of $\mathbf{a}$. But in many real-life contexts, such homogeneous connections among the inputs may not exist rather the inputs are related to each other in a heterogeneously related fashion. To capture such heterogeneous connections among the inputs, Dutta *et al.* [25] developed a new aggregation operator, which is referred to as extended Bonferroni mean (EBM). Based on heterogeneous connection among the inputs, they classified inputs $\mathbf{a}$ into two categories U and V, where every input of U is related to a subset of the rest of the inputs, i.e., $E_i \subset \mathbf{a} \setminus \{a_i\}$ and the inputs of V are not related to each other. Having this interpretation of the heterogeneous interrelationship pattern, the rule for the EBM aggregation operator is given by

Definition 2. [25] For any $p>0$ and $q \geq 0$, the EBM operator of dimension n is a mapping $EBM\colon [0,1]^n \rightarrow [0,1]$ such that

$$EBM_{p,q}(a_1, a_2, \cdots, a_3) \tag{2}$$

$$= \left(\frac{n-|I'|}{n} \left(\frac{1}{n-|I'|} \sum_{i \in I'} a_i^p \left(\frac{1}{|I_i|} \sum_{j \in I_i} a_j^p \right) \right)^{\frac{p}{p+q}} + \frac{|I'|}{n} \left(\frac{1}{|I'|} \sum_{i \in I'} a_i^p \right) \right)^{\frac{1}{p}}$$

where I_i is the set of indices of the elements of E_i, I' is the collection indices of the inputs of V, $|I'|$ denotes the cardinality of the set I' and empty sum is zero by convention with $\frac{0}{0}=0$.

Partitioned Bonferroni mean (PBM) is another variant of BM, which is capable of capturing partition structure interrelationship pattern among the input set in the aggregation process and reflects it in the aggregated value [24]. In the following, we provide a brief description of the specific partition structure interrelationship pattern and PBM operator.

Let $\mathbf{a}=(a_1, a_2, ..., a_n)$ be the collection of inputs, with a_i's being non-negative real numbers. Suppose, on the basis of the interrelationship pattern, the input set $\mathbf{a}$ is partitioned into d distinct classes $P_1, P_2, ..., P_d$ such that $P_i \cap P_j = \phi$ for all $i \neq j$, $i,j \in \{1,2,...,d\}$, $\cup_{r=1}^{d} P_r = \mathbf{a}$ and $|P_i| \geq 2$ for all $i=1,2,...,d$. We further assume that the inputs of each P_i are interrelated and there is no interrelationship among the inputs of any two partitions P_i and P_j whenever $i,j \in \{1,2,...,d\}$ and $i \neq j$. With these assumptions and notations, the PBM operator of the collection of inputs $(a_1, a_2, ..., a_n)$ is defined as follows:

Definition 3. [24] For $p,q \geq 0$ with $p+q>0$, the PBM operator is a mapping $PBM\colon [0,1]^n \rightarrow [0,1]$ such that

$$PBM(a_1, a_2, ..., a_n)$$

$$= \frac{1}{d} \sum_{r=1}^{d} \left(\frac{1}{|P_r|} \sum_{i \in P_r} a_i^p \left(\frac{1}{|P_r|-1} \sum_{\substack{j \neq i \\ j \in P_r}} a_j^q \right) \right)^{\frac{1}{p+q}} \tag{3}$$

where $|P_r|$ denotes cardinality of P_r.

It is evident from the Definitions 1 and 3 that BM is a special case PBM when all the inputs belong to same class [24]. To establish more concrete link between BM and PBM, we can write Eq. (3) as follows:

$$PBM_{p,q}(a_1, a_2, ..., a_n) \tag{4}$$

$$= \frac{1}{d} \sum_{r=1}^{d} \left(\frac{1}{|P_r|(|P_r|-1)} \sum_{\substack{i,j \in P_r \\ i \neq j}} a_i^p a_j^q \right)^{\frac{1}{p+q}}$$

$$= \frac{1}{d} \sum_{r=1}^{d} BM_r(a_i \in P_r)$$

where,

$$BM_r(a_i \in P_r) = \left(\frac{1}{|P_r|(|P_r|-1)} \sum_{\substack{i,j \in P_r \\ i \neq j}} a_i^p a_j^q \right)^{\frac{1}{p+q}}$$

and $(a_i \in P_r)$ denotes the set of inputs belongs to the partition P_r. With the help of Eq. (4), we can interpret PBM as arithmetic average of BM over different partition of the given input set. Therefore, one can compute the aggregated value of an input set by PBM via computing BM over different partitions.

2.2. Arithmetic Operations of Fuzzy Numbers

In this section, key concepts associated with the fuzzy numbers and their operational laws are briefly described. We start by recalling the definition of a fuzzy set, which is well known to model the concept that does not possess the sharp boundaries. Throughout this article, we will restrict ourselves to the class of fuzzy sets over the universe of discourse X which is a subset of the set of real numbers $\mathbb{R}$.

Definition 4. [3] A fuzzy set $\tilde{A}$ over the universe of discourse X is characterized by a membership function, which associates every element of $x \in X$ to a real number from the interval $[0, 1]$ and denoted as

$$\mu_{\tilde{A}} : X \to [0, 1] \tag{5}$$

A fuzzy set $\tilde{A}$ can also be defined with help of ordered pairs of generic element $x \in X$ and the corresponding membership degree $(\mu_{\tilde{A}}(x))$ and represented as

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \,|x \in X\} \tag{6}$$

Definition 5. [3] The support of the fuzzy set $\tilde{A}$ over the universe of discourse X is the set of all elements $x \in X$, such that, the membership degree is greater than 0, i.e.,

$$Supp\left(\tilde{A}\right) = \{x \in X | \mu_{\tilde{A}}(x) > 0\}. \tag{7}$$

Definition 6. [37] A fuzzy set $\tilde{A}$ is said to be normal if there exists a $x_0 \in X$ such that $\mu_{\tilde{A}}(x_0) = 1$.

Definition 7. [37] A fuzzy set A over a convex universe of discourse X is said to be convex if

$$\mu_A\left(\lambda x + (1 - \lambda) y\right) \geq \min\left\{\mu_A(x), \mu_A\left(y\right)\right\},$$

for all $x, y \in supp(A)$ and $\lambda \in [0, 1]$.

Definition 8. [37] A fuzzy number $\tilde{A}$ over the universe of discourse $X \subset \mathbb{R}$ is a special fuzzy set, which is convex and normal.

As a fuzzy set is completely characterized by its membership function, we can say the membership functions are synonyms of the fuzzy sets. Although any function $f : X \to [0, 1]$ can serve as a membership function, in practice trapezoidal and triangular membership functions are widely used to quantify the fuzzy meaning of the linguistic terms used by the decision maker to express their opinions in natural language.

Definition 9. A trapezoidal fuzzy number (TrFN) $\tilde{A} = (a, b, c, d)$ with four parameters $a, b, c, d \,(a \leq b \leq c \leq d)$ is a fuzzy subset of the real line $\mathbb{R}$ and described by its membership function $\mu_{\tilde{A}}$ as follows:

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{x-a}{b-a} & \text{if} \quad a \leq x < b \\ 1 & \text{if} \quad b < x \leq c \\ \dfrac{d-x}{d-c} & \text{if} \quad c < x \leq d \\ 0, & \text{otherwise} \end{cases} \tag{8}$$

Definition 10. A triangular fuzzy number (TFN) $\tilde{A} = (a, b, c)$ with three parameters $a, b, c \,(a \leq b \leq c)$ is a fuzzy subset of the real line $\mathbb{R}$ and described by its membership function $\mu_{\tilde{A}}$ as follows:

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{x-a}{b-a} & \text{if} \quad a \leq x < b \\ 1 & \text{if} \quad x = b \\ \dfrac{c-x}{c-b} & \text{if} \quad b < x \leq c \\ 0, & \text{otherwise} \end{cases} \tag{9}$$

The obvious motivations behind the use of trapezoidal and TFNs come from the simplicity of the membership functions and their characterization requires reasonably limited information about the linguistic term [38,39]. For example, when a triangular $\tilde{A} = (a, b, c)$ is used to quantify a linguistic term, the triplet (a, b, c) represents the lower, most likely and upper values of that linguistic term with varied membership degree, described via membership function $\mu_{\tilde{A}}(x)$.

The fuzzy arithmetic operational laws allow us to facilitate the computation over linguistic information. There are several ways to derive the arithmetic operational laws of the fuzzy numbers based on the Zadeh's *extension principle* [37]. As in the ELICIT computational model [21] the meaning of the primary linguistic term sets are represented by using TFNs or TrFNs, we restrict ourselves on fuzzy arithmetic operational laws, which preserve the shape of the original fuzzy numbers. In this view, we adopt Chen's *function principle* based arithmetic operational laws, which is given as follows [40]:

Definition 11. Let $\tilde{A} = (a_1, b_1, c_1, d_1)$ and $\tilde{B} = (a_2, b_2, c_2, d_3)$ be the two positive TrFNs. Following Chen's function Then arithmetic operations between $\tilde{A}$ and $\tilde{B}$ can be defined as follows:

- Addition: $\tilde{A} \oplus \tilde{B} = (a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2)$
- Multiplication: $\tilde{A} \otimes \tilde{B} = (a_1a_2, b_1b_2, c_1c_2, d_1d_2)$
- Scalar multiplication: $r\tilde{A} = (ra_1, rb_1, rc_1)$, $r > 0$
- Exponent: $\tilde{A}^r = \left(a_1^r, b_1^r, c_1^r\right)$, $r > 0$.

Note that the *function principle* based arithmetic laws differ from extension principle-based arithmetic laws in multiplication operation as the former approximate resultant fuzzy number shape. Further, one may observe that with the increment of the number of aggregated fuzzy numbers in the aggregation process, the difference between *function principle* based aggregation and *extension principle* based aggregation results diminishes.

2.3. ELICIT Information

Despite the evolution of the symbolic approaches over the time [4,14,16], there exists several drawbacks in terms of interpretability and/or accuracy. ELICIT information allows us to keep the interpretability and precision of the results in MADM problems under linguistic environments thanks to the extension of CLEs into a continuous domain. To carry out such extension, the ELICIT expressions are generated by means of a context-free grammar by using the symbolic translation concept used by the 2-tuple linguistic model.

Definition 12. [21] Let G_H be a context-free grammar and $S = \{s_0, \dots, s_g\}$ a linguistic term set. The elements of $G_H = (V_N, V_T, I, P)$ are defined as follows.

$$
\begin{aligned}
V_N &= \{(\textit{continuous primary term}), (\textit{composite term}),\\
&\quad (\textit{unary relation}), (\textit{binary relation}), (\textit{conjunction})\}\\
V_T &= \{\textit{at least, at most, between, and}, (s_0, \alpha)^\gamma, \dots, (s_g, \alpha)^\gamma\}\\
I &\in V_N
\end{aligned}
$$

The production rules defined in an extended Backus–Naur Form are:

$$
\begin{aligned}
&P = \{I ::= (\textit{continuous primary term}) \mid (\textit{composite term})\\
&(\textit{composite term}) ::= (\textit{unary relation})\\
&(\textit{continuous primary term}) \mid (\textit{binary relation})\\
&(\textit{continuous primary term})\ (\textit{conjunction})\\
&(\textit{continuous primary term})\\
&(\textit{continuous primary term}) ::= (s_0, \alpha)^\gamma \mid (s_1, \alpha)^\gamma \mid \dots \mid (s_g, \alpha)^\gamma\\
&(\textit{unary relation}) ::= \textit{at least} \mid \textit{at most}\\
&(\textit{binary relation}) ::= \textit{between}\\
&(\textit{conjunction}) ::= \textit{and}\}
\end{aligned}
$$

Therefore, the possible ELICIT expressions generated according to the previous context-free grammar are: "*at least* $(s_i, \alpha)^\gamma$", "*at most* $(s_i, \alpha)^\gamma$" and "*between* $(s_i, \alpha_1)^{\gamma_1}$ *and* $(s_j, \alpha_2)^{\gamma_2}$" (see Figure 1).

To obtain linguistic results represented by ELICIT information in decision-making processes, a novel approach was introduced in [21]. This approach starts from linguistic preferences provided by the experts modeled by CLEs and/or ELICIT information. Afterward, CLEs and ELICIT information are transformed into TrFNs. Whereas the CLEs are transformed into TrFNs through the computation of their fuzzy envelope [18], the transformation of the ELICIT information into TrFNs is carried by means an *inverse function*.

Definition 13. [21] Let EL_1 be an ELICIT expression and $T(a, b, c, d)$ a TrFN. The function ζ^{-1} is defined as:

$$\zeta^{-1} : EL_1 \rightarrow T(a, b, c, d) \tag{10}$$

Such that, from an ELICIT expression, it returns its equivalent TrFN.

In this point, the *adjustment*, γ, of the ELICIT expression plays a key role. The *adjustment* is an additional parameter included in the ELICIT expression, which will be used to obtain the respective fuzzy number from an ELICIT expression by using its inverse function, ζ^{-1}, preserving as much information as possible in the fuzzy representation and facilitating accurate computations. Depending on the ELICIT expression, the ζ^{-1} function is defined in different ways.

A. *At least expression*: The function ζ^{-1} for an ELICIT expression whose relation is *at least* is defined as follows:

Definition 14. [21] Let *at least* $(s_i, \alpha)^\gamma$ be an ELICIT expression and $T_{ELICIT}(a', b', 1, 1)$ the fuzzy envelope of such ELICIT expression. There is a function ζ^{-1}:

$$
\begin{aligned}
&\zeta^{-1}\left(\textit{at least } (s_i, \alpha)^\gamma\right) = T(a, b, 1, 1)\\
&a = a' + \gamma\\
&b = b'
\end{aligned}
$$

B. *At most expression*: The function ζ^{-1} for an ELICIT expression whose relation is *at most* is defined as follows:

Definition 15. [21] Let *at most* $(s_i, \alpha)^\gamma$ be an ELICIT expression and $T_{ELICIT}(0, 0, c', d')$ the fuzzy envelope of such ELICIT expression. There is a function ζ^{-1}:

$$
\begin{aligned}
&\zeta^{-1}\left(\textit{at most } (s_i, \alpha)^\gamma\right) = T(0, 0, c, d)\\
&c = c'\\
&d = d' + \gamma
\end{aligned}
$$

C. *Between expression*: The function ζ^{-1} for an ELICIT expression whose relation is *between* is defined as follows:

Definition 16. [21] Let *between* $(s_i, \alpha_1)^{\gamma_1}$ and $(s_j, \alpha_2)^{\gamma_2}$ be an ELICIT expression and $T_{ELICIT}(a', b', c', d')$ the fuzzy envelope of such ELICIT expression. There is a function ζ^{-1}:

$$
\begin{aligned}
&\zeta^{-1}\left(\textit{between } (s_i, \alpha_1) \text{ and } (s_j, \alpha_2)\right) = T(a, b, c, d)\\
&a = a' + \gamma_1\\
&b = b'\\
&c = c'\\
&d = d' + \gamma_2
\end{aligned}
$$

Remark 1.

Appendix A.1 has been included in order to show the performance of ζ^{-1} through a practical example.

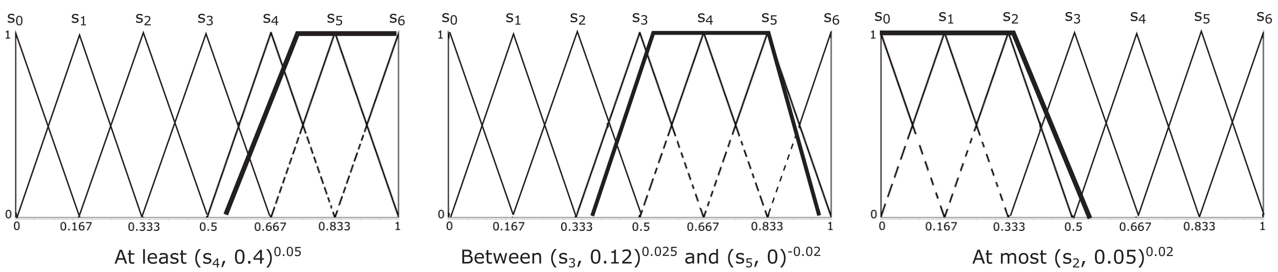

Figure 1 ELICIT information examples.

Once the TrFNs are obtained, they are manipulated and aggregated by means of fuzzy operations that keep the fuzzy parametric representation of such TrFNs [41]. Finally, the resulting TrFNs, noted as β, are retranslated into ELICIT information. This process consists of several steps, which are briefly described below:

1. *Identify relation*: The relation of the ELICIT expression is determined by the fuzzy number $\tilde{\beta}$ and the ζ function, defined in [21] as follows:

   Definition 17. Let $S = \{s_0, \ldots, s_g\}$ be a set of linguistic terms and $\tilde{\beta}$ a fuzzy number. The function ζ is given by Eq. (11) as shown in the beginning of the next page.

   For sake of space, it is assumed that the ELICIT expression is composed by a "*between*" relation (see [21] for further detail about the construction of other ELICIT expressions).

$$\zeta\left(\tilde{\beta}\right) = EL,\ where \begin{cases} EL = at\ least\ (s_i, \alpha)^{\gamma}\ if \tilde{\beta} = T(a, b, 1, 1) \\ EL = at\ most\ (s_i, \alpha)^{\gamma}\ if \tilde{\beta} = T(0, 0, c, d) \\ EL = between\ (s_i, \alpha_1)^{\gamma_1}\ and\ \left(s_j, \alpha_2\right)^{\gamma_2} \\ \qquad if \tilde{\beta} = T(a, b, c, d) \end{cases} \tag{11}$$

2. *2-tuple linguistic terms computation*: The ELICIT expression with the relation "*between*" is composed by two continuous primary terms $(s_i, \alpha_1)^{\gamma_1}$ and $\left(s_j, \alpha_2\right)^{\gamma_2}$. The process of obtaining such terms is divided into different steps:
   (a) *Compute linguistic terms*: To select the linguistic terms s_i and $s_j \in S, i, j \in \{0, \ldots g\}$, whose distance between the coordinates x of their respective centroids [42], $\bar{x}_i$ and $\bar{x}_j$, and the points b and c belonging to $\tilde{\beta}$ is minimal.

$$i = \arg\min_h |b - \bar{x}_h|,\ h \in \{0, \ldots, g\} \tag{12}$$
$$j = \arg\min_h |c - \bar{x}_h|,\ h \in \{0, \ldots, g\}$$

   The ELICIT expression so far is "*between* $(s_i, ?)^?$ *and* $\left(s_j, ?\right)^?$".

   (b) *Compute symbolic translations*: According to [4,43], $1/2g$ represents the distance equivalent to a symbolic translation equal to 0.5 in S, where $g+1$ is the cardinality of S:

$$\alpha_1 = g \cdot \left(b - \bar{x}_i\right) \quad \alpha_1 \in [-0.5, 0.5) \tag{13}$$
$$\alpha_2 = g \cdot \left(c - \bar{x}_j\right) \quad \alpha_2 \in [-0.5, 0.5)$$

   The ELICIT expression so far is "*between* $(s_i, \alpha_1)^?$ *and* $\left(s_j, \alpha_2\right)^?$".

3. *Compute adjustments*: The steps to compute the adjustments for the ELICIT expression are:
   (a) *Compute HFLTS*: The HFLTS of an ELICIT expression whose relation is *between* would be composed by:

$$E_{ELICIT}\left(between\ (s_i, \alpha)\ \text{and}\ \left(s_j, \alpha\right)\right) = \left\{s_k \middle| (s_i, \alpha)\ \text{and}\ \left(s_j, \alpha\right)\ \text{and}\ s_i < s_k < s_j\ \text{where}\ s_k \in S\right\}$$

   (b) *Compute fuzzy envelope*: The fuzzy envelope [18] of the computed HFLTS is computed and noted as $T_{ELICIT} = T(a', b', c', d')$.

   (c) *Compute adjustments γ_1 and γ_2*: The adjustments γ_1 and γ_2 are determined by the subtraction between the points a and d of $\tilde{\beta} = T(a, b, c, d)$ and the points a' and d' of $T_{ELICIT}(a', b', c', d')$, so that:

$$\begin{aligned} \gamma_1 &= a - a' \quad \gamma_1 \in [0, 1] \\ \gamma_2 &= d - d' \quad \gamma_2 \in [0, 1] \end{aligned} \tag{14}$$

   Finally, the ELICIT expression is completed "*between* $(s_i, \alpha_1)^{\gamma_1}$ *and* $\left(s_j, \alpha_2\right)^{\gamma_2}$".

Remark 2.

Appendix B.1 has been included in order to show the retranslation process through a practical example.

3. AGGREGATION OF INTERRELATED ELICIT EXPRESSIONS

The fusion of linguistic information that is represented by CLEs and/or ELICIT expressions according to underlying interrelationship structure of the information is essential to design a variety of linguistic decision-making processes. In this section, we extend the classical interrelated aggregation operators described in the previous section to aggregate the ELICIT expressions with certain underlying interrelationship pattern. From now onward, we are going to use $\mathcal{F}$ to denote the set of all possible ELICIT expressions over a linguistic term set S.

3.1. ELICIT Bonferroni Operators

Based on the Definition 1, the homogeneously interrelated ELICIT expressions can be aggregated as follows:

Definition 18. Let $\mathbf{EL} = (EL_1, EL_2, \ldots, EL_n)$ be the collection of n ELICIT expressions from $\mathcal{F}$. For any $p, q \geq 0$ with $p + q > 0$, the ELICITBM operator is a mapping $ELICITBM: \mathcal{F}^n \rightarrow \mathcal{F}$ and defined as follows:

$$ELICITBM_{p,q}(EL_1, EL_2, \ldots, EL_n) \tag{15}$$
$$= \zeta\left(\frac{1}{n(n-1)} \bigoplus_{\substack{i,j=1 \\ i \neq j}} \left(\zeta^{-1}(EL_i)\right)^p \otimes \left(\zeta^{-1}\left(EL_j\right)\right)^q\right)^{\frac{1}{p+q}}$$

where $\oplus$ represents the addition of fuzzy numbers and $\otimes$ denotes the multiplication of fuzzy numbers.

Based on the arithmetic operational laws of fuzzy numbers, we illustrate the computational formula of ELICITBM in the following theorem:

Theorem 1. *Let* $\mathbf{EL} = (EL_1, EL_2, \ldots, EL_n)$ *be the collection of* n *ELICIT expressions from* $\mathcal{F}$*. For any* $p, q \geq 0$ *with* $p + q > 0$*, the* *aggregated value of ELICIT expressions by ELICITBM is a ELICIT expression and given by*

$$ELICITBM_{p,q}(EL_1, EL_2, ..., EL_n)$$
$$= \zeta\left(\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ j\neq i}}^{n} a_i^p a_j^q\right)^{\frac{1}{p+q}}, \left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ j\neq i}}^{n} b_i^p b_j^q\right)^{\frac{1}{p+q}}, \left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ j\neq i}}^{n} c_i^p c_j^q\right)^{\frac{1}{p+q}}, \left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ j\neq i}}^{n} d_i^p d_j^q\right)^{\frac{1}{p+q}}\right) \quad (16)$$

where $\zeta^{-1}(EL_i) = (a_i, b_i, c_i, d_i)$ is the equivalent fuzzy number of the ELICIT expression EL_i for all $i = 1, 2, ..., n$.

Proof. Please see Appendix C.1

Remark 3.

With the notation of the BM operator, the computational formula for ELICITBM (Eq. 16) can be rewritten as follows:

$$ELICITBM_{p,q}(EL_1, EL_2, ..., EL_n) = \zeta\left(BM_{p,q}(a_1, a_2, ..., a_n), BM_{p,q}(b_1, b_2, ..., b_n), BM_{p,q}(c_1, c_2, ..., c_n), BM_{p,q}(d_1, d_2, ..., d_n)\right) \quad (17)$$

Example 1.

Let us consider the aggregation of homogeneously interrelated ELICIT information: $EL_1 = at\,least\,(s_4, 0)^0$, $EL_2 = at\,least\,(s_5, 0)^0$, $EL_3 = at\,most\,(s_3, 0)^0$ $EL_4 = between\,(s_3, 0)^0$ *and* $(s_4, 0)^0$. To capture the homogeneous interrelation pattern in the aggregation process, we are going to employ ELICITBM operator with parameters $p = q = 1$. As per Theorem 1, we first obtain the fuzzy numbers corresponding to the given ELICIT by utilizing Definitions 14–16 with the semantics of linguistic terms defined in Figure 1 as follows: $\zeta^{-1}(EL_1) = (0.5, 0.86, 1, 1)$, $\zeta^{-1}(EL_2) = (0.67, 0.98, 1, 1)$, $\zeta^{-1}(EL_3) = (0, 0, 0.36, 0.67)$, $\zeta^{-1}(EL_4) = (0.34, 0.5, 0.67, 0.84)$). With the help of Eq. (17), we obtain

$$ELICITBM_{1,1}(EL_1, EL_2, EL_3, EL_4) = \zeta\left(BM_{1,1}(0.5, 0.67, 1, 0.34), BM_{1,1}(0.86, 0.98, 1, 0.5), BM_{1,1}(1, 1, 0.36, 0.67), BM_{1,1}(1, 1, 0.67, 0.84)\right)$$

From Eq. (1), we have

$$ELICITBM_{1,1}(EL_1, EL_2, EL_3, EL_4) = \zeta(0.35, 0.54, 0.74, 0.87)$$

By utilizing Eq. (11) with the retranslation steps of ELICIT information, we obtain

$$ELICITBM_{1,1}(EL_1, EL_2, EL_3, EL_4) = \text{between } (s_3, -0.28)^{0.02} \text{ and } (s_4, 0.42)^{0.04}.$$

Theorem 2. *The ELICIT expressions aggregation operator ELICITBM satisfies the following properties:*

- $ELICITBM: \mathcal{F}^n \to \mathcal{F}$ is commutative, i.e.,

  $$ELICITBM_{p,q}(EL_1, EL_2, ..., EL_n) = ELICITBM_{p,q}\left(EL_{\sigma(1)}, EL_{\sigma(2)}, ..., EL_{\sigma(n)}\right)$$

  where $EL_{\sigma(1)}, EL_{\sigma(2)}, ..., EL_{\sigma(n)}$ is a permutation of the ELICIT expressions $EL_1, EL_2, ..., EL_n$.
- $ELICITBM: \mathcal{F}^n \to \mathcal{F}$ is idempotent, i.e.,

  $$ELICITBM_{p,q}(EL, EL, ..., EL) = EL$$
- $ELICITBM: \mathcal{F}^n \to \mathcal{F}$ is ratio-scale invariant, i.e. for any real number $r > 0$

  $$ELICITBM_{p,q}(rEL_1, rEL_2, ..., rEL_n) = rELICITBM_{p,q}(EL_1, EL_2, ..., EL_n).$$

Proof. Please see Appendix C.2.

Theorem 3. *Let* $\mathbf{EL} = (EL_1, EL_2, ..., EL_n)$ *be the collection of ELICIT expressions and* $\zeta^{-1}(EL_i) = (a_i, b_i, c_i, d_i)$ $(i = 1, 2, ..., n)$ *be the equivalent fuzzy numbers of the ELICIT expression* EL_i $(i = 1, 2, ..., n)$*. Then the operator* $ELICITBM: \mathcal{F}^n \to \mathcal{F}$ *is bounded, i.e.*

$$\zeta\left(\min_i a_i, \min_i b_i, \min_i c_i, \min_i d_i\right) \le ELICITBM_{p,q}(EL_1, EL_2, ..., EL_n) \le \zeta\left(\max_i a_i, \max_i b_i, \max_i c_i, \max_i d_i\right).$$

Proof. Please see Appendix C.3.

In the above, we have not considered the weight of the aggregated ELICIT expressions. But, in many practical applications, we need to consider the weight of input arguments in the aggregation process. In this view, we define the weighted form of ELICITBM as follows:

Definition 19. Let $\mathbf{EL} = (EL_1, EL_2, ..., EL_n)$ be the collection of n ELICIT expressions from $\mathcal{F}$. For any $p, q \ge 0$ with $p + q > 0$, the ELICITWBM operator is a mapping $ELICITWBM: \mathcal{F}^n \to \mathcal{F}$ and defined as follows:

$$ELICITWBM_{p,q}(EL_1, EL_2, ..., EL_n) \quad (18)$$
$$= \zeta\left(\frac{1}{n(n-1)} \oplus_{\substack{i,j=1\\ i\neq j}} \left(w_i\left(\zeta^{-1}(EL_i)\right)^p\right) \otimes \left(\frac{w_j}{1-w_i}\left(\zeta^{-1}(EL_j)\right)^q\right)\right)^{\frac{1}{p+q}}$$

where $(w_1, w_2, ..., w_n)$ be the weights of the input ELICIT expressions and $w_i > 0$ $(i = 1, 2, ..., n)$ with $\sum_{i=1}^{m} w_i = 1$.

With the operational laws of the fuzzy numbers, we derive the computational formula of the *ELICITWBM* as follows:

Theorem 4. *Let* $\mathbf{EL} = (EL_1, EL_2, ..., EL_n)$ *be the collection of n ELICIT expressions from* $\mathcal{F}$*. For any* $p, q \geq 0$ *with* $p + q > 0$*, the aggregated value of ELICIT expressions by ELICITWBM is a ELICIT expression and given by*

$$\begin{aligned} & ELICITWBM_{p,q}(EL_1, EL_2, ..., EL_n) \\ & = \zeta\left(\left(\sum_{\substack{i,j=1\\ j\neq i}}^{n} \frac{w_i w_j}{1-w_i} a_i^p a_j^q\right)^{\frac{1}{p+q}}, \left(\sum_{\substack{i,j=1\\ j\neq i}}^{n} \frac{w_i w_j}{1-w_i} b_i^p b_j^q\right)^{\frac{1}{p+q}}, \right. \\ & \left. \left(\sum_{\substack{i,j=1\\ j\neq i}}^{n} \frac{w_i w_j}{1-w_i} c_i^p c_j^q\right)^{\frac{1}{p+q}}, \left(\sum_{\substack{i,j=1\\ j\neq i}}^{n} \frac{w_i w_j}{1-w_i} d_i^p d_j^q\right)^{\frac{1}{p+q}}\right) \end{aligned} \tag{19}$$

where, $\zeta^{-1}(EL_i) = (a_i, b_i, c_i, d_i)$ *is the equivalent fuzzy number of the ELICIT expression* EL_i *for all* $i = 1, 2, ..., n$ *and* $(w_1, w_2, ..., w_n)$ *is the weight vector of the inputs and* $w_i > 0$ $(i = 1, 2, ..., n)$ *with* $\sum_{i=1}^{m} w_i = 1$.

Proof. It follows in the lines of Theorem 1.

3.2. ELICIT Extended Bonferroni Mean

This section focuses on aggregating ELICIT expressions that are heterogeneously interrelated in the fashion described in Section 2 and define ELICITEBM operator as follows:

Definition 20. Let $\mathbf{EL} = (EL_1, EL_2, ..., EL_n)$ be the collection of n ELICIT expressions from $\mathcal{F}$ such that the input set $\mathbf{EL}$ is heterogeneously interrelated (as described in Section 2). For any $p, q \geq 0$ with $p + q > 0$, the ELICITEBM operator is a mapping $ELICITEBM: \mathcal{F}^n \rightarrow \mathcal{F}$ and defined as follows:

$$\begin{aligned} & ELICITEBM_{p,q}(EL_1, EL_2, ..., EL_n) \\ & = \zeta\left(\frac{n-|I'|}{n}\left(\frac{1}{n-|I'|}\bigoplus_{i\notin I'}\left(\zeta^{-1}(EL_i)\right)^p \otimes \left(\frac{1}{|I_i|}\bigoplus_{j\in I_i}\right.\right.\right. \\ & \left.\left.\left.\left(\zeta^{-1}(EL_j)\right)^q\right)\right)^{\frac{p}{p+q}} \oplus \frac{|I'|}{n}\left(\frac{1}{|I'|}\bigoplus_{i\in I'}\left(\zeta^{-1}(EL_i)\right)^p\right)\right)^{\frac{1}{p}} \end{aligned} \tag{20}$$

where empty sum of fuzzy numbers ($\bigoplus$) is set as fuzzy zero (with TrFN representation (0, 0, 0, 0)) in the lines of convention of classic crisp system with (0, 0, 0, 0) /0 = (0, 0, 0, 0).

For the computational purpose, we derive the explicit mathematical formulae based on the arithmetic operational laws of TrFNs and ELICIT computational model as follows:

Theorem 5. *Let* $\mathbf{EL} = (EL_1, EL_2, ..., EL_n)$ *be the collection of n ELICIT expressions from* $\mathcal{F}$*, which are heterogeneously interrelated. For any* $p, q \geq 0$ *with* $p + q > 0$*, the aggregated value of ELICIT expressions is a ELICIT expression and given by*

$$\begin{aligned} & ELICITEBM_{p,q}(EL_1, EL_2, ..., EL_n) \\ & = \zeta\left(EBM(a_1, a_2, ..., a_n), EBM(b_1, b_2, ..., b_n),\right. \\ & \quad \left. EBM(c_1, c_2, ..., c_n), EBM(d_1, d_2, ..., d_n)\right) \end{aligned} \tag{21}$$

wher, $\zeta^{-1}(EL_i) = (a_i, b_i, c_i, d_i)$ *is the equivalent fuzzy number of the ELICIT expression* EL_i *for all* $i = 1, 2, ..., n$ *and the heterogeneous interrelationship structure of* EL_i*'s is inherited into* $\zeta^{-1}(EL_i)$*'s in component-wise fashion.*

It is not difficult to show that ELICITEBM satisfies commutative, idempotency and ratio-scale invariant properties of the aggregation operator as those properties holds for classic EBM. Further, it is bounded by $\zeta\left(\min_i a_i, \min_i b_i, \min_i c_i, \min_i d_i\right)$ and $\zeta\left(\max_i a_i, \max_i b_i, \max_i c_i, \max_i d_i\right)$. To take into account the relative importance of the aggregated arguments in the aggregation process, we define the weighted form of the ELICITEBM as follows:

Definition 21. Let $\mathbf{EL} = (EL_1, EL_2, ..., EL_n)$ be the collection of n ELICIT expressions from $\mathcal{F}$, which are heterogeneously interrelated in the fashion described Section 2. For any $p, q \geq 0$ with $p + q > 0$ and weight vector $w = (w_1, w_2, ..., w_n)$, such that $w_i > 0$ with $\sum_{i=1}^{n} w_i = 1$, the ELICITWEBM operator is a mapping $ELICITWEBM: \mathcal{F}^n \rightarrow \mathcal{F}$ and defined as follows:

$$\begin{aligned} & ELICITWEBM_{p,q}(EL_1, EL_2, ..., EL_n) \\ & = \zeta\left(\left(\left(1-\sum_{i\in I'} w_i\right)\left(\bigoplus_{i\notin I'}\frac{w_i}{1-\sum_{i\in I'} w_i}\left(\zeta^{-1}(EL_i)\right)^p \otimes\right.\right.\right. \\ & \left.\left.\left(\frac{1}{|I_i|}\bigoplus_{j\in I_i}\frac{w_j}{\sum_{j\in I} w_j}\left(\zeta^{-1}(EL_j)\right)^q\right)\right)^{\frac{p}{p+q}} \\ & \left.\left.\oplus\left(\sum_{i\in I'} w_i \bigoplus_{i\in I'}\frac{w_i}{\sum_{i\in I'} w_i}\left(\zeta^{-1}(EL_i)\right)^p\right)\right)^{\frac{1}{p}}\right) \end{aligned} \tag{22}$$

The explicit computational formula of ELICITWEBM could be obtained by using the arithmetic laws of fuzzy numbers with ELICIT computational model and summarized in the following:

Theorem 6. *Let* $\mathbf{EL} = (EL_1, EL_2, ..., EL_n)$ *be the collection of n ELICIT expressions from* $\mathcal{F}$*, which are heterogeneously related. For any* $p, q \geq 0$ *with* $p + q > 0$ *and weight vector* $w = (w_1, w_2, ..., w_n)$*, such that* $w_i > 0$ *and* $\sum_{i=1}^{n} w_i = 1$*, the aggregated value of ELICIT expressions by ELICITWEBM is a ELICIT expression and given by*

$$\begin{aligned} & ELICITWEBM_{p,q}(EL_1, EL_2, ..., EL_n) \\ & = \zeta\left(WEBM(a_1, a_2, ..., a_n), WEBM(b_1, b_2, ..., b_n),\right. \\ & \quad \left. WEBM(c_1, c_2, ..., c_n), WEBM(d_1, d_2, ..., d_n)\right) \end{aligned} \tag{23}$$

where, $\zeta^{-1}(EL_i) = (a_i, b_i, c_i, d_i)$ *is the equivalent fuzzy number of the ELICIT expression* EL_i *for all* $i = 1, 2, ..., n$ *and the heterogeneous interrelationship structure of* EL_i*'s is inherited into* $\zeta^{-1}(EL_i)$*'s* *in component-wise fashion. The WEBM*$: [0,1]^n \rightarrow [0,1]$ *is the weighted form of EBM aggregation operator, which is given by*

$$WEBM_{p,q}(a_1, a_2, ...a_n) = \left(\left(1-\sum_{i\in I'} w_i\right)\left(\sum_{i\notin I'} \frac{w_i}{1-\sum_{i\in I'} w_i} a_i^p \left(\frac{1}{|I_i|}\sum_{j\in I_i} \frac{w_j}{\sum_{j\in I} w_j} a_j^q\right)\right)^{\frac{p}{p+q}} \oplus \sum_{i\in I'} w_i \left(\sum_{i\in I'} \frac{w_i}{\sum_{i\in I'} w_i} a_i^p\right)\right)^{\frac{1}{p}} \tag{24}$$

3.3. ELICIT Partitioned Bonferroni Mean

In this section, we consider the aggregation of ELICIT expressions, which follows a partitioned structure interrelationship pattern described in Section 2. Based on the fact in Eq. (4) and Definition 18, we define ELICITPBM operator in the following:

Definition 22. Let $\mathbf{EL} = (EL_1, EL_2, ..., EL_n)$ be the collection of n ELICIT expressions from $\mathcal{F}$ such that the input set $\mathbf{EL}$ is partitioned into d distinct classes $P_1, P_2, ..., P_d$ (as described in Section 2). For any $p, q \geq$ with $p + q > 0$, the ELICITPBM operator is a mapping $ELICITPBM: \mathcal{F}^n \rightarrow \mathcal{F}$ and defined as follows:

$$ELICITPBM_{p,q}(EL_1, EL_2, ..., EL_n) = \zeta\left(\frac{1}{d}\bigoplus_{r=1}^{d} \zeta^{-1}(ELCITBM(EL_i : i \in P_r))\right) \tag{25}$$

where $(EL_i : i \in P_r)$ denotes the set of ELICIT expressions EL_is that belong to the partition P_r.

From the Definition 22, we note that by repeated application of ELICITBM over the partitions of the input set we can obtain the aggregated value of ELICITPBM. The more explicit computational formula to find the aggregated value of the ELICICTPBM in terms of BM is given below:

Theorem 7. *Let* $\mathbf{EL} = (EL_1, EL_2, ..., EL_n)$ *be the collection of* n *ELICIT expressions from* $\mathcal{F}$*,which are partitioned into* d *classes* P_1, P_2,..., P_d. *For any* $p, q \geq 0$ *with* $p + q > 0$, *the aggregated value of ELICIT expressions is a ELICIT expression and given by*

$$ELICITPBM_{p,q}(EL_1, EL_2, ..., EL_n) = \zeta\left(\frac{1}{d}\left(\sum_{r=1}^{d} BM_{p,q}(a_i : i \in P_r), \sum_{r=1}^{d} BM_{p,q}(b_i : i \in P_r), \sum_{r=1}^{d} BM_{p,q}(c_i : i \in P_r), \sum_{r=1}^{d} BM(d_i : i \in P_r)\right)\right) \tag{26}$$

where, $\zeta^{-1}(EL_i) = (a_i, b_i, c_i, d_i)$ *is the equivalent fuzzy number of the ELICIT expression* EL_i *for all* $i = 1, 2, ..., n$ *and the partitioned structure interrelationship of* EL_i*'s is inherited into* $\zeta^{-1}(EL_i)$*'s in component-wise fashion.*

As the ELICITPBM operator is composed of a set of ELICITBM operators with different dimensions, we can easily exhibit that the ELICITPBM operator satisfies commutative, idempotent and ratio-scale invariant properties with help of Theorem 2. Further, the ELCITPBM operator is bounded as follows:

$$\zeta\left(\min_i a_i, \min_i b_i, \min_i c_i, \min_i d_i\right) \leq ELICITPBM_{p,q}(EL_1, EL_2, ..., EL_n) \leq \zeta\left(\max_i a_i, \max_i b_i, \max_i c_i, \max_i d_i\right).$$

When the inputs ELICIT expressions have different relative importance, we need to take account it in the aggregation process and to reflect on the aggregated value. In this view, the weighted form of the ELICITPBM can be defined as follows:

Definition 23. Let $\mathbf{EL} = (EL_1, EL_2, ..., EL_n)$ be the collection of n ELICIT expressions from $\mathcal{F}$ such that the input set $\mathbf{EL}$ is partitioned into d distinct classes P_1, P_2, ..., P_d (as described in Section 2). For any $p, q \geq 0$ with $p + q > 0$ and weight vector $w = (w_1, w_2, ..., w_n)$, such that $w_i > 0$ and $\sum_{i=1}^{n} w_i = 1$, the ELICITWPBM operator is a mapping $ELICITWPBM: \mathcal{F}^n \rightarrow \mathcal{F}$ and defined as follows:

$$ELICITWPBM_{p,q}(EL_1, EL_2, ..., EL_n) = \zeta\left(\frac{1}{d}\bigoplus_{r=1}^{d} \zeta^{-1}(ELCITWBM(EL_i \in P_r))\right) \tag{27}$$

where $(EL_i : i \in P_r)$ denotes the set of ELICIT expressions EL_is that belong to the partition P_r.

Theorem 8. *Let* $\mathbf{EL} = (EL_1, EL_2, ..., EL_n)$ *be the collection of* n *ELICIT expressions from* $\mathcal{F}$. *For any* $p, q \geq 0$ *with* $p + q > 0$ *and weight vector* $w = (w_1, w_2, ..., w_n)$, *such that* $w_i > 0$ *and* $\sum_{i=1}^{n} w_i = 1$, *the aggregated value of ELICIT expressions by ELICITWPBM is a ELICIT expression and given by*

$$ELICITWPBM_{p,q}(EL_1, EL_2, ..., EL_n) = \zeta\left(\frac{1}{d}\left(\sum_{r=1}^{d} WBM_{p,q}(a_i : i \in P_r), \sum_{r=1}^{d} WBM_{p,q}(b_i : i \in P_r), \sum_{r=1}^{d} WBM_{p,q}(c_i : i \in P_r), \sum_{r=1}^{d} WBM_{p,q}(d_i : i \in P_r)\right)\right) \tag{28}$$

where $\zeta^{-1}(EL_i) = (a_i, b_i, c_i, d_i)$ *is the equivalent fuzzy number of the ELICIT expression* EL_i *for all* $i = 1, 2, ..., n$ *and*

$$WBM_{p,q}(a_i : i \in P_r) = \left(\sum_{\substack{i,j\in P_r \\ j\neq i}} \frac{w_i w_j}{\left(\sum_{i\in P_r} w_i\right)\left(\sum_{\substack{j\in P_r \\ j\neq i}} w_j\right)} a_i^p a_j^q\right)^{\frac{1}{p+q}}$$

4. APPROACHES TO MADM WITH ELICIT ASSESSMENTS

In this section, we develop an approach based on ELICIT expressions aggregation operators to solve MADM problem in which attributes follow a typical interrelationship pattern, and the decision maker provides his/her assessments by using CLEs and/or ELICIT expressions.

We consider a typical MADM problem, where a finite set of alternatives are evaluated against a predefined set of performance measuring attributes in the aim of ranking the alternatives from best to worst on their suitability. In such a decision-making problem two pieces of information are required to find the ranking of the alternatives. One is assessment information of the alternatives against the criteria, which we often refer to as decision information. Another one is related to the relative importance of the criteria that is referred to as weight information. Mathematically, we can describe the MADM problem with all the relevant information as follows:

- A finite set of $m\,(\geq 2)$ alternatives: $X = \{X_i | i \in I\}$, where $I = \{1, 2, ..., m\}$
- A fixed set of criteria: $A = \left\{A_j | j \in J\right\}$ where $J = \{1, 2, ..., n\}$
- The weight vector of the criteria: $w = (w_1, w_2, ..., w_n)$ such that $w_j \geq 0$ and $\sum_{j=1}^{n} w_j = 1$.
- The alternatives are assessed over criteria and evaluations are summarized in the following decision matrix:

$$D = \begin{array}{c c} & \begin{array}{cccc} A_1 & A_2 & \cdots & A_n \end{array} \\ \begin{array}{c} X_1 \\ X_2 \\ \vdots \\ X_m \end{array} & \begin{pmatrix} EL_{11} & EL_{12} & \cdots & EL_{1n} \\ EL_{21} & EL_{22} & \cdots & EL_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ EL_{m1} & EL_{m2} & \cdots & EL_{mn} \end{pmatrix} \end{array}$$

  where EL_{ij} is the ELICIT expression that has been obtained from the decision maker's linguistic opinions to provide his/her assessment for the alternative X_i against the criteria A_j. Specifically, decision maker uses CLEs to express his/her assessments against the alternatives under different attributes.

Apart from these binding pieces of information, the decision maker needs to provide the typical pattern of the interrelationship among the attributes. As interrelationship is vital in the selection of an appropriate aggregation operator, this information is crucial to make a reliable decision.

$$\mathrm{WEBM}_{\mathrm{p,q}}\left(a_1, a_2, ...a_n\right) = \left(\left(\left(1 - \sum_{s\in I'} w_s\right)\left(\sum_{s\notin I'} \frac{w_s}{1 - \sum_{s\in I'} w_s} a_{is}^{p}\left(\frac{1}{|I_s|}\sum_{t\in I_s} \frac{w_t}{\sum_{t\in I_s} w_j} a_{it}^{q}\right)\right)\right)^{\frac{p}{p+q}} + \sum_{s\in I'} w_s \left(\sum_{s\in I'} \frac{w_s}{\sum_{s\in I'} w_s} a_{is}^{p}\right)\right)^{\frac{1}{p}} \tag{29}$$

With this available information in hand, we intend to design an algorithm based on the aggregation operators, developed in the previous section, to find the most desirable alternative(s) from the alternatives' pool $\{X_1, X_2, ..., X_m\}$. Our proposed algorithm takes following steps to find ranking order of the alternatives:

Step 1

Give the decision maker's preference summarized in the decision matrix $D = \left(EL_{ij}\right)_{m\times n}$ and weight information $w = (w_1, w_2, ..., w_n)$.

Step 2

Provide the interrelationship patter among the attributes, i.e., whether, the attributes follows homogeneous interrelationship pattern, heterogeneously interrelation patter or partitioned structured interrelationship pattern. In the cases of heterogeneous and partitioned interrelationship, specific structure of interrelationship data need to be provided.

Step 3

Based on the interrelationship pattern, the suitable aggregation operator is selected to obtain the overall performance of the alternative X_i from the alternative's individual performances under different attributes $E_{ij}\ \left(j = 1, 2, ..., n\right)$. Specifically, three scenarios arise here:

- *attributes are homogeneously related* in this case, we utilize ELICITBM operator to find the alternatives X_i overall performance $r_i\ (i = 1, 2, ..., m)$ as follows:

$$\begin{aligned} r_i &= ELICITBM\left(EL_{i1}, EL_{i2}, ..., EL_{in}\right) \\ &= \zeta\left(\left(\sum_{\substack{s,t=1\\ t\neq s}}^{n} \frac{w_s w_t}{1-w_s} a_{is}^{p} a_{it}^{q}\right)^{\frac{1}{p+q}}, \left(\sum_{\substack{s,t=1\\ t\neq s}}^{n} \frac{w_s w_t}{1-w_s} b_{is}^{p} b_{it}^{q}\right)^{\frac{1}{p+q}}, \left(\sum_{\substack{s,t=1\\ t\neq s}}^{n} \frac{w_s w_t}{1-w_s} c_{ts}^{p} c_{it}^{q}\right)^{\frac{1}{p+q}}, \left(\sum_{\substack{s,t=1\\ t\neq s}}^{n} \frac{w_s w_t}{1-w_s} d_{is}^{p} d_{it}^{q}\right)^{\frac{1}{p+q}}\right) \end{aligned} \tag{30}$$

  where $\zeta^{-1}\left(EL_ij\right) = \left(a_{ij}, b_{ij}, c_{ij}, d_{ij}\right)$ is the equivalent fuzzy number of the ELICIT expression EL_ij for all $i = 1, 2, ..., m$.

- *attributes are heterogeneously interrelated*, in this case, we employ ELCITWEBM operator to obtain overall performance r_i of the alternative X_i as follows:

$$\begin{aligned} & ELICITWEBM_{p,q}\left(EL_{i1}, EL_{i2}, ..., EL_{in}\right) \\ & = \zeta\left(WEBM\left(a_{i1}, a_{i2}, ..., a_{in}\right),\right. \\ & WEBM\left(b_{i1}, b_{i2}, ..., b_{in}\right), \\ & WEBM\left(c_{i1}, c_{i2}, ..., c_{in}\right), \\ & \left. WEBM\left(d_{i1}, d_{i2}, ..., d_{in}\right)\right) \end{aligned} \tag{31}$$

  where, $\zeta^{-1}\left(EL_{ij}\right) = \left(a_{ij}, b_{ij}, c_{ij}, d_{ij}\right)$ is the equivalent fuzzy number of the ELICIT expression EL_{ij} for all $j = 1, 2, ..., n$ and $WEBM\left(a_{i1}, a_{i2}, ..., a_{in}\right)$ is given by Eq. (29).

- *attributes are partitioned structured*, in this case, WELCITPBM operator is utilized to obtain overall performance r_i of the alternative X_i as follows:

$$\begin{aligned} r_i &= WELICITPBM\left(EL_{i1}, EL_{i2}, \ldots, EL_{in}\right) \\ &= \zeta\left(\frac{1}{d}\left(\sum_{r=1}^{d} WBM\left(a_{ij} \in P_r\right), \sum_{r=1}^{d} WBM\left(b_{ij} : j \in P_r\right), \right.\right. \\ &\left.\left. \sum_{r=1}^{d} WBM\left(c_{ij} : j \in P_r\right), \sum_{r=1}^{d} WBM\left(d_{ij} : j \in P_r\right)\right)\right) \end{aligned} \tag{32}$$

where, $\zeta^{-1}\left(EL_i\right) = \left(a_{ij}, b_{ij}, c_{ij}, d_{ij}\right)$ is the equivalent fuzzy number of the ELICIT expression EL_{ij} for all $j = 1, 2, \ldots, n$ and

$$WBM\left(a_{ij} : j \in P_r\right) = \left(\sum_{\substack{k,j \in P_r \\ j \neq k}} \frac{w_k w_j}{\left(\sum_{k \in P_r} w_k\right)\left(\sum_{\substack{j \in P_r \\ j \neq k}} w_j\right)} a_k^p a_j^q\right)^{\frac{1}{p+q}}$$

Step 4

The overall performance of the alternatives $r_i\,(i = 1, 2, \ldots, m)$ are ELICIT expressions. To facilitate the comparisons, we first transformed them into fuzzy numbers $T_{r_i} = \zeta^{-1}\left(r_i\right) = \left(t_{i1}, t_{i2}, t_{i3}, t_{i4}\right)$ for $i = 1, 2, \ldots, m$ and then defuzzified them into real number $Mag\left(T_{r_i}\right)\,(i = 1, 2, \ldots, m)$ by using the approach proposed by Abbasbandy and Hajri [44].

Step 5

Based on the $Mag\left(T_{r_i}\right)\,(i = 1, 2, \ldots, m)$, we rank the alternatives $X_i\,(i = 1, 2, \ldots, m)$ in the sense that better the magnitude, better the rank.

5. PRACTICAL EXAMPLE

In this section, we provide a practical example to demonstrate the working and feasibility of the proposed decision-making algorithm.

In the face of a trade war, a major company is considering to shift its manufacturing plant from the current location. After, initial screening the company has identified five possible locations around the world to step up the new manufacturing plant. We name this potential locations as $\{X_1, X_2, X_3, X_4, X_5\}$. To prioritize further these locations, the company has identified seven assessment attributes: market (A_1), business climate (A_2), labour characteristic (A_3), infrastructure (A_4), availability of raw materials (A_5), investment cost (A_6) and possibility for the further extensions (A_7). These performance measuring attributes have some intrinsic connections/interrelations and that could be described as follows: A_1 is interrelated with A_4; A_2 with $\{A_6, A_7\}$; A_3 with A_7; A_4 with $\{A_1, A_6\}$; A_5 with A_7; A_6 with $\{A_2, A_4\}$ and A_7 with $\{A_3, A_5\}$. The information regarding the attributes for all possible options are collected and presented to the key managerial responsible for taking a decision.

Due to the presence of vagueness and uncertainty, the decision maker uses linguistic information to assess the locations against the attributes. According to the expertise of the decision maker, a linguistic term set with 7 labels is provided, S = {s_0: *unfeasible (UF), s_1: very unsuitable (VUS), s_2: unsuitable (US), s_3: fair (F), s_4: suitable (S), s_5: very suitable (VS), s_6: excellent (E)*}.

Decision maker uses a single linguistic term or complex linguistic expression, modeled by CLEs to rate the alternatives against the attributes. The decision maker's preferences are represented by CLEs (Table 1 Rating in CLEs) that are transformed into ELICIT information and modeled by the decision matrix D and presented as follows:

$$D = \begin{array}{c} \\ X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \end{array} \begin{array}{c} \begin{array}{ccccccc} A_1 & A_2 & A_3 & A_4 & A_5 & A_6 & A_7 \end{array} \\ \left(\begin{array}{ccccccc} at\ least\ (s_4,0)^0 & at\ least\ (s_5,0)^0 & (s_4,0)^0 & (s_4,0)^0 & at\ least\ (s_3,0)^0 & at\ least\ (s_4,0)^0 & bt\ (s_3,0)^0\ and\ (s_4,0)^0 \\ at\ most\ (s_1,0)^0 & (s_3,0)^0 & bt\ (s_3,0)^0\ and\ (s_4,0)^0 & bt\ (s_0,0)^0\ and\ (s_1,0)^0 & at\ least\ (s_3,0)^0 & (s_3,0)^0 & (s_3,0)^0 \\ (s_5,0)^0 & at\ least\ (s_5,0)^0 & (s_4,0)^0 & (s_5,0)^0 & bt\ (s_2,0)^0\ and\ (s_3,0)^0 & at\ most\ (s_3,0)^0 & (s_3,0)^0 \\ (s_0,0)^0 & (s_0,0)^0 & (s_1,0)^0 & bt\ (s_3,0)^0\ and\ (s_4,0)^0 & at\ most\ (s_2,0)^0 & (s_3,0)^0 & (s_2,0)^0 \\ (s_6,0)^0 & (s_3,0)^0 & (s_6,0)^0 & at\ least\ (s_4,0)^0 & (s_2,0)^0 & bt\ (s_4,0)^0\ and\ (s_5,0)^0 & (s_5,0)^0 \end{array}\right) \end{array}$$

$$\overline{D} = \begin{array}{c} \\ X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \end{array} \begin{array}{c} \begin{array}{ccccccc} A_1 & A_2 & A_3 & A_4 & A_5 & A_6 & A_7 \end{array} \\ \left(\begin{array}{ccccccc} T(0.5,0.86,1,1) & T(0.67,0.98,1,1) & T(0.5,0.67,0.84) & T(0.5,0.67,0.84) & T(0.34,0.64,1,1) & T(0.5,0.86,1,1) & T(0.34,0.5,0.67,0.84) \\ T(0,0,0.03,0.34) & T(0.34,0.5,0.67) & T(0.34,0.5,0.67,0.84) & T(0,0,0.17,0.34) & T(0.34,0.65,1,1) & T(0.34,0.5,0.67) & T(0.34,0.5,0.67) \\ T(0.67,0.84,1) & T(0.67,0.98,1,1) & T(0.5,0.67,0.84) & T(0.67,0.84,1) & T(0.17,0.34,0.5,0.67) & T(0,0,0.36,0.67) & T(0.34,0.5,0.67) \\ T(0,0,0.17) & T(0,0,0.17) & T(0.67,0.84,1) & T(0.4,0.5,0.67,0.84) & T(0,0,0.15,0.5) & T(0.34,0.5,067) & T(0.17,0.34,0.5) \\ T(0.84,1,1) & T(0.34,0.5,0.67) & T(0.84,1,1) & T(0.5,0.84,1,1) & T(0.17,0.34,0.5) & T(0.5,0.67,0.84,1) & T(0.67,0.84,1) \end{array}\right) \end{array}$$

Further all performance measuring attributes are not equally important. To take into account the variation in relative importance of the attributes, weight information is set as $w = (0.2, 0.1, 0.15, 0.15, 0.2, 0.1, 0.1)$.

With this available information about the locations' choices problem, we employ the proposed decision-making algorithm to prioritize the locations and to find the most suitable one.

Step 1

To carry out the linguistic computations, all the ELICIT expressions are required to transform into machine manipulative format, i.e., TrFNs. Decision maker's opinions in terms of ELICIT expressions given in D are converted into TrFNs and summarized in the matrix $\overline{D}$ and given in the previous page, where the first entry of $\overline{D}$, $T(0.5, 0.86, 1, 1)$ is the equivalent TrFN corresponding to the ELICIT expression at least $(s_4, 0)^0$, i.e., $\zeta^{-1}\left(\text{at least } (s_4, 0)^0\right) = T(0.5, 0.86, 1, 1)$.

Step 2

From the description of the attributes interrelationship pattern, it is quite evident that the attributes are heterogeneously related with no independent arguments. In the aim of capturing this heterogeneous interaction among the attributes and its reflection in the aggregated value, we choose ELICITWEBM (Eq. 29), to compute the overall performance of the alternatives. We set the associated parameter p and q to 1 in ELICITWEBM and compute the overall performance with the translated information $\overline{D}$ and weight information w. The results are summarized in the following Table 2. From Table 2 decision maker obtains the overall performance of alternatives expressed in terms of linguistic ELICIT expressions, which is quite intuitive to interpret. It is also clear to the decision maker from the Table 2 that X_3 is better than $\{X_2, X_4\}$ and X_2 is better than X_4. Undoubtedly, X_1 and X_5 are better than rest of the alternatives but it is not very clear about the order of the X_1 and X_5 from the linguistic overall performances. We are going to the next step for finding the exact ranking order of the alternatives.

Step 3

From the overall performances $r_i\,(i = 1, 2, 3, 4, 5)$, we compute the magnitude of the corresponding TrFNs, $T_{r_i}\,(i = 1, 2, 3, 4, 5)$ of the r_i as follows: $Mag\left(T_{r_1}\right) = 0.7416$, $Mag\left(T_{r_2}\right) = 0.4486$, $Mag\left(T_{r_3}\right) = 0.6242$, $Mag\left(T_{r_4}\right) = 00.3144$ and $Mag\left(T_{r_5}\right) = 0.7928$. Based on the $Mag\left(T_{r_i}\right)\,(i = 1, 2, 3, 4, 5)$, the ranking of the alternatives are as follows: $X_5 \succ X_1 \succ X_3 \succ X_2 \succ X_4$. Hence the location X_5 is the most suitable to set up the manufacturing plant followed by location X_1.

In the above analysis, we have set the parameters associated with ELICITWEBM as $(p, q) = (1, 1)$. But this choice of the parameters p and q associated with ELICITWEBM may have an impact on the final ranking of the locations. Thus, it is necessary to check the robustness of the ranking result concerning the parameters. For this purpose, we adopt the simulation-based approach, specifically, the framework of stochastic multi-criteria acceptability analysis [45]. As there is no preference over the parameters' values, we assume that the parameters are uniformly distributed in the space $[0.1, 100]^2$. By randomly drawing the parameters from the space $[0.1, 100]^2$, we solve the decision-making problem and find the ranking of the locations. Further, repeating this process for the sufficient numbers of times (10,000) within Monte Carlo framework, we collect the evidence in terms of probability of occupying a ranking position by an alternative. We report the result of the Monte Carlo in the Table 3, where $\mathbf{b}^r$ corresponding the alternative X_i denotes the probability of occupying r-th ranking position by X_i. It is quite evident that for the almost all configuration of the parameters from the space $[0.1, 100]^2$, the X_5 occupied the first ranking positions followed by X_1. Unanimously, X_3 is always occupied the third-ranking positions followed by X_2 and X_4. But there is a possibility of switching the ranking position between X_2 and X_4 for some configurations of the parameters. In nutshell, we can conclude that present ranking results are robust and not much sensitive to the parameters. Note that the exact estimation of the appropriate parameters associated with ELICITEBM could also be stem from the decision maker's perceived view towards aggregation process [22,46].

Table 1 Alternatives rating under different criteria.

	A_1	A_2	A_3	A_4	A_5	A_6	A_7
X_1	*at least S*	*at least VS*	*S*	*S*	*at least F*	*at least S*	*bt F and S*
X_2	*at most VUS*	*F*	*bt F and S*	*bt UF and VUS*	*at least F*	*F*	*F*
X_3	*VS*	*at least VS*	*S*	*VS*	*bt US and F*	*at most F*	*F*
X_4	*UF*	*UF*	*VUS*	*bt F and S*	*at most US*	*F*	*US*
X_5	*E*	*F*	*E*	*at least S*	*US*	*bt S and VS*	*VS*

bt = between.

Table 2 Alternatives overall performance.

Alternative	T_{r_i} (TrFN)	$r_i = \zeta^{-1}\left(T_{r_i}\right)$
X_1	$T(0.4545, 0.6963, 0.8106, 0.9102)$	*between* $(s_4, 0.1758)^{-0.0455}$ *and* $(s_5, -0.1362)^{-0.0898}$
X_2	$T(0.2612, 0.4084, 0.4889, 0.6351)$	*between* $(s_2, 0.4524)^{0.0942}$ *and* $(s_3, -0.0666)^{0.1351}$
X_3	$T(0.4401, 0.5869, 0.6567, 0.8315)$	*between* $(s_4, -0.4806)^{-0.0599}$ *and* $(s_4, -0.0618)^{-0.0015}$
X_4	$T(0.1862, 0.2874, 0.3241, 0.5291)$	*between* $(s_2, -0.2736)^{0.0192}$ *and* $(s_2, -0.0540)^{0.0291}$
X_5	$T(0.5639, 0.7797, 0.8286, 0.9084)$	*between* $(s_5, -0.3198)^{-0.1031}$ *and* $(s_5, -0.0264)^{-0.0916}$

Table 3 Percentage of occupying different ranking positions by alternatives.

Alternative	b^1	b^2	b^3	b^4	b^5
X_1	0.0040	99.9960	0	0	0
X_2	0	0	0	88.7300	11.2700
X_3	0	0	100	0	0
X_4	0	0	0	11.2700	88.7300
X_5	99.9960	0.0040	0	0	0

As we have emphasized on the fact that capturing the underlying interrelationship pattern in the aggregated ELICIT information is vital to make a reliable decision, it is worthy here to investigate the consequence if we do not consider the interrelationship in the information fusion process. For this purpose, we use the weighted ELICIT arithmetic mean operators, which assume that the input arguments are independent, in place of ELICITWBM in the proposed decision-making algorithm to compute the overall performances of the alternatives. Rest of the steps in our proposed MADM algorithm to find the ranking of the alternatives is kept unaltered. With this new configuration of the algorithm, we re-execute the step of the MADM algorithms and found the following ranking order of the alternatives $X_1 \succ X_5 \succ X_3 \succ X_2 \succ X_4$. It is evident that the ranking positions for X_1 and X_5 are reversed, which due to not capturing the underlying interrelationship structure among the attributes.

6. CONCLUSION

In this study, we have investigated the aggregation of linguistic information that is represented by ELICIT expressions and followed some specific interrelationship patterns. Specifically, we have considered three types of interrelationship patterns, namely, heterogeneous, homogeneous and partition structure among the aggregated arguments and such relationships are captured via direct conjunctions among the aggregated arguments with the core of three classical aggregation operators: BM, EBM, and PBM. In this view, we have extended these classical operators in ELICIT information environment and developed three new aggregation operators for aggregation ELICIT expressions, which we have referred to as ELICITBM, ELICITEBM, and ELICITPBM. Furthermore, we have investigated the properties of these aggregation operators and proposed the weighted form of these aggregation operators to deal with the situations where inputs have different relative importance. Using these aggregation operators as an information fusion tool, an algorithm for solving the MADM problems, in which attributes follow some specific interrelationship patterns, has been develped. Finally, we have presented numerical examples to illustrate the feasibility and applicability of our proposed approach.

In the future, it would be interesting to investigate the more complex interaction among the ELICIT expressions via Choquet integral [47]. Further, one may consider extending the aggregation of ELICIT expressions for other class of averaging aggregation operators, such as ordered weighted average operators [48] , power averaging operator [49], prioritize aggregation operator [50] and their different variants.

CONFLICT OF INTEREST

The authors declare that they have no conflict of interest.

AUTHORS' CONTRIBUTIONS

Bapi Dutta, Alvaro Labella, Rosa M. Rodriguez, and Luis Martinez did the conceptualization, methodology, validation, review writing, and editing. Bapi Dutta and Alvaro Labella wrote the original draft.

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APPENDIX A

A.1 ELICIT Inverse Function Example

In order to facilitate the understanding of the inverse function, ζ^{-1}, for ELICIT information, let us suppose a linguistic term set with seven labels, $S = \{s_0 : \textit{horrible}, s_1 : \textit{very bad}, s_2 : \textit{bad}, s_3 : \textit{medium}, s_4 : \textit{good}, s_5 : \textit{very good}, s_6 : \textit{perfect}\}$ and an ELICIT expression *between* $(s_3, 0.432)^{0.024}$ *and* $(s_4, 0.144)^{-0.023}$ (see Figure A.1).

First, it is necessary to compute the fuzzy envelope [18] of the ELICIT expression. To do that, the HFLTS of the expression is obtained through the transformation function defined in [21]:

$$E_{ELICIT}\left(\textit{between }(s_i,\alpha)\text{ and }\left(s_j,\alpha\right)\right) = \left\{s_k \middle| (s_i,\alpha)\text{ and }\left(s_j,\alpha\right)\text{ and } s_i < s_k < s_j \text{ where } s_k \in S\right\}$$

For our example:

$$E_{ELICIT}\left(\textit{between }(s_3,0.432)\text{ and }(s_4,0.144)\right) = \{s_k| (s_3,0.43)\text{ and }(s_4,0.14)\text{ and } s_3 < s_k < s_4 \text{ where } s_k \in S\} = \{(s_3,0.432),(s_4,0.144)\}$$

Once the HFLTS is computed, the different fuzzy memberships functions of the linguistic terms that belong to the HFLTS are aggregated with the OWA operator [48]. The OWA operator assigns different importance to the linguistic terms that compose the HFLTS through the *orness measure* thus, the way of computing the OWA weights affect directly to the resulting fuzzy envelopes. This process is carried out in [21] by means of a parameter, noted as $\epsilon \in [0, 1]$, which allows modifying the way to compute the OWA weights. The variation of ϵ modifies the importance of the linguistic terms of the HFLTS, in order to reduce the interval whose height is 1 in the fuzzy envelope. In [21], several fixed orness values provided by ϵ are used in order to compute fuzzy envelopes that preserve as much information as possible. The fixed values of ϵ are: $\epsilon = 0$ for *at least* relations, $\epsilon = 1$ for *at most* relations and $\epsilon_1 = 0$ and $\epsilon_2 = 1$ for *between* relations. Following this process, the resulting fuzzy envelope for the ELICIT expression is $T(0.405, 0.572, 0.691, 0.857)$.

Finally, the corresponding TrFN of the respective ELICIT expression is obtained by applying Prop. 16:

$$\begin{aligned}
&\zeta^{-1}\left(\textit{between }(s_3,0.432)^{0.024}\text{ and }(s_4,0.144)^{-0.023}\right)\\
&= T(0.429, 0.572, 0.691, 0.834)\\
&a = 0.405 + 0.024 = 0.429\\
&b = 0.572\\
&c = 0.691\\
&d = 0.857 + (-0.023) = 0.834
\end{aligned} \tag{A.1}$$

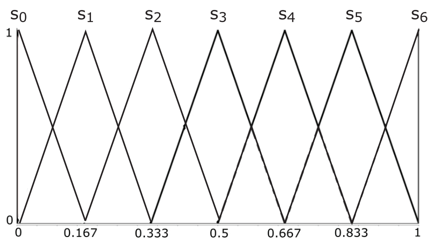

Figure A.1 Extended Comparative LInguistiC Expressions with SymbolIc Translation (ELICIT) information examples.

APPENDIX B

B.1 ELICIT Retranslation Process Example

In order to facilitate the understanding of the retranslation process to obtain an ELICIT expression from a TrFN, let us suppose the TrFN computed in A.1, $\tilde{\beta} = T(0.429, 0.572, 0.691, 0.834)$. The process to obtain an ELICIT expression is composed by several steps:

1. *Identify relation*: The relation of the ELICIT expression is determined by the fuzzy number $\tilde{\beta} = T(0.429, 0.572, 0.691, 0.834)$ and the ζ function (see Eq. 11).

$$\zeta\left(T(0.429, 0.572, 0.691, 0.834)\right) = EL, \tag{B.1}$$
$$\textit{where}\begin{cases} EL = \textit{at least }(s_i,\alpha)^{\gamma}\ \textit{if}\ \tilde{\beta} = T(a,b,1,1)\\ EL = \textit{at most }(s_i,\alpha)^{\gamma}\ \textit{if}\ \tilde{\beta} = T(0,0,c,d)\\ EL = \textit{between }(s_i,\alpha_1)^{\gamma_1}\textit{ and }\left(s_j,\alpha_2\right)^{\gamma_2}\\ \qquad \textit{if}\ \tilde{\beta} = T(a,b,c,d)\end{cases}$$

According to the fuzzy number $\tilde{\beta}$, the relation of the ELICIT expression is “between”.

2. *2-tuple linguistic terms computation* (see Figure B.1): The ELICIT expression with the relation “*between*” is composed by two continuous terms, $(s_i, \alpha_1)^{\gamma_1}$ and $\left(s_j, \alpha_2\right)^{\gamma_2}$.
   (a) *Compute linguistic terms*: First, we select the linguistic terms s_i and $s_j \in S, i, j \in \{0, 1, 2, 3, 4, 5, 6\}$, whose distance between the coordinates x of their respective centroids [42], $\overline{x}_i$ and $\overline{x}_j$, and the points $b = 0.572$ and $c = 0.691$ belonging to $\tilde{\beta}$ is minimal. In this case, such centroids are $\overline{x}_3$ and $\overline{x}_4$:

$$\begin{aligned} i &= \underset{h\in\{0,1,2,3,4,5,6\}}{\arg\min} \left|0.572 - \overline{x}_h\right| = 3\\ j &= \underset{h\in\{0,1,2,3,4,5,6\}}{\arg\min} \left|0.691 - \overline{x}_h\right| = 4 \end{aligned} \tag{B.2}$$

   The ELICIT expression so far is “*between* $(s_3, ?)^?$ *and* $(s_4, ?)^?$.”

   (b) *Compute symbolic translations*: Once the linguistic terms have been selected, the symbolic translations of the continuous terms are computed as follows:

$$\begin{aligned} &\alpha_1 = 6 \cdot (0.57 - 0.5) = 0.432\\ &\alpha_2 = 6 \cdot (0.691 - 0.667) = 0.144\\ &\alpha_1, \alpha_2 \in [-0.5, 0.5), \end{aligned} \tag{B.3}$$

   The ELICIT expression so far is “*between* $(s_3, 0.432)^?$ *and* $(s_4, 0.144)^?$.”

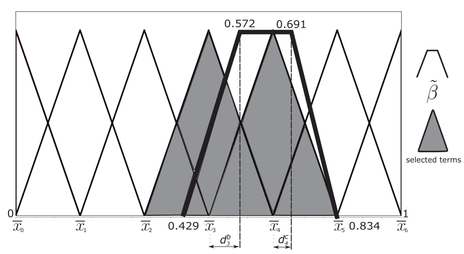

Figure B.1 Select linguistic terms.

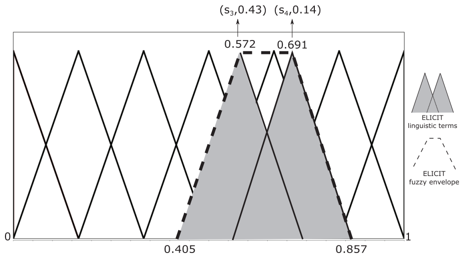

Figure B.2 | Extended Comparative LInguistiC Expressions with SymbolIc Translation (ELICIT) fuzzy envelope.

3. *Compute adjustments*: Finally, to complete the ELICIT expression, we compute the adjustments for the ELICIT expression following the steps below:
   (a) *Compute HFLTS*:

$$\begin{aligned} & E_{ELICIT}\,(between\ (s_3, 0.432)\ \text{and}\ (s_4, 0.144)) \\ & = \{s_k | \, (s_3, 0.432)\ \text{and}\ (s_4, 0.144)\ \text{and} \\ & \quad s_3 < s_k < s_4 \text{ where } s_k \in S\} \\ & = \{(s_3, 0.432), (s_4, 0.144)\} \end{aligned}$$

   (b) *Compute fuzzy envelope* (see Figure B.2): The fuzzy envelope [18] of the HFLTS $\{(s_3, 0.432), (s_4, 0.144)\}$ is:

$$T_{ELICIT} = T\,(0.405, 0.572, 0.691, 0.857)$$

   (c) *Compute adjustments γ_1 and γ_2*:

$$\begin{aligned} \gamma_1 &= 0.429 - 0.405 = 0.024 \\ \gamma_2 &= 0.834 - 0.857 = -0.023 \\ gamma_1&, \gamma_2 \in [0, 1] \end{aligned} \tag{B.4}$$

Finally, the ELICIT expression is completed "*between* $(s_3, 0.432)^{0.024}$ *and* $(s_4, 0.144)^{-0.023}$."

APPENDIX C

C.1 Proof of Theorem 1

By using operational laws of fuzzy numbers, we have

$$\left(\zeta^{-1}(EL_i)\right)^p \otimes \left(\zeta^{-1}\left(EL_j\right)\right)^q = \left(a_i^p a_j^q, b_i^p b_j^q, c_i^p c_j^q, d_i^p d_j^q\right) \tag{C.1}$$

Clearly, the right-hand side of Eq. (C.1) is a TrFN due to the assumption $0 \le a_i \le b_i \le c_i \le d_i\,(i = 1, 2, ..., n)$ on the parameters of the envelope of ELICIT expression $\zeta^{-1}(EL_i)$. Further the Eq. (C.1) is true for any pair of ELICIT expressions $\left(EL_i, EL_j\right)$ $\left(i, j \in \{1, 2, ..., n\}\right)$. As the addition of TrFNs is associative, we can extend easily to the addition of $n\,(n-1)$ TrFNs of the form $\left(\zeta^{-1}(EL_i)\right)^p \otimes \left(\zeta^{-1}\left(EL_j\right)\right)^q$ $\left(i, j \in \{1, 2, ..., n\}, i \neq j\right)$ and obtain

$$\bigoplus_{\substack{i,j=1 \\ i \neq j}} \left(\zeta^{-1}(EL_i)\right)^p \otimes \left(\zeta^{-1}\left(EL_j\right)\right)^q = \left(\sum_{\substack{i,j=1 \\ j \neq i}}^{n} a_i^p a_j^q, \sum_{\substack{i,j=1 \\ j \neq i}}^{n} b_i^p b_j^q, \sum_{\substack{i,j=1 \\ j \neq i}}^{n} c_i^p c_j^q, \sum_{\substack{i,j=1 \\ j \neq i}}^{n} d_i^p d_j^q\right) \tag{C.2}$$

With the help of scalar multiplication laws of TrFNs, we get

$$\begin{aligned} & \frac{1}{n\,(n-1)} \bigoplus_{\substack{i,j=1 \\ i \neq j}} \left(\zeta^{-1}(EL_i)\right)^p \otimes \left(\zeta^{-1}\left(EL_j\right)\right)^q \\ & = \left(\frac{1}{n\,(n-1)} \sum_{\substack{i,j=1 \\ j \neq i}}^{n} a_i^p a_j^q, \frac{1}{n\,(n-1)} \sum_{\substack{i,j=1 \\ j \neq i}}^{n} b_i^p b_j^q, \right. \\ & \left. \frac{1}{n\,(n-1)} \sum_{\substack{i,j=1 \\ j \neq i}}^{n} c_i^p c_j^q, \frac{1}{n\,(n-1)} \sum_{\substack{i,j=1 \\ j \neq i}}^{n} d_i^p d_j^q \right) \end{aligned} \tag{C.3}$$

Finally by using exponential operational laws of TrFN from Eq. (C.3), we obtain

$$\begin{aligned} & \left(\frac{1}{n\,(n-1)} \bigoplus_{\substack{i,j=1 \\ i \neq j}} \left(\zeta^{-1}(EL_i)\right)^p \otimes \left(\zeta^{-1}\left(EL_j\right)\right)^q\right)^{\frac{1}{p+q}} \\ & = \left(\left(\frac{1}{n\,(n-1)} \sum_{\substack{i,j=1 \\ j \neq i}}^{n} a_i^p a_j^q\right)^{\frac{1}{p+q}}, \left(\frac{1}{n\,(n-1)} \sum_{\substack{i,j=1 \\ j \neq i}}^{n} b_i^p b_j^q\right)^{\frac{1}{p+q}}, \right. \\ & \left. \left(\frac{1}{n\,(n-1)} \sum_{\substack{i,j=1 \\ j \neq i}}^{n} c_i^p c_j^q\right)^{\frac{1}{p+q}}, \left(\frac{1}{n\,(n-1)} \sum_{\substack{i,j=1 \\ j \neq i}}^{n} d_i^p d_j^q\right)^{\frac{1}{p+q}} \right) \end{aligned} \tag{C.4}$$

Since $a_i \le b_i \le c_i \le d_i$ for all $i = 1, 2, ..., n$, the monotonocity property of the $BM_{p,q}: [0, 1]^n \to [0, 1]$ implies Eq. (C.5).

$$\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ j\neq i}}^{n} a_i^p a_j^q\right)^{\frac{1}{p+q}} \leq \left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ j\neq i}}^{n} b_i^p b_j^q\right)^{\frac{1}{p+q}} \quad (C.5)$$

$$\leq \left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ j\neq i}}^{n} c_i^p c_j^q\right)^{\frac{1}{p+q}} \leq \left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ j\neq i}}^{n} d_i^p d_j^q\right)^{\frac{1}{p+q}}$$

$$\begin{aligned}&ELICITBM_{p,q}\left(EL_{\sigma(1)}, EL_{\sigma(2)}, ..., EL_{\sigma(n)}\right)\\ &= \zeta\left(BM_{p,q}\left(a_{\sigma(1)}, a_{\sigma(2)}, ..., a_{\sigma(n)}\right),\ BM_{p,q}\left(b_{\sigma(1)}, b_{\sigma(2)}, ..., b_{\sigma(n)}\right),\right.\\ &\left.BM_{p,q}\left(c_{\sigma(1)}, c_{\sigma(2)}, ..., c_{\sigma(n)}\right), BM_{p,q}\left(d_{\sigma(1)}, d_{\sigma(2)}, ..., d_{\sigma(n)}\right)\right)\end{aligned} \quad (C.6)$$

It infers that $\left(\frac{1}{n(n-1)} \oplus_{\substack{i,j=1\\ i\neq j}} \left(\zeta^{-1}(EL_i)\right)^p \otimes \left(\zeta^{-1}\left(EL_j\right)\right)^q\right)^{\frac{1}{p+q}}$ is TrFN and therefore $ELICITBM_{p,q}(EL_1, EL_2, ..., EL_n)$ is an ELICIT expression. Hence the results.

C.2 Proof of Theorem 2

(i) First we will show that *ELICITBM* is commutative. Let $EL_{\sigma(1)}, EL_{\sigma(2)}, ..., EL_{\sigma(n)}$ is a permutation of the ELICIT expressions $EL_1, EL_2, ..., EL_n$. With the help of computational formula Eq. (17), we can express $ELICITBM\left(EL_{\sigma(1)}, EL_{\sigma(2)}, ..., EL_{\sigma(n)}\right)$ in the form of Eq. (C.6)

The values of the parameters p and q, and the underlying interrelationship structure among the aggregated ELICIT expressions remain intact in the permutation $\left(EL_{\sigma(1)}, EL_{\sigma(2)}, ..., EL_{\sigma(n)}\right)$. Further such interrelationship is also inherited in the parameters of the TrFNs $\left(\left(a_{\sigma(i)}, b_{\sigma(i)}, c_{\sigma(i)}, d_{\sigma(i)}\right)(i = 1, 2, ..., n)\right.$, which are the envelope of the ELICIT expression $EL_{\sigma(i)}$, $(i = 1, 2, ..., n)$. Thus, the components of the envelopes of $EL_{\sigma(i)}$, $(i = 1, 2, ..., n)$ become connected. Under this circumstance, BM exhibits the commutative property, i.e.,

$$BM_{p,q}\left(a_{\sigma(1)}, a_{\sigma(2)}, ..., a_{\sigma(n)}\right) = BM_{p,q}(a_1, a_2, ..., a_n)$$

It follows that

$$\begin{aligned}&ELICITBM_{p,q}\left(EL_{\sigma(1)}, EL_{\sigma(2)}, ..., EL_{\sigma(n)}\right)\\ &= \zeta\left(BM_{p,q}(a_1, a_2, ..., a_n), BM_{p,q}(b_1, b_2, ..., b_n),\right.\\ &\quad\left.BM_{p,q}(c_1, c_2, ..., c_n), BM_{p,q}(d_1, d_2, ..., d_n)\right)\\ &= ELICITBM_{p,q}(EL_1, EL_2, ..., EL_n)\end{aligned} \quad (C.7)$$

(ii) Now we will show that *ELICITBM* operator is idempotent. Let $\zeta^{-1}(EL) = (a, b, c, d)$ be the envelope of the ELICIT expression EL. From the Eq. (17), we have

$$\begin{aligned}&ELICITBM_{p,q}(EL, EL, ..., EL)\\ &= \zeta\left(BM_{p,q}(a, a, ..., a), BM_{p,q}(b, b, ..., b),\right.\\ &\quad\left.BM_{p,q}(c, c, ..., c), BM_{p,q}(d, d, ..., d)\right)\end{aligned} \quad (C.8)$$

Since the BM operator is idempotent, i.e., $BM_{p,q}(e, e, ..., e) = e$, we obtain from Eq. (C.8)

$$ELICITBM_{p,q}(EL, EL, ..., EL) = \zeta(a, b, c, d) = \zeta\left(\zeta^{-1}(EL)\right) = EL$$

(iii) Now we will prove that *ELICITBM* is ratio-scale invariant. Let $r > 0$ be a scalar. From the scalar multiplication law of TrFN, we have $r\zeta^{-1}(EL_i) = (ra_i, rb_i, rc_i, rd_i)$. From the definition of ELICITBM, we obtain

$$\begin{aligned}&ELICITBM_{p,q}(rEL_1, rEL_2, ..., rEL_n) \quad (C.9)\\ &= \left(\frac{1}{n(n-1)} \oplus_{\substack{i,j=1\\ i\neq j}} \left(\zeta^{-1}(rEL_i)\right)^p \otimes \left(\zeta^{-1}\left(rEL_j\right)\right)^q\right)^{\frac{1}{p+q}}\\ &= \zeta\left(BM_{p,q}(ra_1, ra_2, ..., ra_n), BM_{p,q}(rb_1, rb_2, ..., rb_n),\right.\\ &\quad\left.BM_{p,q}(rc_1, rc_2, ..., rc_n), BM_{p,q}(rd_1, rd_2, ..., rd_n)\right)\end{aligned}$$

As the BM operator is ratio-scale invariant i.e. $BM_{p,q}(re_1, re_2, ..., re_n) = rBM_{p,q}(e_1, e_2, ..., e_n)$, from Eq. (C.9) we have

$$\begin{aligned}&ELICITBM_{p,q}(rEL_1, rEL_2, ..., rEL_n)\\ &= \zeta\left(rBM_{p,q}(a_1, a_2, ..., a_n), rBM_{p,q}(b_1, b_2, ..., b_n),\right.\\ &\left.rBM_{p,q}(c_1, c_2, ..., c_n), rBM_{p,q}(d_1, d_2, ..., d_n)\right)\\ &= r\zeta\left(BM_{p,q}(a_1, a_2, ..., a_n), BM_{p,q}(b_1, b_2, ..., b_n),\right.\\ &\left.BM_{p,q}(c_1, c_2, ..., c_n), BM_{p,q}(d_1, d_2, ..., d_n)\right)\\ &= rELICITBM_{p,q}(EL_1, EL_2, ..., EL_n)\end{aligned}$$

C.3 Proof of the Theorem 3

We will show that ELICITBM is bounded. Since $a_i \geq \min_i a_i$ for all i, the monotonocity and idempotency of properties of the BM operator implies that

$$BM_{p,q}(a_1, a_2, ..., a_n) \geq BM\left(\min_i a_i, \min_i a_i, ..., \min_i a_i\right) = \min_i a_i$$

Similarly, we can obtain

$$\begin{aligned}BM_{p,q}(b_1, b_2, ..., b_n) &\geq \min_i b_i\\ BM_{p,q}(c_1, c_2, ..., c_n) &\geq \min_i c_i\\ BM_{p,q}(d_1, d_2, ..., d_n) &\geq \min_i d_i.\end{aligned}$$

From these inequalities, we have

$$\begin{aligned}&\left(BM_{p,q}(a_1, a_2, ..., a_n), BM_{p,q}(b_1, b_2, ..., b_n),\right.\\ &\left.BM_{p,q}(c_1, c_2, ..., c_n), BM_{p,q}(d_1, d_2, ..., d_n)\right)\\ &\geq \left(\min_i a_i, \min_i b_i, \min_i c_i, \min_i d_i\right)\end{aligned} \quad (C.10)$$

Note that the inequality Eq. (C.10) is in the sense of lexicographic ordering of TrFNs, i.e., $(a_1, b_1, c_1, d_1) \geq (a_2, b_2, c_2, d_2)$ iff $a_1 \geq a_2$, $b_1 \geq b_2$, $c_1 \geq c_2$ and $d_1 \geq d_2$. From Eq. (C.10), we have

$$\begin{aligned} &ELICITBM_{p,q}(EL_1, EL_2, ..., EL_n) \\ &= \zeta\left(BM_{p,q}(a_1, a_2, ..., a_n), BM_{p,q}(b_1, b_2, ..., b_n),\right. \\ &\quad \left. BM_{p,q}(c_1, c_2, ..., c_n), BM_{p,q}(d_1, d_2, ..., d_n)\right) \\ &\geq \zeta\left(\min_i a_i, \min_i b_i, \min_i c_i, \min_i d_i\right) \end{aligned}$$

Similarly, we can show that

$$\begin{aligned} &ELICITBM_{p,q}(EL_1, EL_2, ..., EL_n) \\ &\leq \zeta\left(\max_i a_i, \max_i b_i, \max_i c_i, \max_i d_i\right) \end{aligned}$$

Hence the result.

4.4. Análisis del Desempeño de los Modelos Clásicos de Consenso en la Toma de Decisión en Grupo a Gran Escala: Un Estudio Comparativo

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Analyzing the performance of classical consensus models in large scale group decision making: A comparative study

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ABSTRACT

Consensus reaching processes (CRPs) in group decision making (GDM) attempt to reach a mutual agreement among a group of decision makers before making a common decision. Different consensus models have been proposed by different authors in the literature to facilitate CRPs. Classical CRP models focus on achieving an agreement on GDM problems in which few decision makers participate. However, nowadays, societal and technological trends that demand the management of larger scale of decision makers add new requirements to the solution of consensus-based GDM problems. This paper presents a comparative study of different classical CRPs applied to large-scale GDM in order to analyze their performance and find out which are the main challenges that these processes face in large-scale GDM. Such analyses will be developed in a java-based framework (AFRYCA 2.0) simulating different scenarios in large scale GDM.

1. Introduction

Group decision making (GDM) problems, in which multiple individuals/experts with their own attitudes/opinions need to achieve a common solution to a decision problem consisting of several alternatives or possible solutions, have become the focus of a large body of research [1–4]. GDM problems widely exist in diverse application areas that require the joint participation of multiple experts, such as management, engineering, politics and so on [5–7]. In the traditional resolution process of GDM problems [8], the best alternative/alternatives should be chosen after each expert provides his/her own preference over alternatives, disregarding the level of agreement among the preferences of different experts. This often leads to the shortcoming that some experts may not accept the decision result [2], because they might consider that their opinions have not been considered. For this reason, consensus reaching processes (CRPs), in which individuals/experts discuss and modify their preferences in order to reach a collective agreement before making decisions [9], have become an increasingly prominent research topic in GDM problems [10–12].

Classically, GDM problems have been solved by a few number of experts. However, the expansion of technological paradigms, such as e-democracy [6], social networks [13], and marketplace selection for group shopping [14], call for the public attention for the so-called large scale GDM (LGDM) problems, in which a larger number of experts take part in the decision process and responsibility for the decision result. Chen and Liu [15] classified the GDM problems in which the decision makers exceed 20 into LGDM problems. It is noticed that experts have to face a lot of new challenges in terms of the resolution of LGDM problems, such as the higher resources consuming and the time invested for decision making. It requires a higher complexity with respect to the analysis of experts' preferences in LGDM problems, for instance, to detect the conflicts and the closeness amongst experts' opinions, identify the scale of experts that agree/disagree with each other and find coalitions/subgroups of the same or similar interests in the group, etc.

Thorough the study on CRPs over the past few decades, different theoretical consensus models have been proposed [16–22]. On the other hand, in order to provide groups with computer-based decision support systems focused on supporting CRPs, some researches have been done in the development of consensus support systems (CSSs) [20,23–25], based on the implementation of different consensus models.

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Despite the research already conducted on CRPs, there are still some aspects that require improvement. One of them is the demand for managing large groups in such processes. Managing LGDM problems makes more frequent the existence of strong disagreement cases among some experts in the group, therefore the necessity of applying a CRP is higher [26]. As far as we know, most of the existing CRPs are focused on GDM problems with few experts. There is no any depth and systematic study about their performance dealing with LGDM problems yet. Even though, specific proposals for CRPs in LGDM have been introduced [26–28], it seems necessary to make a study about the performance of classical CRPs developed for GDM with few decision makers to evaluate their ability and shortages in the new contexts of LGDM. Consequently, this paper aims at developing a comparative study of different classical CRPs widely used in the literature by using AFRYCA 2.0 [29], a framework which allows to simulate different scenarios for GDM in which decision makers can adapt different behaviors regarding the CRP.

With this study our goal is to answer the following questions:

1. Which is the performance of different types of classical CRPs in the context of LGDM?
   This question is two-fold:
   - The number of experts involved in the GDM can influence the performance of the consensus model, if so, at what extent?
   - A large number of experts make easier to break the collaboration contract to achieve an agreement and non-cooperative behaviors can appear and bias the agreement. Can classical consensus models reach consensus in such LGDM contexts?
2. Is time cost crucial in all classical CRPs to deal with a LGDM problem?
   It also implies a two-fold view:
   - The number of experts involved in the GDM can imply an increasing of time cost in the CRP, can classical consensus models manage the time cost in LGDM?
   - What kind of consensus models deal better with the time cost in LGDM to achieve the agreement?

By a comparative study on the performance of different existing classical consensus models in LGDM problems, the answer of the previous questions could be achieved, and provided some suggestions and necessary conditions that should be added to consensus models in order to manage CRPs in LGDM problems.

This paper is structured as follows: in Section 2, some basics about GDM, LGDM, CRPs and a taxonomy of classical consensus models are reviewed. In Section 3, the framework, AFRYCA 2.0, for the analysis of consensus approaches is briefly introduced. Based on this framework Section 4 introduces and develops a comparative study on performance of different consensus models in LGDM. Section 5 shows new challenges that CRPs should face to deal with LGDM inferred from previous study. Finally, some concluding remarks are provided in Section 6.

2. Background

In this section, GDM problems and several main concepts related to CRPs and a taxonomy about them are reviewed. The notion of large-scale GDM is then revised, as well as some main challenges which experts may encounter during the CRP of LGDM problems.

2.1. Group decision making

GDM is the process of reaching a common judgment or a common solution for a decision making problem, which consists of a set of alternatives or possible solutions, with the participation of multiple individuals. Decision making results made by multiple experts with various types of knowledge and experience are usually supposed to be better compared with those made by only one expert [3].

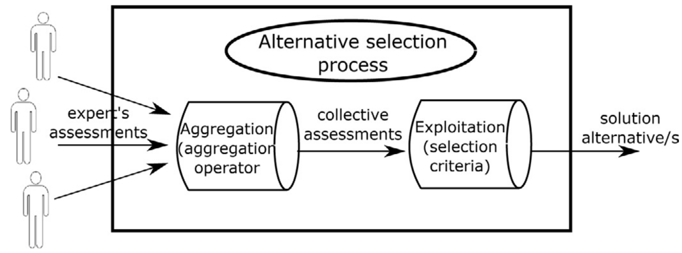

Fig. 1. Selection process for the solution of GDM problems.

A GDM problem can be formally defined as a decision situation in which there are [4]:

1. A group of m individuals/experts, $E = \{e_1, e_2, \ldots, e_m\}$, each one of them has his/her own knowledge and attitude.
2. A decision problem containing n alternatives or possible solutions, which is denoted by $X = \{x_1, x_2, \ldots, x_n\}$.
3. The individuals/experts try to achieve a common solution.

In the common process of a GDM problem, each expert in E expresses his/her preferences over different alternatives in X, by means of a certain kind of preference structure. *Preference Ordering of the Alternatives* [30], *Utility Values/ Utility Function* [31] and *Preference Relation* [32] are some widely used preference representation formats. *Preference Relation* is briefly reviewed below.

For each expert $e_i \in E$, construct a function $\mu_{pi} : X \times X \to D$ where D is the information representation domain and $\mu_{pi}(x_l, x_k) = p^i_{lk} (l, k \in \{1, 2, \ldots, n\})$ denotes the preference degree or intensity of the alternative x_l over x_k in D. Then, these expert's preferences on all alternatives in X can be described as a matrix $P^i = (p^i_{lk})_{n\times n}$. Depending on the information representation domain D, different types of preference relations can be used, such as fuzzy preference relations [4,33], multiplicative preference relations [34] and linguistic preference relations [35–39].

The most commonly used preference structure in GDM approaches is the fuzzy preference relation associated to expert e_i represented by matrix $P^i = (p^i_{lk})_{n\times n}$, where:

1. p^i_{lk} denotes the preference degree associated to expert e_i of alternative x_l to x_k;
2. $D = [0, 1]$, that is, $p^i_{lk} \in [0, 1]$;
3. $p^i_{lk} = 0.5$ indicates indifference between x_l and x_k;
4. $p^i_{lk} > 0.5$ indicates that x_l is preferred over x_k. Especially, $p^i_{lk} = 1$ indicates that x_l is absolutely preferred over x_k ;
5. In order to obtain the consistent preference relations, it is usual to assume the additive reciprocity property, i.e. $p^i_{lk} + p^i_{kl} = 1$ ($\forall l, k \in \{1, \ldots, n\}$).

Regarding GDM solving approaches, there are two common approaches to solve a GDM problem: a direct approach or an indirect approach [8]. In the former approach, the solution can be directly obtained based on the individual preferences of experts, rather than constructing a social opinion first. Meanwhile in the latter approach, a social opinion or a collective preference is computed first, and it is then utilized to achieve a solution for the problem. The classical alternative selection process for reaching a solution to GDM problems contains two phases [40], as shown in Fig. 1: (i) Aggregation phase: by using an aggregation operator, the experts' preferences are combined. (ii) Exploitation phase: by using

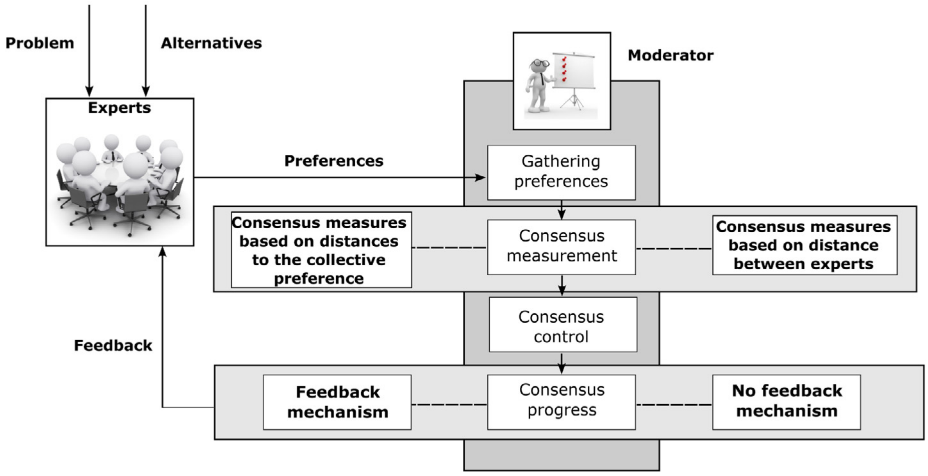

Fig. 2. General CRP scheme.

a selection criterion, an alternative or a subset of alternatives will be obtained as the solution for the problem.

2.2. Consensus in group decision making

If a GDM problem is solved only by the selection process, the existence of agreement amongst experts cannot be guaranteed, which may lead to a solution which cannot be accepted by some experts who feel that their individual opinions have not been taken into consideration [2]. Since a high level of acceptance degree by the whole group could be critical in a number of real-life GDM problems, it is necessary to add a phase so-called "consensus" to the resolution process for GDM problems. A CRP is a dynamic and iterative process consisting of several rounds of discussion, it is designed to reach a compromise before making a decision [2,9]. Reaching consensus implies that experts should modify their initial opinions throughout the CRP in order to bring them closer to the opinions of the rest of the group. The term consensus can be defined to refer to "the mutual agreement produced by consent of all memberships in a group or between several groups" [9]. The concept of consensus has been interpreted from various perspectives, from unanimity to some more flexible interpretations considering different degrees of partial agreement [41]. As one of the most accepted approaches to soften the concept of consensus, the notion of *soft consensus* which is defined as "most of the important individuals agree with almost all of the relevant opinions", was introduced by Kacprzyk et al. based on the concept of "fuzzy majority" [4].

The process of reaching consensus is usually coordinated by a human figure known as moderator. The moderator takes responsibility for supervising and guiding the discussion amongst experts [2,9]. A general CRP scheme followed by a large number of consensus models consists of four main phases (see Fig. 2):

1. Gathering preferences The preferences of each expert are provided and collected in this phase.
2. Consensus measurement The moderator makes use of experts' individual preferences to estimate the current group agreement level by consensus measures. Based on the type of computations and information fusion procedures applied to measure consensus, the existing consensus measures have been classified by Palomares et al. [42] into two categories:
   - Consensus measures based on distances to the collective preference: In this case, firstly a collective preference should be computed by aggregating all individual preferences of experts, then the consensus degrees are obtained by computing the distances between each individual preference and the collective preference [37,43,44].
   - Consensus measures based on distances between experts: In this case, firstly the similarity values between each different pair of experts in the group should be calculated based on the similarity/distance metrics, then the consensus degrees are obtained by aggregating these similarity values [18,45–47].
3. Consensus control The consensus degree obtained previously is compared with a *threshold value* $\boldsymbol{\mu} \in [0, 1]$, which indicates the minimum value of acceptable agreement. If the consensus degree exceeds the threshold value, μ, means that the desired consensus has been achieved, the group moves into the selection process; otherwise, another discussion round should be carried out. It is worth noticing that another threshold value *maxrounds* $\in \mathbb{N}$, which indicates the maximum number of allowed rounds can be introduced in order to prevent a never ending process.
4. Consensus progress A procedure should be adopted to increase the level of agreement throughout the discussion rounds of the CRP. The procedure can also be classified into two categories [42]:
   - Traditionally, such a procedure incorporates a feedback generation process, in which the moderator identifies the farthest assessments from consensus and then advises them to modify their assessments in the direction to increase the consensus degree in the following rounds [9,41]. Each expert has the responsibility to modify his/her own assessments to get close to the collective preference.
   - Some other consensus models employ a procedure without a feedback generation process, by implementing approaches in which the experts' assessments can be updated automatically to increase consensus in the group [44,48,49].

A lot of different consensus approaches have been proposed during the past decades. So far, various criteria have been used to categorize different consensus approaches, such as the reference domain used to compute the soft consensus measures, the coincidence method used to compute the soft consensus measures, the generation method of recommendations supplied to the experts and the kind of measures used to guide the CRP [11]. In this paper it is utilized the categorization introduced in [42] that considers two types of consensus measures and two classes of consensus progress procedures to propose a taxonomy for consensus models, graphically shown in Fig. 3:

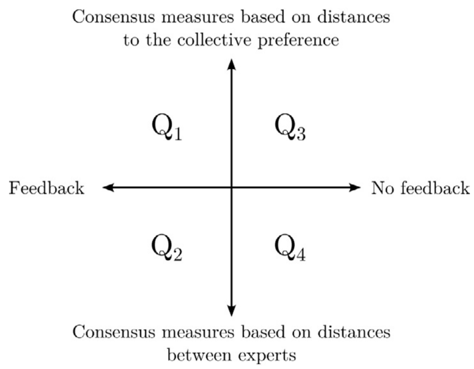

Fig. 3. A taxonomy of approaches for consensus reaching.

Q_1: Consensus models with feedback mechanism and a consensus measure based on computing distances to the collective preference.

Q_2: Consensus models with feedback mechanism and a consensus measure based on computing pairwise similarities.

Q_3: Consensus models without a feedback mechanism and with a consensus measure based on computing distances to the collective preference.

Q_4: Consensus models without a feedback mechanism and with a consensus measure based on computing pairwise similarities.

2.3. Large-scale decision making and its challenge in consensus

Current technological and societal demands have made necessary to make decisions in which a huge amount of participants take part. As a result, LGDM which indicates GDM with a larger number of individuals/experts, attain a greater importance. The presence of a larger number of participants could definitely increase the complexity of a given problem. So far, studies on LGDM concentrate on four categories, i.e., cluster methods in LGDM, CRP in LGDM, LGDM methods, and LGDM support systems [50].

Two main differences between classical GDM and LGDM are: (1) the number of decision makers and the amount of information in the latter case is larger; (2) in LGDM, more time is needed to achieve a final decision, especially when agreement is required.

Some of the challenges that CRPs should face caused by LGDM problems are the following ones:

1. *Non-cooperative behaviors*: Since the amount of decision makers is very large in a LGDM problem, experts cannot cooperate to achieve an agreement. Two typical non-cooperate experts' behaviors in a LGDM problem are described below and noted in this paper as follows;
   - *Refuse behavior* (see Fig. 4): After receiving some suggestions to get closer to the group opinion, the individuals/experts may refuse to change his/her initial preference.
   - *Defense behavior* (see Fig. 5): In this case, the individuals/experts may change his/her initial preference in an opposite direction in order to bias the consensus.

   This paper also refers to the cooperative behavior of experts as *accept behavior*, which indicates the expert will accept the suggestions to get closer to the group.
2. *Subgroup behaviors*: Non-cooperative behavior may be no longer just a personal behavior in LGDM. In other words, when CRPs are carried out in large-scale contexts, there may exist some subgroups of experts who have similar interests and do not want to change their initial positions. They may collaborate to break the collaboration contract [41] at some stage, by refusing to modify their preferences [27], or by moving their preferences on the contrary way in order to bias the final solution for the GDM problem [51]. Hence, it is critical to identify timely and dispose effectively these subgroup non-cooperative behaviors to ensure correct CRPs development.

Fig. 4. Refuse behavior.

Fig. 5. Defense behavior.

3. *Minority opinions*: In order to ensure a correct decision result, Xiong et al. [52] spoke highly of the importance of minority opinions in the CRPs and proposed a consensus mechanism to protect such opinions. However, it will be much more difficult to take into account all the minority opinions in a large group situation.
4. *Supervision*: The need for constant human supervision for preferences by either the moderator or experts during the CRP will be much more complex in a LGDM problem [22,26,53,54].

Other difficulties caused by time cost in a LGDM problem which must be considered in consensus models may be the following ones:

1. Some emergency decision problems ask for a relatively satisfactory result within a short time, which requires effective coordination of the non-cooperative behaviors mentioned above [55]. In LGDM, the existence of non-cooperative behaviors and group non-cooperative behaviors indicate higher time cost in

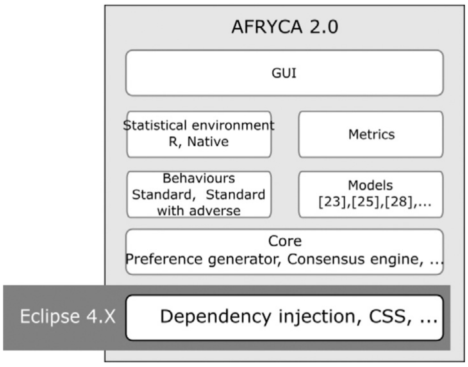

Fig. 6. AFRYCA 2.0 architecture.

CRPs. Then the issue of balancing the relationship between decision quality and time invested emerges.

2. Under a consensus model with feedback mechanism, time cost of supervising and modifying opinions might not only increase the CRP's discussion rounds considerably in LGDM, but also lead to a result that some experts may lose their motivation and interest and then eventually abandon the discussion process [41].
3. The phenomenon that human moderator may tend to consider the opinions of his/her own interest would be more apparent and serious in large-scale decisions, since they need to save time cost. This phenomenon implies that real consensus cannot be reached by the whole group [41]. Although some existing CSSs have took place of human moderator in order to prevent constant supervision by the human moderator [21,22,53], dealing with large-scale CRPs still requires the development of more appropriate architectures that manage the large amount of information efficiently.

3. A framework for the analysis of consensus approaches: AFRYCA 2.0

Our paper aims to analyze the performance of different classical consensus models in LGDM problems and the development of this task is not simple, specially when it is necessary to take into account a large number of experts in the CRP. The necessity of a suitable tool which allows to simulate the performance of the distinct consensus models and the behavior of the experts who take part in the CRP, is clear to achieve our objective. For this reason, this section revises briefly a software so-called, *A Framework for the analYsis of Consensus Approaches* (AFRYCA) [42], that will be used to carry out the simulation of CRPs and the solving process of GDM problems by using different consensus models proposed in the literature. Specifically, the latest version of this software, AFRYCA 2.0 [29], is used to simulate different experts behavior patterns during the CRPs. In technological terms, AFRYCA 2.0 is a component-based application which has been developed by using Eclipse Rich Client Platform (Eclipse RCP) [56], a platform to build and deploy desktop rich client applications easy to maintain and extend. AFRYCA 2.0 [29] uses more than 40 components which are grouped in six types (see Fig. 6):

- *Graphical User Interface (GUI)*: Components which allow to interact with the framework.
- *Statistical environments*: Two statistical environments are included in AFRYCA 2.0, R[1] and a native statistical environment. They are able to carry out Multi-Dimensional Scaling (MDS) of the preferences and the simulation of behavior patterns by means of probability distributions. The statistical environment can be selected during the runtime of the program.
- *Metrics*: Components to analyze several consensus models and the CRPs performance.
- *Behavior patterns*: Components which simulate expert's behavior regarding the advice received. AFRYCA 2.0 includes two behavior patterns: (1) the *standard behavior pattern* (see Fig. 7), which simulates behaviors of experts accept/refuse suggestions; (2) the *standard with adverse behavior pattern* (see Fig. 8), which allows to simulate behaviors of experts accept/refuse/defense recommendations.
- *Models*: Components which implement consensus models proposed in the literature. Each component corresponds to a consensus model and it includes the different phases and parameters considered in such a model. AFRYCA 2.0 implements eight consensus model components [26,28,31,57–61]. Furthermore, to carry out this paper, another consensus model has been included in AFRYCA 2.0 [62].
- *Core*: Components which implement the main features of AFRYCA 2.0 such as, preference generator, consensus engine, etc.

Fig. 7. Optional behaviors of experts in *standard behavior pattern*.

Fig. 8. Optional behaviors of experts in *standard with adverse behavior pattern*.

Therefore AFRYCA components provide different functionalities that can be used for:

[1] https://www.r-project.org/.

Table 1
Behaviors default values.

	Standard	Standard with adverse
p	0.5	0.5
c	–	0.25
μ	0.05	0.05
Size	0.2	0.2

- The performance analyses of consensus models to analyze their advantages and weaknesses.
- Performance analysis of a consensus model under different situations by using its setting configuration.
- Selection of the most suitable consensus model for a specific type of GDM problem through its results reporting.
- Easy comparison of different consensus models by using the graphical interface.

AFRYCA 2.0 allows to carry out experiments with different consensus models implemented in the framework. It is possible to evaluate and compare the performances of different consensus models in LGDM by AFRYCA 2.0, since it provides important information such as initial consensus degree, final consensus degree, ranking of alternatives and final solutions. Furthermore, AFRYCA 2.0 is able to show graphically the state of the experts' preferences for each round by means of a graphical 2-D representation, with MDS [63] (see Fig. 9).

When using AFRYCA 2.0 to simulate the resolution of a GDM problem with a consensus model implemented, the methodology can be divided into 5 steps:

1. *Framework defining*: A specific example of GDM problem should be settled, to be solved by applying the pre-selected consensus model.
2. *Model choosing*: A consensus model is chosen from those included in the framework.
3. *Parameters configuration*: Configure the parameters for the consensus model and behaviors of experts, such as consistency of generated preference relations, consensus thresholds, aggregation operators, etc.
4. *Simulation of the CRP*: Once the consensus model settings are fixed, the CRP should be carried out.
5. *Alternative selection process and analysis of the results*.

In AFRYCA 2.0, two behaviors patterns can be simulated (see Figs. 7 and 8). In the *standard behavior pattern*, the experts are allowed to accept/refuse suggestions. In the *standard with adverse behavior pattern*, the experts are allowed to accept/refuse/defense suggestions. To carry out such behaviors different aspects are taken into account in AFRYCA 2.0:

- In the standard behavior pattern, the probability for experts to accept suggestions has been simulated by a binomial probability distribution, which is configured by a parameter p.
- In the standard with adverse behavior pattern, besides the refuse behavior, the defense behavior has also been taken into consideration. Hence, besides parameter p mentioned above, a new parameter c will be added to configure another binomial probability distribution, which is used to simulate the probability for experts to move into an opposite direction of suggestions.

Although all parameters can be configured in AFRYCA 2.0, this framework has been defined with some default values (see Table 1).

4. Comparative study on the performances of classical CRPs models in LGDM

In this section, a comparative study on the performance of classical CRPs models in LGDM is carried out. First, different representative consensus models with different features are selected for the study. Second, it is necessary to describe the LGDM scenarios in which the comparative study will be developed. Afterwards, the simulation by using AFRYCA 2.0 will be carried out for all models in each scenario defined previously; obtaining different results that will be analyzed for each consensus model in order to find out necessary conditions to reach consensus in LGDM problems. And from such individual analyses a comparative analysis among all models is performed. Eventually, previous study will support us to obtain key characteristics that may be necessary to add to classical CRPs for dealing successfully with LGDM problems. In this way, if it is possible, managers/decision makers will be able to select suitable classical consensus models for LGDM, and even construct some new appropriate consensus models which fit such a type of problems.

4.1. Choosing classical CRPs for study

Due to the multiple proposals introduced in the specialized literature to carry out CRPs in GDM before developing our comparative study it is necessary to choose several classical CRPs to show their performance in LGDM. Therefore for such a selection and according to the taxonomy revised in Fig. 3 from [42], one representative model from each quadrant is selected:

- *Representative model in* Q_1: consensus model with a feedback mechanism and a consensus measure based on computing distances to the collective preference. The model selected was proposed by Herrera-Viedma et al. in [31], it has been selected because:
  - ○ It follows the soft consensus view [11].
  - ○ It is the first attempt to use proximity measures taking place of the moderator.
  - ○ Both consensus measure and proximity measures are based on the comparison of the individual solutions and the collective solution.
  - ○ The comparison for alternatives is done by comparing the position of the alternatives in each solution, which allows us to know the real consensus situation in each moment during the consensus process.
  - ○ It allows experts to express their preferences by using different preference structures and then uniform diverse preferences into fuzzy preference relations.

  The Herrera-Viedma et al.'s consensus model needs several parameters, for its implementation in the simulation framework, which are briefly introduced here (see [31] for further detailed descriptions):
  - β: parameter to control the OR-LIKE of the aggregation operator that computes the global consensus degree.
  - Aggregation quantifiers: parameters of the linguistic quantifier used to compute the collective preference by means of the OWA operator.
  - Exploitation quantifiers: parameters of the linguistic quantifier used to compute dominance and non-dominance degrees and conduct preferences of experts into preference orderings.
- *Representative model in* Q_2: consensus models with a feedback mechanism and a consensus measure based on computing pairwise similarities. The model selected is the proposed by Chiclana et al. in [57], because:
  - Initially it was introduced as a framework for integrating individual consistency into a consensus model.

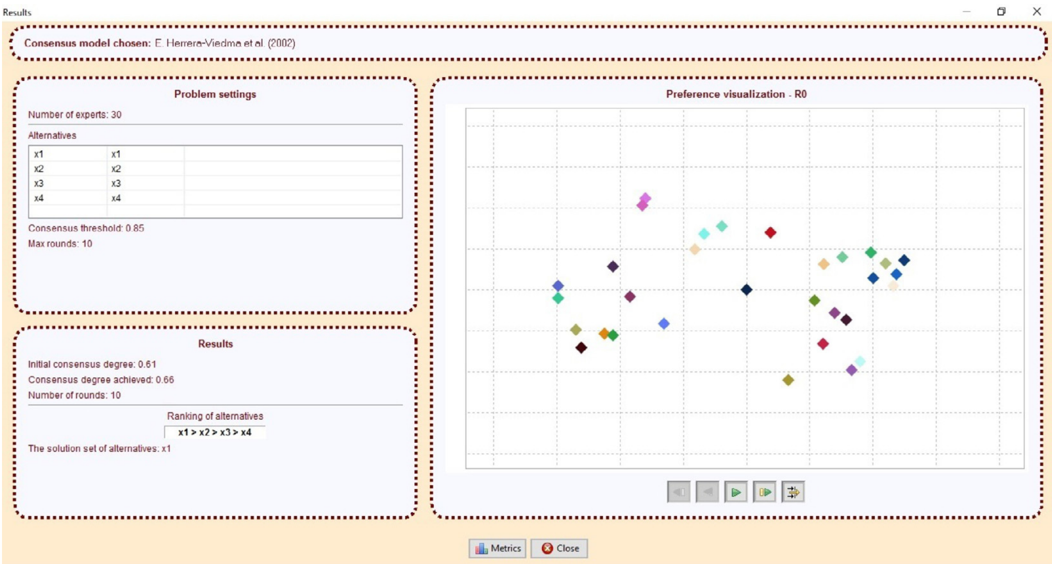

Fig. 9. AFRYCA 2.0 results.

- It has been the basis for further extensions for AHP consensus models introduced by Dong et al. [64].
- Also it has been the basis for a linear optimization model for reaching consensus proposed by Zhang et al. [62].

Similarly to the previous model, Chiclana et al.'s one needs also several parameters for the simulation (further detail about parameters in [57]):

– B: consistency threshold for preferences.
– θ_1, low consensus threshold: if consensus degree is lower than this value, a low consensus preference search is applied.
– θ_2, medium consensus threshold: a medium or high consensus level is applied depending on whether consensus degree is lower or higher than this value, respectively.

- *Representative model in* Q_3: consensus models without a feedback mechanism and with a consensus measure based on computing distances to the collective preference. In this case, two models proposed by Wu et al. in [58] and Xu et al. in [60] are selected because they have similar characteristics, but the former one deals with individual consistency and it is worthy to analyze in this type of consensus models. Therefore, we preferred in this case to study both of them because due to the lack of feedback mechanism their simulation will be easier (see Remark 1):
  - Both models considered are simple and straightforward.
  - These two consensus models can be easily extended with different features generalizing them.
  - In the achievement of a predefined consensus level, each individual preference relation is still ensured to be of acceptable consistency [58].
  - Both the individual consistency and the group consensus are stressed in the consensus process introduced in [58].

  Some parameters in Wu et al.'s consensus model are necessary for its simulation (please refer to [58] to see detailed descriptions):

  – CI: individual consensus threshold.
  – β: update coefficient for assessments.
  – W_i: experts weights.

  The parameters for Xu et al.'s consensus model are shown below (see [60] for further detail):

  – CI: individual consensus threshold.
  – γ: group consensus threshold.
  – W_i: experts weights.

- *Representative model in* Q_4: consensus models without a feedback mechanism and with a consensus measure based on computing pairwise similarities. The model selected in this case is proposed by Zhang et al. in [62], because:
  - It optimally preserves the original preference information when constructing individual consistency and reaching consensus.
  - This model extends the consistency-driven consensus model of Chiclana et al. to ensure a minimum cost of modifying preferences.
  - It can be used not only for conducting the CRP, but also to reach a high level of consistency for each individual preference relation.

  The parameters necessary in Zhang et al.'s model for the simulation are (see [62] for detailed descriptions):

  – cl: consistency level for each preference. The expert's preferences change and each one has to reach this minimal consistency threshold.
  – ccl: consensus consistency level. The consensus among the different preferences have to reach this minimal consensus threshold.

4.2. LGDM scenarios

Earlier it was pointed out that LGDM problems present several challenges. One of the most important is the different behaviors which appear in the CRP, due to the large numbers of experts involved in it. It is vital to take into account that many experts can present a non-cooperate behavior in real life and, although these experts can refuse the suggestions provided or even go in an opposite direction of the suggestions, they can never been ignored in the evaluation of CRP in LGDM. For this reason, it is necessary to define different *scenarios* which adjust to these challenges by simulating different behaviors. In this way, different simulations as real as possible are proposed. AFRYCA 2.0 first generates consistent fuzzy preference relations, according to [65], for the experts involved in the LGDM and then it will develop the consensus simulation in the following three scenarios (initial preferences are the same for all simulations):

- *Scenario 1*: In this scenario, all experts accept all the recommendations. This kind of scenario is the *ideal* one but not very common in real world problems. It is interesting check how classical consensus models work with favorable conditions.

- *Scenario 2*: In this scenario, 80% of the experts accept all the recommendations. On the other hand 20% of the experts present a defense behavior.
- *Scenario 3*: In this scenario, 70% of the experts accept all the recommendations. On the other hand 20% of the experts refuse the suggestions and the 10% of the experts present a defense behavior.

Remark 1. It is important to highlight that those consensus models without feedback will perform similarly in the three scenarios because they do not consider the experts' opinions after round one and then, the experts' behavior is not meaningful in their performance.

4.3. Results and analysis

This section presents an experimental study to compare the performance of different consensus models in LGDM, during the resolution of GDM problems with a large amount of experts.

Therefore, let us suppose the following LGDM problem: the International Olympic Committee organizes a special committee which is composed of 30 members from all over the world $E=\{e_1, e_2, \ldots, e_{30}\}$, to make a decision on the place where the Olympic Games in 2040 will be held. It is final selection round and there are only four candidate cities: $X=\{x_1$: Paris, x_2: Tokyo, x_3: Madrid, x_4: New York$\}$.

All preferences are expressed as fuzzy preference relations generated by AFRYCA, the corresponding data sets are available in the public access of AFRYCA website.[2] To find a satisfactory solution for this problem, the consensus threshold and the maximum consensus discussions rounds in the CRP are set as $\boldsymbol{\mu}=\mathbf{0.85}$ and **maxround = 30** respectively. *Maxround* has been selected for sake of clarity about consensus models performance, but usually is much smaller. Hence, if the consensus threshold, μ, is not reached after 30 discussion rounds, the simulation stops and then the results at that round are shown indicating that the consensus has not been reached.

This comparative study is carried out on the previous LGDM problem, such that the five consensus models selected in Section 4.1 will be applied to it taking into account the different scenarios of application.

Remark 2. Wu et al.'s model measures consensus with Individual Consensus Indices $ICI(P_i)=d(P_i, P_c)$ for each $e_i \in E$ [58], and Xu et al.'s model measures consensus with Group Consensus Index (GCI) [60]. To facilitate the comparative analysis in this section, benefiting from the idea in [42], the consensus degrees for Wu et al.'s model and Xu et al.'s model are given by $1-max_i\, ICI(P_i)$ and $1-GCI$, respectively.

For each simulation performed, experts behaviors have been configured with the parameter values shown in Table 1. The consensus models have been configured with the parameter values shown in Table 2. Results of the LGDM problem resolution with different consensus models are shown in Tables 3–5, keeping in mind that the results in Table 5 are not sensitive to experts' behaviors (see Remark 1).

4.3.1. Analysis for each representative model

Here a single analysis for each consensus model according to its performance in the different scenarios for the LGDM problem is developed. Such an analysis consists of a brief explanation of the results obtained with their graphical visualization together an analysis of its performance inferring the main advantages and disadvantages of each model.

- Herrera-Viedma et al.'s model [31]
  - ○ Simulation results:

    This model reaches consensus in the three scenarios evaluated, even when there exists non-cooperative behaviors such as in scenarios 2 and 3 (see Fig. 10). Evidently such non-cooperative behaviors may imply more discussions rounds (Scenario 3 needs 8 discussion rounds, others only 6).

    The ranking of alternatives and the solution set of alternatives in all the scenarios are the same which shows that model is robust and coherent in their consensus process.
  - ○ Analysis:

    It is worth noticing that this model weights the alternatives for computing the consensus measure by using S-OWA OR-LIKE operator [66]. By using a parameter β, that bounds the impact of non-cooperative behaviors to a certain degree.

    That is the reason why the simulation results in Scenarios 1, 2 and 3 have similar performances. However, it should be remarked that experts' consensus degree on each alternative is based on an average operator that does not weight expert's behavior in the CRP process. Hence, the impact of non-cooperative behavior is limited to some extent but not in a general way. If we look carefully at Fig. 10 some experts, in Scenarios 2 and 3, seems to be quite far away from mutual agreement. Therefore, to show the good performance of the model is limited, we carried out a new simulation in which the consensus threshold was fixed as $\mu=0.9$, in such a case the scenario 2 could not reach consensus after *maxrounds* = 30 (see Fig. 11), due to the averaging process is not enough for this situation.

    Based on previous analysis, in order to guarantee a robust and correct performance of this model in LGDM, it is necessary the weighting of the set of alternatives and include some penalization in the computation of the consensus degree to decrease the impact of behaviors in Scenarios 2 and 3.
  - ○ Advantages:

    Benefiting from the simulation results and the analysis, it can be seen that the performance of Herrera-Viedma et al.'s model in this LGDM could be good because:
    - The existence of refuse and defense behaviors can be managed by using S-OWA OR-LIKE operator but not in all situations;
    - The decision results tend to be robust in different scenarios.
    - The number of discussion rounds necessary to reach consensus is relatively small taking into account the LGDM problem.
  - Disadvantages:
    - As is shown in Fig. 10, although the model reaches the consensus, there are some experts far away from the mutual agreement, which indicates that the final consensus is reached by *ignoring* some experts' opinions.
    - The weighting of alternative set versus the weighting of experts regarding their behavior can lead to deadlock situations in which agreement is not reaching.
- Chiclana et al.'s model [57]
  - Simulation results:

    Unlike the previous one, this model just reaches the consensus within the *maxrounds* in Scenario 1 with 13 rounds, but not in Scenarios 2 and 3 in which not all experts accept suggestions from feedback process. Additionally, the ranking obtained by the model in different scenarios and solution set are not robust.

[2] http://sinbad2.ujaen.es/afryca/.

Table 2
Consensus models parameters.

Wu et al. [58]	Xu et al. [60]	Zhang et al. [62]	Herrera-Viedma et al. [31]	Chiclana et al. [57]
$\mu = 0.85$ $\beta = 0.8$ $\bar{CI} = 0.15$ $w_i = \frac{1}{30}, i = 1, \ldots, 30$	$\mu = 0.85$ $\gamma = 0.2$ $\bar{CI} = 0.15$ $w_i = \frac{1}{30}, i = 1, \ldots, 30$	$\mu = 0.85$ $cl = 0.95$ $ccl = 0.85$	$\mu = 0.85$ $\beta = 0.8$ $Aggregation\ quantifier = F_{most}$ $Exploitation\ quantifier = F_{as\ many\ as\ possible}$	$\mu = 0.85$ $B = 0.8$ $\theta_1 = 0.7$ $\theta_2 = 0.8$

Table 3
CRP simulations results with Herrera-Viedma et al.'s model [31].

Herrera-Viedma et al. [31]	Initial consensus degree	Final consensus degree	Number of rounds	Ranking	Solution
Scenario 1	0.61	0.87	6	$x_1 \succ x_2 \succ x_3 \succ x_4$	x_1
Scenario 2	0.61	0.86	8	$x_1 \succ x_2 \succ x_3 \succ x_4$	x_1
Scenario 3	0.61	0.85	8	$x_1 \succ x_2 \succ x_3 \succ x_4$	x_1

Table 4
CRP simulations results with Chiclana et al.'s model [57].

Chiclana et al. [57]	Initial consensus degree	Final consensus degree	Number of rounds	Ranking	Solution
Scenario 1	0.603	0.855	13	$x_1 \succ x_4 \succ x_2 \succ x_3$	x_1
Scenario 2	0.603	0.72	–	$x_4 \succ x_2 \succ x_1 \succ x_3$	x_4
Scenario 3	0.603	0.703	–	$x_4 \succ x_2 \succ x_1 \succ x_3$	x_4

Table 5
CRP simulations results with no feedback consensus models.

Models without feedback	Initial consensus degree	Final consensus degree	Number of rounds	Ranking	Solution
Wu et al. [58]	0.568 (0.432)	0.72 (0.28)	–	$x_1 \succ x_2 \succ x_4 \succ x_3$	x_1
Xu et al. [60]	0.303 (0.697)	0.876 (0.124)	4	$x_1 \succ x_2 \succ x_4 \succ x_3$	x_1
Zhang et al. [62]	0.605	0.85	1	$x_1 \succ x_2 \succ x_3 \succ x_4$	x_1

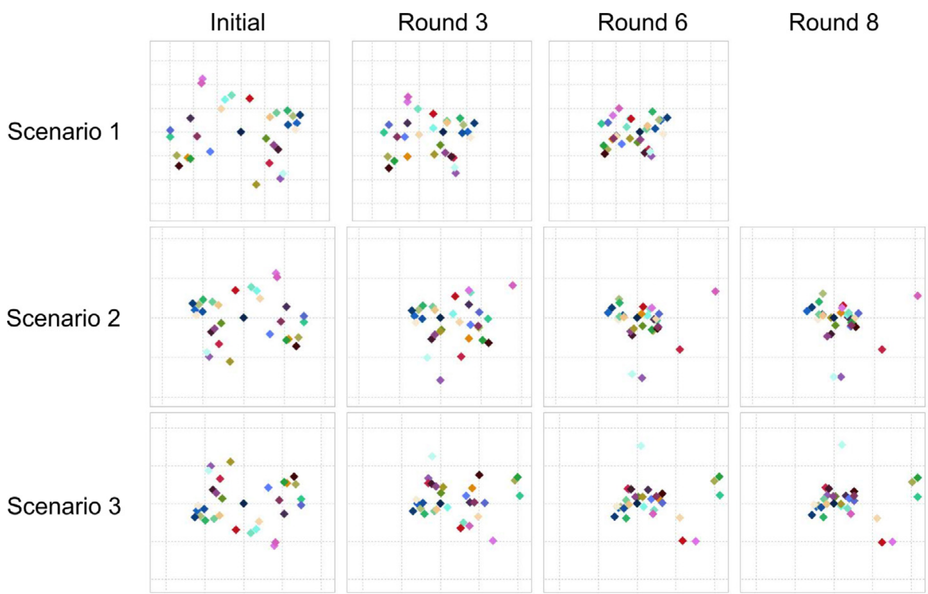

Fig. 10. MDS visualization of CRP using Herrera-Viedma et al.'s model [31].

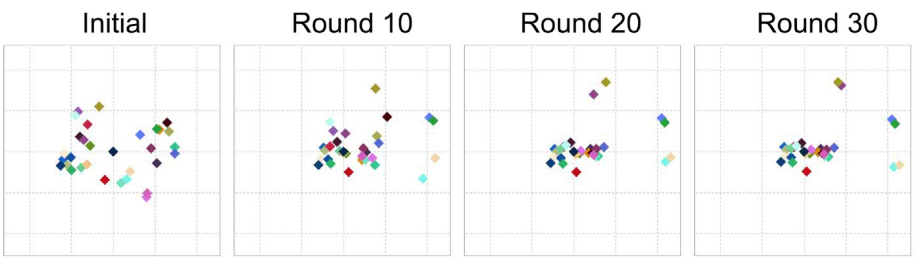

Fig. 11. MDS visualization of CRP using Herrera-Viedma et al.'s model [31] with a consensus threshold 0.9.

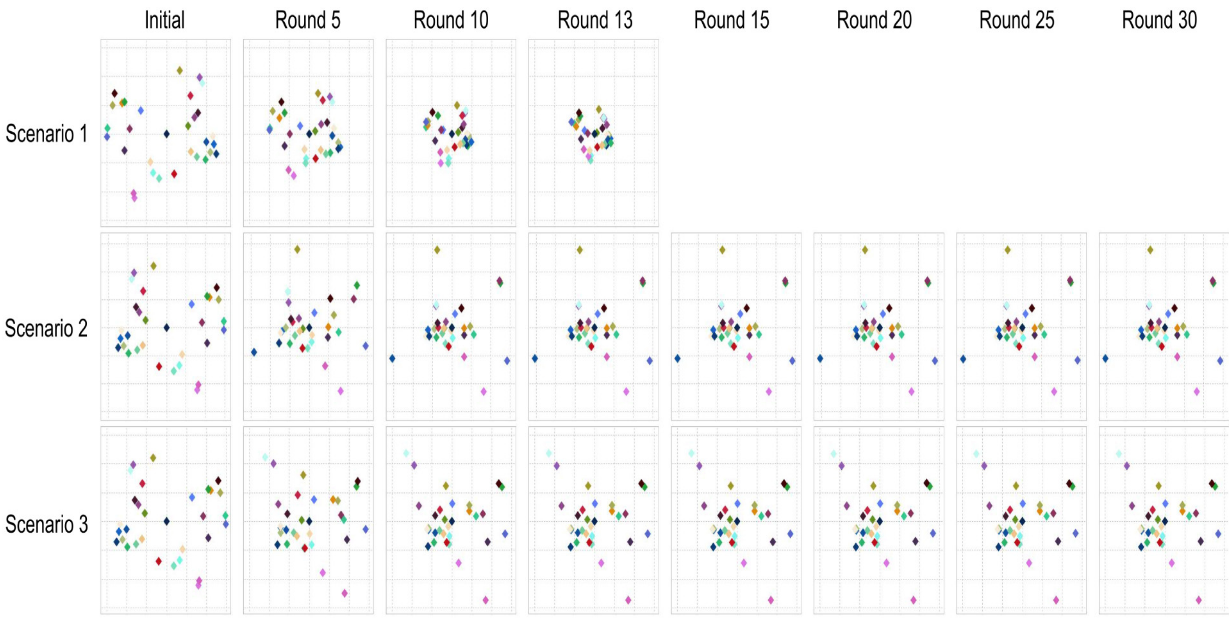

Fig. 12. MDS visualization of CRP using Chiclana et al.'s model [57].

In Fig. 12 can be seen that model cannot effectively manage non-cooperative behaviors of Scenarios 2 and 3.

- Analysis:

  This model deals with consensus at different levels: relation, alternatives and pair of alternatives. The *consensus on the relation* is calculated based on the average of all alternatives, and the *consensus on alternatives* is calculated based on the average of the *consensus on pairs of alternatives*. The weights of experts have been neither determined nor updated based on their behaviors during the CRP when people calculate the consensus. Besides, the proximity degree is calculated in a similar way of the consensus degree based on an average operator, there are not any mean to detect and deal with non-cooperative behaviors during the feedback process, that is the reason why the existence of non-cooperative behaviors leads to deadlock in the consensus process. However, in lots of practical situations of LGDM problems, experts' non-cooperative behaviors cannot be avoided. Hence, the current model need improvements to fit real-world LGDM problems.
- Advantages:
  - In the ideal situation when all experts accept suggestions, the consensus can be successfully reached within several discussion rounds but more than previous model;
  - Determined by the construction of the model which adopts different feedback methods when reaching different consensus degrees, the CRP saves human-being efforts to a certain degree by limiting the rounds for specific experts to change their preferences;
- Disadvantages:
  - The existence of non-cooperative behaviors is not well managed by the model and leads to situations in which the consensus cannot be reached;

- Wu et al.'s model [58]
  - Simulation results:

    Taking into account Remark 1, this model does not consider experts' behaviors because there is not a feedback mechanism in the model. Therefore, the results shown Fig. 13 are the same for the three scenarios, and it can be seen that the model cannot reach the consensus threshold, μ, in any of them within *maxrounds*.
  - Analysis:

    Due to the fact that in this model just one expert's preferences are changed in each round, the consensus process is very slow for LGDM and then a large amount rounds of changing will be needed to reach the consensus threshold by the group.
  - Advantages:
    - Behaviors not affect to the CRP;
    - This model considers not only the group consensus, but also the individual consistency at the same time.
  - Disadvantages:
    - Each round changes only one expert's preferences, which result in a slow process to achieve agreement, especially in large-group problems.
    - Due to the consensus process in this model, it might happen that expert's preferences close to the collective preference should be changed, because the expert is the farthest from the group.
    - This model ignores real experts' preferences because there is not a feedback mechanism that guides experts to express their genuine modified preferences.
- Xu et al.'s model [60]
  - Simulation results:

    Similarly to the previous model, the results in Fig. 14 are valid for all scenarios (see Remark 1). In this case the consensus model reaches the consensus threshold, μ, with just four rounds.
  - Analysis:
    - This consensus model carries out the consensus progress without feedback mechanism but unlike the Wu et al.'s model, in this case the experts' preferences changed in each round are much more than in [58].
    - These changes are carried out based on a group and individual indexes that optimize the distances among experts by means of a quadratic program, which makes the CRP more efficient to reach the consensus threshold.
  - Advantages:
    - Its efficiency to reach consensus within few rounds due to the mathematical programming process.
    - Behaviors not affect to the CRP.
  - Disadvantages:

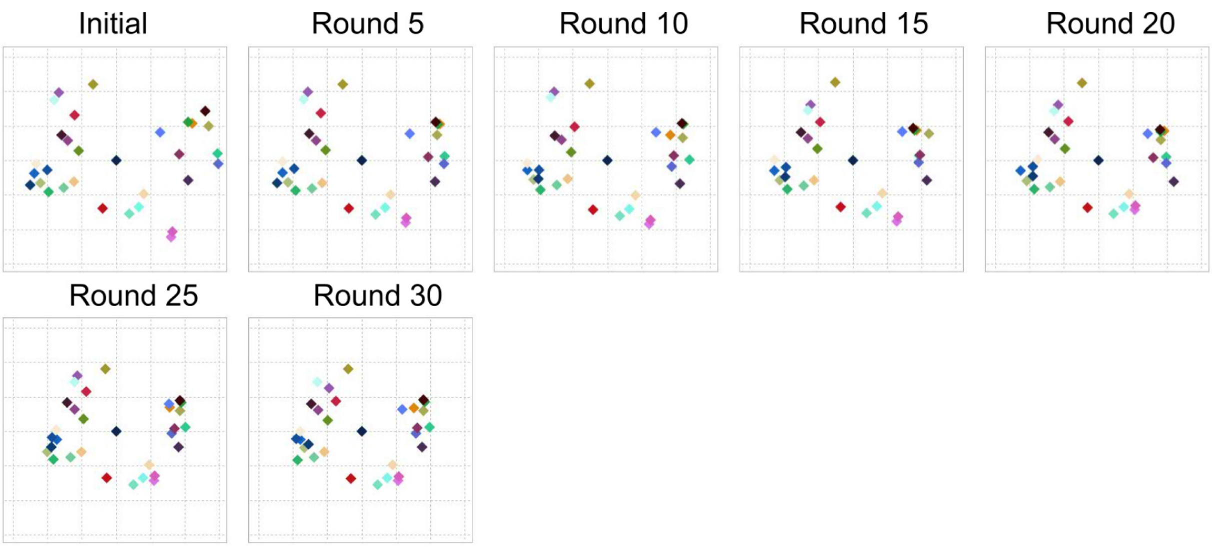

Fig. 13. MDS visualization of CRP using Wu et al.'s model [58].

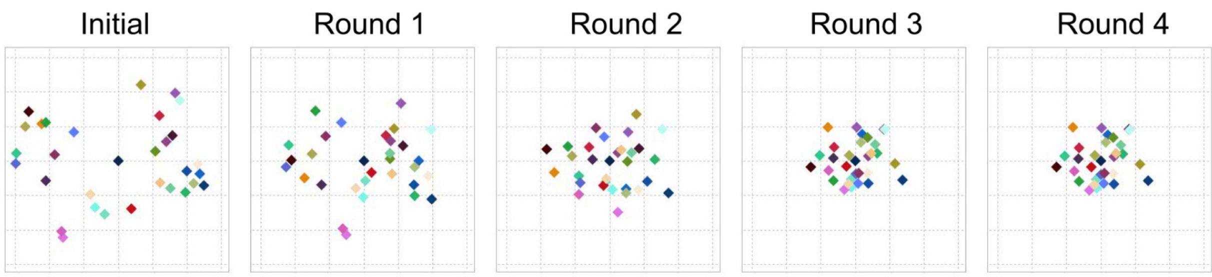

Fig. 14. MDS visualization of CRP using Xu et al. model [60].

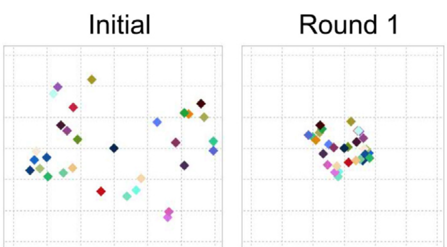

Fig. 15. MDS visualization of CRP using Zhang et al. model [62].

- It does not consider the individual consistency to reach the consensus despite no experts' uncertainty is involved in the revised preferences.
- As a consensus model without feedback, it ignores real experts' preferences.

- Zhang et al.'s model [62]
  - Simulation results:
    This model is the last one with no feedback and again all scenarios obtain the same results graphically shown in Fig. 15. It is remarkable that this model reaches the consensus threshold, μ, just in 1 round, because it just looks for the preferences that achieve an agreement by means of a linear optimization model.
  - Analysis:
    In spite of this simulation the model performs quite well, we should be aware that this model presents an important risk, because the linear optimization model utilized for computing and controlling the consensus process, might be irresolvable and hence other model should be applied to achieve the agreement.
  - Advantages:
    - If the linear optimization consensus model can be solved for the LGDM problem the consensus threshold can certainly be reached within one round.
    - Zhang et al.'s model takes into account not only group consensus but also individual consistency.
  - Disadvantages:
    - Since the restrictions of linear optimization model are very strict, it is hard to determine when a consensus threshold can be reached a priori.
    - Despite some experts' preferences are substantially changed, it ignores real experts' preferences, that it is a common drawback of consensus models without feedback;
    - In Zhang et al.'s model, the time cost is highly dependent on the number of experts, so this model presents an important problem of scalability.

4.3.2. *Comparative analysis*

Taking into account research questions introduced in Section 1 and looking at the previous results as a whole. There are some important issues that should be stressed:

- Even though consensus models without feedback mechanism are not affected by non-cooperative behaviors like the models with feedback, the former ones with their automatic changing strategies highly impact on the expert's preferences changing many of them in each round that can initially seem more suitable for the context of LGDM to reach the consensus threshold, but even these models might not be able to achieve the consensus threshold, μ established in the LGDM, either. They face the scalability problem with more difficulties in LGDM than the latter models because

of their mathematical background to carry out the consensus progress. Eventually the large modification of experts' preferences in such a type of problems without considering experts' genuine opinions can lead to decisions not accepted by the large group.
- Due to the fact of unaccepted decisions by experts within the large group, it should be considered the use of consensus models for LGDM despite could be less efficient to achieve the agreement. But in such cases experts can perform refuse/defense behaviors not only just in one round, but also across the whole consensus process. Therefore, the performance of classical consensus models with feedback mechanisms in LGDM with this real-world circumstances in which the consensus model cannot reach an agreement because these experts does not follow the collaboration contract [41] such as it happens in [31,57] although the Herrera-Viedma's model shows a better management of LGDM, even with non-cooperative behaviors, and seems promising to deal with any LGDM problem with just some adjustments.
- Comparing the performances of Herrera-Viedma et al.'s model and Chiclana et al's model in Scenarios 2 and 3, it seems clear that the use of weighting processes for computing consensus fits better LGDM problems, hence it appears the opportunity to revise/penalize weights for experts based on their behaviors during the CRP when calculating the collective preference can improve the performance to ensure the reaching of the consensus threshold in these models.
- Analyzing Figs. 10–12 it is easy to see that models with feedback from experts face the non-cooperative behaviors and when they are able to reach the consensus threshold, several experts are still far away from the mutual agreement and that is the reason that sometimes agreement is not possible to reach. However, when models without feedback reach the consensus threshold the cohesion of the different experts is higher. This issue is quite interesting for further analysis later on.

5. New challenges

As it is revised in Section 2, CRPs needs to deal with several challenges and difficulties when they are applied to LGDM problems that have already been noticed by experts, such as the higher time consuming, the need for more time on constant preferences supervision and the higher complexity with respect to dealing with experts' non-cooperative behaviors. All these challenges have been clearly visualized in the case study. To overcome these challenges, Palomares et al. [27] provided several tools to detect and manage the non-cooperative behaviors in the context of LGDM:

- A fuzzy clustering-based scheme was used to detect non-cooperating individuals or subgroups in their research. In [28] was proposed an extended method to manage participators' behavior in CRP in LGDM, in which, a weighting approach coorperates with uninorm aggregation operator which determines the importance weights of participators according to their overall behavior across the CRP, and thus overcomes the shortage in [27] in which the participators' importance weights cannot be increased again, even though they change their attitudes and decide to adopt more cooperating behaviors.
- Palomares et al. [26] proposed a semi-supervised multi-agent system which reduces time cost of preference supervision and allows experts to revise preferences manually when human supervision is convenient and necessary, which can be regarded as a consensus model with semi-feedback mechanism.

According to our previous study, it is clear that not all the classical consensus models are appropriate for managing LGDM problems. Therefore, although new consensus models are necessary to deal with LGDM, first it should be analyzed if the improvement of classical existing models is a better way to face the challenges of LGDM. Some models can be easily improved to fit the context of LGDM, whereas others can be too much complex and maybe it is better to design other type of specific consensus models for LGDM.

By studying the different performances of the consensus models in the comparative study, it can be observed several key conditions that can be added to consensus models in order to manage LGDM problems in a suitable way. These new conditions can be summarized as below:

1. For consensus models with feedback mechanism:
   - *Weighting measures*: Consensus and proximity measures based on the distance offer an easier and effective way to weight the experts based on their behaviors during the CRPs when calculating the collective preference.
   - *Weighting alternatives*: If alternatives are also weighted when calculating the consensus measure the convergence to consensus threshold could be quicker.
2. For consensus models without feedback mechanism:
   - *Automatic changing scheme*: It should be able to manage multiple experts' opinions at each round otherwise it is not adequate for LGDM problems.
   - *Flexibility*: The conditions of optimization models should be flexible enough to reach consensus when they are used to deal with LGDM problems.

From the results and analyses obtained in the comparative study together the previous conditions, we can figure out several new challenges which should be faced by consensus models within LGDM problems in the future:

1. *Weighting processes*:
   - Within consensus models with feedback mechanism, different weighting mechanisms not only for experts but also for alternatives can provide suitable ways to penalize non-cooperative behaviors in LGDM.
2. *Optimization models*:
   - Within consensus models with feedback mechanism, they should consider seriously time cost for consensus models in LGDM.
   - The restrictions should be decreased or make more flexible in order to adapt them to LGDM, otherwise they become irresolvable.
3. *Hybrid consensus models*:
   - According to our intuition and the visualization of the different models in the previous study, it makes sense to think that in real-world LGDM, it may be useful the use of models without feedback mechanism when there is a cohesive group/subgroup and models with feedback when the group/subgroup is diverse.
4. *Time cost versus experts' willingness*:
   - Setting up consensus models for LGDM considering experts' real willing and keeping or decreasing the time cost at the same time is a promising research topic.

6. Conclusions

The need of solving LGDM under agreement demands CRPs able to deal with these problems. Even though a few new specific proposals of CRPs for LGDM have been done, there have not been carried out so far a study about the performance of classical CRP models designed for GDM problems with a small number of experts within LGDM problem. Therefore, this paper has utilized the consensus simulation framework AFRYCA 2.0 to carry out a comparative study of different types of classical CRPs in different scenarios that are similar to the ones that can be found in real-world LGDM.

From the results obtained, it is clear that the straightforward application of such classical consensus models to LGDM is not always working well, but some models can be easily adapted to deal with LGDM with some improvements that have been pointed out in the analyses provided across the paper.

Finally, some new challenges, that consensus models should cope with in LGDM problems, have been elicited to show their needs if they want to obtain successful results in their performance.

As future research it should be interesting carry out specific analysis of consensus models, such as [12,21,39,48], in addition to the a general study.

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4.5. Proceso de Consenso a Gran Escala Gestionando las Dudas del Grupo

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A large scale consensus reaching process managing group hesitation

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ABSTRACT

Nowadays due to the social networks and the technological development, large-scale group decision making (LS-GDM) problems are fairly common and decisions that may affect to lots of people or even the society are better accepted and more appreciated if they agreed. For this reason, consensus reaching processes (CRPs) have attracted researchers attention. Although, CRPs have been usually applied to GDM problems with a few experts, they are even more important for LS-GDM, because differences among a big number of experts are higher and achieving agreed solutions is much more complex. Therefore, it is necessary to face some challenges in LS-GDM. This paper presents a new adaptive CRP model to deal with LS-GDM which includes: (i) a clustering process to weight experts' sub-groups taking into account their size and cohesion, (ii) it uses hesitant fuzzy sets to fuse expert's sub-group preferences to keep as much information as possible and (iii) it defines an adaptive feedback process that generates advice depending on the consensus level achieved to reduce the time and supervision costs of the CRP. Additionally, the proposed model is implemented and integrated in an intelligent CRP support system, so-called AFRYCA 2.0 to carry out this new CRP on a case study and compare it with existing models.

1. Introduction

A recent and challenging problem in the decision making field, driven by the current technological developments (social networks, P2P) and societal demands (e-group shopping, group marketing), is the engagement of a large number of people in different decision problems. Consequently, large-scale group decision making (LS-GDM) is becoming an important topic in the decision making field [26–28,47]. Unlike classical GDM problems in which a decision framework with a few number of experts is assumed, LS-GDM problems deal with a large number of experts (in [10] was pointed out more than 20 experts, but here we may assume several hundreds even thousands). This situation implies new challenges pointed out in previous researches in this topic [24,33,35], such as: (i) Scalability, (ii) Time cost, (iii) Constant preference supervision, iv) Stronger disagreement positions, v) Difficulties to understand/visualize current state of agreement, etc.

The study of LS-GDM has been mainly focused on four major topics:

- Clustering methods in LS-GDM [26,53].
- Consensus reaching processes in LS-GDM [34,48,49].
- LS-GDM methods [27,28].
- LS-GDM support systems [8,35].

Due to the fact that, consensual decisions for conflicting problems that may affect groups of people are better adopted and much more appreciated [13], the study and development of consensus reaching processes (CRPs) for GDM has been then a fruitful, interesting and necessary area of research in recent years [16,33,36]. However, most of results presented in this area are focused on GDM problems assuming just a few number of experts involved in the decision process. Notwithstanding, in LS-GDM this type of process seems to be even more important, because opinions among a larger number of people tend to be easily controversial and conflicting. Main shortcomings of classical CRPs when they are applied to LS-GDM problems have been identified [24] and initial CRP proposals for LS-GDM do not have overcame these shortcomings yet [34,49].

In light of the multiple challenges and shortcomings of classical CRPs for LS-GDM problems [24], this paper introduces a new *adaptive* CRP model for LS-GDM to overcome *scalability* problems and *experts' preference supervision* that is highly related to *time cost*. Therefore, to achieve these goals, our proposal incorporates to the CRP applied to LS-GDM the following novelties:

- *Clustering process for weighting experts' sub-groups*: the large number of experts in the LS-GDM problem are clustered into sub-groups according to their preferences and the *importance* of each sub-group

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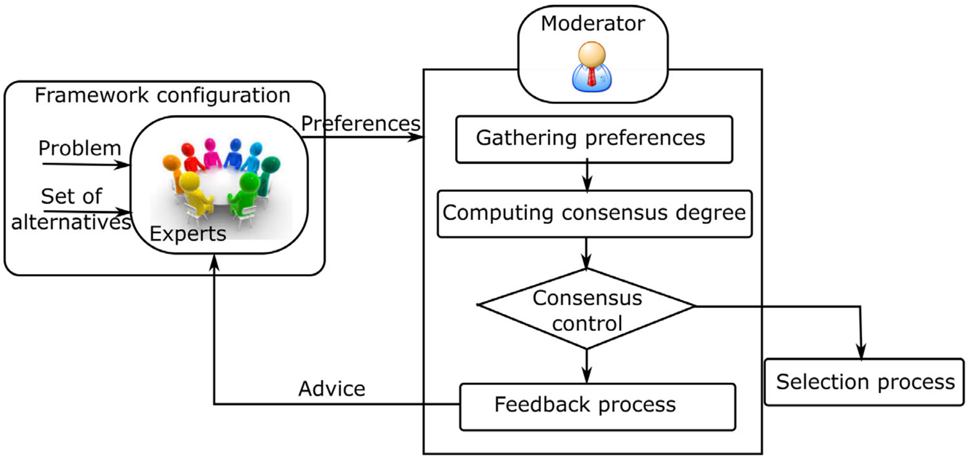

Fig. 1. General scheme of a consensus reaching process.

in the CRP that is computed considering two features such as its ***size*** and its ***cohesion***.

- *Grouping Opinions*: so far, most of CRPs aggregate experts' preferences from early stages of the process, the aggregation may result in a loss of important features of the information, such as distribution or shape [18]. In order to avoid such situations, our proposal will model experts' sub-group preferences by means of hesitant fuzzy sets (HFS), introduced by Torra [45] for representing the expert's hesitation to assign a degree of membership in a fuzzy set; so, it will be assumed that experts' preferences in a sub-group represent the group hesitation to express its fuzzy preference [9,39,52].
- *Adaptive feedback process*: last but not least, the negotiation process in a CRP is usually driven by a feedback mechanism [31] that is often time consuming even more in LS-GDM [33]; therefore, our proposal develops a new adaptive feedback mechanism process that guides the consensus process according to the level of agreement achieved by softening experts' preference supervision and reducing the time cost of the CRP.

Finally, the proposed CRP is implemented and integrated in the intelligent CRP support system so-called AFRYCA 2.0 [23,33] to compute the results of the case study, visualize the CRP and carry out a comparison with other consensus models.

The remainder of this paper is structured as follows: Section 2 revises some preliminary concepts about LS-GDM problems, CRPs and hesitant fuzzy information. Section 3 presents a novel adaptive consensus model based on clustering and hesitant fuzzy information to deal with LS-GDM problems. Section 4 introduces a case study to show the utility and applicability of the proposed model using an intelligent CRP support system and presents a comparison with other models. Finally, in Section 5 some concluding remarks are pointed out.

2. Preliminaries

This section revises different concepts about LS-GDM, CRPs and HFS that will be used in the proposed consensus model for LS-GDM.

2.1. Large-Scale group decision making

Even though the concept of GDM has been widely studied in decision theory [4,6,19,29], recently the concept of LS-GDM has risen because of the societal demand of involving crowds in important decision processes [12,13] that is facilitated by current technologies and tools [43]. Hence, the concept of LS-GDM is quite similar to GDM, but differs because in the former the number of experts eliciting their preferences on a set of alternatives, X, is much greater than in the latter. Formally, a LS-GDM problem consists of: (i) a set of alternatives $X = \{x_1, \ldots, x_n\}$, $(n \geq 2)$, which can be selected as possible solutions for the problem, and (ii) a set of experts $E = \{e_1, \ldots, e_m\}$, $(m >> n)$, who express their judgements on the set of alternatives X. Fuzzy preference relations [32], $P = (p_{ij})_{n \times n} \subset X \times X$, $p_{ij} \in [0, 1]$, are a common structure for eliciting preferences in both types of group decision problems [7].

Due to its similar structure, LS-GDM can be solved by a selection process similar to the one used in GDM [41] with an aggregation and exploitation phase. In such a case, this selection process does not always guarantee that the solution obtained would be accepted by all experts involved in the decision problem, because several of them might consider that their opinions were not taken sufficiently into account [42]. A usual solution to overcome this drawback and obtain agreed decisions accepted by the whole group is the application of a CRP [5,46]. In spite of the existence of different interpretations of consensus [30], in this paper it is understood as "*a state of mutual agreement among members of a group in which the decision made satisfies all of them*" [42]. Usually, achieving a consensus requires that experts modify their preferences bringing them closer to each other toward a collective opinion which is satisfactory for all of them [19,37].

A consensus process is an iterative and dynamic discussion process that can be carried out in different ways, Palomares et al. introduced in [33] a deep revision and a taxonomy of the different types of models for performing it, and a general scheme of a CRP sketched in Fig. 1 that is briefly described below:

- *Framework configuration*: it sets up the GDM problem determining the set of alternatives, the set of experts engaged in the decision making and fixing the consensus threshold to reach.
- *Gathering preferences*: the preferences provided by experts are gathered.
- *Computing the consensus degree*: by using a consensus measure [17] which is based on distance measures and aggregation operators [2,15]. This degree reflects the level of agreement in the group.
- *Consensus control*: if the obtained consensus degree is greater than the consensus threshold, a selection process is applied, otherwise more discussion rounds are required.
- *Feedback process*: the preferences causing disagreement are identified and advice is generated to guide experts how to modify their preferences and make them closer. Afterwards, another round starts by gathering preferences again.

In order to cope with the necessity of achieving agreed solutions in LS-GDM problems, several proposals have been introduced in the literature. Palomares et al. [34] proposed a consensus model to detect and manage non-cooperative behaviors and developed a visual tool based on self-organizing maps to facilitate the monitoring of the process performance [35]. Taking into account such a model, Xu et al. [49] proposed a consensus model for multi-criteria LS-GDM dealing with emergency problems that considers non-cooperative behaviors and minority opinions. Quesada et al. [38] introduced a weighting method for CRPs dealing with LS-GDM which includes the use of uninorm aggregation operators to compute experts weights taking into account their behaviors.

Previous proposals aggregate the experts' preferences in early stages of the decision process that may imply disregarding important information [18] and not considering the different levels of agreement across the CRP that can provoke a high time cost due to a greater experts' preference supervision during the feedback and discussion processes.

Therefore, to overcome these drawbacks our proposed model first, will include an approach to detect and weight sub-groups. Second, to keep as much information and avoid the loss of information, the sub-groups preferences will be fused by using HFS instead of aggregating them. Finally, a new adaptive feedback process based on previous inputs will be defined.

2.2. Hesitant information

The concepts of HFS and hesitant fuzzy preference relation have been widely applied to decision making [39], in this section these are briefly reviewed to facilitate the understanding of their use in our proposal for modelling experts sub-group preferences in order to keep as much information as possible during the proposed CRP.

HFSs [45] are an extension of fuzzy sets with the aim at modelling the uncertainty provoked by the doubt that an expert can have when she/he wants to assign the membership degree of an element in a fuzzy set. A HFS allows assigning several membership degrees of an element to a fuzzy set. Formally, a HFS is defined in terms of a function that obtains a set of membership degrees for each element in the domain.

Definition 1 (*[45]*). Let X be a reference set, a HFS on X is a function $\mathfrak{h}$ that returns a subset of values in [0,1]:

$$\mathfrak{h}: X \to \wp([0, 1]) \tag{1}$$

Previous definition was completed with the following mathematical representation of a HFS:

$$A = \{\langle x, h_A(x)\rangle : x \in X\},$$

where $h_A(x)$ is called Hesitant Fuzzy Element (HFE) that is a set of some values in [0,1], denoting the possible membership degrees of the element $x \in X$ to the set A. A HFS can also be seen as a mapping of HFEs, one for each element in the reference set. Therefore, if $h(x)$ is the HFE associated to x, $\cup_{x \in X} h(x)$ is then a HFS.

By using the concepts of fuzzy preference relation and HFS, the concept of Hesitant Fuzzy Preference Relation (HFPR) was proposed [56].

Definition 2 (*[56]*). Let X be a reference set, a HFPR on X is represented by a matrix $H = (h_{ij})_{n\times n} \subset X \times X$, where $h_{ij} = \left\{ p_{ij}^s \middle| s = 1, 2, ..., \#h_{ij} \right\}$ ($\#h_{ij}$ is the number of elements in h_{ij}) is a HFE that indicates the membership degrees that denote to which extent x_i is preferred to x_j. Additionally, h_{ij} should satisfy the following conditions: $p_{ij}^{\sigma(s)} + p_{ji}^{\sigma'(s)} = 1$, $p_{ii} = \{0.5\}$, $\#h_{ij} = \#h_{ji}$, $i, j = \{1, 2, ..., n\}$ $p_{ij}^{\sigma(s)} < p_{ij}^{\sigma(s+1)}$, $p_{ji}^{\sigma'(s+1)} < p_{ji}^{\sigma'(s)}$, where $\{\sigma(1), ..., \sigma(\#h_{ij})\}$ is a permutation of $\{1, ..., \#h_{ij}\}$, i.e., $p_{ij}^{\sigma(s)}$ is the smallest element in h_{ij}, and $\{\sigma'(1), ..., \sigma'(\#h_{ji})\}$ is a permutation of $\{1, ..., \#h_{ji}\}$, i.e., $p_{ji}^{\sigma'(s)}$ is the largest element in h_{ji}.

During the CRP in LS-GDM with HFSs, it might happen that the cardinality of HFEs in HFPRs would be different, i.e, $h_{ij}^a \in H^a = (h_{ij}^a)_{n\times n}$ and $h_{ij}^b \in H^b = (h_{ij}^b)_{n\times n}$ with $\#h_{ij}^a \neq \#h_{ij}^b$ (e.g., $\#h_{ij}^a < \#h_{ij}^b$). In such a case, it is necessary to normalize the h_{ij}^a with smaller cardinality until both have the same cardinality to operate correctly between them. Xu and Zhang [51] proposed the β-normalization, based on the optimization parameter, η.

Definition 3 (*[51]*). Let h_i be the HFE with the smaller cardinality and $h_i^- = min\{\gamma|\gamma \in h_i\}$ and $h_i^+ = max\{\gamma|\gamma \in h_i\}$, then the value γ' to add in the HFE h_i, is computed as:

$$\gamma' = \eta h_i^+ + (1 - \eta) h_i^-, \tag{2}$$

where $\eta (0 \leq \eta \leq 1)$.

The value of the optimization parameter, η, relies on experts' risk attitudes. If $\eta = 1$, the value added is $\gamma' = h_i^+$, which indicates an optimistic point of view; if $\eta = 0$, the value added is $\gamma' = h_i^-$, which indicates a pessimistic point of view; and if $\eta = 1/2$, then $\gamma' = 1/2(h_i^+ + h_i^-)$, which means that expert is neutral. Consequently, by using η, if $\#h_{ij}^a < \#h_{ij}^b$ the HFPR, H^a, is normalized as:

Definition 4 (*[55]*). Let $H^a = (h_{ij}^a)_{n\times n} \subset X \times X$ be a HFPR and $\eta\,(0 \leq \eta \leq 1)$ an optimization parameter to add values into $h_{ij}^a (i < j)$, moreover $1 - \eta$ is used to add values into $h_{ji}^a (i < j)$, a normalized HFPR $\overline{H}^a = (\overline{h}^a{}_{ij})_{n\times n}$, is obtained satisfying the following conditions, $\#\overline{h}^a{}_{ij} = max\{\#h_{ij}^a | i, j = 1, 2, ..., n;\ i \neq j\}$ $\overline{\gamma}_{ij}^{\sigma(s)} + \overline{\gamma}_{ji}^{\sigma'(s)} = 1$, $\overline{h}_{ii}^a = \{0.5\}$, $\overline{\gamma}_{ij}^{\sigma(s)} \leq \overline{\gamma}_{ij}^{\sigma(s+1)}$, $\overline{\gamma}_{ji}^{\sigma'(s+1)} \leq \overline{\gamma}_{ji}^{\sigma'(s)}$, where $\{\sigma(1), ..., \sigma(\#h_{ij}^a)\}$ is a permutation of $\{1, ..., \#h_{ij}^a\}$, i.e., $\overline{\gamma}_{ij}^{\sigma(s)}$ is the smallest element in h_{ij}^a, and $\{\sigma'(1), ..., \sigma'(\#h_{ji}^a)\}$ is a permutation of $\{1, ..., \#h_{ji}^a\}$, i.e., $\overline{\gamma}_{ji}^{\sigma'(s)}$ is the largest element in h_{ji}^a.

Even though, our proposal avoids aggregation operations in early stages, there are several procedures in the CRP that need to aggregate and compute distances with HFSs. Despite there exist multiple proposals to carry out such operations [40]. Here, the Hesitant Fuzzy Weighted Average (HFWA) operator and the Euclidean distance which are used for sake of clarity in the proposed consensus model for LS-GDM are just revised.

Definition 5 (*[54]*). Let H be a HFS and $h_i (i = 1, ..., n)$ be a collection of HFEs, $h_i \in H$, the Hesitant Fuzzy Weighted Average operator is a mapping $H^n \to H$ such that

$$HFWA(h_1, ..., h_n) = \oplus_{i=1}^n (w_i h_i) = \bigcup_{\gamma_1^{\sigma(s)} \in h_1, ..., \gamma_n^{\sigma(s)} \in h_n} \left\{ \sum_{i=1}^n w_i \gamma_i^{\sigma(s)} \right\}, \tag{3}$$

where $w = (w_1, w_2, ..., w_n)^T$ is the weighting vector of $h_i (i = 1, ..., n)$ with $w_i \in [0, 1]$ and $\sum_{i=1}^n w_i = 1$.

Definition 6 (*[50]*). Let H_1 and H_2 be two HFSs on $X = \{x_1, ..., x_n\}$, the hesitant normalized Euclidean distance is defined as follows,

$$d_{hne}(H_1, H_2) = \left[\frac{1}{n} \sum_{i=1}^n \left(\frac{1}{\#h} \sum_{s=1}^{\#h} |\gamma_2^{\sigma(s)}(x_i) - \gamma_1^{\sigma(s)}(x_i)|^2 \right) \right]^{1/2} \tag{4}$$

where $\#h$ is the cardinality of any HFE $h_i \in H_1, H_2$, considering that all of them are equal cardinality.

Additionally, during the CRP will be necessary to compare HFEs of the HFS. Therefore, one suitable function will be the below one:

Definition 7 (*[14]*). Let h be a HFE, the score function of h is given by,

$$score(h) = \frac{\sum_{s=1}^{\#h} \gamma^s \tau(s)}{\sum_{s=1}^{\#h} \tau(s)} \tag{5}$$

where $\{\tau(s)\}_{s=1}^{\#h}$ is a positive-valued monotonic increasing sequence of index s.

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3. An adaptive consensus model for large scale group decision making based on group hesitation

The goal of this paper is to introduce a novel CRP for LS-GDM problems able to tackle the *scalability* and *time cost* challenges of a CRP in this type of decision problems.

- To cope with the former one, a clustering process to detect experts sub-groups based on their preference similarity is done. And such sub-groups' preferences are modelled as the group's hesitation by means of HFSs; eventually the hesitant preferences are weighted according to the *size* and *cohesion* of the group.
- On the other hand, the latter challenge is managed by an *adaptive process* that varies the feedback procedure in the CRP between two levels according to the level of consensus achieved at each discussion round.

The proposed adaptive consensus model based on group hesitation for LS-GDM extends the general scheme shown in Fig. 1 by introducing two new phases:

- *Sub-groups management* that clusters similar experts' opinions, maintaining the maximum possible information by HFSs and computing the relevance of the sub-groups.
- A new *adaptive feedback process* that adapts the feedback to the current agreement among experts.

Besides, these new phases, other two of the general scheme are modified (dashed lines):

- *Framework configuration* in which a new parameter to deal with the adaptivity is introduced.
- *Computing the consensus degree* to deal with hesitant information.

So, the proposed model consists of six main phases (see Fig. 2), but only the new and modified ones (previously enumerated) will be further detailed below.

3.1. Framework configuration

In a LS-GDM problem there are two important elements (Section 2.1): a set of alternatives $X = \{x_1, \ldots, x_n\}$ and a large number of experts $E = \{e_1, \ldots, e_m\}$ who are involved in the problem, being $m > > n$.

Classically, two parameters are established, the consensus threshold and the maximum number of discussion rounds. However, in our proposal a new parameter is necessary to introduce the adaptivity during the consensus process. Therefore, three parameters are defined in our adaptive CRP:

- $\vartheta \in [0, 1]$: It is the consensus threshold established to achieve the consensus among experts.
- $\delta \in [0, 1], \delta < \vartheta$: It is a parameter used in the adaptive feedback process to determine the level of consensus reached (high or low), such that different rules for the advice generation can be applied.
- *Maxround*: This parameter controls the maximum allowed number of discussion rounds for the LS-GDM problem.

3.2. Sub-groups management: Managing scalability in LS-GDM

To tackle the scalability problem in LS-GDM, we consider that among a large number of experts there will be sub-groups of them with similar preferences. Therefore, with this idea in mind, this phase reduces the number of preferences to manage by means of a three-step process (further detailed in the coming subsections):

1. ***Detection***: A clustering process is applied to detect experts' groups with similar opinions.

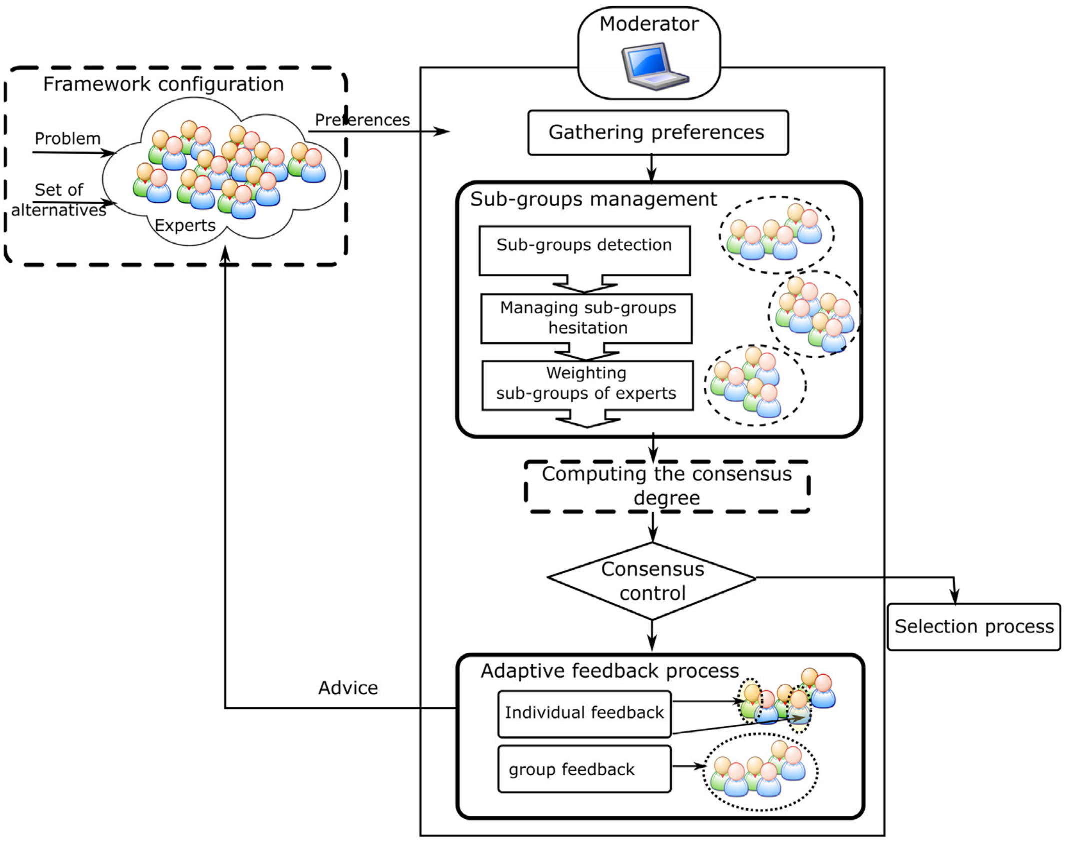

Fig. 2. Scheme of the proposed consensus model.

```
Require: Map preferences
Require: List alternatives
Require: List experts
Ensure: List clusters
n ← lenght(alternatives)
m ← lenght(experts)
for l=1 to n do
    clusters(l) ← generateCluster(l) //Including initialization of centroid
iteration ← 0
variation ← 1
repeat
    for r=1 to m do
        for l=1 to n do
            degree ← computeMembership(getPreferences(experts(r)), getCentroid(clusters(l))
            membershipDegrees(l) ← degree
        matrixMembershipDegrees(r) ← membershipDegrees
        idCluster ← maximumMembershipDegree(membershipDegrees)
        for l=1 to n do
            if (idCluster == getIdCluster(clusters(l))) then
                if (contain(clusters(l),experts(r)))==false then
                    assignPreferencesCluster(getPreferences(experts(r)), clusters(idCluster))
    allMembershipDegrees(iteration) ← matrixMembershipDegrees
    if (iteration > 0) then
        variation ← compareMembershipDegrees(allMembershipDegrees(iteration), allMembershipDegrees(iteration −
1))
    if (variation > ϵ) then
        for l=1 to n do
            updateCentroid(clusters(l))
    iteration ← iteration + 1
until (variation ≤ ϵ) return clusters
```

Algorithm 1. Fuzzy c-means algorithm applied to expert fuzzy preference relations.

2. ***Hesitation modelling***: The experts' opinions in each sub-group are modelled by means of a HFS that represents the hesitation of the group
3. ***Weighting***: The importance of the sub-group's opinion should reflect its features, in our case the size and cohesion of a subgroup is considered.

3.2.1. Sub-groups detection

To detect experts' groups with similar opinions, an adapted fuzzy c-means based algorithm [3] that assigns a membership degree to each data object for each cluster according to the distance between the data object and the corresponding centroid is presented. The nearer the data object is to the centroid, the higher its membership degree with respect to this centroid is. Both centroids and memberships degrees are iteratively updated until an optimal solution is found.

1. The number of clusters can be randomly selected, in this proposal, the initial number of clusters is the number of different alternatives, $C=\{C^1, \ldots, C^n\}$, because we want to find the clusters of experts supporting each different alternative.
2. A centroid represents each cluster c^l, $l \in \{1, \ldots, n\}$. Centroids can be either randomly initialized or assigned to a value from the dataset, but their initialization is very sensitive to converging [1,21]. In this case, as the problem is known, each centroid is initialized with a fuzzy preference relation that ideally prefers the corresponding alternative over all the others, i.e. for alternative x_k, the centroid c^k contains $c^{kj}=1$, $c^{jk}=0$ ($j \in \{1, \ldots, n\}$) and for the remaining ones the preference is 0.5 that representing indifference.

$$c^1=\begin{pmatrix} - & 1 & 1 & \cdots & 1\\ 0 & - & 0.5 & \cdots & 0.5\\ 0 & 0.5 & - & \cdots & 0.5\\ \vdots & \cdots & \cdots & \cdots & \vdots\\ 0 & 0.5 & 0.5 & \cdots & - \end{pmatrix},\ c^2=\begin{pmatrix} - & 0 & 0.5 & \cdots & 0.5\\ 1 & - & 1 & \cdots & 1\\ 0.5 & 0 & - & \cdots & 0.5\\ \vdots & \cdots & \cdots & \cdots & \vdots\\ 0.5 & 0 & 0.5 & \cdots & - \end{pmatrix}\ \cdots$$

$$c^n=\begin{pmatrix} - & 0.5 & 0.5 & \cdots & 0\\ 0.5 & - & 0.5 & \cdots & 0\\ 0.5 & 0.5 & - & \cdots & 0\\ \vdots & \cdots & \cdots & \cdots & \vdots\\ 1 & 1 & 1 & \cdots & - \end{pmatrix}$$

3. Centroids are computed in each iteration t, and the membership degree of each experts' fuzzy preference relation P^r to each centroid $c^{l,\,t}$, $\mu_{c^{l,t}}(P^r) \in [0, 1]$, is calculated by:

$$\mu_{c^{l,t}}(P^r)=\frac{(1/d(P^r, c^{l,t}))^{1/(b-1)}}{\sum_{u=1}^{n}(1/d(P^r, c^{u,t}))^{1/(b-1)}} \tag{6}$$

where $d(P^r, c^{l,\,t})$ is the Minkowski distance, t is the current iteration, and b indicates the fuzziness degree of the clusters. The larger b, the fuzzier the cluster [3]. A common value for this parameter is $b=2$.

Definition 8 (*[25]*). Let P^r be a fuzzy preference relation provided by the expert e_r, and $c^{l,\,t}$ be the centroid for the cluster C^l at iteration t, the Minkowski distance is defined as follows,

$$d(P^r, c^{l,t})=\left(\frac{1}{n(n-1)}\sum_{i=1}^{n}\sum_{j=1,i\neq j}^{n}\left|p_{ij}^r-c_{ij}^{l,t}\right|^{\lambda}\right)^{1/\lambda} \tag{7}$$

being $\lambda > 0$.In our proposal, $\lambda=2$ that is the Euclidean distance.

4. The preference relation P^r of expert e_r is assigned to the cluster for which, the membership degree is maximum.

$$C^{l,t}(P^r)=\mathrm{argmax}_l\, \mu_{c^{l,t}}(P^r) \tag{8}$$

5. New centroids are computed according to the experts preference relations included in each cluster.

$$c_{ij}^{l,t+1}=\frac{1}{|C^{l,t}|}\sum_{P^r\in C^{l,t}} p_{ij}^r,\ i, j \in \{1,\cdots,n\}, \tag{9}$$

where $|C^{l,\,t}|$ is the number of preference relations that belong to the cluster C^l at iteration t.

6. The algorithm stops when all clusters stabilize. This happens when the variation of the membership degrees between two consecutive iterations approaches to zero. Formally, the iterative process stops when

$$\frac{\sum_{r=1}^{m}\sum_{l=1}^{n}|\mu_{c^{l,t}}(P^r)-\mu_{c^{l,t-1}}(P^r)|}{m\cdot n}\leq \epsilon \tag{10}$$

where ϵ is a threshold value that should be close to zero.

Algorithm 1 formalizes previous steps. The outcome provides clusters, C^l, containing a sub-group of experts, G^l, with similar opinions.

3.2.2. Sub-groups hesitation modelling

The classification into sub-groups according to preferences similitude aims at reducing scalability problems, however it is necessary to establish how to model the sub-group preferences. There exist several possibilities, from using the centroid that represents the sub-group's cluster to aggregate all the expert's preferences in the sub-group. But bearing in mind our goal of keeping as much information as possible in the CRP, unlike of oversimplifying the preferences modelling with aggregation procedures, our proposal considers that the different experts' preferences elicited in the sub-group despite of being similar, show a kind of hesitation in the group regarding such preferences.

Therefore, let $G^l=\{e_1^l, \ldots, e_k^l\}$ be the sub-group of experts belonging to cluster, C^l, whose preference relations are, $P^{lk}=\left(p_{ij}^{lk}\right)_{n\times n}$. From such preference relations a HFPR, $HP^l=(h_{ij}^l)_{n\times n}$, $l \in \{1, \ldots, n\}$ is built, that fuses all experts' preferences in G^l such that, $h_{ij}^l=\{p_{ij}^{lk}\,|k=1, 2, \ldots, |G^l|\}$, $|G^l|$ is the cardinality of G^l and will be the number of preferences in the HFE $\#h_{ij}^l$ which represents the sub-group's preference over the pair of alternatives (x_i, x_j) provided by all experts in G^l.

At this moment, the large number of experts $E=\{e_1, \ldots, e_m\}$ and their respective preference relations, P^r, have been replaced by a smaller number of sub-groups, G^l, and their respective HFPRs, HP^l, that will be the input for the CRP in the LS-GDM.

3.2.3. Sub-groups weighting

To conduct a fair CRP taking into account the previous elements, G^l and HP^l, it is necessary to characterize the sub-groups by computing their importance. Our proposal takes into account their *size* and *cohesion* [44] to reflect their weight:

- *Size*: number of experts in the sub-group.
- *Cohesion*: level of togetherness among the experts' preferences in a sub-group.

Therefore, the importance of the sub-groups is based on the two following statements:

- The greater the group the more important.
- The more cohesive the more important.

Hence, the weight of a sub-group of experts will be based on the *size* and *cohesion*. The former is directly obtained from the *sub-group detection* process, and the latter needs further computation. So, to obtain the weights for all sub-groups of experts it is necessary to carry out three steps: a) to compute the cohesion of each group, b) to compute the size of each group and c) to obtain the group's weight. These steps are further detailed below:

(a) Computing the cohesion of a sub-group. For the sake of clarity, a geometric description of the *cohesion* of experts' preferences, $HP^l=(h_{ij}^l)_{n\times n}$, in a sub-group G^l is introduced. First, the area

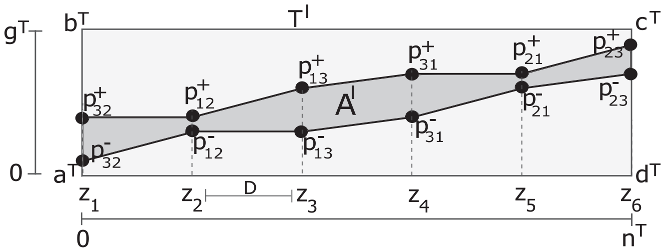

Fig. 3. Graphical representation for computing the cohesion of a sub-group.

delimited by the maximum and minimum assessments in h^l_{ij}, over the set of alternatives X is computed. For instance, let $G^l = \{e_1, e_2\}$ be a sub-group of experts, $X = \{x_1, x_2, x_3\}$ a set of alternatives, and HP^l the HFPR representing the preferences of the sub-group G^l.

$$HP^l = \begin{pmatrix} - & \{0.3, 0.4\} & \{0.3, 0.6\} \\ \{0.7, 0.6\} & - & \{0.7, 0.9\} \\ \{0.7, 0.4\} & \{0.3, 0.1\} & - \end{pmatrix}$$

The preference degrees in the HFE, h^l_{ij}, elicited by experts on (x_i, x_j) are shown in Fig. 3. The X-axis represents a discrete set Z formed by all pair of alternatives over X where each pair $z_t = (x_i, x_j)$ $i, j \in \{1, 2, 3\}$, $i \neq j$ is positioned equidistantly on the X-axis. In order to compute the area, the maximum, p^+_{ij}, and minimum, p^-_{ij}, assessments, for each pair of alternatives are obtained. To do so, it is necessary to establish the order in which the pairs of alternatives are located across the X-axis. In this approach, we have considered the minimum assessments in increasing order.

The *cohesion* of, G^l, is related to the dark shadowed area, A (the larger, A, the lower the cohesion), that is computed as follows:

(i) Let T^l be the total area of the rectangle formed by the points a^T, b^T, c^T and d^T (see Fig. 3), i.e., $T^l = g^T \times n^T$ in which g^T corresponds to the height of the rectangle and $n^T = (n^2 - n) - 1$, corresponds to the number of pairs of alternatives (considering p_{ii} is not assessed) minus 1, because an area needs at least two pairs in A.

(ii) Let $I = \bigcup_{i,j \in n, i \neq j} \{(i, j)\}$ be the n^T pairs over the set of alternatives $X = \{x_1, \ldots, x_n\}$. The p^+_{ij}, and p^-_{ij}, assessments for each p_{ij} taking into account all the preferences in G^l are obtained as:

$$p^-_{ij} = min\left\{p^1_{ij}, p^2_{ij}, \ldots, p^s_{ij}\right\}, \ \forall \ (i, j) \in I \tag{11}$$

$$p^+_{ij} = max\left\{p^1_{ij}, p^2_{ij}, \ldots, p^s_{ij}\right\}, \ \forall \ (i, j) \in I \tag{12}$$

The first and last pair of alternatives considered in the X-axis are obtained by,

$$p^-_{ab} = min_{i,j \in I}\left\{p^-_{ij}\right\}, \ (a, b) \in I \tag{13}$$

$$p^+_{cd} = max_{i,j \in I}\left\{p^-_{ij}\right\}, \ (c, d) \in I \tag{14}$$

A function f is defined to obtain the indexes of the pairs of alternatives.

Definition 9. Let f be a function that returns the indexes of a pair of alternatives,

$$f: \{z_1, z_2, \ldots, z_{n(n-1)}\} \rightarrow I \tag{15}$$

being $f(z_1) = (a, b) \in I$ such that, $p^-_{ab} = \min\limits_{i,j \in I}\left\{p^-_{ij}\right\}$,

$f(z_t) = (e, f) \in I$ where $p^-_{ef} = \min\limits_{i,j \in I / \{f(z_1), f(z_2), \ldots, f(z_{t-1})\}}\left\{p^-_{ij}\right\}$,

$f(z_{n(n-1)}) = (e, f) \in I$ with $p^-_{ef} = \min\limits_{i,j \in I / \left\{f(z_1), f(z_2), \ldots, f\left(z_{n(n-1)-1}\right)\right\}}\left\{p^-_{ij}\right\} = \max\limits_{i,j \in I}\left\{p^-_{ij}\right\}$,

therefore, $f(z_{n(n-1)}) = (c, d) \in I$. The area A^l, between the maximum and minimum assessments ordered in the X-axis by the minimum is computed by,

$$A^l = \left[\sum_{i,j \in I}\left(p^+_{ij} - p^-_{ij}\right) - \frac{(p^+_{ab} - p^-_{ab}) + (p^+_{cd} - p^-_{cd})}{2}\right] \cdot D \tag{16}$$

where D is the distance between z_i and z_{i+1}, that in our case it is 1.

(iii) Finally, the cohesion of a sub-group of experts G^l is given by,

$$cohesion(G^l) = 1 - \frac{A^l}{T^l} \in [0, 1], \tag{17}$$

(b) Computing the size of a sub-group. The value of the *size* of the group, G^l, is directly obtained from the *sub-group detection* process, but its representation should be adjusted and adapted to the number of experts involved in the LS-GDM problem. Therefore, a adaptation process based on computing with words [38] is proposed in which, the *size* is modelled by a fuzzy membership function μ_{size} shown in Fig. 4, such that the universe of discourse is the number of experts in a sub-group and the membership degree reflects group's influence regarding all the experts involved in the LS-GDM.

The points a and b of this membership function depend on the number of alternatives and experts in the LS-GDM problem, where the highest membership degree is for values above b and the lowest membership degree is for values below a and different importance is assigned in between.

(c) Computing the relevance of a sub-group. Eventually, for weighting the sub-groups, the values of their *size* and *cohesion* are aggregated, our proposal defines a function to fuse both values making such a computation more flexible according to the specific LS-GDM.

Definition 10. Let $Y_{G^l} = \{y_1, y_2\}$ be the values obtained for cohesion and size, respectively, $y_1, y_2 \in [0, 1]$, of the sub-group G^l which are aggregated as follows,

$$\varphi(Y_{G^l}) = (1 + y_2)^{y_1 \beta} \tag{18}$$

being $\beta > 0$ a parameter to increase/decrease the impact of the cohesion in the computation of the sub-group's weight.

The aggregated values, $\varphi(Y_{G^l})$, reflects the relevance of the sub-group, G^l. Finally, such values are normalized.

$$w_l = \frac{\varphi(Y_{G^l})}{\sum_{z=1}^{n} \varphi(Y_{G^z})}, \ \forall \ l \in \{1, \ldots, n\}. \tag{19}$$

Below, an example shows how the aggregation function performs and the influence of parameter β on the computation of the sub-groups' weight.

Let suppose a LS-GDM problem with 80 experts distributed into four sub-groups, $G = \{G^1, G^2, G^3, G^4\}$ whose size, membership degree and

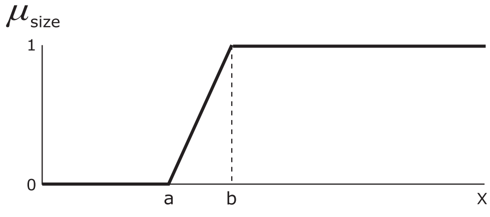

Fig. 4. Membership function for the sub-group *size*.

Table 1
Weights for different values of β.

	Size	Memb. Degree	Coh.	Weigths			
				$\beta = 1$	$\beta = 1.5$	$\beta = 2$	$\beta = 3$
G^1	11	0.25	0.55	0.219	0.205	0.191	0.166
G^2	14	0.5	0.55	0.243	0.238	0.234	0.224
G^3	23	1	0.49	0.273	0.284	0.295	0.318
G^4	32	1	0.45	0.265	0.272	0.279	0.292

cohesion are depicted in Table 1. Different values for the parameter β have been used to solve Eq. (18).

Note that the weights in Table 1 are already normalized. We can observe that the sub-groups $\{G^1, G^2\}$ have different size but equal cohesion, therefore, when the value of β increases, the sub-group's weight with higher size, G^2, decreases slower than G^1. On the other hand, sub-groups $\{G^3, G^4\}$ have the same membership degree, therefore, when the value of β increases, the sub-group's weight G^3 increases more than the sub-group G^4, because its cohesion is higher. Thus, the parameter β allows to increase/decrease the impact of the cohesion in the computing weights.

3.3. Computing the consensus degree

Our CRP model modifies the way of computing the level of agreement among experts shown in Fig. 1 by adapting the three-step process introduced in [31], to deal with the HFSs obtained in the previous phase.

1. *Pairwise similarity matrix*: For each pair of sub-groups G^l and G^k, a similarity matrix $SM^{lk} = (sm_{ij}^{lk})_{n\times n}$ is obtained, being $sm_{ij}^{lk} \in [0, 1]$ the similarity between h_{ij}^l and h_{ij}^k:

$$sm_{ij}^{lk} = 1 - d(h_{ij}^l, h_{ij}^k) \tag{20}$$

being d a distance measure for HFEs [40] (see Remark 1). In this proposal d is the Euclidean distance (see Def. 6).

Remark 1. The number of values in the HFEs of each HFPR, HP^l, might be different. In such a case, based on Definition 4 and using the optimization parameter η^l, all HFPRs, $HP^l(h_{ij}^l)_{n\times n}$, are normalized, $\overline{HP}^l = (\bar{h}_{ij}^l)_{n\times n}$, before carrying out the computations.

2. *Consensus matrix*: The similarity matrices are aggregated to obtain a consensus matrix $CM = (cm_{ij})_{n\times n}$. Though, different aggregation operators may be used, without loss of generality in this proposal the arithmetic mean is applied:

$$cm_{ij} = \frac{\sum_{u=1}^{l-1}\sum_{k=u+1}^{l} sm_{ij}^{lk}}{l(l-1)/2} \tag{21}$$

with $l(l-1)/2$ the number of pairwise sub-group comparisons.

3. The *consensus degree* is calculated at two different levels using the consensus matrix CM:

- Level of alternatives (ca_i): the consensus degree of each alternative $x_i \in X$ is computed as,

$$ca_i = \frac{1}{n-1}\sum_{j=1, i\neq j}^{n} cm_{ij} \tag{22}$$

- Level of preference relation (cr): the consensus degree among all experts participating in the LS-GDM problem is computed by,

$$cr = \frac{1}{n}\sum_{i=1}^{n} ca_i \tag{23}$$

Require: cr
Require: δ
Require: ϑ
if $cr > \vartheta$ **then**
  Consensus achieved
else
  if $cr \geq \delta$ **then**
    Consensus level high
    Individual feedback process
  else
    Consensus level low
    Group feedback process

Algorithm 2. Adaptive feedback process.

3.4. Adaptive feedback process: Time cost supervision

When the consensus degree, cr, achieved in a consensus round is not high enough, i.e. $cr < \vartheta$, another discussion round is necessary to increase the agreement among experts. This new discussion round is usually guided by a feedback process [33]. So far, CRPs introduced for dealing with LS-GDM [34,49] do not consider the agreement achieved in each round to adapt the feedback process making the expert's preference supervision harder and the consensus process longer. Due to the fact that, one goal of our proposal is to reduce the time cost and soften the preference supervision, this CRP model for LS-GDM proposes an *adaptive* procedure that adapts the feedback process according to the rules for the advice generation based on the consensus level achieved (see Algorithm 2). According to such a level, the generated feedback is intended for the whole *group* or for several *individuals*. The adaptivity of the feedback is based on the consensus threshold, ϑ, fixed to achieve the consensus, and the parameter δ that distinguishes between the two feedback processes. From this definition, the adaptive feedback process consists of three steps:

1. A *collective matrix* that represents the collective opinion of the experts involved in the LS-GDM problem is computed by aggregating the normalized HFPRs $\{\overline{HP}^1, \ldots, \overline{HP}^n\}$. Different hesitant fuzzy aggregation operators [40,54] can be used, here the Hesitant Fuzzy Weighted Average operator is used. This operator is revised in Definition 5, and adapted for our proposal.

Definition 11. Let $\overline{HP}^l = (\bar{h}_{ij}^l)_{n\times n}$, $(l = 1, \ldots, n)$, be the normalized HFPRs of the n sub-groups G^l, and $w = (w_1, w_2, \ldots, w_n)^T$ the weighting vector for those sub-groups (see Section 3.2.3), the collective HFPR, $HP^C = (h_{ij}^C)_{n\times n}$, is computed as,

$$h_{ij}^C = \oplus_{l=1}^{n}(w^l \bar{h}_{ij}^l) = \bigcup_{\bar{\gamma}_{ij}^{l,s} \in \bar{h}_{ij}^l}\left\{\sum_{l=1}^{n} w^l \bar{\gamma}_{ij}^{l,s}\right\}, \quad \forall\, i, j \in \{1, \ldots, n\} \tag{24}$$

being HP^C a normalized HFPR.

2. The *proximity* between each sub-group represented by a normalized HFPR $\{\overline{HP}^1, \ldots, \overline{HP}^n\}$, and the collective matrix HP^C, is calculated by using a similarity measure like in Eq. (20).

$$pr^l = sim(HP^C, \overline{HP}^l) = 1 - d_{hne}(HP^C, \overline{HP}^l) \tag{25}$$

Proximity values, pr^l, are used to identify the sub-groups that are furthest from the collective opinion.

3. *Adapting the feedback*: depending on the consensus level reached cr, the feedback process will be aimed at all experts of the furthest sub-groups or just for several further experts. Both processes are explained in more detail:

i) Group feedback process. Low consensus level

In this case $cr < \delta$, that means the consensus level is “low” and consensus is still far away, therefore quite a lot more changes are necessary, consequently all experts of the furthest sub-groups will obtain suggestions for modifying their preferences over the pair of alternatives identified in disagreement. To identify the furthest sub-groups, the proximity value of each sub-group pr^l is compared with the average of the proximity values $\overline{pr}$, such that,

$$\overline{pr} = \frac{1}{n}\sum_{l=1}^{n} pr^l \tag{26}$$

and to select the pair of alternatives to be changed, the proximity value of each pair of alternatives, pr^l_{ij}, is compared with the average of the proximity value for such an alternative $\overline{pr_i}$, such that,

$$\overline{pr_i} = \frac{1}{n}\sum_{j=1}^{n} pr^l_{ij}, \;\; with\; pr^l_{ij} = 1 - d(h^C_{ij}, h^l_{ij}) \tag{27}$$

where $h^C_{ij} \in HP^C$ and $h^l_{ij} \in \overline{HP}^l$.

Therefore,

1. If $pr^l \leq \overline{pr}$ then the sub-group G^l is selected.
2. If $ca_i \leq \vartheta$ then the alternative x_i is selected and it is necessary to look for the pair of alternatives,
   (a) if $pr^l_{ij} \leq \overline{pr_i}$, then the pair of alternatives (x_i, x_j) is selected.

Once the sub-groups and pair of alternatives have been identified, a suggestion indicating the right direction of the preference changes (increase or decrease) to improve the agreement among experts is provided, according to the following direction rules:

- If $score(h^l_{ij}) < score(h^C_{ij})$, then all experts who belong to the sub-group G^l should increase their preferences degrees for the pair of alternatives (x_i, x_j).
- If $score(h^l_{ij}) > score(h^C_{ij})$, then all experts who belong to the sub-group G^l should decrease their preferences degrees for the pair of alternatives (x_i, x_j).

Being $score(h^l_{ij})$ and $score(h^C_{ij})$ the score function for the HFEs $h^l_{ij} \in \overline{HP}^l$ and $h^C_{ij} \in HP^C$, respectively (see Eq. (5)).

ii) Individual feedback process. High consensus level

In this case $\delta \leq cr < \vartheta$, that means the consensus level is "high" but not enough yet. Therefore not many changes should be necessary, hence those experts whose opinion differs most from the collective opinion will obtain advice to modify their opinions. Thus, it would be necessary to identify the sub-group, G^l, the pair of alternatives (x_i, x_j) and experts e_r who should modify their preferences in disagreement:

1. If $pr^l \leq \overline{pr}$ then the sub-group G^l is selected.
2. If $ca_i \leq \vartheta$ then the alternative x_i is selected and,
   (a) If $pr^l_{ij} \leq \overline{pr_i}$, then the pair of alternatives (x_i, x_j) is selected.
3. If $\left(1 - d\left(h^C_{ij}, p^{lr}_{ij}\right) \leq \overline{pr_i}\right)$, then the expert e_r is selected to change his/her preference.

The direction in which the selected expert should change his/her preferences is determined as follows:

- If $\left(p^{lr}_{ij}\right) < score(h^C_{ij})$, then expert $e_r \in G^l$ should increase his/her preference degree for the pair of alternatives (x_i, x_j).
- If $\left(p^{lr}_{ij}\right) > score(h^C_{ij})$, then the expert $e_r \in G^l$ should decrease his/her preference degree for the pair of alternatives (x_i, x_j).
- If $\left(p^{lr}_{ij}\right) = score(h^C_{ij})$, then it is not necessary to make changes.

Being $score(h^C_{ij})$ the score function of the HFE h^C_{ij}, in the collective matrix HP^C calculated by Eq. (5). After this process, the CRP will go to the *sub-groups management* phase again.

4. Case study

This section presents a case study to show the usefulness of the proposed CRP for LS-GDM. To do so, firstly the LS-GDM problem is described. Afterwards, the problem is solved by means of the proposed model which has been implemented and integrated into the intelligent CRP support system, AFRYCA 2.0 [23,33]. A comparison with some existing models is then shown, and finally an analysis to display distinctive characteristics regarding existing approaches is introduced.

4.1. Definition of the LS-GDM problem

The GDM problem is formulated as follows: let $E = \{e_1, e_2, ..., e_{50}\}$, be the students of the course of basic programming of Computer Science degree. The professor asks them which programming language they would like to use for the practices in the laboratory and he provides four options, $X = \{x_1: C, x_2: C++, x_3: Java, x_4: Python\}$. The professor wants an agreed solution because once the language is selected, they cannot change it for another one. Students provide their preferences by fuzzy preference relations over the four options. For the sake of space, the preferences have been included as a supplementary material document which is available at http://sinbad2.ujaen.es/afryca/sites/default/files/app/computerScienceDegree-programmingLanguage.pdf.

Additionally to the experts and alternatives, it is necessary to establish the following parameters:

- Consensus threshold: $\vartheta = 0.85$
- Level of consensus for the advice generation: $\delta = 0.7$
- Maximum number of rounds allowed: max_round = 15

4.2. Resolution of the LS-GDM problem

In order to solve the problem and achieve the consensus, the new adaptive CRP is applied and the intelligent CRP support system is used to carry out the computations and visualize the CRP.

1. *Framework configuration*: all the parameters necessary in this phase have been already defined previously.

2. *Sub-groups management*: The fuzzy c-means based algorithm explained in Section 3.2.1 is applied to obtain the clusters containing the sub-groups of experts with similar opinions. Table 2 shows the sub-groups of experts $G = \{G^1, G^2, G^3, G^4\}$ in the first round.

Afterwards, a HFPR for each sub-group of experts is built and they are the input for the proposed CRP for LS-GDM.

The points a and b to define the membership function for the sub-group size are computed according to the number of experts m involved in the LS-GDM problem and the number of alternatives n. In this case study, we have considered 10% of experts to define the point a and the number of experts divided by the number of alternatives to define the point b, i.e. experts are equally distributed in the clusters obtained, but any other technique can be used.

$$a = Round(m\cdot 10/100), \quad b = Round(m/n)$$

Therefore, the points are $a = Round(50\cdot 10/100) = 5$ and $b = Round(50/4) = 13$, (see Fig. 5), where $Round(\cdot)$ is the round function.

The weights of each sub-group of experts considering its size and

Table 2
Sub-group of experts in the first round.

G^1	$e_1, e_9, e_{24}, e_{27}, e_{30}, e_{32}, e_{33}, e_{37}, e_{48}$
G^2	$e_6, e_8, e_{11}, e_{21}, e_{25}, e_{28}, e_{35}, e_{47}, e_{50}, e_{39}, e_{36}, e_4, e_{19}$
G^3	$e_2, e_{12}, e_{13}, e_{15}, e_{34}, e_{40}, e_{45}, e_5, e_{43}, e_{46}, e_{44}, e_{29}, e_{41}, e_3, e_{20}, e_{26}, e_{38}$
G^4	$e_7, e_{10}, e_{14}, e_{16}, e_{18}, e_{22}, e_{31}, e_{42}, e_{49}, e_{23}, e_{17}$

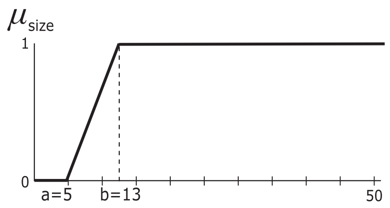

Fig. 5. Membership function for the sub-group *size*.

Table 3
Weights of the sub-groups of experts in the first round.

Sub-groups	G^1	G^2	G^3	G^4
Size	9	13	17	11
Membership degree sub-group size	0.5	1	1	0.75
Cohesion	0.51	0.49	0.44	0.59
Weights	0.19	0.29	0.25	0.27

Table 4
Consensus degree and consensus level for each round.

round	0	1	2	3	4	5	6	7	8
cr	0.64	0.66	0.69	0.72	0.74	0.75	0.79	0.83	**0.85**
level	low	low	low	high	high	high	high	high	high

cohesion are computed by using Eq. (18), in which the parameter β has been established after several experiments as $\beta = 3.0$ to increase the impact of the cohesion. Table 3 shows the size, cohesion and weight for each sub-group of experts in the first round.

3. Computing the consensus degree: The consensus degree obtained in the first round is $cr = 0.64$.

4. Adaptive feedback process: As the consensus degree achieved is not enough, another discussion round is necessary. Applying the Algorithm 2, it is easy to see that the consensus level is low, because $0.64 < \delta = 0.7$, thus a group feedback process is carried out to identify the furthest sub-groups and suggest them to modify their preferences and increase the consensus degree in the next round.

This adaptive CRP is repeated until the consensus threshold is achieved. Table 4 shows the consensus degrees obtained for each round and indicates the consensus level reached in such rounds. Fig. 6 shows the visualization of the CRP obtained by using the statistics tool implemented in AFRYCA 2.0 that is able to carry out Multi-Dimensional Scaling (MDS) [22] of preferences.

4.3. Comparison with previous CRP models

Even though, previous results provide a good performance according to our goals. It should seem convenient to compare such results with other previous proposals for CRP. First, we compare our model with two well-known and widespread CRP proposals, Chiclana's approach [11] and Kacprzyk's approach [20]. Our hypothesis was that our

Table 5
Classical approaches.

	Chiclana's approach	Kacprzyk's approach
Initial consensus degree	0.63	0.53
consensus degree achieved	0.85	0.86
rounds	12	15

proposal should reduce the cost to achieve the consensus (rounds, supervisions,...) and in both cases it is necessary to carry out more rounds to achieve the consensus (see Table 5). Fig. 7 shows the visualization of the CRP for each approach.

Second, a fairer comparison would consist of comparing our proposal with other CRPs for LS-GDM [34,38,48,49], but most of them [34,38,49] are focused on managing non-cooperative behaviours, therefore the comparison with our proposal is not fair, because their main feature is useless in our case study. And the CRP proposal in [48] for LS-GDM is incomparable, because the constrains imposed in it (see remark 2).

Remark 2. This approach represents the group preferences by using a possibility distribution based on hesitant fuzzy elements. The use of this type of information limits the elicitation of preferences, because experts have to use a discrete scale such as {0.1,0.2,0.3,0.4,0.5,0.6,0.7,0.8,0.9} and in the feedback process, a set of values from the same scale is computed as possible suggestions for experts to change their preferences. This implies another limitation in the feedback process because experts cannot change their preferences as they want, and the minimum change is ± 0.1. It is remarkable that in the feedback process the decision problem shows changes of 0.3 regarding the original preference in just one round which is not realistic, because usually experts do not want to make big changes in their preferences. Additionally, we have found some errors in the resolution process of the emergency decision making problem presented in the paper which makes difficult a comparison.

4.4. Analyzing the results

As result of the previous sections the following points must be highlighted:

- The proposed model performs effectively the CRP in LS-GDM as can be seen in Fig. 7 by adapting the process to the consensus degree in each round and reducing the preferences by a clustering process.
- Classical models compared, Chiclana's approach and Kacprzyk's approach, obtain from the initial round a consensus degree lower than the proposed model.
- The necessary number of rounds to achieve the required consensus degree with classical approaches is greater than in our proposal, therefore the latter reduces the time cost.
- The use of cohesion in the proposed model facilitates that experts are close to each other in the solution achieved unlike Kacprzyk's

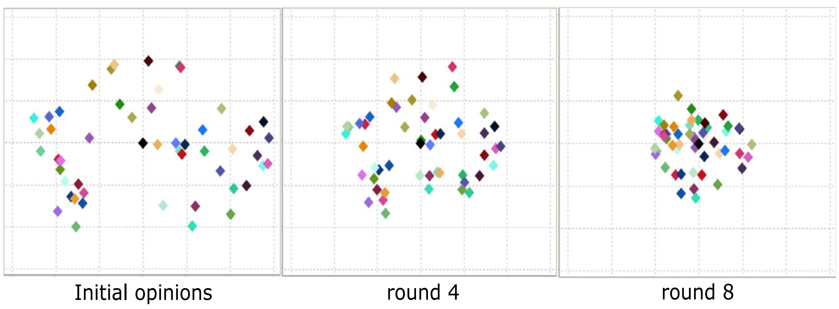

Fig. 6. MDS visualization of proposed CRP.

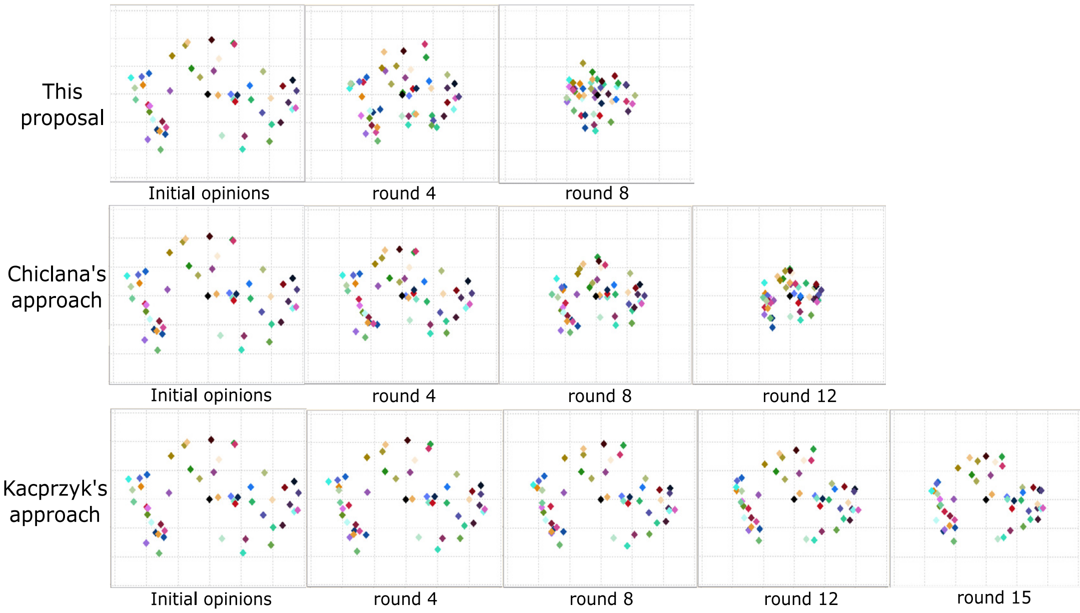

Fig. 7. MDS visualization of the CRPs.

approach and quicker than in Chiclana's one.

- The proposed CRP does not need to impose any limitation regarding the elicitation of preferences to achieve its goal, meanwhile others in the literature as Wu and Xu's approach limits the values of preferences elicited.

5. Conclusions and future research

Consensual decisions is a growing societal demand nowadays that becomes harder and more challenging in those decision making problems that involve a large number of experts. Despite its importance, most of current proposals in specialized literature are still focused on group decision situations with a few number of experts that present scalability and time cost limitations.

A novel CRP for LS-GDM based on a clustering process for weighting experts' preferences by using the size and cohesion of the clusters together a preference modelling with HFS and an adaptive feedback process has been introduced and compared with previous CRPs models. The results obtained show that the new CRP model for LS-GDM can effectively deal with these types of problems overcoming challenges proper of LS-GDM. This model has been implemented and integrated in an intelligent CRP support system.

As future research, we will study how the minimum cost can be used in the CRP to decrease the number of rounds to achieve the consensus and how to manage experts' behaviour that can make difficult to reach the consensus.

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4.6. Proceso de Alcance de Consenso Tratando con Expresiones Lingüísticas Comparativas en la Toma de Decisión en Grupo: Un Enfoque Difuso

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A consensus reaching process dealing with comparative linguistic expressions for group decision making: A fuzzy approach

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Abstract. Group Decision Making (GDM) deals with decision problems in which multiple experts, with their own attitudes and knowledge, evaluate different alternatives or solutions with the aim of achieving a common solution. In such cases disagreements can appear, which might led to failed solutions. To manage such conflicts, Consensus Reaching Processes (CRPs) have been added to the GDM solving process. GDM problems under uncertainty often model uncertainty by linguistic descriptors, being most of linguistic based CRPs based on the use of single linguistic terms for modelling experts' opinions, which cannot be expressive enough in some situations because of either the uncertainty involved or the experts' hesitancy. Therefore, this paper aims to fill this gap by proposing a novel *consensus* model dealing with GDM problems in which experts' preferences are elicited by means of Comparative Linguistic Expressions (CLEs) based on Hesitant Fuzzy Linguistic Term Sets, which allow to model the experts' hesitancy in a flexible way. Furthermore, CLEs are modelled by fuzzy membership functions in order to keep the fuzzy representation in the whole CRP and preserve as much information as possible. Additionally, the proposed model is implemented and integrated in an intelligent CRP support system, so-called AFRYCA 3.0 to carry out a case study about this new CRP and compare it with previous models.

Keywords: group decision making, comparative linguistic expressions, hesitant fuzzy linguistic term sets, consensus reaching process

1. Introduction

Human beings, firms, organizations, companies and so forth, cope with decision situations in their daily tasks. Nowadays, these decision situations become more and more complex due to the tighter time constraints, the increasing uncertainty surrounding the problems and the lack of information of experts regarding the problem. The increasing complexity in real world decision problems usually implies that a group of experts replaces the role of a single expert by assuming that the former is smarter than the latter. In these situations, we talk about Group Decision Making (GDM) problems in which a group of experts aims at obtaining a solution for the decision problem that usually consists of choosing the best alternative from a set of possible alternatives, by eliciting their preferences [12, 43, 54] and applying a decision rule [17]. Generally, GDM problems have been solved by applying a selection process [14, 27, 56] that can lead to solutions in which one or several experts of the group do not agree. When it happens, Consensus Reaching Processes (CRPs) have become an additional and necessary task in GDM, in which an iterative discussion process supervised by a human figure so-called *moderator* guides the experts by providing advice to overcome conflicts among them and obtain agreed solutions by all experts [5, 32, 50, 53].

Real-world GDM problems are usually *ill-structured* and hence defined under uncertainty

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and incomplete information hard to model by probabilistic models [4]. When decision problems are defined under non-probabilistic uncertainty context, fuzzy linguistic information [21, 41, 58] has provided successful results in different fields [9, 19, 28, 55]. Hence, experts involved GDM problems provide their opinions according to their own knowledge and use the fuzzy linguistic information whenever its fuzzy representation would be adequate for the decision situations [29, 46].

Across specialized literature different fuzzy linguistic based approaches for modelling preferences in GDM can be found [3, 24, 42, 51, 62]. However, these approaches offer to assess experts' opinions just by single linguistic terms that hardly matches experts' real knowledge in complex decision situations. Hence, the need of enhancing the elicitation of linguistic preferences by expressions more elaborated than single labels has been deeply studied [30]. Among the different proposals introduced, it should be highlighted the use of Hesitant Linguistic Term Sets (HFLTSs) and Comparative Linguistic Expressions (CLEs) [33], because both facilitate the modelling of experts' hesitation and provides a flexible and powerful linguistic modelling close to human beings cognitive process. Recently, multiple proposals have introduced different CRPs to deal with HFLTSs in GDM [6, 40, 48, 52, 60].

Therefore, due to the fact that the use of more than a single linguistic term by experts is often necessary in GDM as well as achieve agreed solutions, this paper aims at developing a novel consensus model for GDM problems that models experts' information by means of CLEs that enrich the assessment of experts' opinions and model experts' hesitation in a closer way to human cognition than HFLTSs. This consensus model will manage such CLEs based on HFLTS by using a fuzzy representation for keeping the fuzzy view of experts' preferences. Eventually, the CRP for dealing with GDM problems with CLEs proposed in this problem will be included in the consensus based software AFRYCA [13, 27] and then applied to a GDM problem in order to clarify the performance of the CRP.

This paper is structured as follows: Section 2 revises some preliminary concepts about GDM, CRP, linguistic modelling with HFLTS and CLEs and reviews some related works with the proposal. Section 3 presents the consensus model dealing with CLEs modelled by HFLTS. Section 4 shows the performance of the CRP with CLEs and Section 5 concludes this paper.

2. Preliminaries

This paper aims at introducing a consensus model capable of dealing with CLEs as preference assessments in hesitant decision situations keeping the fuzzy representation of such preferences. Before presenting such a model, this section briefly reviews some basic and necessary concepts about GDM and CLEs based on HFLTS to facilitate the understanding of our proposal. Afterwards, some proposals about CRPs dealing with HFLTS are revised in short.

2.1. Group decision making

Recently, real world decision problems become more and more complex due to their scalability, uncertainties involved, lack of information, time constraints and so forth. Consequently, the participation of several experts in their resolution is becoming increasingly common, resulting in GDM problems.

A GDM problem is formally defined as a decision situation in which a group of experts $E = \{e_1, \ldots, e_m\}$ $(m \geq 2)$ provide their preferences over a finite set of alternatives $X = \{x_1, \ldots, x_n\}$ $(n \geq 2)$ in order to obtain the best either solution or set of solutions among all alternatives. Each expert e_i provides his/her preferences according to their own attitudes and motivations with the aim of reaching a collective decision. Such preferences can be modelled by using different preference structures being the most usual one in these problems the elicitation of preference relations. A preference relation P_i, is composed by assessments, p_i^{lk}, provided by the expert e_i, that represents the preference degree of the alternative x_l over the alternative x_k, $l, k \in \{1, \ldots, n\}$:

$$P_i = \begin{pmatrix} p_i^{11} & \ldots & p_i^{1n} \\ \vdots & \ddots & \vdots \\ p_i^{n1} & \ldots & p_i^{nn} \end{pmatrix}$$

According to information domain in which the experts elicit their information, different types of preference relations can be used such as, fuzzy preference relation [26], linguistic preference relation [31] and hesitant fuzzy linguistic preference relation [64].

Once experts have elicited their preferences, the classical resolution scheme for a GDM problem is composed by two phases (see Fig. 1) [36]:

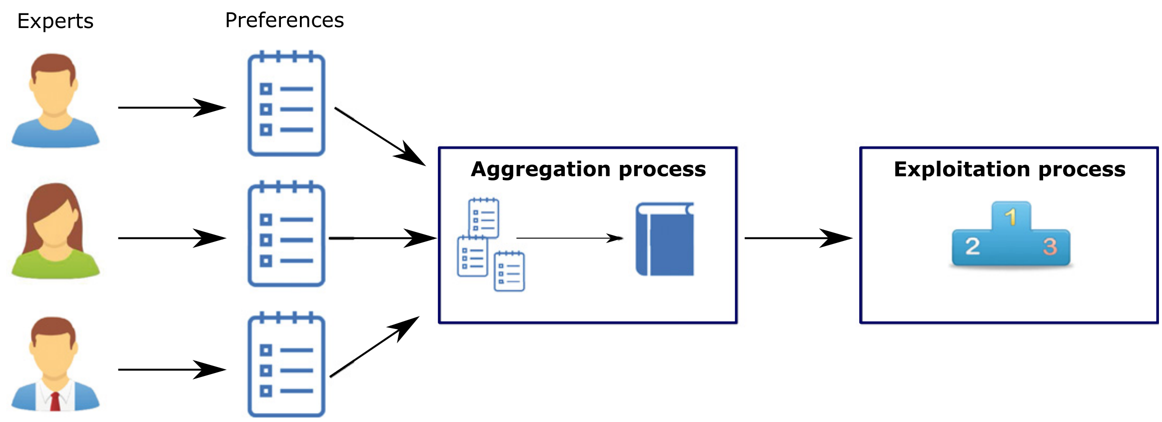

Fig. 1. GDM classical resolution scheme.

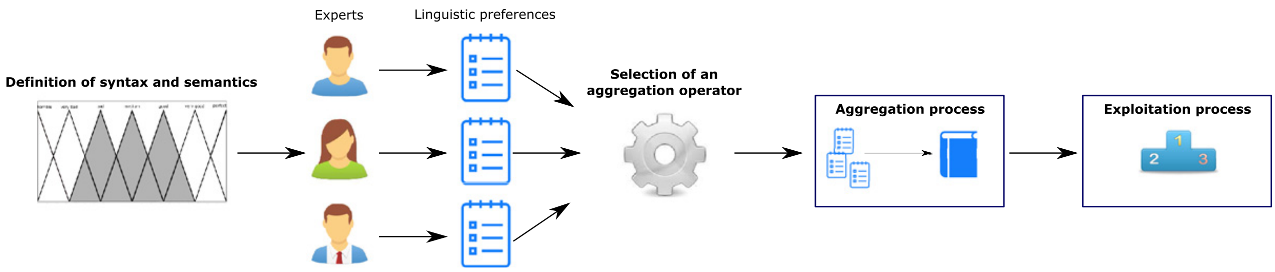

Fig. 2. LDM classical resolution scheme.

– *Aggregation*: Experts' preferences are aggregated by using an aggregation operator.
– *Exploitation*: Selecting a set of criteria, an alternative or a set of them are obtained, being such alternative/s the solution/s of the problem.

When experts provide their preferences by using linguistic expression domains, the resolution scheme includes two additional phases (see Fig 2) [10]:

– *Definition of syntax and semantics*: The linguistic expression domain in which experts provide their assessments about alternatives and criteria is defined.
– *Selection of an aggregation operator*: A linguistic aggregation operator suitable to aggregate the assessments provided by experts is chosen.

The classical resolution schemes for GDM problems reach a solution but do not guarantee that such a solution would be an agreed one, since the scheme does not consider the agreement across all the resolution phases. In order to avoid any conflict in the GDM resolution process a previous stage so called Consensus Reaching Process has been added [37].

2.2. *Consensus reaching process*

The CRP is an iterative process in which experts try to make their opinions closer to each other, it means to reach a consensus. It is convenient to clarify the meaning of *consensus* in this context, because it has been used from different points of view. Some researchers have defined consensus as the *unanimity*, almost impossible to reach in real-world GDM problems [18, 22]. Therefore, other softer views have been presented. *Soft consensus* is one of the most accepted consensus definitions based on the fuzzy majority concept introduced by Kacprzyk [12], closer to the perception that human beings have about consensus. According to this concept, the consensus is reached when "*most of the important individuals agree (as to their testimonies concerning) almost all of relevant opinions*".

The CRPs based on soft consensus often involve several key aspects: preference representation, consensus measure, feedback adjustment mechanism, decision context, and behaviors of decision-makers. Different CRP models often focus on different aspects of the CRP. Regarding the main aspects of focus in the CRP study, we list different types of CRPs based on soft consensus reported in the literature. Notably, a CRP study may focus on multiple

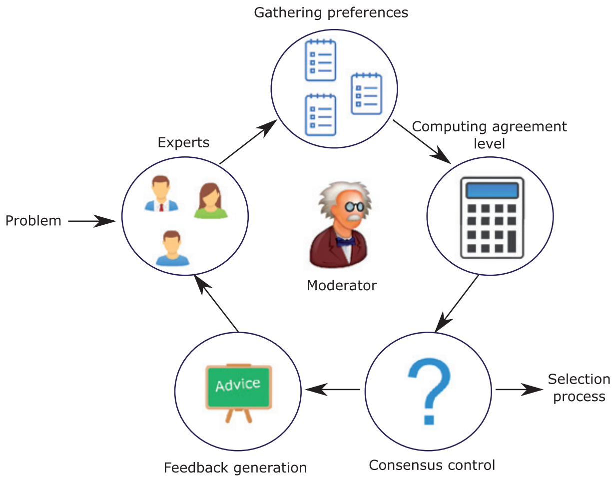

Fig. 3. General CRP scheme.

combinations of the aspects introduced above, so the listed categories of CRPs based on soft consensus are not mutually exclusive.

CRPs aim at reaching a high level of agreement after several discussion rounds [37]. The discussion process is generally supervised by a person, so-called *moderator*, whose main task is to guide to the experts who participate in the process, specially, those whose opinions are farther apart from of the remaining ones of the group. Fig. 3 shows a general CRP scheme according to the definition proposed in [27] and whose main phases are further detailed below:

1. *Gathering preferences*: Each expert provides his/her opinion over the alternatives by using the corresponding preference relation.
2. *Computing agreement level*: This phase computes the existing level of agreement among experts. It can be obtained by applying different consensus measures [2].
3. *Consensus control*: The CRP aims at achieving a minimum agreement among experts, hence this stage compares the consensus degree with a consensus threshold, defined a priori. If the agreement desired has been reached, the GDM process moves onto the selection, otherwise, another round of discussion is carried out. The number of rounds allowed is also limited.
4. *Feedback generation*: When the agreement reached is lower than the required, it is necessary to increase the level of agreement in the coming round of the CRP. To do so, a *feedback generation process* is carried out. Classically, the moderator identified experts' assessments which were farthest from consensus, and advises them to modify them [22, 37]. However, there exist other proposals in which the consensus is achieved without considering experts' opinion changes but rather by applying automatic changes [6, 32, 59, 61].

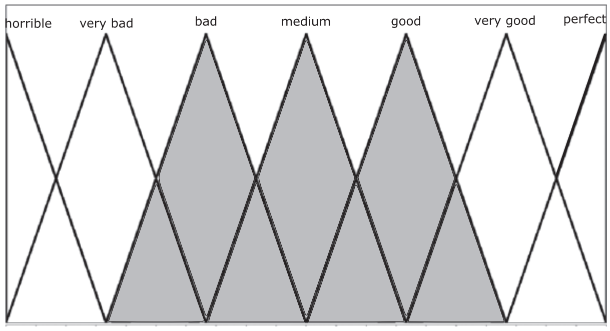

Fig. 4. HFLTS example.

2.3. *Hesitant fuzzy linguistic terms set and comparative linguistic expressions*

Our aim of developing a new consensus model dealing with CLEs so it is necessary to review different concepts related to CLEs and HFLTSs.

The use of linguistic information has provided successful results when modelling uncertainty and vagueness, which usually appear in GDM problems [23, 44]. Classically, in LDM, experts provided their preferences by using single linguistic terms, that may imply an important drawback when experts have lack of information about the problem or do not have enough knowledge. Such situations might drive experts to hesitate among different linguistic terms. The concept of HFLTS was introduced to facilitate the experts' preferences elicitation by using linguistic expressions in those cases in which they hesitate among several linguistic terms (see Fig. 4).

Definition 1. [33] Let $S = \{s_0, \ldots, s_g\}$ be a linguistic term set, a HFLTS, H_S, is an ordered finite subset of the consecutive linguistic terms of S.

$$H_S = \{s_i, s_{i+1}, \ldots, s_j\}, s_k \in S, k \in \{i, \ldots, j\}$$

Even though HFLTSs facilitate the modelling of experts' hesitancy by using multiple linguistic terms, they are far from the human cognition in which human beings express their knowledge/information/preferences. Therefore, several proposals related to the generation of CLEs were reviewed in [30]. Among them, stands out the preference modelling presented in [33, 34] based on CLEs generated by a context-free grammar and based on HFLTSs that model experts' hesitancy and are closer to human cognition.

The context-free grammar, G_H, defined to generate CLEs was defined as:

Definition 2. [34] Let G_H be a context-free grammar and $S = \{s_0, \ldots, s_g\}$ a linguistic term set. The elements of $G_H = (V_N, V_T, I, P)$ are defined as follows.

$$V_N = \{\langle primary\ term\rangle, \langle composite\ term\rangle, \langle unary\ relation\rangle, \langle binary\ relation\rangle, \langle conjunction\rangle\}$$
$$V_T = \{at\ least,\ at\ most,\ between,\ and,\ s_0,\ s_1, \ldots, s_g\}$$
$$I \in V_N$$

Whose production rules defined in an extended Backus-Naur form are:

$$P = \{I\ ::=\ \langle primary\ term\rangle|\langle composite\ term\rangle$$
$$\langle composite\ term\rangle\ ::=\ \langle unary\ relation\rangle\langle primary\ term\rangle|\ \langle binary\ relation\rangle\langle primary\ term\rangle\langle conjunction\rangle\langle primary\ term\rangle$$
$$\langle primary\ term\rangle\ ::=\ s_0|s_1|\ldots|s_g$$
$$\langle unary\ relation\rangle\ ::=\ at\ least|at\ most$$
$$\langle binary\ relation\rangle\ ::=\ between$$
$$\langle conjunction\rangle\ ::=\ and\}$$

Therefore, some examples of the CLEs that can be generated by G_H, and used by experts are the following ones: $at\ most\ s_i$, $at\ least\ s_j$ or $between\ s_i\ and\ s_{i+1}$.

These CLEs are easy to elicit and understand in the GDM process, but still it is necessary to accomplish the computations with such CLEs. Therefore, in [33, 34] was defined a transformation function, E_G, of CLEs to HFLTSs to facilitate such computations.

Definition 3. [34] Let E_{G_H} be a function that transforms CLEs, $ll \in S_{ll}$, obtained by G_H, into HFLTSs, H_S. S is the linguistic term set used by G_H and S_{ll} is the expression domain generated by G_H.

$$E_{G_H} : S_{ll} \to H_S$$

The CLEs generated by the context-free grammar G_H are transformed into HFLTSs H_S as follows:

$$E_{G_H}(s_i) = \{s_i | s_i \in S\}$$
$$E_{G_H}(at\ most\ s_i) = \{s_j | s_j \leq s_i\ and\ s_j \in S\}$$
$$E_{G_H}(at\ least\ s_i) = \{s_j | s_j \geq s_i\ and\ s_j \in S\}$$
$$E_{G_H}(between\ s_i\ and\ s_j) = \{s_k | s_i \leq s_k \leq s_j\ and\ s_k \in S\}$$

Eventually, to carry out the linguistic computations with CLEs, several computational models have been proposed [16, 43, 65], which mainly consist of merging the HFLTS by means of an envelope [16, 33]. In our proposal, we will use the concept of *fuzzy envelope* [16], which represents the semantics of the CLEs by means of trapezoidal fuzzy membership functions, by keeping a fuzzy representation of such preferences.

Definition 4. [16] The *fuzzy envelope*, $env_F(H_S)$, is defined as a trapezoidal fuzzy membership function as follows:

$$env_F(H_S) = T(a, b, c, d)$$

where H_S is a HFLTS and $T(a, b, c, d)$ is a fuzzy trapezoidal membership function (see [16] for further detail).

Note that a trapezoidal fuzzy membership function can be defined as:

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & \text{if} \quad x \leq a \\ \dfrac{x-a}{b-a}, & \text{if} \quad x \in (a, b] \\ 1 & \text{if} \quad x \in (b, c] \\ \dfrac{d-x}{d-c}, & \text{if} \quad x \in (c, d) \\ 0, & \text{if} \quad x \geq d \end{cases}$$

with a and d being the lower and upper limits respectively, for the not null values of $\mu_{\tilde{A}}(x)$.

2.4. Related works

The use of HFLTS in CRPs has attracted the attention of some scholars who have proposed different consensus models. Dong et al. defined in [7] a distance-based consensus measure for HFLTS that is used to develop an optimization-based consensus model by means of a linear programming model. The consensus model provides the optimal solutions minimizing the number of changes between the original and adjusted experts' opinions. In a similar way, Zhang et al. [60] proposed another distance measure for HFLTS and developed a consensus model for multi-atribute GDM which minimizes the adjustment distance between the initial and adjusted experts' opinions in the CRP. Monserrat-Adell et al. studied in [25] the degree of agreement among multiple decision makers in a GDM problem using an algebraic extension of HFLTS and introduced several individual and collective hesitant consensus measures which allow to measure the polarization within the group's opinions.

The study of the consistency it is a very important process because it ensures the opinions provided by experts are neither random nor illogical. Since, several proposals include a consistency process before applying the CRP. Xu et al. [52] defined the additive consistency of a hesitant fuzzy linguistic preference relation (HFLPR) and introduced a consistency index based on a deviation measure for HFLPRs to decide whether a HFLPR is the acceptable consistency. An algorithm to improve the consistency of a HFLPR was proposed. Moreover, an individual and group consensus indexes are defined to compute the consensus degree of a GDM problem and a consensus reaching algorithm was designed to reach the consensus. Similarly, Zhao et al. [63] defined a consistency measure for HFLPR based on discrete fuzzy numbers and proposed an optimization algorithm to increase the consistency degree of a HFLPR. They also defined a consensus measure to calculate the consensus level based on the distance between individual preference relations and developed a CRP. Considering the additive consistency for HFLPR, Song et al. [38] introduced two approaches to complete the missing elements of a HFLPR, thus, the completed HFLPRs are consistent. They also developed a CRP that adjusts only the maximum deviation elements in each round. Wu et al. [48] used a possibility distribution and the 2-tuple linguistic model to deal with HFLTSs and presented a novel algorithm to improve the consistency of a HFLPR. A consensus model based on the distance between experts was also introduced. In [47], Wu et al, defined another consensus model that transforms the HFLTS into a possibility distribution by means of the numerical scale model [8]. In such a model, computations are carried out by means of the possibility distribution and the consensus measure introduced is based on it. This model uses the distance between experts and the collective preference to obtain the consensus level. A similar consensus process was introduced in [49], but the consensus measure is based on the distance between experts.

A different challenge was studied by Tian et al. in [40]. In this paper, the consensus model introduced is able to deal with multi-granular and unbalanced linguistic term set for HFLTS. A distance measure based on ordinal semantics and possibility distribution is defined and used in the consensus reaching algorithm.

3. A consensus model for GDM with CLEs based on a fuzzy representation

The use of HFLTSs in GDM problems for modelling experts' preferences has been recently and widely applied [1, 15, 34, 39, 57] because of its advantages [35]. Hence, the need to achieve agreed solutions has been object of interest in multiple researches as it was aforementioned, but all of them neglect the fuzzy representation of the HFLTSs, losing valuable information in the process in many cases. Therefore, in this section we propose a novel CRP for dealing with GDM problems in which experts' preferences will be first elicited by means of CLEs that are close to human cognition and then transformed to HFLTSs and the latter will be modelled by means of their fuzzy envelopes as fuzzy trapezoidal membership functions, keeping a fuzzy representation that will be used to achieve the level of agreement required in the CRP. The proposal modifies and adds new phases to the classical CRP model (see Fig. 3) that are shown in Fig. 5 and further explained in the coming subsections.

3.1. Gathering preferences

The novel consensus model deals with GDM problems defined in a framework in which experts express their preferences by CLEs instead of HFLTSs to make the elicitation of preferences closer to the way that human beings express their opinions. In particular, each expert e_i expresses his/her preferences by using a preference relation P_i, $X \times X \rightarrow S_{ll}$, where S is a linguistic terms set. Considering S the linguistic terms set represented in Fig. 4, an example of preference may be as follows:

$$P_i = \begin{pmatrix} - & \textit{bt bad and medium} & \textit{good} \\ \textit{at least good} & - & \textit{at most bad} \\ \textit{bad} & \textit{bt medium and good} & - \end{pmatrix}$$

Remark 1. bt stands for between.

3.2. Unification

Due to the fact that the context free grammar presented in Def. 2 allows experts to provide their preferences by either single linguistic terms or CLEs, in which the former is represented by a single fuzzy membership function and the latter for multiple ones. It is necessary to conduct both representations into a unified expression domain. Hence, first

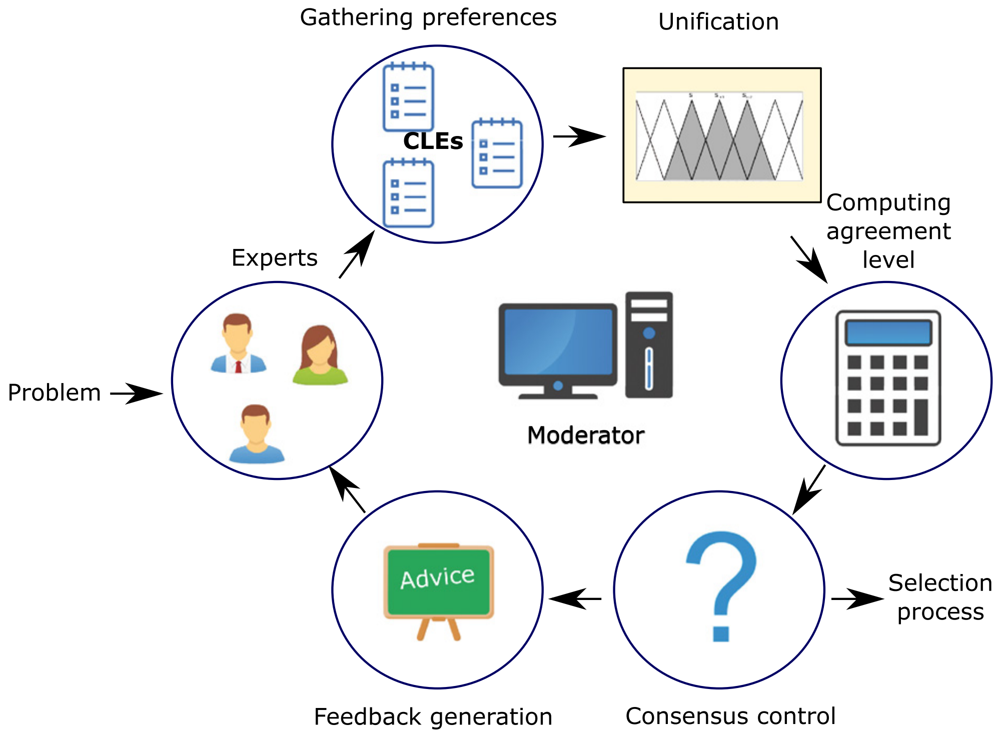

Fig. 5. Proposed CRP scheme.

the CLEs will be represented by their fuzzy envelope (see Def. 4) so both are represented by means of parametric fuzzy membership functions. After that, the unification process introduced in [11] is adapted to conduct the gathered information into fuzzy sets in a basic linguistic term set, S_T, as follows:

Definition 5. [11, 20] Let $s_i = T(a, b, c, d)$ and $S_T = s_0^T, \dots, s_g^T$ be either the semantics of a linguistic term in S_{ll} or the fuzzy envelope of a CLE and the basic linguistic term set respectively. The unification transformation function $\tau_s s_T$ is then defined as:

$$\tau_s s_T : s_i \to F(S_T)$$
$$\tau_s s_T(s_i) = \sum_{k=0}^{g_T} s_k^T / \gamma_k^j \qquad (1)$$
$$\gamma_k^i = \max_y \min\{\mu_{s_i}(y), \mu_{s_k^T}(y)\}$$

being $F(S_T)$ the set of fuzzy sets defined in S_T, $\mu_{s_i}(y)$ and $\mu_{s_k^T}(y)$ the membership functions of the fuzzy sets associated to the terms s_i and s_k^T, respectively.

For sake of clarity and similarly to [11], it is assumed that each unified assessment is represented just by the degrees of membership to each term of S_T, $p_i^{lk} = (\gamma_{i0}^{lk}, \dots, \gamma_{ig}^{lk})$. Therefore, each unified expert's preference relation models each preference as a fuzzy set as follows:

$$\begin{pmatrix} - & \dots & (\gamma_{i0}^{1n}, \dots, \gamma_{ig}^{1n}) \\ \vdots & \ddots & \vdots \\ (\gamma_{i0}^{n1}, \dots, \gamma_{ig}^{n1}) & \dots & - \end{pmatrix}$$

3.3. Computing consensus degree

Once unification process has been carried out, the next step in the CRP is to compute the consensus degree, $cr \in [0, 1]$, that measures the current level of agreement in the group of experts [12]. The more the consensus degree, the better the agreement among experts. Our consensus model needs to adapt its computation to the unified information obtained in the previous step as follows:

– For each unified assessment $p_i^{lk} = (\gamma_{i0}^{lk}, \dots, \gamma_{ig}^{lk})$ a representative central value $cv_i^{lk} \in [0, g]$ is computed as follows:

$$cv_i^{lk} = \frac{\sum_{j=0}^{g} index(s_j) \cdot \gamma_{ij}^{lk}}{\sum_{j=0}^{g} \gamma_{ij}^{lk}}, s_j \in S_T \qquad (2)$$

being $index(s_j) = j \in \{0, \dots, g\}$.

– For each pair of experts e_i, e_t, $(i < t)$, a similarity matrix $SM_{it} = (sm_{it}^{lk})_{n\times n}$ is obtained. Each similarity value $sm_{it}^{lk} \in [0, 1]$ represents the agreement level between the experts e_i and e_t about the pair of alternatives (x_l, x_k) and it is computed as follows:

$$sm_{it}^{lk} = 1 - \left|\frac{cv_i^{lk} - cv_t^{lk}}{g}\right| \tag{3}$$

– The similarity values are aggregated by means of an aggregation operator, ρ, by obtaining a consensus matrix $CM = (cm^{lk})_{n\times n}$.

$$\begin{aligned} cm^{lk} &= \rho(SIM^{lk}) \\ SIM_{lk} &= \{sm_{12}^{lk}, \ldots, sm_{1m}^{lk}, \ldots, \\ &sm_{(m-1)m}^{lk}\} \end{aligned} \tag{4}$$

SIM_{lk} represents the set of all pairs of experts' similarities about the pair of alternatives (x_l, x_k) with $\left|SIM^{lk}\right| = \binom{m}{2}$, and cm^{lk} the consensus degree achieved by the group of experts about the pair of alternatives (x_l, x_k).

– Starting from CM, a consensus degree ca^l is computed for each alternative x_l.

$$ca^l = \frac{\sum_{k=1,k\neq l}^{n} cm^{lk}}{n-1} \tag{5}$$

– Finally, overall consensus degree, cr, is computed as follows:

$$cr = \frac{\sum_{l=1}^{n} ca^l}{n} \tag{6}$$

3.4. Consensus control

Analogously to the general CRP scheme presented in Fig. 5, our proposal determines in this phase if the required agreement is achieved or it is still necessary to keep discussing to make closer the experts' opinions. Therefore, the consensus degree cr, is compared with a consensus threshold $\alpha \in [0, 1]$, that defines the required consensus. If $cr > \alpha$, the CRP ends and the group moves on the selection process [34], otherwise, the process requires another discussion round. The number of discussion rounds is usually limited by the parameter $Maxrounds \in \mathbb{N}$.

3.5. Feedback generation

When consensus required α is not reached, moderator usually advises experts about where are the conflicts and how to change their preferences, specially those whose opinions are further away from the collective opinion, with the aim to increase the level of agreement in the next round.

Classically, a human has played the role of the moderator. However, our model replaces the role of the moderator [28] by an automatic process. Thus, the proposed feedback generation process identifies the furthest preferences from the collective opinion for each pair of alternatives and then, it is advised to modify them according to a specific direction (increase/decrease) determined by different rules. This process consists of the following steps:

1. Compute a collective preference and proximity matrices: A collective preference $P_c = (p_c^{lk})_{n\times n}$, $p_c^{lk} \in [0, g]$, is computed for each pair of alternatives by aggregating preference relations:

$$p_c^{lk} = \nu(cv_1^{lk}, \ldots, cv_m^{lk}) \tag{7}$$

Then, a proximity matrix PP_i between the expert e_i and the collective preference P_c is computed as follows:

$$PP_i = \begin{pmatrix} - & \ldots & pp_i^{1n} \\ \vdots & \ddots & \vdots \\ pp_i^{n1} & \ldots & - \end{pmatrix}$$

Afterwards, proximity values $pp_i^{lk} \in [0, 1]$ are computed for each pair of alternatives (x_l, x_k):

$$pp_i^{lk} = 1 - \left|\frac{cv_i^{lk} - p_c^{lk}}{g}\right| \tag{8}$$

Proximity values identify the furthest preferences from the collective opinion and which one should be modified by the experts.

2. Identify preferences to change: Consensus degrees ca^l and cp^{lk} of each pair of alternatives (x_l, x_k) are compared with the overall consensus degree cr in order to identify the alternatives that should be changed.

$$CC = \{(x_l, x_k)\big| ca^l < cr \wedge cp^{lk} < cr\}$$

Once the pairs of alternatives are identified, the model looks for the experts who should change their preferences on each of these pairs of alternatives. The identified experts will be those whose assessment cv_i^{lk} on the pair alternatives $(x_l, x_k) \in CC$ is furthest to p_c^{lk}. To do so, an average proximity $\overline{pp}^{lk}$ is calculated by means of an aggregation operator.

$$\overline{pp}^{lk} = \lambda(pp_1^{lk}, \ldots, pp_m^{lk}) \tag{9}$$

Thus, expert e_i whose $pp_i^{lk} < pp^{lk}$ is advised to modify his/her assessments p_i^{lk} on (x_l, x_k).

3. Establish change directions: Once the modifications for the experts are identified, the following step is to determine in which direction experts should modify their assessments. Several direction rules are applied to suggest the direction of the changes in order to increase the level of agreement in the group. To do so, an acceptability threshold $\epsilon \geq 0$, a positive value close to zero, defines a margin of acceptability when cv_i^{lk} and p_c^{lk} are close to each other.
   - **RULE 1**: If $cv_i^{lk} - p_c^{lk} < -\epsilon$ then e_i should *increase* his/her assessments p_i^{lk} on (x_l, x_k).
   - **RULE 2**: If $cv_i^{lk} - p_c^{lk} > \epsilon$ then e_i should *decrease* his/her assessments p_i^{lk} on (x_l, x_k).
   - **RULE 3**: If $-\epsilon \leq cv_i^{lk} - p_c^{lk} \leq \epsilon$ then e_i should *not modify* his/her assessments p_i^{lk} on (x_l, x_k).

The previous rules identify the direction of the change but do not determine how much should the change be carried out by experts. When the expression domain in which experts express their preferences is continuous, the change can highly vary but, in a discrete domain as the current one used in our proposal, such changes are much less because will not be greater than the granularity of the linguistic term set.

However, in order to show the performance of our proposal, it will be implemented in the CRP support system, AFRYCA [27] and for sake of clarity we will fix in such an implementation the modifications that can be carried out by experts according to the direction of the change received, his/her current assessment and considering experts accept the suggestion provided by the consensus model, as follows:

- Expert e_i should increase his/her assessment p_i^{lk}.
  * If $p_i^{lk} = s_p$, where s_p is a single linguistic term, then the recommendation for the expert is to change his/her assessment so that $p_i^{lk} = s_{p+\theta}, \theta \in [1, g-1], p+\theta \leq g$. In case that $s_p = s_g$ no change will be applied.
  * If $p_i^{lk} = \textit{at least } s_p$ or $\textit{at most } s_p$, where s_p is a linguistic term, then the recommendation for the expert is to change his/her assessment so that $p_i^{lk} = \textit{at least } s_{p+\theta}$ or $\textit{at most } s_{p+\theta}$ respectively, $\theta \in [1, g-1]$, $p+\theta \leq g$. In case that $s_p = s_g$ no change will be applied.
  * If $p_i^{lk} = \textit{between } s_p \textit{ and } s_q$, where s_p, s_q are linguistic terms $p > q$, then the recommendation for the expert is to change his/her assessment so that $p_i^{lk} = \textit{between } s_{p+\theta} \textit{ and } s_q, \theta \in [1, g-1]$, $p+\theta \leq q$. In case that $s_{p+\theta} = s_q$, the new assessment is $p_i^{lk} = s_q$.
- Expert e_i should decrease his/her assessment p_i^{lk}.
  * If $p_i^{lk} = s_p$, where s_p is a single linguistic term, then the recommendation for the expert is to change his/her assessment so that $p_i^{lk} = s_{p-\theta}, \theta \in [1, g-1], p-\theta \geq 0$. In case that $s_p = s_0$ no change will be applied.
  * If $p_i^{lk} = \textit{at least } s_p$ or $\textit{at most } s_p$, where s_p is a linguistic term, then the recommendation for the expert is to change his/her assessment so that $p_i^{lk} = \textit{at least } s_{p-\theta}$ or $\textit{at most } s_{p-\theta}$ respectively, $\theta \in [1, g-1]$, $p-\theta \geq 0$. In case that $s_p = s_0$ no change will be applied.
  * If $p_i^{lk} = \textit{between } s_p \textit{ and } s_q$, where s_p, s_q are linguistic terms $p > q$, then the recommendation for the expert is to change his/her assessment so that $p_i^{lk} = \textit{between } s_{p-\theta} \textit{ and } s_q, \theta \in [1, g-1]$, $p-\theta \geq 0$. In case that $s_p = s_0$, no change will be applied.

Remark 2. The parameter $\theta \in \mathbb{N}$ and whose possible values are in the range $[1, g-1]$, expresses the change degree to apply, which can be adjust depending on the desired degree.

Remark 3. Note that, the proposed consensus model is focused on GDM problem with a few number of experts and non-cooperative behavior of experts has not been considered.

4. Case study

This section presents a case study, which is applied to the proposed CRP, in order to demonstrate its usefulness and advantages. Furthermore, among the different proposals for CRP dealing with HFLTS we make a comparison with the CRP model presented in [48] because of similar phases and tasks but not fuzzy representation. To facilitate such comparison, the illustrative example presented by Wu et al. in [48] will be simulated for both CRP models. Note that our CRP model has been implemented and integrated into the intelligent CRP support system, AFRYCA 3.0 [13, 27].

4.1. Definition of the case study

The GDM problem formulated by Wu et al. in [48] describes a situation in which an investment company wants to invest a sum of money in the best industrial sector. There are four possible alternatives or sectors $X = \{x_1, x_2, x_3, x_4\}$ and four experts $E = \{e_1, e_2, e_3, e_4\}$ who make the decision. Each expert provides their preferences using the following linguistic term set $S = \{s_0 = \textit{Extremely poor}, s_1 = \textit{Very poor}, s_2 = \textit{Poor}, s_3 = \textit{Slightly poor}, s_4 = \textit{Fair}, s_5 = \textit{Slightly good}, s_6 = \textit{Good}, s_7 = \textit{Very good}, s_8 = \textit{Extremely good}\}$.

The preferences provided by the experts were represented in [45] by HFLTS as follows:

$$P_1 = \begin{pmatrix} \{s_4\} & \{s_4, s_5\} & \{s_5, s_6\} & \{s_6, s_7\} \\ \{s_3, s_4\} & \{s_4\} & \{s_4, s_5\} & \{s_5\} \\ \{s_2, s_5\} & \{s_3, s_4\} & \{s_4\} & \{s_4, s_5\} \\ \{s_1, s_2\} & \{s_3\} & \{s_3, s_4\} & \{s_4\} \end{pmatrix}$$

$$P_2 = \begin{pmatrix} \{s_4\} & \{s_5, s_6, s_7\} & \{s_2, s_3\} & \{s_6\} \\ \{s_1, s_2, s_3\} & \{s_4\} & \{s_2\} & \{s_4, s_5\} \\ \{s_5, s_6\} & \{s_6\} & \{s_4\} & \{s_6, s_7\} \\ \{s_2\} & \{s_3, s_4\} & \{s_1, s_2\} & \{s_4\} \end{pmatrix}$$

$$P_3 = \begin{pmatrix} \{s_4\} & \{s_5\} & \{s_6, s_7\} & \{s_6, s_7\} \\ \{s_3\} & \{s_4\} & \{s_4, s_5\} & \{s_5, s_6\} \\ \{s_1, s_2\} & \{s_3, s_4\} & \{s_4\} & \{s_5\} \\ \{s_1, s_2\} & \{s_2, s_3\} & \{s_3\} & \{s_4\} \end{pmatrix}$$

$$P_4 = \begin{pmatrix} \{s_4\} & \{s_4, s_5\} & \{s_3, s_4\} & \{s_1, s_2\} \\ \{s_3, s_4\} & \{s_4\} & \{s_1, s_2\} & \{s_0, s_1\} \\ \{s_4, s_5\} & \{s_6, s_7\} & \{s_4\} & \{s_3, s_4\} \\ \{s_6, s_7\} & \{s_7, s_8\} & \{s_4, s_5\} & \{s_4\} \end{pmatrix}$$

For our proposal, the experts would provide their preferences by using preference relations based on CLEs, whose transformations into HFLTSs by Def 3 are equivalent to the above. The equivalent experts' preferences by using CLEs can be expressed as follows:

$$P_1 = \begin{pmatrix} s_4 & bt\ s_4\ and\ s_5 & bt\ s_5\ and\ s_6 & bt\ s_6\ and\ s_7 \\ bt\ s_3\ and\ s_4 & s_4 & bt\ s_4\ and\ s_5 & s_5 \\ bt\ s_2\ and\ s_5 & bt\ s_3\ and\ s_4 & s_4 & bt\ s_4\ and\ s_5 \\ bt\ s_1\ and\ s_2 & s_3 & bt\ s_3\ and\ s_4 & s_4 \end{pmatrix}$$

$$P_2 = \begin{pmatrix} s_4 & bt\ s_5\ and\ s_7 & bt\ s_2\ and\ s_3 & s_6 \\ bt\ s_1\ and\ s_3 & s_4 & s_2 & bt\ s_4\ and\ s_5 \\ bt\ s_5\ and\ s_6 & s_6 & s_4 & bt\ s_6\ and\ s_7 \\ s_2 & bt\ s_3\ and\ s_4 & bt\ s_1\ and\ s_2 & s_4 \end{pmatrix}$$

$$P_3 = \begin{pmatrix} s_4 & s_5 & bt\ s_6\ and\ s_7 & bt\ s_6\ and\ s_7 \\ s_3 & s_4 & bt\ s_4\ and\ s_5 & bt\ s_5\ and\ s_6 \\ bt\ s_1\ and\ s_2 & bt\ s_3\ and\ s_4 & s_4 & s_5 \\ bt\ s_1\ and\ s_2 & bt\ s_2\ and\ s_3 & s_3 & s_4 \end{pmatrix}$$

$$P_4 = \begin{pmatrix} s_4 & bt\ s_4\ and\ s_5 & bt\ s_3\ and\ s_4 & bt\ s_1\ and\ s_2 \\ bt\ s_3\ and\ s_4 & s_4 & bt\ s_1\ and\ s_2 & bt\ s_0\ and\ s_1 \\ bt\ s_4\ and\ s_5 & bt\ s_6\ and\ s_7 & s_4 & bt\ s_3\ and\ s_4 \\ bt\ s_6\ and\ s_7 & bt\ s_7\ and\ s_8 & bt\ s_4\ and\ s_5 & s_4 \end{pmatrix}$$

Additionally, it is necessary to establish the following parameters for our consensus model:

- Consensus threshold, $\alpha = 0.8$.
- Acceptability threshold, $\epsilon = 0.05$.
- Change degree, $\theta = 1$.
- Maximal number of rounds, $Maxrounds = 15$.

Finally, another relevant aspect to take into account in a CRP, is the experts' behaviour. AFRYCA 3.0 allows to configure and simulate the behaviour of the experts by means of the modification of several parameters, being able to define if experts will be receptive to accept recommendations provided for the CRP models or not. For this case study, and to subsequently carry out a proper comparison between our proposal and the above mentioned CRP model, we consider that experts are always receptive to the suggestions.

4.2. Resolution of the case study

In order to solve the problem and reach the consensus, the novel CRP is applied and included in the intelligent CRP support system AFRYCA 3.0 [13, 27], which is used to carry out the computations and visualize the CRP. Fig. 6 shows the evolution of the experts' preferences along to the CRP, by representing the collective opinion in the center of the plot and the experts preferences around it. In the figure, it can be appreciated that experts come closer as the CRP progresses. The consensus is reached just in 2 round, which means that the feedback process works properly, being able to identify the experts whose opinions are further away from the collective one

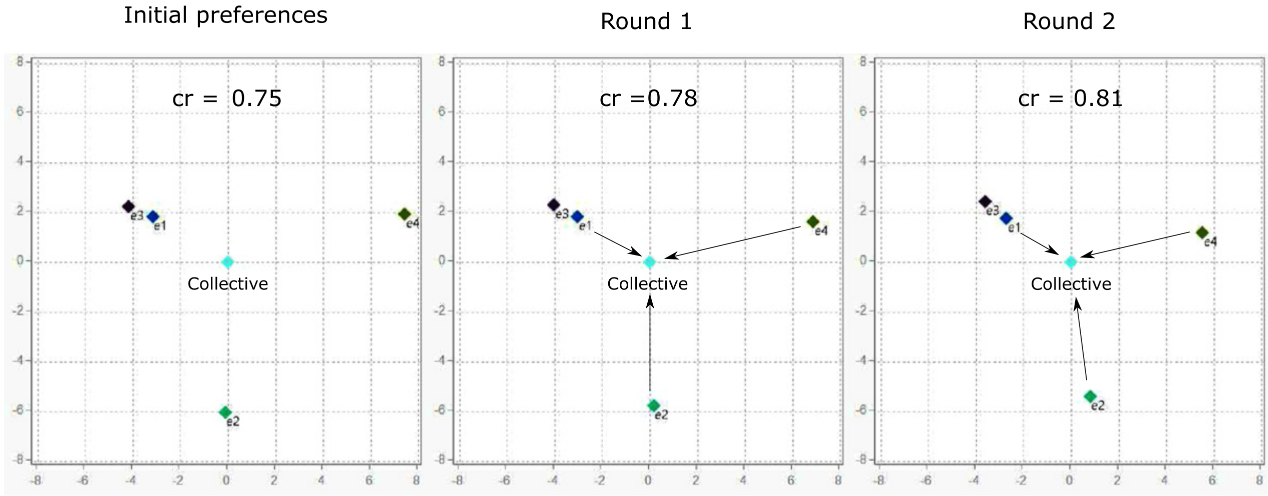

Fig. 6. Visualization of the CRP.

Table 1
Simulation results

cr	Rounds required	Ranking	Solution
0.81	2	$x_1 \succ x_3 \succ x_2 \succ x_4$	x_1

and change their assessments according to the rules and the change process which were aforementioned in Section 3.5. Apart from the visualization, Table 1 shows the results obtained when the CRP finishes. According to the results obtained, the best sector to invest money is the car industry, x_1.

4.3. Comparison analysis

Even though, previous results provide a good performance according to our goals, it should seem convenient to compare such results with other proposals for CRP. In our case we will compare with the CRP proposed by Wu et al. [45] for sake of clarity because both share phases and tasks. Hence, it is interesting to compare both models in order to analyse differences and similarities among both.

The first significant difference is the way in which experts provide their preferences. In spite of Wu et al. refer to the CLEs to provide the experts' preferences, they do not provide such preferences by using these expressions.

Regarding to the CRPs results, both models need the same number of rounds to reach the predefined consensus threshold, 2, and reach a similar consensus degree, 0.8021 for Wu et al. proposal and 0.81 for ours. Nevertheless, Wu et al. proposal does not present a systematic and formalized mechanism to change the experts preferences and suppose the changes applied to such preferences. However, our proposal provides a formalized method to change the assessments of the experts and none of applied changes is assumed but computed in a formal way. Furthermore, the changes provided by the Wu et al. CRP model are directly applied to the HFLTS, in our proposal, the changes are applied to the CLEs, an important advantage, since the experts preserve the type of expressions that they have used at the beginning of the CRP and, being these expressions much more understandable than HFLTS and thus, easier to change for them.

According to the results for the ranking of the alternatives, Wu et al. proposal provide the same global ranking $x_1 \succ x_3 \succ x_2 \succ x_4$, demonstrating the robustness of the performance of both models.

The consistency of the experts' preferences of both models is also analyzed. In this proposal is applied the consistency index, $(CI) \in [0, 1]$, proposed by Zhu et al. in [64]. This CI evaluates the consistency of HFLPRs, so that the lower CI the better the consistency. Table 2 shows the experts' preferences consistency along to the CRP. Both models provide appropriate values of consistency with slightly variations. However, the lack of a systematic process of change in the Wu et al. proposal does not guarantee that the consistency keeps proper values as could be inferred from Table 2, since the changes applied to the assessments are supposed not computed. Our proposal, on the contrary, does propose a formalized method for changing the experts' assessments and thus, it can guarantee that such assessments keep a acceptable level of consistency along to the not only the analysed CRP but any CRP, as shown in Table 2.

Table 2
Consistency

	Preferences	Initial	Round 1	Round 2
Wu et al. [45]	P_1	0.03	0.03	0.03
	P_2	0.04	0.04	0.02
	P_3	0.03	0.03	0.02
	P_4	0.05	0.09	0.09
Proposal	P_1	0.03	0.03	0.03
	P_2	0.04	0.07	0.07
	P_3	0.03	0.03	0.04
	P_4	0.05	0.05	0.04

5. Conclusions

Consensual decisions for GDM problems is a growing societal demand that becomes harder and more challenging by provoking the apparition of experts' hesitancy, which cannot be modeled by single linguistic terms. However, there are not many linguistic-based CRP that consider the latter issue.

A novel CRP for GDM in which experts' preferences are expressed by means of CLEs for representing experts' hesitancy has been introduced. The model takes advantage of the HFLTS representation of the CLEs for keeping a fuzzy representation of the information in the whole CRP and, in this way, avoiding the possible loss of relevant information. Furthermore, a comparative case study has been presented, showing the capacity of the model to achieve an agreement in a quick, formalized and consistent way. This model has been implemented and integrated in an intelligent CRP support system.

As future research, we will study how to apply the proposed consensus model for large-scale group decision making problems and face challenges related to this kind of problems such as scalability, non-cooperative behavior or time cost.

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4.7. Modelo de Consenso para Expresiones Lingüísticas Comparativas con Translación Simbólica

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A Consensus Model for Extended Comparative Linguistic Expressions with Symbolic Translation

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Abstract: Consensus Reaching Process (CRP) is a necessary process to achieve agreed solutions in group decision making (GDM) problems. Usually, these problems are defined in uncertain contexts, in which experts do not have a full and precise knowledge about all aspects of the problem. In real-world GDM problems under uncertainty, it is usual that experts express their preferences by using linguistic expressions. Consequently, different methodologies have modelled linguistic information, in which computing with words stands out and whose basis is the fuzzy linguistic approach and their extensions. Even though, multiple consensus approaches under fuzzy linguistic environments have been proposed in the specialized literature, there are still some areas where their performance must be improved because of several persistent drawbacks. The drawbacks include the use of single linguistic terms that are not always enough to model the uncertainty in experts' knowledge or the oversimplification of fuzzy information during the computational processes by defuzzification processes into crisp values, which usually implies a loss of information and precision in the results and also a lack of interpretability. Therefore, to improving the effects of previous drawbacks, this paper aims at presenting a novel CRP for GDM problems dealing with Extended Comparative Linguistic Expressions with Symbolic Translation (ELICIT) for modelling experts' linguistic preferences. Such a CRP will overcome previous limitations because ELICIT information allows both fuzzy modelling of the experts' uncertainty including hesitancy and performs comprehensive fuzzy computations to, ultimately, obtain precise and understandable linguistic results. Additionally, the proposed CRP model is implemented and integrated into the CRP support system so-called A FRamework for the analYsis of Consensus Approaches (AFRYCA) 3.0 that facilitates the application of the proposed CRP and its comparison with previous models.

Keywords: fuzzy linguistic approach; computing with words; extended comparative linguistic expression with symbolic translation; group decision making; consensus reaching process

1. Introduction

Human beings are continuously facing decision making problems in their daily life, some of them so simple that we do not even notice their presence. However, not all decision problems are so easy to solve and the engagement of several people or experts with different knowledge may be necessary to reach the solution, giving rise to Group Decision Making (GDM) [1–3]. Obviously, the participation of several experts implies different points of view and consequently, conflicting opinions on the solution to the problem. A GDM classical resolution scheme ignores the latter aspect and usually computes the solution based on a simple aggregation of the initial experts' preferences, disregarding the conflicts on the solution. It can result in several experts feeling that their opinions have been completely omitted [4], decreasing the support for the solution and the resolution scheme in this or future decisions.

To overcome the previous drawback of the GDM process, a Consensus Reaching Process (CRP) has been added to the GDM resolution scheme [5]. In brief, a CRP is a cyclical process in which the experts discuss with each other and modify their initial preferences in order to achieve a satisfactory and agreed solution. This process is usually guided by a moderator who identifies the experts whose opinions are furthest from the rest of the group and advises them with the aim of bringing their positions closer to the rest of the group. CRP has attracted the attention of many researchers and many consensus models that support CRPs have been developed [4,6,7].

Most real-world GDM problems and their correspondent CRPs deal with uncertain and vague information that should be properly modelled and managed to obtain reliable solutions. In such cases, experts usually elicit their information by means of linguistic values or expressions that make them more comfortable to represent their vague assessments. The inherent uncertainty of such linguistic values has been successfully modelled by the fuzzy linguistic approach [8–11] resulting in Linguistic Decision Making (LDM) [12,13]. Such a type of modelling implies processes of Computing with Words (CW) [14,15], which is one of the most used methodologies for operating with linguistic assessments (words in a natural or artificial language) and not numbers, thus emulating human cognitive processes [16]. The input in CW processes are represented by linguistic values that are manipulated to, finally, obtain results represented by linguistic information that is easy to understand [17]. Most classical LDM proposals in the literature model linguistic information by means of single linguistic terms [12], in which the linguistic 2-tuple model has a prominent position [18,19]. However, it provokes some limitations to experts during the elicitation of their knowledge [20]; thus, several proposals to model multiple linguistic terms for experts' assessments have been proposed, with Hesitant Fuzzy Linguistic Term Sets standing out (HFLTSs) [21].

Accordingly, many consensus models that deal with LDM problems have been proposed in the specialized literature [6,22–25], but each presents significant, different drawbacks as follows:

1. *Limitation to model expert's uncertain knowledge*: some models use single linguistic terms to represent the experts' preferences [22]. However, it is common that experts often have doubt among several linguistic terms when providing their opinions due to the complexity of the problem and such hesitancy cannot be modelled by using just a single linguistic term.
2. *Closeness to human reasoning*: other models represent more complex linguistic assessments [6,26] but their preference modelling does not provide expressions close to humans' way of thinking.
3. *CW integrity and Interpretability*: in many linguistic CRPs, the fuzzy linguistic inputs are oversimplified, transforming fuzzy representation into interval or crisp values [27,28], which disrupts the CW process [16,17] suffering loss of information and lack of interpretability.

Even though, some recent improvements modeled linguistic expressions closer to human cognitive process, for instance, by means of the use of context-free grammars to generate richer and flexible comparative linguistic expressions (CLEs) based on HFLTSs [21,29]. This improves the interpretability and allows for the two previous drawbacks to be overcome but still, the consensus models for such a representation [26,30] cannot maintain an appropriate fuzzy representation during CW processes and they use linguistic discrete representation domains which produce bias during the CRP. Therefore, this paper takes advantage of a novel fuzzy linguistic modelling that hybridizes the main ideas of the linguistic 2-tuple model [18] and the HFLTSs [21] resulting in the Extended Comparative Linguistic Expressions with Symbolic Translation (ELICIT) information [31]. It provides the following advantages regarding the previous mentioned drawbacks: (i) their representation is based on the CLEs [21,29]; thus, they can model the experts' hesitancy, (ii) the ELICIT information is transformed into fuzzy numbers with the premise of keeping as much information as possible by accomplishing fuzzy computations without loss of information and then, the fuzzy numbers are retranslated into linguistic expressions, which means the results are represented linguistically. Consequently, ELICIT information facilitates the representation of a continuous linguistic domain even in complex contexts with multiple linguistic-term-based expressions and provides a fuzzy operational computational approach to accomplish CW processes in a precise way, obtaining comprehensible results in decision making problems.

Therefore, the aim of this research is to introduce a new consensus model dealing with ELICIT that overcomes the previous limitations of an existing consensus model in LDM. This new consensus model presents a key novelty, the use of ELICIT information. As far as we know, there is no other proposal that uses this type of information in a CRP. Furthermore, the use of ELICIT provides relevant advantages related to CW processes, expressiveness, loss of information and interpretability. Then, contrary to other proposals, our consensus model performs precise fuzzy computations in a continuous domain thanks to the symbolic translation of the ELICIT information, avoiding the loss of information and, in turn, obtaining more accurate results that are easy to understand. In addition, the proposed consensus model is implemented and integrated in the consensus support software AFRYCA 3.0 (A FRamework for the analYsis of Consensus Approaches) [4,32,33] in order to simulate the performance of the CRP and solve real world LDM problems dealing with ELICIT information.

To sum up, this proposal aims to achieve the following goals:

1. Define a new consensus model to deal with fuzzy linguistic information modelled by means of ELICIT information to overcome the limitations of the existing consensus model.
2. Such a model will apply CW processes to ELICIT information that will obtain precise linguistic results that are easy to understand.
3. Application of the ELICIT-based consensus model to a real-world GDM problem to show its performance validity and advantages in comparison with other approaches by its integration in the software AFRYCA 3.0 [32].

The remainder of this contribution is structured as follows: Section 2 reviews some basic concepts related to the proposal. Section 3 presents a novel consensus model based on ELICIT information. Section 4 introduces an LDM problem to show the performance of the proposal and includes a comparative analysis with another approach with similar characteristics. To conclude this work, Section 5 draws several conclusion and proposes future research directions.

2. Preliminaries

This section briefly revises the main concepts related to GDM, CRP and ELICIT information that are necessary to understand our proposal.

2.1. Group Decision Making

GDM consists of the participation of several experts in the resolution of a decision problem. By definition, a GDM problem is characterized by a finite set of experts, $E = \{e_1, e_2, \ldots, e_m\}$, who provide their opinions over a finite set of possible alternatives/solutions, $A = \{a_1, a_2, \ldots, a_n\}$ [1,5,34]. In GDM, each expert e_i expresses her/his opinion by using a preference structure P^k, a $A \times A$ matrix so that

$$P^k = \begin{pmatrix} - & \cdots & p^k_{1n} \\ \vdots & \ddots & \vdots \\ p^k_{n1} & \cdots & - \end{pmatrix}$$

where p^k_{ij} represents the preference of the expert e_k over the alternative a_i regarding the alternative a_j.

The classical resolution scheme for this kind of problem is formed by two phases [35]: (i) *aggregation*: the experts' preferences obtained are aggregated by using an aggregation operator and (ii) *exploitation*: one or several alternatives are selected as solutions to the problem (see Figure 1).

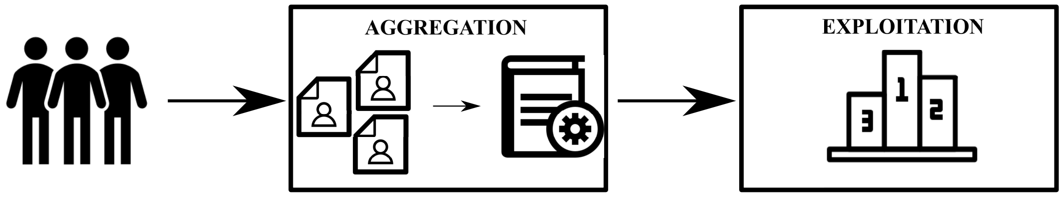

Figure 1. GDM resolution scheme.

The definition of GDM problems under uncertainty is fairly common in real-world scenarios because of pressure to make quick decisions and the lack of information and knowledge about the problem. Therefore, the experts have to deal with incomplete and vague information and, as a result, expressing their knowledge may become an extremely complex task. Under these conditions, linguistic information and its modelling by linguistic variables [8–10] has obtained successful results [15] with the use of CW processes [14]. The resolution scheme for LDM problems varies slightly regarding the classical one shown in Figure 1—it includes, as the first step, the definition of the expression domain that experts use to provide their linguistic preferences [36] (see Figure 2).

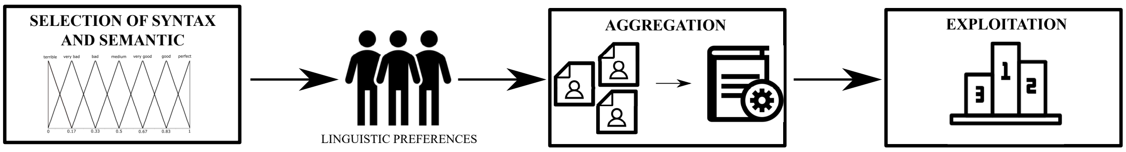

Figure 2. LDM resolution scheme.

The Figure 2 shows the need to accomplish computations with linguistic information to solve LDM problems. The CW methodology has been successfully applied to compute and reason by means of words, obtaining linguistic outputs from linguistic inputs [17,37]. Recently, CW has been intensively and comprehensively applied in decision making [15,38] and thus, multiple CW schemes have been proposed in the literature [39,40] to reinforce the need of easy computations to obtain accurate and understandable linguistic results. The CW scheme introduced by Yager in [17,40] includes two main processes in CW, *translation* and *retranslation*. The translation process transforms the linguistic assessments into a format based on fuzzy tools to accomplish the computations. Then, the retranslation process transforms the manipulated information into linguistic values that are easily to understand.

Multiple fuzzy-based linguistic modelling approaches together with their computational models have been developed for CW [41,42]. One of the most remarkable is the 2-tuple linguistic model proposed by Herrera and Martínez [18] due to its advantages in terms of interpretability and accuracy [43].

2.2. 2-Tuple Linguistic Model

The 2-tuple linguistic model [44] is one of the most widely used linguistic models thanks to its great qualities both in terms of interpretability and precision related to symbolic translation. This model represents the information by a 2-tuple (s_p, α) in which s_p is a linguistic term belonging to a predefined linguistic term set $S = \{s_0, s_1, \ldots, s_g\}$ and $\alpha \in [-0.5, 0.5)$ is so-called symbolic translation, a numerical value that represents the translation of the fuzzy membership function of s_p in a continuous domain (see Figure 3).

$$\alpha = \begin{cases} [-0.5, 0.5) & if\ s_p \in \{s_1, s_2, \ldots, s_{g-1}\} \\ [0, 0.5) & if\ s_p = s_0 \\ [-0.5, 0] & if\ s_p = s_g \end{cases}$$

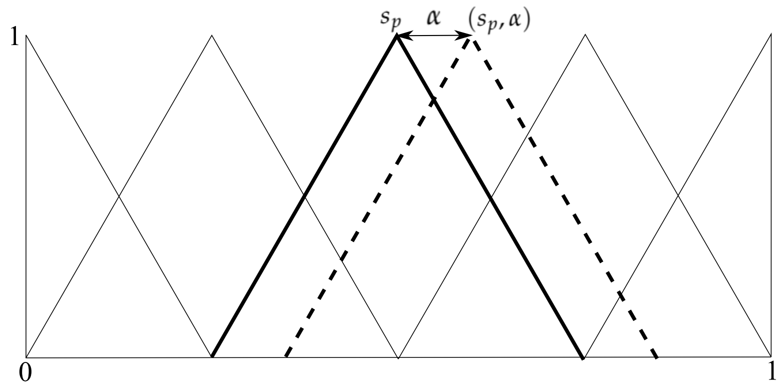

Figure 3. Symbolic translation.

Note that, the symbolic translation computation in linguistic terms in S provides a value $\beta \in [0, g]$. This value can be translated into its corresponding 2-tuple linguistic value, (s_p, α) using the function Δ_S:

Definition 1. *[44] Let $S = \{s_0, \ldots, s_g\}$ be a set of linguistic terms and $\overline{S}$ the 2-tuple set associated with S defined as $\overline{S} = S \times [-0.5, 0.5)$. The function $\Delta_S : [0, g] \rightarrow \overline{S}$ is given by:*

$$\Delta_S(\beta) = (s_p, \alpha), \; with \begin{cases} p = round(\beta) \\ \alpha = \beta - p \end{cases}$$

with $round(\cdot)$ being the function that assigns the closest integer number $i \in \{0, \ldots, g\}$ to β.

Therefore, a 2-tuple linguistic value (s_p, α) can be represented by its equivalent numerical value β in the interval of granularity of S, $[0, g]$.

Proposition 1. *Let $S = \{s_0, \ldots s_g\}$ be a linguistic term set and $(s_p, \alpha) \in \overline{S}$ be a 2-tuple linguistic value. There is a function, Δ^{-1}:*

$$\Delta^{-1} : \overline{S} \rightarrow [0, g]$$
$$\Delta_S^{-1}(s_p, \alpha) = \alpha + p = \beta$$

Remark 1. *Note that according to Definition 1 and Preposition 1, the transformation of a linguistic term $s_p \in S$ into a 2-tuple linguistic value in $\overline{S}$ is obtained by adding a zero as a symbolic translation to the linguistic term:*

$$s_p \in S \rightarrow (s_p, 0) \in \overline{S}$$

2.3. Consensus Reaching Process

The classical GDM resolution schemes shown in Figures 1 and 2 directly aggregate the experts' preferences and do not guarantee a solution that is accepted by all the experts because agreement on it is not considered. Therefore, some experts may disagree with the solution and feel that their opinions have not been sufficiently considered during the decision process, which can result in either a lack of support for the solution or lack of confidence in the GDM process. In such cases, to avoid such drawbacks, an additional CRP has been added to the GDM process [7].

A CRP is an iterative discussion process among experts involved in the GDM problem in which they discuss with each other, provide their different opinions and points of view and try to achieve a higher collective level of agreement by adjusting their initial preferences and seeking a common point of agreement [45]. A CRP is classically formed by four steps:

1. *Gathering preferences*: the experts analyze the GDM problem and provide their opinions over the different alternatives by using preference relations.

2. *Consensus level*: the level of agreement (cl) within the group is computed.
3. *Consensus control*: cl is compared with a predefined consensus threshold (μ), which represents the desired level of agreement to be achieved by the group. If the consensus threshold is reached, the CRP finishes and a selection process of the best alternative starts, otherwise a new consensus round begins. In order to avoid an endless CRP, the number of consensus rounds is limited with another threshold (r_{max}).
4. *Feedback generation*: the moderator identifies the experts whose opinions are furthest from the rest of the group and advises them to change their preferences in order to reach a higher level of agreement.

CRPs have attracted great attention from many researchers in recent years and a large number of consensus models to support groups in CRPs have been presented in the specialized literature [7,23,46] and several metrics have been proposed to study their performance [31].

2.4. ELICIT Information

Labella et al. proposed in [31] a new fuzzy linguistic representation model so-called ELICIT with the aim of overcoming the drawbacks of existing linguistic representation models in terms of interpretability and precision. The ELICIT information has two main advantages:

- *Interpretability*: ELICIT information is generated by a context-free grammar [43]; thus, flexible and rich linguistic expressions are built that are able to model the experts' hesitancy with expressions such as *between*, *at least* or *at most*. Furthermore, in spite of the ELICIT information being manipulated using fuzzy operations, the ELICIT computational model allows for the fuzzy numbers to be translated again into ELICIT information by obtaining interpretable linguistic results and following a CW approach [14].
- *Accuracy*: a key aspect in the ELICIT information is the representation in a continuous domain of the linguistic terms that compose the expressions, thanks to the symbolic translation value introduced in the 2-tuple linguistic model (see Section 2.2).

The different complex linguistic expressions that compose the ELICIT information are generated by means of the following context-free grammar:

Definition 2. *[31] Let G_H be a context-free grammar and $S = \{s_0, \ldots, s_g\}$ a linguistic terms set. The elements of $G_H = (V_N, V_T, I, P)$ are defined as follows.*

$$\begin{aligned}
&V_N = \{\langle continuos\ primary\ term\rangle,\ \langle composite\ term\rangle,\\
&\langle unary\ relation\rangle, \langle binary\ relation\rangle, \langle conjunction\rangle\}\\
&V_T = \{at\ least,\ at\ most,\ between,\ and,\ (s_0,\alpha)^{\gamma}, (s_1,\alpha)^{\gamma}, \ldots, (s_g,\alpha)^{\gamma}\}\\
&I \in V_N
\end{aligned}$$

The production rules defined in an extended Backus-Naur Form are:

$$\begin{aligned}
&P = \{I\ ::=\ \langle continuos\ primary\ term\rangle | \langle composite\ term\rangle\\
&\langle composite\ term\rangle\ ::=\ \langle unary\ relation\rangle\langle continuous\ primary\ term\rangle|\\
&\langle binary\ relation\rangle\langle continuous\ primary\ term\rangle\langle conjunction\rangle\langle continuous\ primary\ term\rangle\\
&\langle continuous\ primary\ term\rangle\ ::=\ (s_0,\alpha)^{\gamma}|(s_1,\alpha)^{\gamma}|\ldots|(s_g,\alpha)^{\gamma}\\
&\langle unary\ relation\rangle\ ::=\ at\ least|at\ most\\
&\langle binary\ relation\rangle\ ::=\ between\\
&\langle conjunction\rangle\ ::=\ and\}
\end{aligned}$$

Some examples of ELICIT information may be: "at least $(s_p, \alpha)^{\gamma}$", "at most $(s_p, \alpha)^{\gamma}$" and "between $(s_p, \alpha_1)^{\gamma_1}$ and $(s_q, \alpha_2)^{\gamma_2}$".

Remark 2. *Note that the parameter γ, so-called adjustment, preserves relevant information about the parametric form of the corresponding fuzzy number of a ELICIT and it is key to obtain results without loss of information [31,47].*

The ELICIT representation model was proposed together with a CW approach based on the fuzzy linguistic approach. This approach allows for fuzzy information to be computed in a precise way and return linguistic and understandable results represented by ELICIT information. To carry out these fuzzy operations, first the initial linguistic assessments modelled by complex linguistic expressions are translated into trapezoidal fuzzy numbers (TrFNs) that represent their corresponding fuzzy envelopes [31]. Then, fuzzy arithmetic operations are applied to the fuzzy envelopes in order to preserve the fuzzy representation and guarantee that the new fuzzy numbers can be translated into ELICIT information.

The fuzzy arithmetic operations are based on the work introduced by Rezvani and Molani [48]. They proved that, by means of the fuzzy numbers shape function and $\alpha - cuts$, it is possible to accomplish arithmetic operations that preserve the fuzzy parametric representation. Here, we present the addition of the fuzzy operation because it will be used later in the contribution.

Definition 3. *Let $T_{\tilde{A}}(a_1, a_2, a_3, a_4)$ and $T_{\tilde{B}}(a'_1, a'_2, a'_3, a'_4)$ be two fuzzy envelopes modelled by two TrFNs. Suppose the normal shape functions of $\tilde{A}$, $\tilde{B}$ as follows:*

$$\mu_{\tilde{A}} = \begin{cases} (\frac{x-a_1}{a_2-a_1})^n & when \quad x \in [a_1, a_2), \\ 1 & when \quad x \in [a_2, a_3], \\ (\frac{a_4-x}{a_4-a_3})^n & when \quad x \in (a_3, a_4], \\ 0 & otherwise \end{cases} \qquad \mu_{\tilde{B}} = \begin{cases} (\frac{x-a'_1}{a'_2-a'_1})^n & when \quad x \in [a'_1, a'_2), \\ 1 & when \quad x \in [a'_2, a'_3], \\ (\frac{a'_4-x}{a'_4-a'_3})^n & when \quad x \in (a'_3, a'_4], \\ 0 & otherwise \end{cases}$$

Supposing $\tilde{A}_{\overline{\alpha}}$, $\tilde{B}_{\overline{\alpha}}$ are the $\alpha - cuts$ of $\tilde{A}$ and $\tilde{B}$ [49], respectively:

$$\begin{aligned} \tilde{A}_{\overline{\alpha}} &= [a_1 + \overline{\alpha}^{1/n}(a_2 - a_1), a_4 - \overline{\alpha}^{1/n}(a_4 - a_3)] \\ \tilde{B}_{\overline{\alpha}} &= [a'_1 + \overline{\alpha}^{1/n}(a'_2 - a'_1), a'_4 - \overline{\alpha}^{1/n}(a'_4 - a'_3)] \end{aligned}$$

Definition 4. *[48] The addition of two fuzzy envelopes modelled by two TrFNs $\tilde{A}$, $\tilde{B}$ can be defined with a shape function $\mu_{\tilde{A}+\tilde{B}}$ as*

$$\mu_{\tilde{A}+\tilde{B}} = \begin{cases} \dfrac{(x-(a_1+a'_1))^n}{(a_2+a'_2)-(a_1+a'_1)} & a_1+a'_1 \le x \le a_2+a'_2, \\ 1 & a_2+a'_2 \le x \le a_3+a'_3, \\ \dfrac{((a_4+a'_4)-x)^n}{(a_4+a'_4)-(a_3+a'_3)} & a_3+a'_3 \le x \le a_4+a'_4, \\ 0 & otherwise \end{cases}$$

The fuzzy arithmetic operations play a key role in the ELICIT computational model, since they allow for the retention of the fuzzy parametric form when the fuzzy envelopes are manipulated and make it possible to transform the fuzzy numbers back into linguistic information, fulfilling the basic premise of the CW approach. This retranslation process into linguistic information is accomplished by the function ζ.

Definition 5. *[31] Let $S = \{s_0, \ldots, s_g\}$ be a set of linguistic terms and $\tilde{A}$ a fuzzy number. The function ζ is given by*

$$\zeta(\tilde{A}) = x, \text{ where} \begin{cases} x = \text{at least } (s_p, \alpha)^{\gamma} \text{ if } \tilde{A} = T(a_1, a_2, 1, 1) \\ x = \text{at most } (s_p, \alpha)^{\gamma} \text{ if } \tilde{A} = T(0, 0, a_3, a_4) \\ x = \text{between } (s_p, \alpha_1)^{\gamma_1} \text{and } (s_q, \alpha_2)^{\gamma_2} \text{ if } \tilde{A} = T(a_1, a_2, a_3, a_4) \end{cases} \tag{1}$$

Another key function in the ELICIT CW approach is ζ^{-1}, which transforms the ELICIT information into TrFNs based on the fuzzy envelope computation:

Definition 6. *[31] Let x an ELICIT expression and $T(a_1, a_2, a_3, a_4)$ a TrFN. The function ζ^{-1} is defined as follows:*

$$\zeta^{-1} : x \rightarrow T(a_1, a_2, a_3, a_4) \tag{2}$$

such that, from an ELICIT expression, it returns its equivalent TrFN.

For the sake of clarity, the previous functions have not been fully described, see [31] for further details.

3. Consensus Model with ELICIT Information

The need for dealing with complex GDM problems defined under uncertainty in real-world scenarios demands new preference modelling that facilitates the flexible and correct elicitation of experts' knowledge. We have pointed out that CLEs based on HFLTSs [43] provide such a flexibility and are similar to human cognitive processes. Different CRPs have been developed based on CLEs and HFLTSs [26,50,51]; however, as has been pointed out previously, the use of a discrete representation of the linguistic domain produces biases and problems in the evolution of the experts agreements across the CRP. Therefore, this section introduces a new CRP that is able to deal with ELICIT information that facilitates linguistic assessment elicitation, maintains the fuzzy representation across the CW processes, uses a continuous representation of the linguistic domain that results in a proper evolution of the agreement across the CRP and, finally, obtains precise and understandable results.

The ELICIT consensus model proposed follows the general scheme shown in Figure 4 but with additional tasks, which are highlighted in Figure 5.

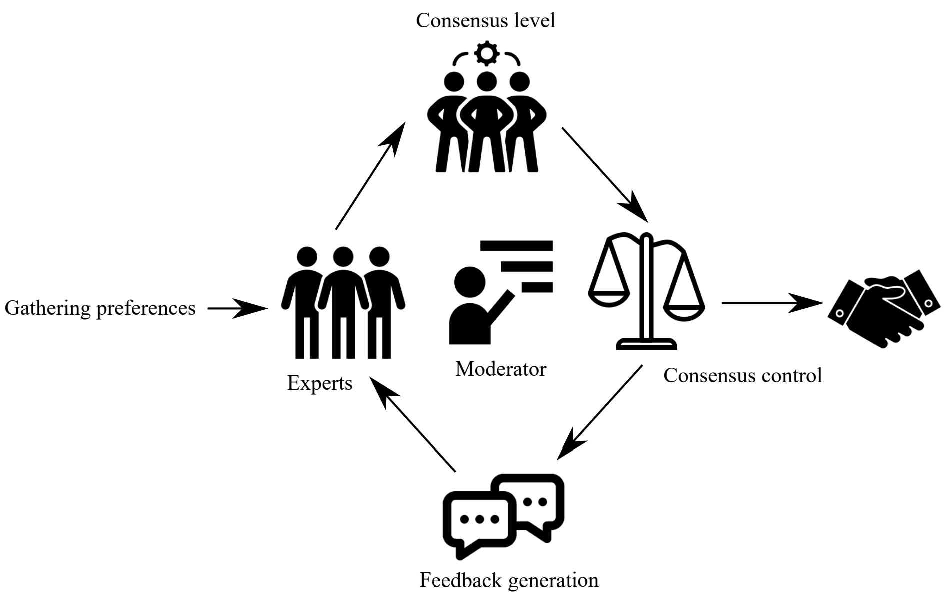

Figure 4. CRP resolution scheme.

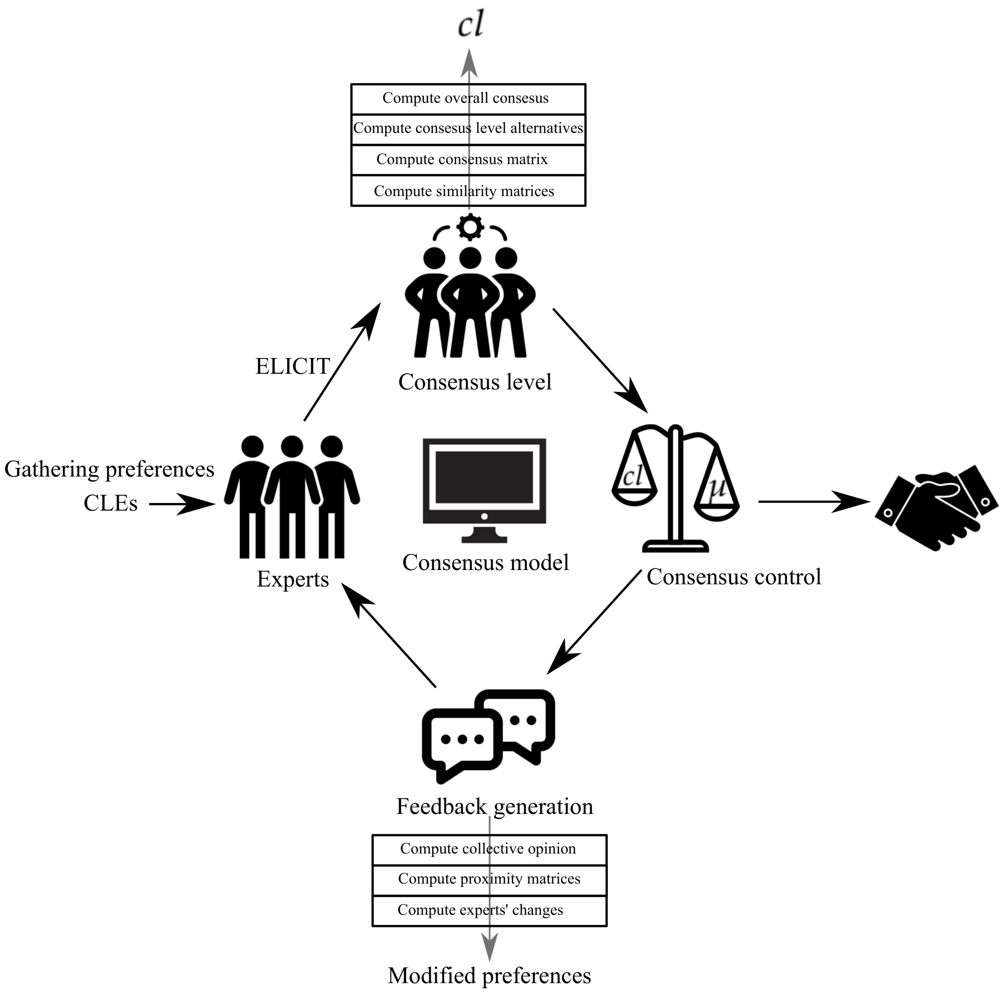

Figure 5. Consensus model resolution scheme.

The resolution scheme shown in Figure 5 includes additional steps regarding the classical CRP resolution scheme shown in Figure 4. First, the experts' preferences are modelled by CLEs that are lately transformed into ELICIT information. Then, the consensus level is computed by the following four consecutive steps—(i) *Compute similarity matrices*; (ii) *Compute consensus matrix*; (iii) *Compute consensus level of alternatives*; (iv) *Compute overall consensus*. The overall consensus, cl, is compared with the predefined consensus threshold, μ, in the consensus control step. Finally, if needed, a feedback process composed by three processes—(a) *Compute collective opinion*; (b) *Compute proximity matrices*; (c) *Compute experts' changes*; will provide the modified preferences according to the suggestions provided by the consensus model. The previous steps and processes are described in further detail in the following subsections and summarized in Algorithm 1.

Algorithm 1: Proposal steps

Data: The experts preferences, $P^k, X \times X \rightarrow S_{ll}$, the predefined consensus threshold μ, the maximum number of consensus rounds r_{max}, the acceptability threshold ε, the change direction parameter θ and the change degree parameters θ_1 and θ_2.

Result: The adjusted experts' preferences $\overline{P}^k, X \times X \rightarrow S_{ll}$ and the consensus degree cl.

The preferences p_{ij}^k for each expert $e_k, k \in \{1, 2, \ldots, m\}$ modelled by CLEs are transformed into ELICIT information. Afterwards, the fuzzy envelopes of the latter are computed using Definition 6

cl is derived by using the computation of the similarity matrices, SM^{kt} for each pair of experts $(e_k, e_t), k < t$ by Equation (3). Then, a consensus matrix, CM is obtained by Equation (6). CM is used to obtain the consensus degree ca_i for each alternative a_i using Equation (7). Finally, the overall consensus degree cl is calculated using Equation (8).

while $cl < \mu$ *and* $round < r_{max}$ **do**
    The collective opinion, C, of the experts' group is obtained with Equation (9). Then, a set of proximity matrices PM^k regarding the collective opinion are derived using Equation (12).
    if $ca_i < \mu$ *and* $cm_{ij} < \mu$ *and* $pm_{ij}^k < \overline{pm}_{ij}$ **then**
        if $\phi(\tilde{p}_{ij}^k) - \phi(c_{ij}) < -\varepsilon$ **then**
            Increase p_{ij}^k on (a_i, a_j).
            if $cen(\tilde{p}_{ij}^k) - cen(c_{ij}) > \theta$ **then**
                Significant change.
            else
                Slight change.
            end
        else if $\phi(\tilde{p}_{ij}^k) - \phi(c_{ij}) > \varepsilon$ **then**
            Decrease p_{ij}^k on (a_i, a_j). **if** $cen(\tilde{p}_{ij}^k) - cen(c_{ij}) > \theta$ **then**
                Significant change.
            else
                Slight change.
            end
        else
            No change.
        end
    else
        No change.
    end
end

3.1. Input Information

In this initial step of the proposed CRP, each expert e_k can elicit their preferences into a linguistic preference relation whose values could be any of the generated ones for the context-free grammar introduced in Definition 2. Initially, it is reasonable that the elicited assessments by experts would be CLEs or linguistic terms to model her/his opinions in a matrix $P^k = (p_{ij}^k)_{n \times n}$, where p_{ij}^k is either a CLE or a linguistic term. Assuming the linguistic terms set, $S = \{very\ weak, weak, fair, strong, very\ strong\}$, an example of such an input matrix may be the following:

$$P^k = \begin{pmatrix} - & at\ most\ weak & strong \\ at\ least\ strong & - & between\ weak\ and\ fair \\ weak & between\ fair\ and\ strong & - \end{pmatrix}$$

3.2. Transformation into ELICIT Information and Fuzzy Numbers

The initial CLEs expressions p_{ij}^k provided by the experts are transformed into ELICIT information. Depending on the type of CLE, the corresponding ELICIT information is obtained according to Remark 1 as follows:

- single linguistic term: the CLE s_p is transformed into $(s_p, 0)^0$.
- *at least* expression: the CLE *at least* s_p is transformed into *at least* $(s_p, 0)^0$.
- *at most* expression: the CLE *at most* s_p is transformed into *at most* $(s_p, 0)^0$.
- *between* expression: the CLE *between* s_p *and* s_q is transformed into *between* $(s_p, 0)^0$ *and* $(s_q, 0)^0$.

Once the initial CLEs are transformed into ELICIT information, the fuzzy representation of the latter is obtained by the function ζ^{-1} (see Definition 6). The experts' preferences transformed into TrFNs are noted as $\tilde{p}_{ij}^k$.

3.3. Compute Consensus Level

In this step, the current consensus level within the group is computed. This process is divided into several sub-steps:

3.3.1. Compute Similarity Matrices

A similarity matrix $SM^{kt} = (sm_{ij}^{kt})_{n \times n}$ for each pair of experts (e_k, e_t) is computed:

$$sm_{ij}^{kt} = \sum_{k=1}^{m-1} \sum_{t=k+1}^{m} \sum_{i=1}^{n-1} \sum_{j=n+1}^{n} sim(\tilde{p}_{ij}^k, \tilde{p}_{ij}^t) \tag{3}$$

where $\tilde{p}_{ij}^k$ and $\tilde{p}_{ij}^t$ represents the TrFNs of the preferences of the expert e_k and e_t over the pair of alternatives (a_i, a_j) and $sim(\cdot)$ computes the similarity between two TrFNs.

Definition 7. *Let $\tilde{A} = T(a_1, a_2, a_3, a_4)$ and $\tilde{B} = T(a_1', a_2', a_3', a_4')$ two fuzzy numbers, the similarity measure between them is computed as follows*

$$sim(\tilde{A}, \tilde{B}) = 1 - dist(\tilde{A}, \tilde{B}) \tag{4}$$

where $dist(\tilde{A}, \tilde{B})$ represents the distance between fuzzy numbers computed as follows

$$dist(\tilde{A}, \tilde{B}) = \frac{1}{4} \sum_{i=1}^{4} (|a_i - a_i'|) \tag{5}$$

3.3.2. Compute Consensus Matrix

From the aggregation of the similarity values, a consensus matrix $CM = (cm_{ij})_{n \times n}$ is computed:

$$cm_{ij} = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{m(m-1)}{2} \sum_{k=1}^{m-1} \sum_{t=k+1}^{m} sm_{ij}^{kt} \tag{6}$$

3.3.3. Compute Consensus Level for Alternatives

The degree of consensus ca_i for each alternative a_i is computed:

$$ca_i = \frac{\sum_{j=1, j \neq i}^{n} cm_{ij}}{n-1} \tag{7}$$

3.3.4. Compute Overall Consensus

The overall consensus cl is computed as:

$$cl = \frac{\sum_{i=1}^{n} ca_i}{n} \tag{8}$$

3.4. Consensus Control

The overall consensus degree cl is compared with the predefined consensus threshold μ. If the latter is achieved, a selection process of the best alternative starts, otherwise, the CRP requires another discussion round to increase the level of agreement.

3.5. Feedback Mechanism

The feedback mechanism requires the identification of the experts who are furthest from the rest of the group and the assessments over the alternatives to change.

3.5.1. Compute Collective Opinion

The collective opinion $C = (c_{ij})_{n\times n}$ of the experts' group is obtained from:

$$c_{ij} = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \overline{\psi}(\tilde{p}_{ij}^{1} \ldots, \tilde{p}_{ij}^{m}) \tag{9}$$

where $\overline{\psi}$ represents the fuzzy arithmetic mean aggregation operator defined as follows (see Definition 4):

Definition 8. *[31] Let $\{\tilde{A}_1, \ldots, \tilde{A}_m\}$ be a set of fuzzy numbers, the fuzzy arithmetic mean $\overline{\psi}$ is computed as follows:*

$$\overline{\psi}\{\tilde{A}_1, \ldots, \tilde{A}_m\} = \frac{1}{m}(\mu_{\tilde{A}_1+\tilde{A}_2+\ldots+\tilde{A}_m}) \tag{10}$$

where the division between a TrFN, $\tilde{A} = T(a_1, a_2, a_3, a_4)$, and a scalar o is computed as follows:

$$\frac{\tilde{A}}{o} = T(\frac{a_1}{o}, \frac{a_2}{o}, \frac{a_3}{o}, \frac{a_4}{o}) \tag{11}$$

3.5.2. Compute Proximity Matrices

For each expert e_k her/his proximity matrix $PM^k = (pm_{ij}^k)$ regarding the collective opinion is computed so:

$$pm_{ij}^k = \sum_{k=1}^{m} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} sim(\tilde{p}_{ij}^k, c_{ij}) \tag{12}$$

3.5.3. Compute Experts' Changes

To apply the changes to the experts' preferences, it is necessary to identify the assessments of the pair of alternatives to change, and which experts have to modify such assessments.

- *Identify alternatives*: the pair of alternatives (a_i, a_j) will be modified if $ca_i < \mu$ and $cm_{ij} < \mu$.
- *Identify experts*: the expert e_k is candidate to modify their preferences if $pm_{ij}^k < \overline{pm}_{ij}$, where $\overline{pm}_{ij}$ is the average proximity value of all the experts for each pair of alternatives (a_i, a_j) selected to be modified.

Once the alternatives and experts have been identified, the next step consists of defining the direction of change (increase/decrease). To determine the direction, an acceptability threshold of change, ε, is introduced. According to this value, several direction rules are applied:

- **RULE 1**: If $\phi(\tilde{p}_{ij}^k) - \phi(c_{ij}) < -\varepsilon$ then e_k should *increase* her/his assessments p_{ij}^k on (a_i, a_j).
- **RULE 2**: If $\phi(\tilde{p}_{ij}^k) - \phi(c_{ij}) > \varepsilon$ then e_k should *decrease* her/his assessments p_{ij}^k on (a_i, a_j).

where $\phi(\cdot)$ denotes the defuzzified value of a TrFN $T(a_1, a_2, a_3, a_4)$ such that:

$$\phi(T(a_1, a_2, a_3, a_4)) = \frac{(a_1 + 2a_2 + 2a_3 + a_4)}{6} \tag{13}$$

Finally, we define how the change in the preference will take place. The degree of change to apply is a very relevant aspect in a CRP, since an excessive/insignificant modification in the experts' preferences could lengthen the CRP more than necessary.

Our proposal includes an adaptive process to deal with the latter issue so a greater or slighter change is applied depending on the distance between the expert's preference to be modified and the collective opinion. This is a key aspect of our contribution since, contrary to other existing proposals, the ELICIT information allows for the modification of the experts' preferences in a continuous domain. Whereas other consensus models that use HFLTSs or CLEs apply the change in the experts' preferences by means of "jumps" between the linguistic terms belonging to a predefined linguistic term set and thus, in a discrete domain, our proposal can use the symbolic translation of the ELICIT information to apply the changes in intermediate continuous values between linguistic terms. This facilitates the reaching of a consensus since excessive modifications in the experts' preferences that may provoke a deadlock in the consensus process are avoided, as it will be shown in the comparative analysis introduced in Section 4.3.

To identify the degree of change needed in the expert's preferences, we use the concept of centroid of a fuzzy number [52]. If the distance between the centroid of the fuzzy number that represents the expert's preferences, noted as $cen(\tilde{p}_{ij}^k)$, and the fuzzy number that represents the collective opinion, $cen(c_{ij})$, for the pair of alternatives (a_i, a_j) is greater than a predefined closeness threshold θ, the change to apply will be greater, otherwise, it will be less. This is summarized in two cases:

- **CASE 1**: If $|cen(\tilde{p}_{ij}^k) - cen(c_{ij}))| > \theta$ then a significant change is applied. This change will be applied directly over the linguistic terms that compose the ELICIT expression.
- **CASE 2**: If $|cen(\tilde{p}_{ij}^k)) - cen(c_{ij}))| \leq \theta$, then a slight change is applied. This change will be applied over the symbolic translation of the terms of the ELICIT expression.

Remark 3. *The function $cen(\cdot)$ represents the coordinate x of the centroid of a fuzzy number and the parameter $\theta > 0$ defined as a closeness threshold between the expert's preference and the collective opinion.*

Depending on the case, we studied two changes direction, increase and decrease:

- **CASE 1**
  - *Increase assessment*
    - If $p_{ij}^k = (s_p, \alpha)$, then the advice for the expert is $p_{ij}^k = (s_{p+\theta_1}, \alpha)$, $\theta_1 \in [1, g-1]$, $p + \theta_1 \leq g$. In case that $s_p = s_g$ no change will be applied.
    - If $p_{ij}^k =$ *at least* (s_p, α) or *at most* (s_p, α), then the advice for the expert is $p_{ij}^k =$ *at least* $(s_{p+\theta_1}, \alpha)$ or *at most* $(s_{p+\theta_1}, \alpha)$, respectively, $\theta_1 \in [1, g-1]$, $p + \theta_1 \leq g$. In case that $s_p = s_g$ no change will be applied.
    - If $p_{ij}^k =$ *between* (s_p, α_1) *and* (s_q, α_2) then the advice for the expert is $p_{ij}^k =$ *between* $(s_{p+\theta_1}, \alpha_1)$ *and* (s_q, α_2), $\theta_1 \in [1, g-1]$, $p + \theta_1 \leq g$ and $p + \theta_1 \leq q$. In case that $s_{p+\theta_1} = s_q$ and $\alpha_1 \geq \alpha_2$, the new assessment is $p_{ij}^k = (s_{p+\theta_1}, \alpha_1)$.

- *Decrease assessment*
    * If $p_{ij}^k = (s_p, \alpha)$, then the advice for the expert is $p_{ij}^k = (s_{p-\theta_1}, \alpha)$, $\theta_1 \in [1, g-1]$, $p - \theta_1 \geq 0$. In case that $s_p = s_0$ no change will be applied.
    * If $p_{ij}^k =$ *at least* (s_p, α) or *at most* (s_p, α), then the advice for the expert is $p_{ij}^k =$ *at least* $(s_{p-\theta_1}, \alpha)$ or *at most* $(s_{p-\theta_1}, \alpha)$, respectively, $\theta_1 \in [1, g-1]$, $p - \theta_1 \geq 0$. In case that $s_p = s_0$, no change will be applied.
    * If $p_{ij}^k =$ *between* (s_p, α_1) *and* (s_q, α_2), then the advice for the expert is $p_{ij}^k =$ *between* (s_p, α_1) *and* $(s_{q-\theta_1}, \alpha_2)$, $\theta_1 \in [1, g-1]$, $q - \theta_1 \geq 0$ and $q - \theta \geq p$. In case that $s_{q-\theta_1} = s_p$ and $\alpha_2 \leq \alpha_1$, the new assessment is $p_{ij}^k = (s_{q-\theta_1}, \alpha_2)$.

- **CASE 2**
  - *Increase assessment*
    * If $p_{ij}^k = (s_p, \alpha)$, then the advice for the expert is $p_{ij}^k = (s_p, \alpha + \theta_2)$, $\theta_2 \in [0, 0.5]$.
    * If $p_{ij}^k =$ *at least* (s_p, α) or *at most* (s_p, α), then the advice for the expert is $p_{ij}^k =$ *at least* $(s_p, \alpha + \theta_2)$ or *at most* $(s_p, \alpha + \theta_2)$, respectively, $\theta_2 \in [0, 0.5]$.
    * If $p_{ij}^k =$ *between* (s_p, α_1) *and* (s_q, α_2) then the advice for the expert is $p_{ij}^k =$ *between* $(s_p, \alpha_1 + \theta_2)$ *and* (s_q, α_2), $\theta_2 \in [0, 0.5]$. In case that $(s_p, \alpha_1 + \theta_2) \geq (s_q, \alpha_2)$, the new assessment is $p_{ij}^k = (s_p, \alpha_1 + \theta_2)$.
  - *Decrease assessment*
    * If $p_{ij}^k = (s_p, \alpha)$, then the advice for the expert is $p_{ij}^k = (s_p, \alpha - \theta_2)$, $\theta_1 \in [0, 0.5]$.
    * If $p_{ij}^k =$ *at least* (s_p, α) or *at most* (s_p, α), then the advice for the expert is $p_{ij}^k =$ *at least* $(s_p, \alpha - \theta_2)$ or *at most* $(s_p, \alpha - \theta_2)$, respectively, $\theta_2 \in [0, 0.5]$.
    * If $p_{ij}^k =$ *between* (s_p, α_1) *and* (s_q, α_2), then the advice for the expert is $p_{ij}^k =$ *between* (s_p, α_1) *and* $(s_q, \alpha_2 - \theta_2)$, $\theta_2 \in [0, 0.5]$. In case that $(s_q, \alpha_2 - \theta_2) \leq (s_p, \alpha_1)$, the new assessment is $p_{ij}^k = (s_q, \alpha_2 - \theta_2)$.

Remark 4. *The adjustment parameters θ_1 and θ_2 represent the change degree to apply.*

4. Case Study

This section introduces a real world GDM problem to show the performance of the proposed consensus model together with its advantages and novelties. Furthermore, a comparative performance analysis with another consensus approach is introduced. Note that both the GDM problem below and the consensus approaches have been integrated into the AFRYCA 3.0 software [4,32,33].

Let us suppose a panel of eight experts, $E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$, who have to decide between three action plans to increase the flow of tourists in a given city. The three action plans are $A = \{a_1 : TV\ advertisement, a_2 : sport\ event, a_3 : commercial\ products\}$. Due to the complexity of the the decision, the experts express their preferences by using the linguistic expression domain shown in Figure 6.

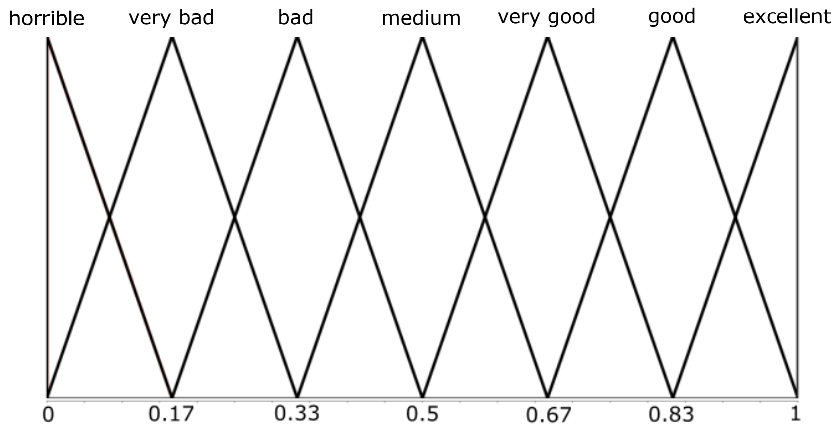

Figure 6. Expression domain.

The adjustment parameters to solve this problem are as follows:

Remark 5. *Note that the values of the parameters (see Table 1) μ and r_{max} have been assigned with the aim of showing clearly that our proposal is able to reach a high level of agreement in decision situations in which the time pressure is key. The parameters ε, θ_1 and θ_2 have been evaluated taking into account the multiple experiments that we have carried out using the AFRYCA software. Finally, the value for θ represents the distance between two consecutive linguistic labels in a linguistic term set. We consider that when the distance between the centroids of the expert's preference and the collective opinion is greater than θ, we should apply a significant change. Otherwise, the distance would be smaller than the one between two consecutive linguistic labels and the change should be slighter.*

Table 1. Parameters.

Parameter	Value
μ	0.9
r_{max}	5
ε	0.05
θ	1/g
θ_1	2
θ_2	0.2

Finally, a key aspect in any CRP is the experts' behavior in the face of the changes advised by the model. AFRYCA 3.0 allows for the configuration and simulation of different experts' behavior, which means experts may accept the recommendations provided for the consensus approach or refuse them. Keeping in mind our idea of doing a fair comparison between our proposal and another CRP model and showing the advantages of the former, we considered that experts always accept the recommendations provided by both models.

4.1. Resolution Scheme

In order to solve the GDM problem by using the CRP support system AFRYCA 3.0 and according to the resolution scheme introduced in Section 3:

1. Input information: the experts provide their assessments by means of HLPR using the expression domain shown in Figure 6. These preferences are shown below:

$$e_1 = \begin{pmatrix} - & \textit{very bad} & \textit{good} \\ \textit{very good} & - & \textit{at most very bad} \\ \textit{bad} & \textit{at least very good} & - \end{pmatrix}$$

$$e_2 = \begin{pmatrix} - & \textit{very good} & \textit{very bad} \\ \textit{very bad} & - & \textit{very good} \\ \textit{very good} & \textit{very bad} & - \end{pmatrix}$$

$$e_3 = \begin{pmatrix} - & \textit{bt horrible and very bad} & \textit{good} \\ \textit{bt very good and excellent} & - & \textit{very bad} \\ \textit{bad} & \textit{very good} & - \end{pmatrix}$$

$$e_4 = \begin{pmatrix} - & \textit{bt medium and good} & \textit{medium} \\ \textit{bt bad and medium} & - & \textit{very good} \\ \textit{bad} & \textit{very bad} & - \end{pmatrix}$$

$$e_5 = \begin{pmatrix} - & \textit{very good} & \textit{medium} \\ \textit{very bad} & - & \textit{bad} \\ \textit{medium} & \textit{good} & - \end{pmatrix}$$

$$e_6 = \begin{pmatrix} - & \textit{good} & \textit{horrible} \\ \textit{bad} & - & \textit{at most very bad} \\ \textit{excellent} & \textit{at least very good} & - \end{pmatrix}$$

$$e_7 = \begin{pmatrix} - & \textit{bad} & \textit{horrible} \\ \textit{good} & - & \textit{bt medium and good} \\ \textit{excellent} & \textit{bt bad and medium} & - \end{pmatrix}$$

$$e_8 = \begin{pmatrix} - & \textit{medium} & \textit{good} \\ \textit{medium} & - & \textit{at most very bad} \\ \textit{bad} & \textit{at least very good} & - \end{pmatrix}$$

2. Transformation into ELICIT information and Fuzzy Numbers: the assessments modelled by CLEs are transformed into ELICIT information and, finally, into TrFN, $\tilde{p}^k_{ij}$.
3. Compute the consensus level: initially, the level of agreement within the group is $cl = 0.72$.
4. Consensus control: taking into account that $\mu = 0.9$, the desired level of consensus is not achieved thus, a consensus round is necessary.
5. Feedback mechanism: the pair of alternatives to be changed and the expert candidates to modify their assessments are identified:

   - Pair of alternatives to change: $(a_1, a_2), (a_1, a_3), (a_2, a_3)$
   - Experts' assessments to change: $e_1 = \{(a_1, a_2), (a_1, a_3), (a_2, a_3)\}, e_2 = \{(a_1, a_2), (a_2, a_3)\}, e_3 = \{(a_1, a_2), (a_1, a_3)\}, e_4 = \{(a_2, a_3)\}, e_5 = \{(a_1, a_2)\}, e_6 = \{(a_1, a_3), (a_2, a_3)\}, e_7 = \{(a_1, a_3)\}, e_8 = \{(a_1, a_3), (a_2, a_3)\}$

   Then, depending on the direction of change and the degree of change needed, the assessments are modified. Figure 7 represents the evolution of the experts' preferences across the CRP by using the multi-dimensional scaling technique [53]. In the center of each plot, the collective opinion of the experts is represented and around it, the experts' preferences. The closer the experts are to the collective opinion, the greater the level of agreement in the group.

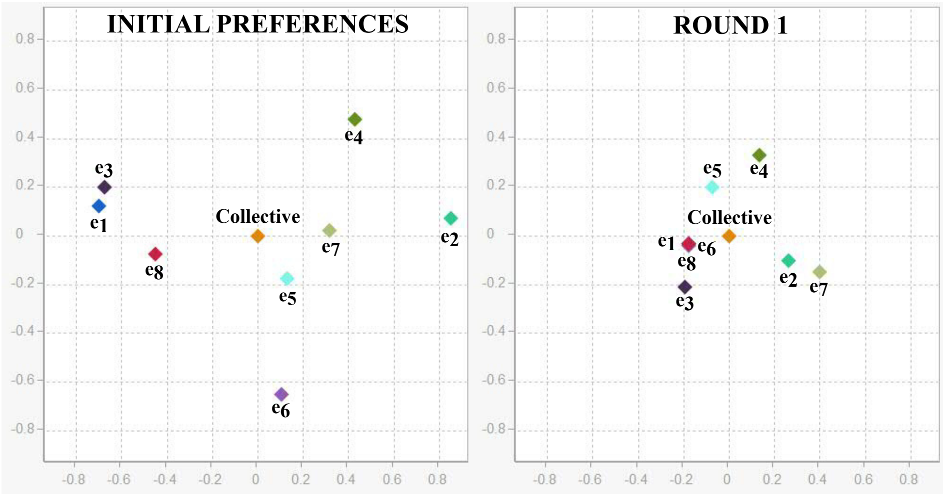

Figure 7. CRP evolution.

After the first discussion round, the level of agreement achieved in the group is $cl = 0.9$. Due to $cl \geq \mu$, the CRP finishes and the selection process of the best alternative starts. For this problem, the ranking of alternatives is $a_3 \succ a_2 \succ a_1$, thus a_3 is selected as the solution to the problem. The ranking of the alternatives is obtained from the collective opinion computed by Equation (9) and a dominance process [54].

4.2. Discussion

The results obtained in the previous section demonstrate the good performance of the proposed consensus model. Despite the desired level of consensus being high ($\mu = 0.9$) and the initial consensus degree in the group being far from this value ($cl = 0.72$), our consensus model needs only one discussion round to achieve the desired level of consensus. That means, that our consensus model is able to achieve a high level of agreement rapidly, thus, it can be applied perfectly both to LDM problems where a high level of consensus is required within the group and to problems where the time pressure is key, such as emergency decision situations. The ELICIT information and its modelling in a continuous domain are key to achieving these excellent results, since both precise fuzzy computations and changes in experts' preferences can be carried out. The computations by means fuzzy operations avoid the loss of information in the resolution process by obtaining more reliable solutions and the experts' preferences are modified in the right measure, discarding excessive changes that negatively influence the achievement of consensus. Furthermore, the initial preferences are represented linguistically as well as the final preferences, which facilitates the elicitation task and the understanding of the CRP. This is of great importance since experts should be able to understand the results that the consensus model provides, otherwise, it is meaningless. To further highlight the good performance of our proposal, in the following section, we will detail a comparative analysis with another model similar to ours.

Inevitably, our proposal presents some limitations that may be fixed in future works. For instance, the values for the parameters introduced in Table 1 have been assigned according to several experiments carried out with AFRYCA but, undoubtedly, on many occasions, these values will depend on the decision problem to deal with. Although many of the values have a good performance in any decision situation, it would be interesting to provide a formal methodology to set them accordingly. Additionally, we have focused our proposal on decision making problems with few experts but, today, decision problems with hundreds or thousands of experts are common too [7,23]. We should adapt our consensus model to deal with the challenges related to this kind of problems, such as scalability or polarized opinions.

4.3. Comparative Analysis

Despite the previous results that show a good performance according to our goals, it is key to perform comparisons with other CRP approaches with similar features. In our case, we compare with the CRP proposed in [55] because of its similarity with our proposal.

In the resolution of the previous GDM problem with the latter approach, we draw interesting conclusions. The model achieves the maximum number of rounds and does not reach the desired level of agreement (see Figure 8). There are two main reasons for this behavior. Firstly, the linguistic information is transformed into numerical values, losing information in the process. Secondly, when the experts express their preferences in a continuous domain, as in the ELICIT assessments case, the change can vary greatly but, in a discrete one, such change is less because it will not be greater than the granularity of the linguistic term set, which limits a lot the feedback process. The latter drawback can be also appreciated in Table 2. The consensus level achieved $cl = 0.86$ in the second round but, in the third one, the level of agreement decreased. This means that the model accomplishes excessive changes in the experts' preferences that decrease the level of agreement within the group.

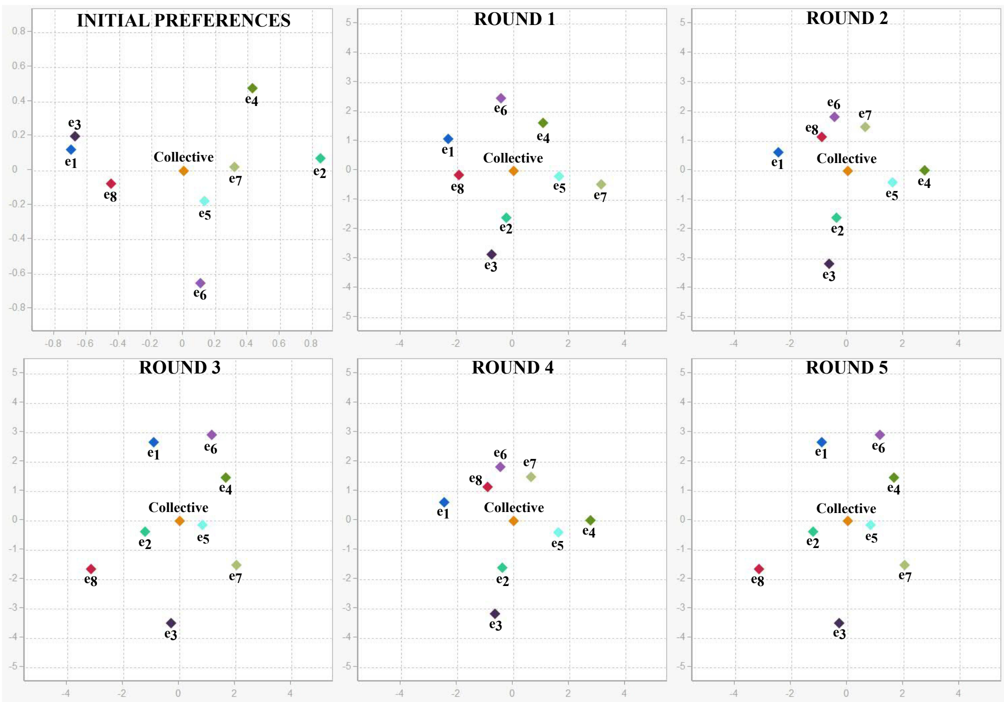

Figure 8. CRP evolution with [55].

Table 2. Consensus level in different rounds.

Round	*cl*
1	0.77
2	0.86
3	0.82
4	0.71
5	0.82

This comparative analysis shows the importance of using ELICIT information in CRPs, since it allows for more accurate computations and precise changes in the experts' preferences to be carried out, which helps a desired level of consensus to be acheived faster.

5. Conclusions and Future Works

This work has introduced a new consensus model under linguistic environments based on ELICIT information. This approach allows for modeling of the experts' preferences by means of linguistic complex expressions to be closer to the experts' way of thinking by facilitating the elicitation task. Furthermore, precise computations with fuzzy operations are carried out together with a linguistic representation of the result that facilitates their understanding by the experts. Finally, a case study has been presented in order to show the performance of the proposal together with a comparative analysis with another proposal to highlight its superior performance. The results obtained from the case study and the comparative analysis show the good performance of our proposal. It is able to achieve a high level of consensus with just a single consensus round. The use of ELICIT information and its modelling in a continuous domain allows for the application of more precise changes to the experts' preferences and positively influences the achievement of the consensus within the group. The latter issue has been widely proved in the comparative analysis with another proposal with similar characteristics in which the desired level of consensus is never reached.

As future research, Large-Scale Group Decision Making (LSGDM) problems are becoming more common and they are drawing the attention of researchers. In this type of problem, it is even more necessary to apply CRP because of the large number of experts involved in the decision process. For this reason, we will study how to apply the proposed consensus approach to deal with LS-GDM problems.

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Abbreviations

The following abbreviations are used in this manuscript:

GDM	Group Decision Making
CRP	Consensus Reaching Process
LDM	Linguistic Decision Making
HFLTS	Hesitant Fuzzy Linguistic Term Set
CLE	Comparative Linguistic Expression
ELICIT	Extended Comparative Linguistic Expressions with Symbolic Translation
AFRYCA	A FRamework for the analYsis of Consensus Approaches
CW	Computing with Words
TrFN	Trapezoidal Fuzzy Number

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Decision Support

A Cost Consensus Metric for Consensus Reaching Processes based on a comprehensive minimum cost model

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ABSTRACT

Consensus Reaching Processes (CRPs) have recently acquired much more importance within Group Decision Making real-world problems because of the demand of either agreed or consensual solutions in such decision problems. Hence, many CRP models have been proposed in the specialized literature, but so far there is not any clear objective to evaluate their performance in order to choose the best CRP model. Therefore, this research aims at developing an objective metric based on the cost of modifying experts' opinions to evaluate CRPs in GDM problems. First, a new and comprehensive minimum cost consensus model that considers distance to global opinion and consensus degree is presented. This model obtains an optimal agreed solution with minimum cost but this solution is not dependent on experts' opinion evolution. Therefore, this optimal solution will be used to evaluate CRPs in which experts' opinion evolution is considered to achieve an agreed solution for the GDM. Eventually, a comparative performance analysis of different CRPs on a GDM problem will be provided to show the utility and validity of this cost metric.

1. Introduction

Decision making is a common process in human being's daily life, characterized by the existence of at least two alternatives and the need of selecting which one is the best solution of the problem. Nowadays, several experts with different points of view often take part in a decision problem with the aim of obtaining a common solution, leading to a Group Decision Making (GDM) problem. Traditionally, the GDM problems have been solved by a selection process (Herrera, Herrera-Viedma, & Verdegay, 1995), but such a process ignores the agreement among experts, which implies that some experts may think that their opinions have not been considered sufficiently. Disagreement among experts is inevitable in most of real world problems, hence it is important to remove the disagreement among experts to obtain an agreed solution that is generally more appreciated by the group and stakeholders as well as demanded by many real world problems. Thus, a Consensus Reaching Process (CRP) has been added in the resolution of GDM problems. In a CRP, experts discuss and modify their preferences to make them closer to each other with the aim of increasing the level of agreement among experts to obtain an acceptable solution for all of them (Butler & Rothstein, 2007; Dong & Xu, 2016; Herrera-Viedma, Cabrerizo, Kacprzyk, & Pedrycz, 2014; Palomares & Martínez, 2014; Palomares, Martínez, & Herrera, 2014b; Palomares, Rodríguez, & Martínez, 2013; Xu, Du, & Chen, 2015). There are different interpretations of consensus, from unanimous agreement among the group to more flexible soft consensus (Cabrerizo, Moreno, Peréz, & Herrera-Viedma, 2010; Kacprzyk & Fedrizzi, 1988; Kacprzyk, Nurmi, & Fedrizzi, 1997; Kacprzyk & Zadrożny, 2010; Kacprzyk, Zadrożny, & Raś, 2010; Zhang, Kou, & Peng, 2019). In the literature there are many consensus models (Herrera-Viedma et al., 2014; Palomares, Estrella, Martínez, & Herrera, 2014a; Zhang, Dong, Chiclana, & Yu, 2019). Palomares et al. (2014a) provided a comprehensive taxonomy of CRPs based on two dimensions (see Fig. 1):

a) Consensus with feedback and without feedback.
b) Consensus measures based on distances to the collective opinions and based on distances between experts.

In the horizontal axis, the CRPs 'without feedback' achieve consensus by modifying initial opinions without consider experts meanwhile, CRPs 'with feedback', involve discussions among experts and they should modify their opinions to reach a consensus.

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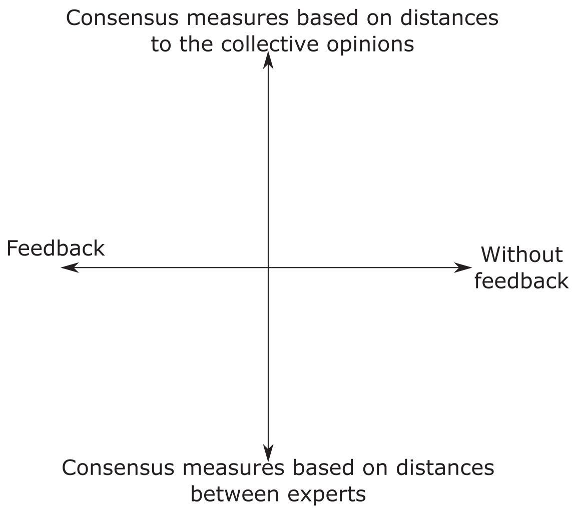

Fig. 1. A taxonomy for consensus approaches.

Despite there are many proposals on CRPs (Cheng, Zhou, Cheng, Zhou, & Xie, 2018; Chiclana, Mata, Martínez, Herrera-Viedma, & Alonso, 2008; Herrera-Viedma, Herrera, & Chiclana, 2002; Kacprzyk & Zadrożny, 2010; Rodríguez, Labella, De Tré, & Martínez, 2018; Wu, Kou, & Peng, 2018), there is no any suitable criteria to evaluate and compare the CRPs, and show which one has a better performance for a given GDM problem. Recently, the software AFRYCA (A FRamework for the analYsis of Consensus Approaches) (Labella, Liu, Rodríguez, & Martínez, 2018; Palomares et al., 2014a) was a first attempt to facilitate the analysis and comparison among different CRPs' performance. This tool uses several measures such as the number of rounds necessary to reach consensus, number of changes carried out across the CRP and different consistency measures as criteria to compare the performance of different CRP models (Labella et al., 2018). However, these criteria are quite simple and cannot objectively and adequately measure the performance of different CRP models. Thus, our objective is to define an objective metric to evaluate the performance of a CRP taking into account the cost of modifying experts' opinion.

Several researchers have pointed out the importance of considering cost of modifying experts' opinion to reach consensus and it has become an attractive challenge to tackle in CRPs. Ben-Arieh and Easton (2007) defined the concept of minimum-cost consensus (MCC) and proposed the first MCC model which uses a linear cost function to achieve consensus. Afterwards, Ben-Arieh, Easton, and Evans (2008) introduced another MCC model by using a quadratic cost function, and taking into account these two models as a base, some new MCC models have been proposed (Gong, Zhang, Forrest, Li, & Xu, 2015; Li, Zhang, & Dong, 2017; Liu, Chan, Li, Zhang, & Deng, 2012; Zhang, Dong, & Xu, 2013; Zhang, Dong, Xu, & Li, 2011; Zhang, Gong, & Chiclana, 2017). Nevertheless, all these models, modify experts' opinions automatically without experts' supervision and consider just a consensus measure, *the distance of each expert to the collective opinion*, ignoring a minimum agreement among all experts. It must be highlighted that small distances among experts and the collective opinion cannot always ensure a required acceptable consensus level. Therefore, it is necessary to consider a more comprehensive cost model that not only uses the distance of each expert to the collective opinion, but also reaches a minimum agreement among experts to obtain better and more acceptable solutions.

In this paper, first several novel MCC models that integrate both consensus measures, (i) distance to collective opinion and (ii) minimum agreement, are introduced. Such MCC models will achieve optimal solutions from each point of view to achieve consensus, but they will not take into account experts for modifying their opinions. Therefore, such an optimal solution will be used to define a Cost Consensus Metric (CCM) that studies the cost performance of CRPs that consider the modification of the experts' opinions to achieve consensus. This CCM will be implemented in the software AFRYCA and a comparative analysis of the cost performance among several CRPs will be carried out to show the results obtained by this new metric.

The remainder of this paper is structured as follows. Section 2 reviews some basic concepts about GDM, CRPs and MCC models. Section 3 presents some new MCC models that consider the distance of each expert to the collective opinion and a minimum agreement among experts to achieve consensus. Section 4 introduces a CCM to evaluate the performance of CRPs. Section 5 provides a comparison experiment of several existing CRP models and analyzes the results by means of AFRYCA. Finally, Section 6 points out some conclusions.

2. Preliminaries

This section makes a short review about basic concepts of GDM, CRP and MCC models, that are necessary to understand our proposal.

2.1. Group Decision Making

GDM problems are very common activities in human's life which consist of a set of experts $E=\{e_1,\dots,e_m\}$, who provide their preferences over a set of possible alternatives or options $X=\{x_1,\dots,x_n\}$, with the aim of obtaining a common solution (Lu, Zhang, Ruan, & Wu, 2007). Each expert $e_k \in E$ expresses his/her opinions over the different alternatives in an information domain by means of a preference structure. There are different preference structures for GDM problems:

- Preference relation: in a preference relation P^k, the assessment p_{ij}^k provided by the expert e_k, represents the preference degree of the alternative x_i over the alternative x_j, $i, j \in \{1,\dots,n\}$. It is shown as follows:

$$P^k = \begin{pmatrix} p_{11}^k & \cdots & p_{1n}^k \\ \vdots & \ddots & \vdots \\ p_{n1}^k & \cdots & p_{nn}^k \end{pmatrix}.$$

- Decision matrix: in a decision matrix, the assessment p_{ij}^k represents the expert e_k's opinion over the alternative x_i based on a certain decision criterion c_j, unlike a preference relation that establishes pairwise comparisons between alternatives. It is expressed as follows:

	c^1	c^2	$\dots$	c^l
x_1	p_{11}	p_{12}	$\dots$	p_{1l}
x_2	p_{21}	p_{22}	$\dots$	p_{2l}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_n	p_{n1}	p_{n2}	$\vdots$	p_{nl}

There are different preference relations depending on the expression domain, such as fuzzy preference relation (FPR), linguistic preference relation, hesitant preference relation (De Baets & Fodor, 1997; Rodríguez, Xu, Martínez, & Herrera, 2018), etc. The use of FPR facilitates the preference elicitation to experts by means of pairwise comparison in a continuous scale in [0,1], due to its simplicity and easy construction it is one of the most widely-used preferences structures in GDM to elicit experts' preferences. Even

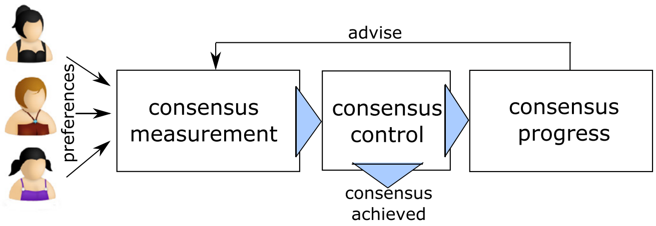

Fig. 2. General scheme of a CRP.

though in the future our research proposals can be studied in other type of preference relations.

Definition 1. Orlovsky (1978) A fuzzy preference relation P^k, associated to an expert e_k on a set of alternatives X, is a fuzzy set on $X \times X$, characterized by the membership function $\mu_{P^k} : X \times X \to [0, 1]$. When the number of alternatives n is finite, P^k is represented by a $n \times n$ matrix of assessments $\mu_{P^k}(x_i, x_j) = p^k_{ij}$ as follows:

$$P^k = \begin{pmatrix} p^k_{11} & \cdots & p^k_{1n} \\ \vdots & \ddots & \vdots \\ p^k_{n1} & \cdots & p^k_{nn} \end{pmatrix},$$

where each assessment p^k_{ij} represents the *degree of preference* of the alternative x_i over x_j according to expert e_k. The fuzzy preference relation is usually assumed to be additive reciprocal, i.e., $p^k_{ij} + p^k_{ji} = 1, \ \forall i, j = 1, 2, \ldots, n, k = 1, 2, \ldots, m$.

The classical selection process to solve a GDM problem is divided into two phases:

- Aggregation: in this phase, the preference relations provided by experts are fused by means of an aggregation operator to obtain a collective opinion.
- Exploitation: it selects the best alternative(s) as solution of the GDM problem by using the result obtained in the previous phase.

Nevertheless, this process does not always guarantee that the decision selected is accepted by all experts involved in the problem, because some of them might consider that their preferences are not taken into account. A common solution to obtain decisions accepted by the whole group of experts, is to remove the disagreement among them. To do so, a CRP is incorporated before the selection process Saint and Lawson (1994).

2.2. Consensus Reaching Process and consensus measures

A CRP is an iterative and dynamic process in which experts discuss and modify their initial preferences with the aim of achieving a collective opinion that satisfies all experts involved in the GDM problem. Such a process is usually guided and supervised by a human figure known as *moderator*. There are many consensus models (Chiclana et al., 2008; Dong, Zhu, & Cooper, 2017; Herrera-Viedma et al., 2002; Kacprzyk & Zadrożny, 2010; Rodríguez et al., 2018), in Palomares et al. (2014a) a taxonomy and a deep revision about some of them were proposed. A general scheme of a CRP is sketched in Fig. 2 and briefly described as follows:

- *Consensus measurement*: the preferences provided by experts are gathered, and the level of agreement in the group is computed by means of consensus measures (Beliakov, Calvo, & James, 2014) which are based on distance measures and aggregation operators (Montserrat-Adell, Agell, Sánchez, & Ruiz, 2018).
- *Consensus control*: the level of agreement computed is compared with the consensus threshold, $\alpha \in [0, 1]$, fixed a priori. If the level of agreement is greater than the consensus threshold, a selection process is applied, otherwise it is necessary to carry out another discussion round. To avoid an excessive number of rounds, a parameter that indicates the maximum number of rounds allowed, $Maxround \in \mathbb{N}$, is considered.
- *Consensus progress*: the moderator identifies the experts' preferences causing disagreement and advises them to modify such preferences to increase the level of agreement in the next round.

A key phase in the scheme is the first one, *Consensus measurement*, that computes the current level of agreement in the group. According to the taxonomy introduced in Palomares et al. (2014a), the consensus measures can be classified in two types Beliakov et al. (2014):

- Consensus measure based on the distance of each expert to the collective opinion given by the following equation:

$$consensus\,(o_1, \ldots, o_m) = 1 - f_2(f_1(d(o_i, o^c))), \tag{1}$$

  where $(o_1, \ldots, o_m)$ are the assessments provided by experts $(e_1, \ldots, e_m)$ over an alternative, o^c is the collective opinion, $d(\cdot, \cdot)$ is a distance measure, and $f_1 : R^+ \to R^+$, $f_2 : R^+ \to [0, 1]$ are functions.
- Consensus measure based on the distances among experts given by the following formula:

$$consensus\,(o_1, \ldots, o_m) = 1 - g_2\big(g_1(d(o_i, o_j))\big), i \neq j, \tag{2}$$

  where $g_1 : R^+ \to R^+, g_2 : R^+ \to [0, 1]$ are functions, and other symbols are the same as in Eq. (1).

2.3. Minimum-cost consensus models

In CRPs the cost of modifying experts' preferences is key in the collective opinion. Ben-Arieh and Easton (2007) introduced the concept of *minimum-cost consensus* and proposed a MCC model that defines the consensus as the minimum distance between each expert and the collective opinion. This model seeks to minimize the overall cost of moving all experts' opinions by using a linear function.

Definition 2. Zhang et al. (2011) Let $(o_1, \ldots, o_m)$ be the original assessments provided by a set of experts $E = \{e_1, e_2, \ldots, e_m\}$ over an alternative. Suppose that after CRP, the experts' assessments are modified into $(\bar{o}_1, \ldots, \bar{o}_m)$, and a collective opinion $\bar{o}$ is obtained based on the modified assessments, and $(c_1, \ldots, c_m)$ are the cost of moving each expert's opinion 1 unit, respectively. The parameter ε is the maximum acceptable distance of each expert to the collective opinion. The MCC model based on a linear cost function is given as follows:

$(\mathbf{M-1})$

$$\min \sum_{k=1}^{m} c_k |\bar{o}_k - o_k|$$
$$s.t. \ |\bar{o}_k - \bar{o}| \leq \varepsilon, k = 1, 2, \ldots, m.$$

It is noteworthy that ε in the model (M–1) measures the absolute deviation between each expert's adjusted opinion and the collective opinion, and it is not necessary to be valued in [0,1]. According to this model, an expert's opinion does not need to be changed if it is in the interval $[\bar{o} - \varepsilon, \bar{o} + \varepsilon]$, and any expert's initial opinion further than ε from $\bar{o}$ should only be changed until that expert's opinion is exactly ε away from $\bar{o}$.

Taking into account (M–1), Zhang et al. (2011) studied how the level of agreement in the group can be different according to the selected aggregation operator for computing the collective opinion, and proposed a new MCC model as follows:

$$(\mathbf{M-2})$$

$$\min \sum_{i=1}^{m} c_i|\bar{o}_i - o_i|$$

$$s.t.\begin{cases} \bar{o} = F(\bar{o}_1, \ldots, \bar{o}_m) \\ |\bar{o}_i - \bar{o}| \le \varepsilon, i = 1, 2, \ldots, m, \end{cases}$$

where F is an aggregation function.

The properties of (M–2) were investigated under the situations that F can be the weighted average operator or the OWA operator Zhang et al. (2011).

Recently, some researchers have paid much attention on the model proposed by Ben-Arieh and Easton (2007) and have introduced some new MCC approaches (Gong et al., 2015; Li et al., 2017; Liu et al., 2012; Zhang et al., 2013; Zhang et al., 2017). However, all these models present a disadvantage, that is, they only consider the distance of each expert to the collective opinion ignoring a minimum agreement among experts to reach consensus that is the main measure considered in many CRPs (Chiclana et al., 2008; Kacprzyk & Zadrożny, 2010; Wu & Xu, 2016; Zhang, Dong, & Xu, 2012). Therefore, the overall opinion obtained cannot guarantee a required consensus degree for all experts involved in the GDM problem and a comprehensive MCC should be developed.

3. New MCC models considering the distance and consensus degree

This section proposes several new MCC models which cope with the previous drawback of the existing MCC models. Therefore, with the aim of defining a comprehensive MCC model that takes into account level of agreement and distance to collective opinion, first a MCC model that deals with single numerical values is defined and it is then extended to deal with FPRs.

3.1. MCC models dealing with numerical values

As it was aforementioned, small distances between experts and the collective opinion cannot always ensure that experts reach a high consensus level. Therefore, it is necessary to define a new MCC model that is able to achieve a minimum agreement among experts to obtain better consensual solutions. Thus, the model (M–2) is modified including the computation of consensus level. The model obtained is the following one:

$$(\mathbf{M-3})$$

$$\min \sum_{i=1}^{m} c_i|\bar{o}_i - o_i|$$

$$s.t.\begin{cases} \bar{o} = F(\bar{o}_1, \ldots, \bar{o}_m) \\ |\bar{o}_i - \bar{o}| \le \varepsilon, i = 1, 2, \ldots, m \\ consensus\,(\bar{o}_1, \ldots, \bar{o}_n) \ge \alpha, \end{cases}$$

where $consensus(\cdot)$ represents the consensus level achieved, $\alpha \in [0, 1]$ is a consensus threshold fixed a priori, F is an aggregation operator, and ε is a parameter measures distance between each expert's adjusted opinion and the collective opinion.

Remark 1. Taking into account that the condition $consensus\,(\bar{o}_1, \ldots, \bar{o}_n) \ge \alpha$, Eqs. (1) and (2) related to the consensus measures can be transformed into the following inequalities:

$$f_2(f_1(d(o_i, o^c))) \le 1 - \alpha, \tag{3}$$

or

$$g_2\big(g_1(d(o_i, o_j))\big) \le 1 - \alpha, i \ne j. \tag{4}$$

Since there are two ways of computing consensus, two MCC models can be defined according to the consensus measures.

- *Consensus measure based on the distance between experts and the collective opinion*. In this case, the distance can be measured by $|\bar{o}_i - \bar{o}|, i = 1, \ldots, m$. Experts might have different importance in the consensus process. Therefore, without loss of generality, the operator to aggregate the distances could be the Weighted Average operator, that is, $\sum_{i=1}^{m} w_i|\bar{o}_i - \bar{o}|$, where $w_i \in [0, 1]$ is the expert e_i's weight and $\sum_{i=0}^{m} w_i = 1$. Therefore, the model $(M-3)$ can be transformed into the following one:

$$(\mathbf{M-4})$$

$$\min \sum_{i=1}^{m} c_i|\bar{o}_i - o_i|$$

$$s.t.\begin{cases} \bar{o} = \sum_{i=1}^{m} w_i\bar{o}_i \\ |\bar{o}_i - \bar{o}| \le \varepsilon, i = 1, 2, \ldots, m \\ \sum_{i=1}^{m} w_i|\bar{o}_i - \bar{o}| \le \gamma, \end{cases}$$

where $\gamma = 1 - \alpha$, $\varepsilon \in [0,\ 1]$ measures the deviation between each expert's adjusted opinion and the collective opinion.
In order to justify the consensus measure used in (M–4), we use the similar consensus measure proposed in Chiclana et al. (2008), then

$$consensus\,(\bar{o}_1, \ldots, \bar{o}_m) = 1 - \frac{1}{m}|\bar{o}_i - \bar{o}|.$$

Thus the condition

$$consensus\,(\bar{o}_1, \ldots, \bar{o}_n) \ge \alpha$$

can be transformed into

$$\frac{1}{m}|\bar{o}_i - \bar{o}| \le 1 - \alpha = \gamma.$$

This approach actually adopts an assumption that each expert has the same contribution to overall consensus. However, we think that the model should offer the view that important experts should contribute more to the consensus. Suppose that in a GDM problem, the expert e_1 has an importance weight $w_1 = 0.9$. It is then reasonable to think that the consensus is almost acceptable if e_1's adjusted opinion $\bar{o}_1$ is close enough to the collective opinion $\bar{o}$. Therefore, we improve the consensus model as follows:

$$consensus\,(\bar{o}_1, \ldots, \bar{o}_n) = \sum_{i=1}^{m} w_i|\bar{o}_i - \bar{o}|.$$

Accordingly, the requirement

$$consensus\,(\bar{o}_1, \ldots, \bar{o}_n) \ge \alpha$$

is transformed into

$$\sum_{i=1}^{m} w_i|\bar{o}_i - \bar{o}| \le 1 - \alpha = \gamma.$$

Remark 2. Note that model (M–4) has been defined by considering that the original values $(o_1, o_2, \ldots, o_m)$ are assessed in [0, 1]. Appendix A shows the transformation of the model (M–4) with non-normalized values.

- *Consensus measure based on the distance among experts*. Given an expert e_i, the distances between e_i and the remaining experts e_j is computed by $|\bar{o}_i - \bar{o}_j|, \forall j = 1, \ldots, m\ j \ne i$, and the average distance among them is obtained as follows:

$$\frac{1}{m-1} \sum_{i=0, j \ne i}^{m} |\bar{o}_i - \bar{o}_j|, i = 1, \ldots, m. \tag{5}$$

Table 1
The costs with different values of ε and γ of (M–4).

	$\gamma=0.3$	$\gamma=0.25$	$\gamma=0.2$	$\gamma=0.15$	$\gamma=0.1$	$\gamma=0.05$
$\varepsilon=0.3$	1.01	1.01	1.01	1.55	2.26	3
$\varepsilon=0.25$	1.13	1.13	1.13	1.55	2.26	3
$\varepsilon=0.2$	1.32	1.32	1.32	1.55	2.26	3
$\varepsilon=0.15$	1.81	1.81	1.81	1.81	2.26	3
$\varepsilon=0.1$	2.47	2.47	2.47	2.47	2.47	3
$\varepsilon=0.05$	3.13	3.13	3.13	3.13	3.13	3.13

Considering that experts might have different importance in the CRP and without loss of generality, the distances can be aggregated by means of the Weighted Average operator as follows:

$$\frac{w_1}{m-1}\sum_{j\neq 1}|\bar{o}_1-\bar{o}_j|+\frac{w_2}{m-1}\sum_{j\neq 2}|\bar{o}_2-\bar{o}_j|+\cdots+\frac{w_m}{m-1}\sum_{j\neq m}|\bar{o}_m-\bar{o}_j|$$
$$=\sum_{i=1}^{m-1}\sum_{j=i+1}^{m}\frac{w_i+w_j}{m-1}|\bar{o}_i-\bar{o}_j|. \quad (6)$$

Therefore, the consensus level can be obtained as follows:

$$consensus\,(\bar{o}_1,\ldots,\bar{o}_n)=\sum_{i=1}^{m-1}\sum_{j=i+1}^{m}\frac{w_i+w_j}{m-1}|\bar{o}_i-\bar{o}_j|. \quad (7)$$

By using Eq. (7), the model (M–3) can be transformed into the following one:

$(\mathbf{M-5})$

$$\min\sum_{i=1}^{m}c_i|\bar{o}_i-o_i|$$
$$s.t.\begin{cases}\bar{o}=\sum_{i=1}^{m}w_i\bar{o}_i\\ |\bar{o}_i-\bar{o}|\leq\varepsilon, i=1,2,\ldots,m\\ \sum_{i=1}^{m-1}\sum_{j=i+1}^{m}\frac{w_i+w_j}{m-1}|\bar{o}_i-\bar{o}_j|\leq\gamma,\end{cases}$$

where ε and γ are the same as in the model (M–4).

Remark 3. Note that model (M–5) has been defined by considering that the original values $(o_1,o_2,\ldots,o_m)$ are assessed in [0, 1]. Appendix A shows the transformation of the model (M–5) with non-normalized values.

In order to show the performance of the proposed models, we present a numerical example.

Example 1. Let us consider a numerical example in which there are five experts $E=\{e_1,\ldots,e_5\}$ with weights $W=(0.375, 0.1875, 0.25, 0.0625, 0.125)$, who provide their assessments over an alternative as $(o_1,\ldots,o_5)$ =(0, 0.09, 0.36, 0.45, 1). The costs are $(c_1,\ldots,c_5)$ =(6, 3, 4, 1, 2), where c_k is the cost of modifying the opinion of the expert e_k. By using the model (M–4), the minimum costs regarding several different values of ε and γ are shown in Table 1. Note that, a high consensus threshold means a small value of γ, therefore the values used for γ are less than 0.30 to show the behaviour of the model (M–4).

By using the model (M–4), Table 2 shows in further detail the optimal solutions and minimum cost with ε =0.2, $\gamma=$ 0.05, 0.1, 0.15, 0.2, 0.25, 0.3 and Table 3 shows the optimal solutions and minimum cost with ε =0.145, $\gamma=$ 0.05, 0.1, 0.15, 0.2, 0.25, 0.3.

Analysis of results

From Table 1, we can see that for a fixed ε, at first the minimum cost is constant, then it increases. For example, for $\varepsilon=0.3$, when $\gamma=$ 0.3, 0.25, 0.2, 0.15, 0.1, 0.05, the minimum costs are 1.01, 1.01, 1.01, 1.55, 2.26, 3, respectively. The reason is because for a big value of γ, the minimum cost is determined by ε, and for a small value of γ, the minimum cost is determined by γ. Furthermore, we can also observe from Tables 2 and 3, that for a same value of γ and different values of ε, the optimal solutions of (M–4) are different even though the minimum costs are identical. For example, when ε=0.2, γ =0.10 in Table 2, the minimum cost is 2.26, and the optimal solution of (M–4) is:

Table 2
The optimal solutions of model (M-4) with $\varepsilon=0.2$.

γ	$\bar{o}_1$	$\bar{o}_2$	$\bar{o}_3$	$\bar{o}_4$	$\bar{o}_5$	$\bar{o}$	*Cost*
0.3	0	0.09	0.36	0.38	0.38	0.18	1.32
0.25	0	0.09	0.36	0.38	0.38	0.18	1.32
0.2	0	0.09	0.36	0.38	0.38	0.18	1.32
0.15	0	0.09	0.31	0.36	0.36	0.16	1.55
0.1	0	0.09	0.22	0.12	0.32	0.12	2.26
0.05	0.02	0.09	0.19	0.09	0.09	0.09	3

Table 3
The optimal solutions of model (M-4) with $\varepsilon=0.145$.

γ	$\bar{o}_1$	$\bar{o}_2$	$\bar{o}_3$	$\bar{o}_4$	$\bar{o}_5$	$\bar{o}$	*Cost*
0.3	0	0.09	0.29	0.29	0.29	0.14	1.88
0.25	0	0.09	0.29	0.29	0.29	0.14	1.88
0.2	0	0.09	0.29	0.29	0.29	0.14	1.88
0.15	0	0.09	0.29	0.29	0.29	0.14	1.88
0.1	0	0.09	0.25	0.12	0.26	0.12	2.26
0.05	0.02	0.09	0.19	0.09	0.09	0.09	3

Table 4
The costs with different values of ε and γ of (M–5).

	$\gamma=0.3$	$\gamma=0.25$	$\gamma=0.2$	$\gamma=0.15$	$\gamma=0.1$	$\gamma=0.05$
$\varepsilon=0.30$	1.01	1.20	1.60	2.14	2.69	3.24
$\varepsilon=0.25$	1.13	1.20	1.60	2.14	2.69	3.24
$\varepsilon=0.20$	1.32	1.32	1.60	2.14	2.69	3.24
$\varepsilon=0.15$	1.81	1.81	1.81	2.14	2.69	3.24
$\varepsilon=0.10$	2.47	2.47	2.47	2.47	2.69	3.24
$\varepsilon=0.05$	3.13	3.13	3.13	3.13	3.13	3.24

$$(\bar{o}_1,\ldots,\bar{o}_5,\bar{o})=(0,0.09,0.22,0.12,0.32,0.12).$$

When ε =0.145, γ =0.10 in Table 3, the minimum cost is also 2.26, but the optimal solution of (M–4) is

$$(\bar{o}_1,\ldots,\bar{o}_5,\bar{o})=(0,0.09,\mathit{0.25},0.12,\mathit{0.26},0.12).$$

We can also use the model (M–5) to solve this example. The minimum costs with respect to several different values of ε and γ are shown in Table 4.

From Table 4, similar conclusions as Table 1 can be obtained and thus they are omitted here.

From Example 1, we can see that both ε and γ play an important role in the models (M–4) and (M–5). In the following we will provide an algorithm to show several rules to select their values.

Algorithm 1

Step 1. Select the value of $\varepsilon\in[0, 1]$ which reflects the permitted maximum deviation between each expert and the collective opinion.

Step 2. Solve the model (M–2) with F being the weighted average operator, and obtain the optimal solution $(\bar{o}_1,\ldots,\bar{o}_n,\bar{o})$. Then calculate γ_0 by using:

$$\gamma_0=\sum_{i=1}^{m}w_i|\bar{o}_i-\bar{o}|,$$

or

$$\gamma_0=\sum_{i=1}^{m-1}\sum_{j=i+1}^{m}\frac{w_i+w_j}{m-1}\cdot|\bar{o}_i-\bar{o}_j|,$$

$\bar{o}_i \in [0,1], i = 1, 2, \ldots, m$.

Step 3. Select an accepted consensus threshold α, and compute $\gamma = 1 - \alpha$. Since $\gamma \geq \gamma_0$, the optimal solutions and minimum costs of the model (M–4) or (M–5) will be the same as the model (M–2). Consequently, we suggest to select the values of γ and α with $\gamma \leq \gamma_0$ and $\alpha \geq 1 - \gamma_0$.

3.2. MCC models dealing with FPRs

One of the preference structures most widely used in GDM and hence, in the corresponding CRPs, is the FPR Orlovsky (1978), therefore, the proposed MCC model (M–3) is modified to deal with FPRs.

Let $P^k = (p_{ij}^k)_{n\times n}$ be a FPR provided by an expert $e_k, k = 1, \ldots, m$. In order to achieve a solution accepted by all experts involved in the GDM problem, P^k is adjusted to $\overline{P}^k = (\overline{p}_{ij}^k)_{n\times n}, k = 1, \ldots, m,$ and the collective FPR of the adjusted FPRs is $\overline{P} = (\overline{p}_{ij})_{n\times n}$.

Depending on the consensus measures to compute consensus level, two different MCC models are introduced.

- Using the consensus measure based on the distance between each expert's opinion and the collective opinion.

$$(\mathbf{M-6})$$

$$\min \sum_{k=1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} c_k |p_{ij}^k - \overline{p}_{ij}^k|$$

$$s.t. \begin{cases} \overline{p}_{ij} = \sum_{k=1}^{m} w_k \overline{p}_{ij}^k \\ |\overline{p}_{ij}^k - \overline{p}_{ij}| \leq \varepsilon, k = 1, \ldots, m, i = 1, \ldots, n-1, j = i+1, \ldots, n \\ \frac{2}{n(n-1)} \sum_{k=1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} w_k |\overline{p}_{ij}^k - \overline{p}_{ij}| \leq \gamma. \end{cases}$$

- Using the consensus measure based on the distance among experts.

$$(\mathbf{M-7})$$

$$\min \sum_{k=1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} c_k |p_{ij}^k - \overline{p}_{ij}^k|$$

$$s.t. \begin{cases} \overline{p}_{ij} = \sum_{k=1}^{m} w_k \overline{p}_{ij}^k \\ |\overline{p}_{ij}^k - \overline{p}_{ij}| \leq \varepsilon, k = 1, \ldots, m, i = 1, \ldots, n-1, j = i+1, \ldots, n \\ \frac{2}{n(n-1)} \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{k=1}^{m-1}\sum_{l=k+1}^{m} \frac{w_k + w_l}{m-1} |\overline{p}_{ij}^k - \overline{p}_{ij}^l| \leq \gamma. \end{cases}$$

Example 2. Let us consider a numerical example in which there are three experts $E = \{e_1, e_2, e_3\}$ with weights W = (0.375, 0.250, 0.375) and costs (c_1, c_2, c_3) =(2, 5, 3), respectively, who provide their assessments over three alternatives $X = \{x_1, x_2, x_3\}$ by means of FPRs (see Defination 1):

$$P^1 = \begin{pmatrix} 0.5 & 0.87 & 0.99 \\ & 0.5 & 0.91 \\ & & 0.5 \end{pmatrix}, P^2 = \begin{pmatrix} 0.5 & 0.14 & 0.03 \\ & 0.5 & 0.14 \\ & & 0.5 \end{pmatrix},$$

$$P^3 = \begin{pmatrix} 0.5 & 0.43 & 0.02 \\ & 0.5 & 0.03 \\ & & 0.5 \end{pmatrix}.$$

Note that FPRs are assumed to be reciprocal and thus elements in the lower triangular matrix are omitted here.

By applying model (M–6) with $\varepsilon = 0.3$ and $\gamma = 0.2$ the resulting modified preferences are:

$$\overline{P}^1 = \begin{pmatrix} 0.5 & 0.66 & 0.47 \\ & 0.5 & 0.52 \\ & & 0.5 \end{pmatrix}, \overline{P}^2 = \begin{pmatrix} 0.5 & 0.14 & 0.02 \\ & 0.5 & 0.14 \\ & & 0.5 \end{pmatrix},$$

$$\overline{P}^3 = \begin{pmatrix} 0.5 & 0.41 & 0.02 \\ & 0.5 & 0.03 \\ & & 0.5 \end{pmatrix}.$$

The resulting cost of modifying the initial preferences is 2.33.

Thus, the preferences $\overline{P}_1$, $\overline{P}_2$ and $\overline{P}_3$ represent the experts' preferences with the minimum necessary modifications to achieve the conditions related to the parameters γ and ε and whose total change cost is 2.33.

4. A Cost Consensus Metric based on minimum cost: measuring Consensus Reaching Processes performance

In spite of the multiple CRPs introduced in the specialized literature (Chiclana et al., 2008; Herrera-Viedma et al., 2002; Kacprzyk & Zadrożny, 2010; Rodríguez et al., 2018), new CRP models are commonly introduced but without a clear advantage over the previous ones. So, in order to support the improvement of CRPs, it is necessary to establish an objective and standard measure to analyse the performance of the CRP models. Consequently, our aim is to define a metric that provides a clear and objective measure about the performance of CRPs to justify the development of new CRPs models as the selection of one model among the multiple ones extant in the literature.

4.1. Cost Consensus Metric

Keeping our aim in mind, we propose to measure the cost incurred by the CRP models to reach the consensus as a metric to evaluate their performance in an objective way. Such a metric is so-called Cost Consensus Metric (CCM), and it will consider the optimal solution the one obtained by models (M–6) or (M–7) because it is the minimum possible cost. Therefore, the metric will compute the difference in cost between the MCC model solution and the solution obtained by different evaluated CRPs.

Suppose that the initial experts' preferences are $P = (P^1, \ldots, P^m)$ and the optimal adjusted FPRs of the MCC model (M–6) or (M–7) are $\overline{P} = (\overline{P}^1, \ldots, \overline{P}^m)$, where P^k and $\overline{P}^k$ are the initial and adjusted FPRs of the expert e_k, $k = 1, 2, \ldots, m$, respectively. The distance between P^k and $\overline{P}^k$ can be computed as

$$d\left(P^k, \overline{P}^k\right) = \frac{2}{n(n-1)} \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \left|p_{ij}^k - \overline{p}_{ij}^k\right|, k = 1, \ldots, m. \tag{8}$$

Based on such distances, the distance factor which measures the relative distance between P and $\overline{P}$, is defined as follows:

$$D\left(P, \overline{P}\right) = \sum_{k=1}^{m} d\left(P^k, \overline{P}^k\right). \tag{9}$$

Similarly, suppose that a CRP model produces the agreed solution as $\widehat{P} = (\widehat{P}^1, \widehat{P}^2, \ldots, \widehat{P}^m)$. The distance between P^k and $\widehat{P}^k$ can be computed as

$$d\left(P^k, \widehat{P}^k\right) = \frac{2}{n(n-1)} \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \left|p_{ij}^k - \widehat{p}_{ij}^k\right|, k = 1, \ldots, m. \tag{10}$$

and

$$D\left(P, \widehat{P}\right) = \sum_{k=1}^{m} d\left(P^k, \widehat{P}^k\right). \tag{11}$$

To evaluate the good performance of a CRP, $\widehat{P}$, its solution is compared with the solution provided by $\overline{P}$ by means of the cost metric, $\phi(\widehat{P},\overline{P})$, which is defined as follows:

$$\phi(\widehat{P},\overline{P}) = \begin{cases} 1-\frac{D(P,\widehat{P})}{D(P,\overline{P})}, & if\ D(P,\widehat{P}) \le D(P,\overline{P}) \\ \frac{D(P,\overline{P})}{D(P,\widehat{P})}-1, & if\ D(P,\widehat{P}) > D(P,\overline{P}). \end{cases} \tag{12}$$

We investigate properties of the previous metric.

1) If $D(P,\widehat{P}) \le D(P,\overline{P})$, then $0 \le \phi(\widehat{P},\overline{P}) \le 1$.
   - If $\widehat{P}=\overline{P}$, then $\phi(\widehat{P},\overline{P})=0$, which means that $\widehat{P}$ provides the minimum cost solution to reach consensus.
   - If $\widehat{P}=P$, then $\phi(\widehat{P},\overline{P})=1$, which means that $\widehat{P}$ provides the worst solution since we assume initial opinions are not under consensus, otherwise, make no sense to apply CRP.
   - If $\phi(\widehat{P},\overline{P})$ increases from $D(P, P)$ to $D(P,\overline{P})$, then $\phi(\widehat{P},\overline{P})$ decreases from 1 to 0.
2) If $D(P,\widehat{P}) > D(P,\overline{P})$, then $-1 \le \phi(\widehat{P},\overline{P}) \le 0$. This case means that, costly changes have been made in the experts' preferences and thus, there is an excessive cost to achieve the consensus regarding the MCC model.
   - If $\widehat{P} \to \overline{P}$, then $\phi(\widehat{P},\overline{P}) \to 0$, which also means that $\widehat{P}$ provides the minimum cost solution to reach consensus. From this result, we can see that the metric $\phi(\widehat{P},\overline{P})$ is continuous at the point $\widehat{P}=\overline{P}$.
   - If $D(P,\widehat{P}) \to +\infty$, then $\phi(\widehat{P},\overline{P}) \to -1$, which means that $\widehat{P}$ provides also the worst solution since there needs infinite cost.
   - If $D(P,\widehat{P})$ increases from $D(P,\overline{P})$ to $+\infty$, then $\phi(\widehat{P},\overline{P})$ decreases from 0 to -1.

From the previous analysis, we know that $\phi(\widehat{P},\overline{P}) \in [-1,1]$, and when $\widehat{P}=\overline{P}$, the CRP solution $\widehat{P}$ is the best. The CRP solution $\widehat{P}$ becomes worse when $\widehat{P}$ goes far away from both sides of $\overline{P}$. Hence, the metric allows to compare the relative closeness of a CRP model to a MCC model.

We provide an example to show the method to evaluate the performance of a CRP model.

Example 3. Suppose that three experts e_1, e_2, e_3 with weights $W=(0.3,0.4,0.3)$ and costs (c_1, c_2, c_3)=(1, 1, 1), respectively, provide their assessments over three alternatives in form of reciprocal FPRs $P=(P^k)_{3\times3}, k=1,2,3$:

$$P^1 = \begin{pmatrix} 0.5 & 0.14 & 0.06 \\ & 0.5 & 0.27 \\ & & 0.5 \end{pmatrix}, P^2 = \begin{pmatrix} 0.5 & 0.55 & 0.13 \\ & 0.5 & 0.11 \\ & & 0.5 \end{pmatrix},$$

$$P^3 = \begin{pmatrix} 0.5 & 0.8 & 0.6 \\ & 0.5 & 0.27 \\ & & 0.5 \end{pmatrix}.$$

We consider the CRP model that is going to be evaluated $\widehat{P}$ and the MCC model $\overline{P}$.

The first CRP model produces the following adjusted FPRs $\widehat{P}=(\widehat{P}^k)_{3\times3}, k=1,2,3$:

$$\widehat{P}^1 = \begin{pmatrix} 0.5 & 0.26 & 0.13 \\ & 0.5 & 0.22 \\ & & 0.5 \end{pmatrix}, \widehat{P}^2 = \begin{pmatrix} 0.5 & 0.55 & 0.13 \\ & 0.5 & 0.11 \\ & & 0.5 \end{pmatrix},$$

$$\widehat{P}^3 = \begin{pmatrix} 0.5 & 0.68 & 0.48 \\ & 0.5 & 0.27 \\ & & 0.5 \end{pmatrix}.$$

The MCC model produces the following adjusted FPRs $\overline{P}=(\overline{P}^k)_{3\times3}, k=1,2,3$:

$$\overline{P}^1 = \begin{pmatrix} 0.5 & 0.5 & 0.11 \\ & 0.5 & 0.26 \\ & & 0.5 \end{pmatrix}, \overline{P}^2 = \begin{pmatrix} 0.5 & 0.56 & 0.13 \\ & 0.5 & 0.2 \\ & & 0.5 \end{pmatrix},$$

$$\overline{P}^3 = \begin{pmatrix} 0.5 & 0.57 & 0.18 \\ & 0.5 & 0.26 \\ & & 0.5 \end{pmatrix}.$$

We want to measure relative closeness of the first CRP model to the MCC model.

We first obtain that

$$\left(d(P^1,\widehat{P}^1), d(P^2,\widehat{P}^2), d(P^3,\widehat{P}^3)\right) = (0.08, 0.0, 0.08).$$

and

$$\left(d(P^1,\overline{P}^1), d(P^2,\overline{P}^2), d(P^3,\overline{P}^3)\right) = (0.14, 0.033, 0.22).$$

We then obtain that

$$D(P,\widehat{P}) = \sum_{k=1}^{3} d(P^k,\widehat{P}^k) = 0.16,$$

and

$$D(P,\overline{P}) = \sum_{k=1}^{3} d\left(P^k,\overline{P}^k\right) = 0.393.$$

As a result the cost metric is computed as

$$\phi(\widehat{P},\overline{P}) = 1-\frac{0.16}{0.393} = 0.5929. \tag{13}$$

Therefore, the CCM shows that the CRP provides a solution, $\widehat{P}$, in which not all experts reach enough consensus. In contrast to the minimum cost solution, $\overline{P}$, in which all experts are close to the overall agreement, the CRP's solution $\widehat{P}$ compensates experts with low degree of agreement with others with high agreement. From our view, such compensatory models should be further penalized because they are not obtaining genuine agreement, therefore, we further study this penalization in our CCM to carry out a proper analysis of the analysed CRP.

4.2. Amplified Cost Consensus Metric

Previous section introduces a novel CCM which allows to evaluate the performance of CRPs. However, Labella et al. shown in Labella et al. (2018) that many of such CRP models are compensatory and their solutions did not fulfil the constraint related to the parameter ε. In our opinion, the metric should apply a greater penalization to these models to reflect such a shortcoming from agreement point of view. Therefore, motivated by the method to amplify extreme values Yager and Petry (2014), we construct an ACCM (Amplified CCM) which can amplify extreme values of an expert. We adopt the amplification factor $f(x)$: $[0, 1] \to [1, \infty)$, which satisfies $f(0)=1$ and $f(a) \ge f(b)$ for $a>b$. In the following we present several forms of f:

$$f(x) = 1+tan\left(\frac{\pi}{2}x\right); \tag{14}$$

$$f(x) = \frac{exp(x)}{1-x}; \tag{15}$$

$$f(x) = \frac{e-1}{e-exp(x)}; \tag{16}$$

$$f(x) = \frac{1+kx}{1-x}, k \ge 0. \tag{17}$$

Table 5
Consensus models' parameters.

Wu and Xu (2012)	Zhang et al. (2012)	Herrera-Viedma et al. (2002)	Chiclana et al. (2008)	Kacprzyk and Zadrożny (2010)	Rodríguez et al. (2018)
$\alpha = 0.85$	$\alpha = 0.85$	$\alpha = 0.85$	$\alpha = 0.85$	$\alpha = 0.85$	$\alpha = 0.85$
$\beta = 0.8$	$cl = 0.95$	$\beta = 0.8$	$B = 0.8$	$\mu = 0.2$	$\omega = 1.5$
$\overline{CI} = 0.15$	$ccl = 0.85$	$Aggregation\ quantifier = F_{most}$	$\theta_1 = 0.7$	$Aggregation\ quantifier = F_{most}$	$\delta = 0.7$
$w_i = \frac{1}{m},\ i \in \{1,\ldots,m\}$		$Exploitation\ quantifier = F_{as\ many\ as\ possible}$	$\theta_2 = 0.8$		$a = 1.0,\ b = 2$

The amplification factor of the expert e_k is then defined as

$$\overline{u}_k = f\left(d(P^k, \overline{P}^k)\right). \tag{18}$$

Based on amplification factor, the distance factor which measures the relative distance between P and $\overline{P}$, is defined as follows:

$$D_A\left(P, \overline{P}\right) = \sum_{k=1}^{m} \overline{u}_k \cdot d\left(P^k, \overline{P}^k\right). \tag{19}$$

Similarly to the previous section, suppose that a CRP model produces the agreed solution as $\widehat{P} = (\widehat{P}^1, \widehat{P}^2, \ldots, \widehat{P}^m)$. The amplification factors are $\widehat{u}_1, \widehat{u}_2, \ldots, \widehat{u}_m$, and the distance factor is

$$D_A\left(P, \widehat{P}\right) = \sum_{k=1}^{m} \widehat{u}_k \cdot d\left(P^k, \widehat{P}^k\right), \tag{20}$$

where $\widehat{u}_k = f\left(d(P^k, \widehat{P}^k)\right),\ k = 1, 2, \ldots, m.$

To evaluate the good performance of a CRP, $\widehat{P}$, its solution is compared with the solution provided by $\overline{P}$ by means of the cost metric, $\phi\left(\widehat{P}, \overline{P}\right)$, which is defined as follows:

$$\phi_A\left(\widehat{P}, \overline{P}\right) = \begin{cases} 1 - \frac{D_A\left(P, \widehat{P}\right)}{D_A\left(P, \overline{P}\right)}, & if\ D_A\left(P, \widehat{P}\right) \le D_A\left(P, \overline{P}\right) \\ \frac{D_A\left(P, \overline{P}\right)}{D_A\left(P, \widehat{P}\right)} - 1, & if\ D_A\left(P, \widehat{P}\right) > D_A\left(P, \overline{P}\right). \end{cases} \tag{21}$$

The properties of the ACCM can be inferred from the CCM definition in the previous section.

Example 4. We solve Example 3 here but applying amplification factors for the different experts:

In order to calculate the amplification factor of experts, we set

$$f(x) = \frac{1 + 1000x}{1 - x}.$$

Based on $d(P^k, \widehat{P}^k)$, the amplification factors of experts $e^k, k = 1, 2, 3$ are obtained as

$$(\widehat{u}_1, \widehat{u}_2, \widehat{u}_3) = (88.0434, 1.0, 88.0434).$$

Based on $d(P^k, \overline{P}^k)$, the amplification factors of experts $e^k, k = 1, 2, 3$ are obtained as

$$(\overline{u}_1, \overline{u}_2, \overline{u}_3) = (163.9535, 35.5172, 283.3333).$$

We then obtain that

$$D_A\left(P, \widehat{P}\right) = \sum_{k=1}^{3} \widehat{u}_k \cdot d\left(P^k, \widehat{P}^k\right) = 14.0870,$$

and

$$D_A\left(P, \overline{P}\right) = \sum_{k=1}^{3} \widehat{u}_k \cdot d\left(P^k, \overline{P}^k\right) = 86.4708.$$

As a result the cost metric is computed as

$$\phi_A\left(\widehat{P}, \overline{P}\right) = 1 - \frac{14.0870}{86.4708} = 0.8371. \tag{22}$$

From this example and comparing with the results obtained in Example 3, we can see that the ACCM evaluates a CRP model by using the MCC model that considers the distances between FPRs, and the importance of each distance, by multiplying each distance with its amplification effect. In such a way those models that compensate experts' opinions to achieve a 'not genuine' agreement are more penalized.

Table 6
Behaviour default values.

Parameter	Value
p	1.0
Change	0.05

Table 7
Minimum distance cost metric parameters.

Parameter	Value
ε	$\varepsilon \in \{0.05, 0.06, 0.1, 0.15, 0.2, 0.3\}$
α	0.85
Cost	$c_i = 1,\ i \in \{1,\ldots,m\}$
Weights	$w_i = \frac{1}{m},\ i \in \{1,\ldots,m\}$

Remark 4. It is noted that any of the functions Eqs. (14)–(17) can be selected to define amplification factors. Here we have selected Eq. (17) with $k = 1000$ since it has a clear amplification effect for small variance of x.

5. Comparative study on the performances of consensus models

This section presents a fair comparative study on the performances of classical consensus models based on the metric proposed in previous section (see Eq. (21)). First, several representative consensus models are selected. Second, the comparative scenarios are described. Afterwards, a simulation process based on AFRYCA 3.0 (Labella et al., 2018; Palomares et al., 2014a) is carried out together with an analysis of the results obtained for each consensus model. Finally, a comparative analysis among all models is also performed.

5.1. Choosing consensus models

Several proposals of CRPs in GDM have been introduced in the specialized literature. For this reason, to carry out our study, a selection of several consensus models is done. Such a selection is composed to a greater extent of classical consensus models and, a more recent consensus model with the aim of carrying out a comparative analysis as complete and diverse as possible. This selection is based on the taxonomy reviewed in Section 1 and graphically represented in Fig. 1. Taking into account the taxonomy, the consensus models selection is divided into 2 groups: consensus models with and without feedback process, that work with the same type of preference relation, FPR. The consensus models selected based on both groups are:

- *Representative models with feedback process*: the selected models are that proposed by Herrera-Viedma et al. (2002), Chiclana et al. (2008) and Kacprzyk and Zadrożny (2010) and Rodríguez et al Rodríguez et al. (2018).

Table 8
Simulation results of CRPs with feedback process.

	Initial consensus degree	Final consensus degree	Number of rounds	Ranking	Solution
Herrera-Viedma et al. (2002)	0.75	0.88	3	$x_2 \succ x_4 \succ x_3 \succ x_1$	x_2
Chiclana et al. (2008)	0.75	0.85	12	$x_2 \succ x_4 \succ x_3 \succ x_1$	x_2
Kacprzyk and Zadrożny (2010)	0.75	0.88	9	$x_4 \succ x_2 \succ x_3 \succ x_1$	x_4
Rodríguez et al. (2018)	0.59	0.85	11	$x_4 \succ x_2 \succ x_1 \succ x_3$	x_4

Table 9
Simulation results of CRPs without feedback process.

	Initial consensus degree	Final consensus degree	Number of rounds	Ranking	Solution
Wu and Xu (2012)	0.75	0.86	16	$x_2 \succ x_4 \succ x_3 \succ x_1$	x_2
Zhang et al. (2012)	0.75	0.85	1	$x_2 \succ x_4 \succ x_3 \succ x_1$	x_2

- *Representative models without feedback process*: the selected models are that proposed by Wu and Xu (2012) and Zhang et al. (2012).

Remark 5. A brief description of the representative consensus models is provided in Appendix B.

5.2. Simulation scenarios

To evaluate the performance of the distinct consensus models selected in Section 5.1 by means of the proposed metric, it is necessary to define the conditions in which the simulations will be carried out. Such conditions will be determined by: (i) maximum number of rounds and consensus threshold in the CRP, (ii) the consensus models' parameters configuration (see Table 5), (iii) the experts' behaviour configuration (see Table 6), and (iv) the metric's parameters configuration (see Table 7).

As it was aforementioned, a CRP finishes when the minimum acceptable agreement, $\alpha \in [0, 1]$, or the maximum numbers of rounds allowed, *Maxround*, are reached, to avoid a never ending process. For the simulations, the predefined values assigned to α and *Maxround* are 0.85 and 30, respectively.

The selected consensus models use different parameters whose predefined values are represented in Table 5.

Thanks to AFRYCA 3.0 (Labella et al., 2018; Palomares et al., 2014a), it is possible to simulate the experts' behaviour. AFRYCA 3.0 includes components which simulates two different experts' behaviours: *standard behaviour* and *adverse behaviour*. The standard behaviour pattern simulates experts who can accept/refuse the suggestion received, otherwise the adverse behaviour simulates experts who can accept, refuse or defend suggestions. The selected behaviour to carry out the simulations is the standard behaviour which is simulated by a binomial probability distribution, configured by a parameter p that defines the probability for experts to accept suggestions. In order to carry out a proper comparison among the selected CRP models, the simulations in this contribution present an ideal scenario in which the experts always accept the suggestion thus, $p = 1$. When an expert accepts a suggestion, the change degree applied to the expert's preference is defined by another parameter noted as *change*. Both parameters are represented in Table 6.

To evaluate the CRP performance by means of the metric, it is necessary to configure its parameters: ε, α, *cost* and experts' weights. Such parameters are shown in Table 7.

5.3. Results and analysis

This section introduces an experimental study to evaluate and compare the performance of the selected classical consensus models.

Let us suppose a group of 8 students, $E = \{e_1, \ldots, e_8\}$. The students plan to organize an end-of-year trip, thus they must make a common decision about choosing the city to travel among 4 possible alternatives, $X = \{x_1, \ldots, x_4\}$, which include: Athens, Portsmouth, Belfast and Chengdu. All students express their preferences as FPRs which are generated by AFRYCA. The corresponding data set is available in the AFRYCA website[1]. The simulations are carried out under the conditions predefined in Section 5.2 and the results are shown below in Tables 8 and 9.

Once the results of each consensus model have been obtained, the evaluation of their performances is carried out by the ACCM (see Table 10). Note that, in order to evaluate the solutions of the representative consensus models with the solutions of the MCC models, those which use a consensus measure based on the

Table 10
ACCM results ($\alpha = 0.85$).

Consensus model	ε	ACCM
Herrera-Viedma et al. (2002)	0.05	0.73
	0.06	0.63
	0.1	0.45
	0.15	0.15
	0.2	-0.29
	0.3	-0.6
Chiclana et al. (2008)	0.05	0.67
	0.06	0.61
	0.1	0.64
	0.15	0.63
	0.2	0.62
	0.3	0.64
Kacprzyk and Zadrożny (2010)	0.05	0.68
	0.06	0.62
	0.1	0.65
	0.15	0.64
	0.2	0.64
	0.3	0.66
Rodríguez et al. (2018)	0.05	0.53
	0.06	0.35
	0.1	0.04
	0.15	-0.32
	0.2	-0.59
	0.3	-0.77
Wu and Xu (2012)	0.05	0.7
	0.06	0.59
	0.1	0.4
	0.15	0.07
	0.2	-0.35
	0.3	-0.63
Zhang et al. Zhang et al. (2012)	0.05	0.77
	0.06	0.72
	0.1	0.74
	0.15	0.74
	0.2	0.73
	0.3	0.75

[1] http://sinbad2.ujaen.es/afryca/.

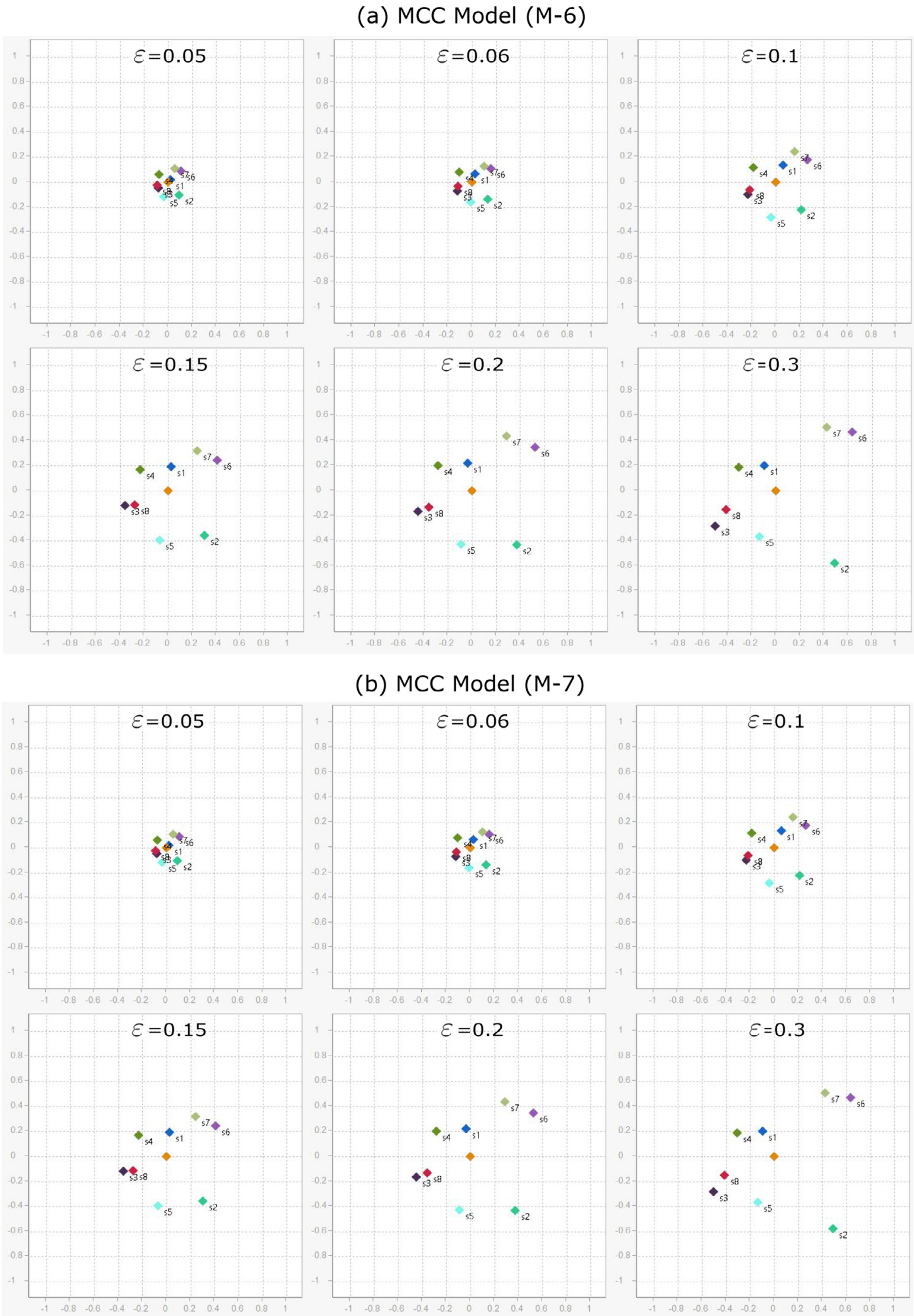

Fig. 3. Visualization of the models (M-6) and (M-7) with different ε.

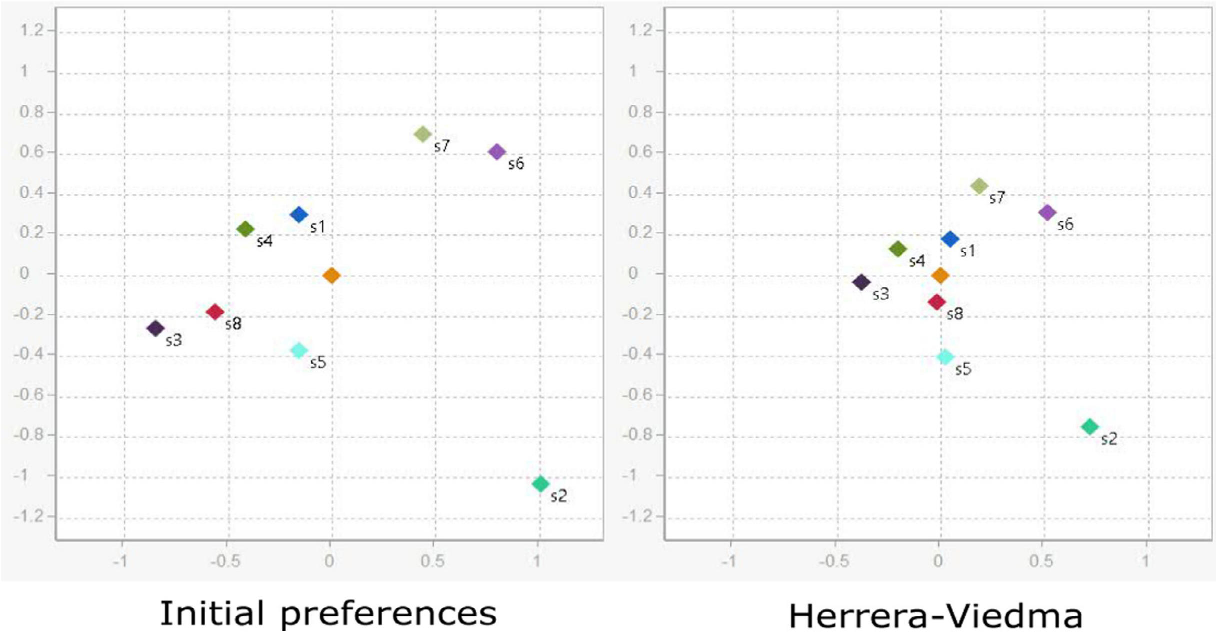

Fig. 4. The final round of Herrera-Viedma et al.'s model.

distance between experts and the collective opinion have been compared with the MCC model (M–6) and those whose consensus measure is based on the distance among experts have been compared with the MCC model (M–7).

5.3.1. Analysis for each representative model

Here a separate analysis for each consensus model according to the ACCM is given. Such an analysis consists of a brief explanation of the results obtained with their graphical visualization together with an analysis of its performance. Furthermore, the graphical visualization of the solutions provided by the MCC models with different values of ε and $\alpha = 0.85$ are also presented in Fig. 3.

- Herrera-Viedma et al.'s model (Herrera-Viedma et al., 2002)
Fig. 4 shows that several experts, for instance, s_2, are far from the rest of the experts, by obtaining a solution in which experts are slightly dispersed. This can also be seen from Table 10. The ACCM shows that the consensus model solution is far from the minimum cost solution with the more restrictive values of ε, such as 0.05, 0.06 or 0.1, as can be appreciated in Figs. 3(a) and 4. The closest solution respect to the minimum cost is provided when $\varepsilon = 0.15$. From $\varepsilon = 0.2$, the bigger ε, the higher the excessive cost. To conclude the analysis, note that it is evident that the consensus model reaches the agreement among experts very fast, but, at the same time, the ACCM shows that the experts do not reach enough agreement degree among them. This situation happens due to experts' consensus degree on each alternative is based on an average operator. Thus, those experts with a high level of agreement compensate those who are further from others.
- Chiclana et al.'s model (Chiclana et al., 2008)
As in the previous model, Fig. 5 shows that the opinions of some experts, for instance, s_2, are far from the other experts. Once again, the ACCM corroborates this situation (Table 10). The experts do not reach enough agreement between them for any value of ε and the opinions of some experts are relatively far from each other, being this situation reflected in Fig. 3(b). Again, those experts with a high level of agreement compensate those who are further from others.
- Kacprzyk et al.'s model (Kacprzyk & Zadrożny, 2010)
Table 10 shows similar results to Chiclana et al.'s consensus model. Thus, the analysis of this consensus model can be inferred from Chiclana et al.'s model in a similar way.
- Rodríguez et al.'s model (Rodríguez et al., 2018)
Rodríguez et al.'s model presents a solution similar to the one provided by the minimum cost model in several values of ε according to the ACCM (see Table 10). Specifically, the model obtains a solution relatively nearby to the minimum cost model for the lowest values of ε, 0.05 and 0.06. In the case of, $\varepsilon = 0.1$, the solution is practically identical to the one provided by the minimum cost, being the closest one to the minimum cost solution among all the models for any value of ε. The similarity among both solutions can be easily appreciated by comparing Figs. 3 and 7. For the rest of ε values the solution present an excessive cost, whereas for $\varepsilon = 0.15$ the excessive cost is not too high, for $\varepsilon = 0.2$ and 0.3 the excessive cost is more accentuated and the solution is far from the minimum cost solution. Therefore, Rodríguez et al.'s model provides a solution in which experts' opinions are closer to each other and there is no any expert far from the opinions of the rest of the group. Furthermore, the experts' opinions are not drastically modified, particularly in certain ε values. However, in spite of obtaining a good solution from the minimum cost's point of view, the model needs 11 rounds to reach the consensus. This shows that the number of rounds is not a good criterion to evaluate the CRPs' performance.
- Wu et al.'s model (Wu & Xu, 2012)
The solution provided by Wu et al.'s model is closer to the minimum cost solutions according to Figs. 3(a) and 8, with several values of ε and it is also certified by the ACCM. By analysing the results with the ACCM (Table 10), we can appreciate that the solution provided by Wu et al.'s model is relatively close to the minimum cost solution when ε is equal to 0.1 and 0.2, the latter with an excessive cost. When $\varepsilon = 0.15$, the solution obtained is quite similar to the one provided by the MCC model. For the rest of the values of ε the solution is far from the minimum cost solution by presenting a high cost. To conclude the analysis, we can ensure that Wu et al.'s model provides a solution for specific values of ε in which the opinions of the experts are close to each other and their preferences have not

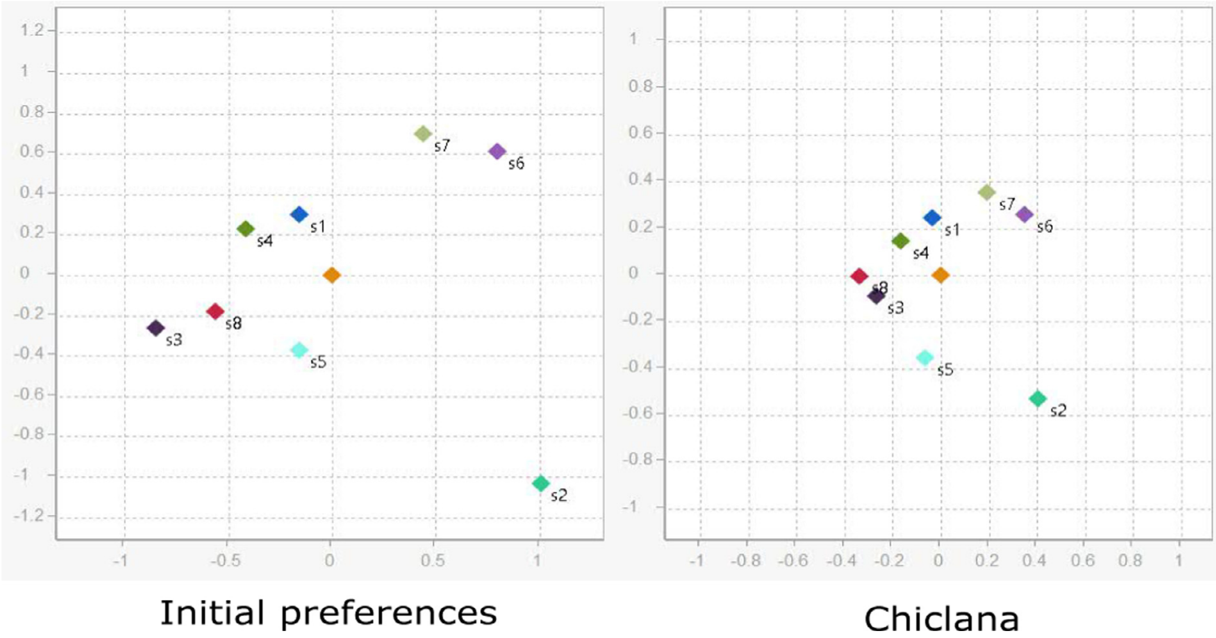

Fig. 5. The final round of Chiclana et al.'s model.

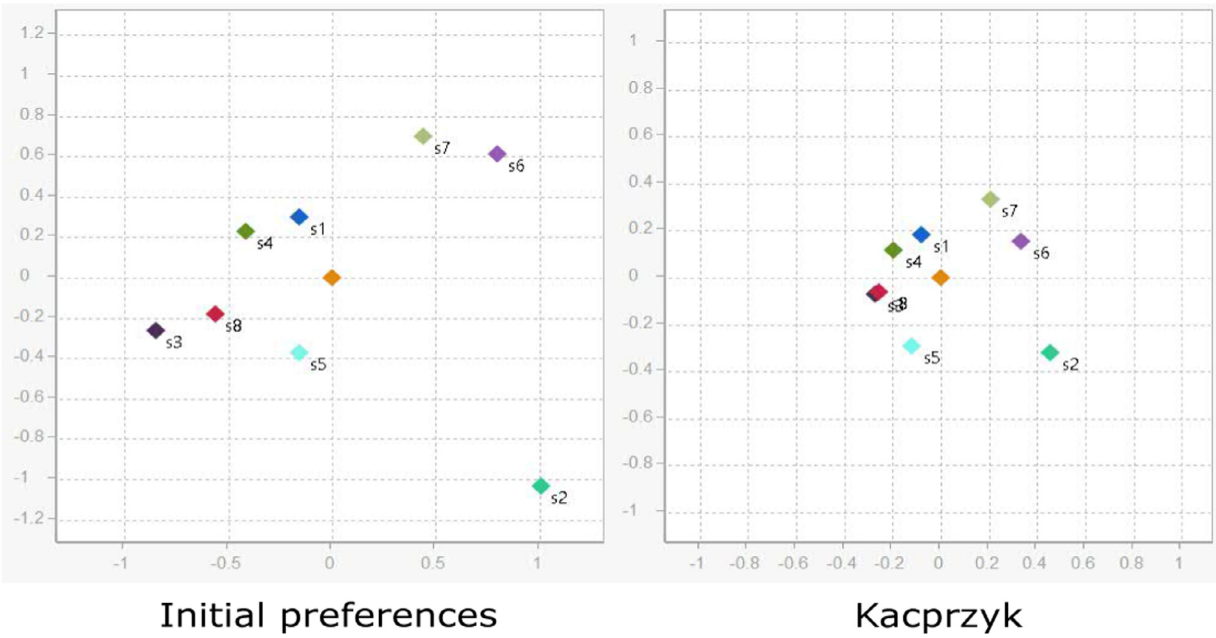

Fig. 6. The final round of Kacprzyk et al.'s model.

been changed excessively to reach consensus. Nevertheless, it should be noted that the consensus model needs 16 rounds to reach the consensus threshold level, due to the fact that just one expert's preference is changed in each round.

- Zhang et al.'s model (Zhang et al., 2012)
  The results of the ACCM show that Zhang et al.'s model provides a solution far from the minimum cost solution with any value of ε. The ACCM shows a considerable distance between the experts' preferences provided by the representative consensus model and the minimum cost model, as it is shown in Figs. 3(b) and 9.

5.3.2. Global analysis

This section introduces a global analysis of the different representative consensus models according to the metric results.

A) Representative consensus models with feedback mechanism: Firstly, we will only take into consideration the consensus models with a feedback process, Herrera-Viedma et al.'s model, Chiclana et al.'s model, Kacprzyk et al.'s model and Rodríguez et al.'s model to do a global analysis. A classical analysis would consider the number of rounds necessary to reach consensus. Regarding this issue, it is evident that Herrera-Viedma's model provides the best performance. However, by analysing the results of the metric for the metric's model (M–6), we can see that Herrera-Viedma's model provides a solution in which experts are far from each other if we consider restrictive values of ε, which means that experts do not reach enough consensus between them. To compensating this situation, it changes some experts' preferences significantly. On the other hand, Herrera-Viedma

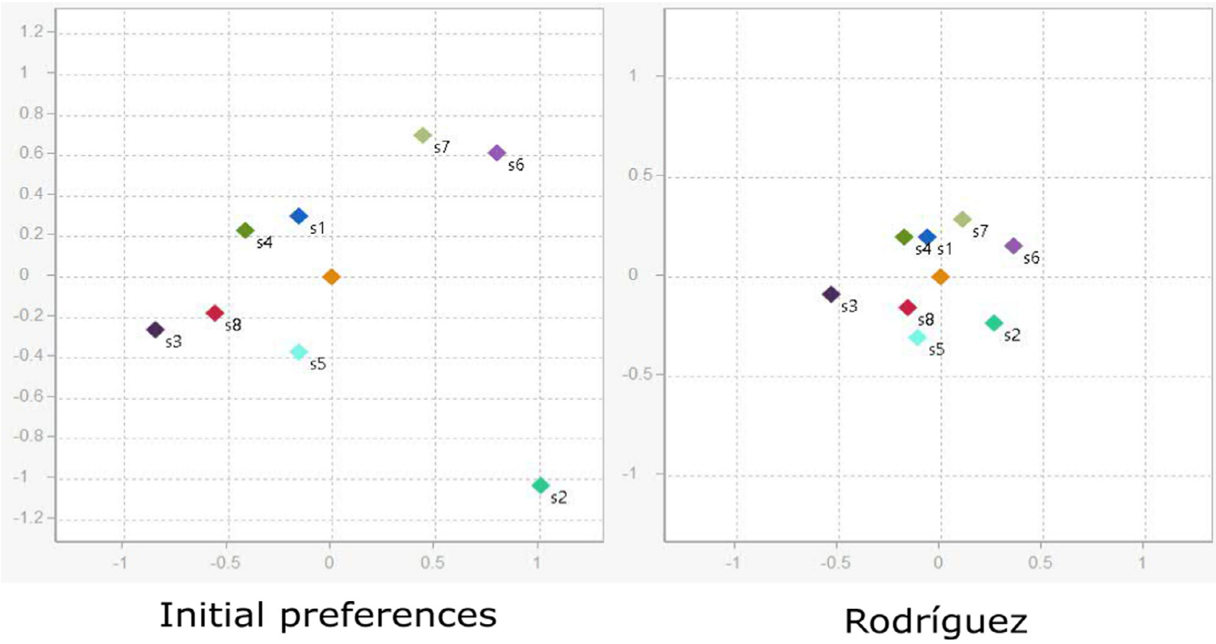

Fig. 7. The final round of Rodríguez et al.'s model.

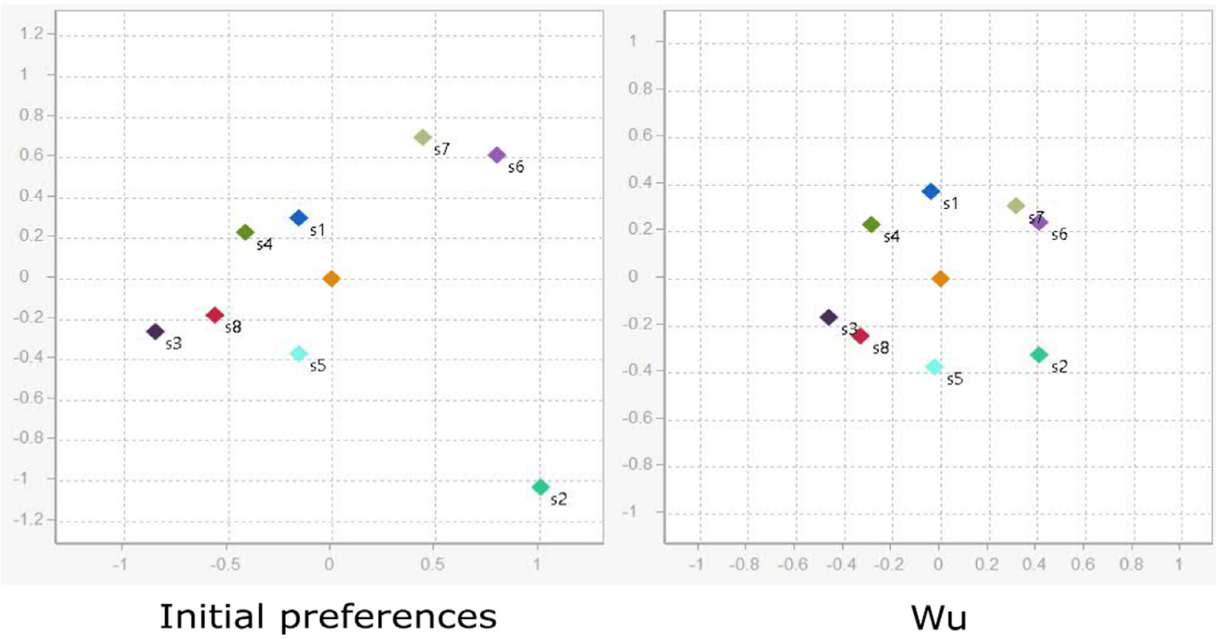

Fig. 8. The final round of Wu et al.'s model.

et al.'s model solution is close to the minimum cost solution for some values of ε, which means that the experts, in this case, are not so far from each other and their preferences have not been modified so much compared with the previous consensus models. Chiclana et al.'s model provides a worse solution than Herrera-Viedma's model for almost any value of ε, thus, once again the experts do not achieve enough consensus among them. Kacprzyk et al.'s model provides similar results to Chiclana's model but it needs less rounds to reach the consensus threshold. However, unlike the rest of the models, Rodríguez et al.'s model does provide a solution in which experts' preferences are closer to each other, even for restrictive values of ε, which means that the solution obtained is the closest to the genuine agreement. To conclude, all the analyzed consensus models except Rodríguez et al.'s model present the same drawback, they provide a compensated solution in which experts with a high level of agreement 'hide' those experts who do not reach enough consensus and thus, it is not obtained a genuine agreement. The facts presented in this analysis prove that evaluating the number of rounds necessary to reach consensus is not enough to carry out a proper analysis of the performance of a consensus model.

B) Representative consensus models without feedback mechanism:
Afterwards, we consider the consensus models without a feedback process, Wu et al.'s model and Zhang et al.'s model. There is no doubt that Wu et al.'s model provides a much better solution than Zhang's model since the former presents a solution closer to the minimum cost solution for

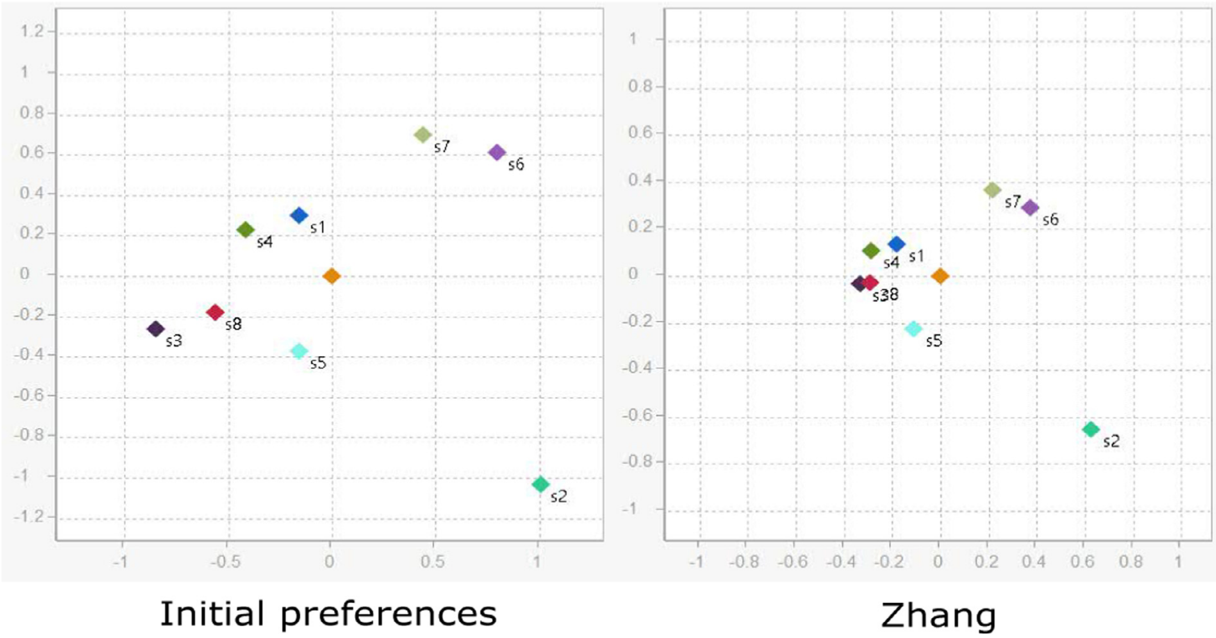

Fig. 9. The final round of Zhang et al.'s model.

several values of ε, which means that experts are close to each other and the modifications of their preferences are not excessive. Note that the number of rounds necessary to reach consensus is greater in Wu et al.'s model than Zhang's model since Zhang's model uses a linear programming model.

C) Both:
Finally, by comparing globally the models regarding the new metric presented, in spite of Zhang et al.'s consensus model is the fastest to reach the consensus, it provides the worst solution for any value of ε regarding the other consensus models. The reason for its less number of rounds is that this model does not use a feedback process and thus, does not consider the participation of the experts to change their opinions but they are modified directly by using a linear programming model. The linear programming model is executed only once, by obtaining the modified experts' preferences in just one round, which does not imply that better solutions are reached, as can be seen in our analysis. Herrera-Viedma's model also reaches consensus in a few number of rounds and, in addition, it presents a feedback mechanism in which experts' opinions are taken into account for changing their preferences. Nevertheless, it has been demonstrated that such a model presents a solution in which experts with a high level of agreement compensate those who are further from others. Furthermore, such a situation is not exclusive of this model, since each analysed consensus model presents the same situation. The model that provides a more homogeneous solution for several values of ε, in which the experts are close to each other and the compensation is not so evident, is Rodríguez et al.'s model, despite being one of the consensus model that needs more rounds to reach consensus.

6. Conclusions

Nowadays, consensual decisions are increasingly important in decision making problems in which it is important to remove the disagreement among experts to obtain a better solution that is more appreciated by the group, giving rise to the CRPs. Due to this fact, there are many proposals on CRPs, but there is no any suitable criterion to evaluate and compare the performance of CRPs.

A novel cost metric to compare the performance of CRPs has been introduced. The metric is based on two novel MCC models that consider the distance of the experts to the collective opinion and also guarantee a minimum agreement among experts and thus, an acceptable and better level of consensus is obtained. The obtained results show that the new metric can be effectively used to make comparison between CRPs, since it allows to reveal anomalous situations in their performance, such as the compensation situations, that cannot be detected by using other criteria. This metric has been implemented and integrated in a decision support system.

In the future research, we will study how to design more efficient MCC models with large-scale GDM problems, and their application in real world problems such as, business management, political negotiation, etc. With the boom of research on CRP in social network, the comparison analysis of these CRP models is an emerging field which deserves to investigate. The proposed cost metric might be one useful criterion for such comparison.

Declarations of Competing interest

None.

Acknowledgments

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Appendix A. MCC models considering non-normalized values

Section 3.1 presents MCC models that take into account level of agreement and distance to collective opinion in which the experts' preferences are provided by numerical values. Nevertheless, these models consider exclusively values valued in [0, 1]. To apply these

MCC models to problems in which the original values are not valued in [0, 1], a normalization process is defined as follows:

$$o_i' = \frac{o_i - \min\limits_{1\le i\le m}\{o_i\}}{\max\limits_{1\le i\le m}\{o_i\} - \min\limits_{1\le i\le m}\{o_i\}}, i = 1, 2, \ldots, m \tag{A.1}$$

Thus, the model (M–4) can be transformed into the following one:

$$(\mathbf{M}-\mathbf{4})(\mathbf{b})$$

$$\min \sum_{i=1}^{m} c_i|\bar{o}_i' - o_i'|$$
$$s.t. \begin{cases} \bar{o} = \sum\limits_{i=1}^{m} w_i \bar{o}_i' \\ |\bar{o}_i' - \bar{o}| \le \varepsilon, i = 1, 2, \ldots, m \\ \sum\limits_{i=1}^{m} w_i|\bar{o}_i' - \bar{o}| \le \gamma, \end{cases}$$

In the same way, the model (M–5) can be transformed into the following one:

$$(\mathbf{M}-\mathbf{5})(\mathbf{b})$$

$$\min \sum_{i=1}^{m} c_i|\bar{o}_i' - o_i'|$$
$$s.t. \begin{cases} \bar{o} = \sum\limits_{i=1}^{m} w_i \bar{o}_i' \\ |\bar{o}_i' - \bar{o}| \le \varepsilon, i = 1, 2, \ldots, m \\ \sum\limits_{i=1}^{m-1} \sum\limits_{j=i+1}^{m} \frac{w_i + w_j}{m-1} |\bar{o}_i' - \bar{o}_j'| \le \gamma, \end{cases}$$

Although the original values have been normalized into [0, 1] and consequently the adjusted values $\bar{o}_i' \in [0, 1]$, the latter can be transformed into the range of the original values by using

$$\bar{o}_i = \left(\max_{1\le i\le m}\{o_i\} - \min_{1\le i\le m}\{o_i\}\right)\bar{o}_i' + \min_{1\le i\le m}\{o_i\}, i = 1, 2, \ldots, m. \tag{A.2}$$

Finally, the resulting cost obtained from the normalized values can also be transformed according to the values in the original range

$$Cost = \sum_{i=1}^{m} c_i|\bar{o}_i - o_i| = \left(\max_{1\le i\le m}\{o_i\} - \min_{1\le i\le m}\{o_i\}\right)Cost'. \tag{A.3}$$

where $Cost'$ is the cost obtained from the normalized values.

Appendix B. Description of the representative consensus models

A brief description of the selected representative consensus models is introduced below.

- *Herrera-Viedma* et al.*'s model (Herrera-Viedma et al., 2002)*: this model follows the *soft consensus* view Herrera-Viedma et al. (2014) and uses proximity measures provided by a moderator. The measures of consensus and proximity are computed through the comparison between the individual experts' preferences and the collective solution. Furthermore, a comparison for alternatives in each consensus round is carried out, computing the current consensus in each moment during the CRP. Another relevant aspect in this model is, which is able to manage distinct preferences relations, unifying all of them into FPR. The parameters of Herrera-Viedma et al.'s consensus model are briefly introduced here (see Herrera-Viedma et al. (2014) for further detailed descriptions):
  - β: parameter related to the control of the OR-LIKE aggregation operator that computes the global consensus degree.
  - Aggregation quantifiers: parameters related to the linguistic quantifier used to compute the collective preference by means of the OWA operator.
  - Exploitation quantifiers: parameters related to the linguistic quantifier used to compute dominance and non-dominance degrees and conduct preferences of experts into preference orderings.
- *Chiclana* et al.*'s model (Chiclana et al., 2008)*: this model integrates individual consistency for the experts' preferences and the consensus measure is based on the computation between pairwise similarities. Several relevant models such as Dong, Zhang, Hong, and Xu (2010) and Zhang et al. (2012) are based on this model. The parameters of Chiclana et al.'s consensus model are briefly introduced here (see Chiclana et al. (2008) for further detailed descriptions):
  - B: parameter related to the consistency threshold for preferences.
  - θ_1: parameter related to the low consensus threshold. If consensus degree is lower than this value, a low consensus preference search is applied.
  - θ_2: parameter related to the medium consensus threshold. If consensus degree is lower or higher than this value, a medium or high consensus level is applied, respectively.
- *Kacprzyk* et al.*'s model (Kacprzyk & Zadrożny, 2010)*: this model is based on the notion of *soft consensus* under fuzzy preference relations. Similarities between pair of experts are computed at level of assessments, as alpha-degrees of sufficient agreement. Distinct consensus degrees are obtained at different levels from such similarities, based on quantifier-guided OWA aggregation. The parameters of Kacprzyk et al.'s consensus model are briefly introduced here (see Kacprzyk and Zadrożny (2010) for further detailed descriptions):
  - μ: parameter related to the non-strict similarity between experts' preferences.
  - Aggregation quantifiers: parameters related to the linguistic quantifier used to compute the collective preference by means of the OWA operator.
- *Wu* et al.*'s model (Wu & Xu, 2012)*: this model deals with individual consistency and computes the consensus measures based on the distance between the individual experts' preferences and the collective opinions. This model is relatively easy and straightforward and guarantees the acceptable consistency (Wu & Xu, 2012) for each individual preference and the whole group in the CRP. The parameters of Wu et al.'s consensus model are briefly introduced here (see Wu and Xu (2012) for further detailed descriptions):
  - CI: parameter related to the individual consensus threshold.
  - β: parameter related to the update coefficients for the preferences.
  - W_i: parameter related to the experts' weights.
- *Zhang* et al.*'s model (Zhang et al., 2012)*: this model preserves the initial preference information and extends the consistency-driven model proposed by Chiclana et al. to guarantee the minimum cost of modifying preferences. Aside from guiding the CRP, this model allows to achieve a high level of consistency for each individual preference relation. The parameters of Zhang et al.'s consensus model are briefly introduced here (see Zhang et al. (2012) for further detailed descriptions):
  - cl: parameter related to the minimal consistency level that each individual preferences have to reach.
  - ccl: parameter related to the minimal consistency level that the different preferences have to reach.

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4.9. AFRYCA 2.0: Un Sistema de Soporte a la Decisión Mejorado para los Procesos de Consenso

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REGULAR PAPER

AFRYCA 2.0: an improved analysis framework for consensus reaching processes

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Abstract Consensus reaching processes (CRPs) are increasingly important in the resolution of group decision-making (GDM) problems. There are many proposals of CRPs models with different characteristics, being difficult either to choose the most adequate for a given GDM problem or for making comparisons among them. For this reason, AFRYCA was proposed as a framework able to carry out comparison analyses and studies of CRPs in GDM problem resolution. This paper presents AFRYCA 2.0 which overcomes some limitations identified in the previous version. This new version incorporates new features for the analysis of CRPs, and increases its functionality, resulting a more powerful framework. Additionally, to show the usefulness and effectiveness of the new functionality of AFRYCA 2.0, an experimental study is carried out.

Keywords AFRYCA · Group decision-making · Consensus model · Consensus reaching process

1 Introduction

Decision-making is a common process in daily life, defined by several options or alternatives and whose goal is to decide which one/s are the best. In group decision-making (GDM) problems, several experts with different points of view participate in a decision problem and they are responsible for achieving a common solution [1,2]. Classically, GDM problems were worked out applying an alternative selection process [3] that consists of two phases: aggregation and exploitation (see Fig. 1) to choose the best alternative/s [4]. This solving scheme can reach solutions in which appears experts conflicts or disagreements. As a result, consensus reaching processes (CRPs) are used as an additional phase in the resolution of GDM problems [5] to avoid such problems. In a CRP, experts discuss, revise and change their preferences, bringing their opinions closer to each other, increasing the level of agreement in the group before making a decision.

CRPs have become a major research topic within the field of GDM and a large number of proposals and approaches have been proposed for this kind of processes [6,7]. Hence, sometimes it is difficult to decide which model is the most suitable or what the best model configuration is for a specific GDM problem. In [7] was proposed a taxonomy that provided an overview and categorization of several existing consensus models for GDM problems defined in a fuzzy context. This taxonomy allowed to group different models that shared characteristics and hence they could be compared to each other. Accompanying that taxonomy, it was presented AFRYCA 1.X, a simulated-based analysis framework for the resolution of GDM problems by means of different consensus models. AFRYCA 1.X aimed to: (1) discover advantages and disadvantages for each model; (2) determine which model is more suitable for a specific GDM problem and (3) make comparisons between different models. To carry out these tasks, AFRYCA 1.X was able to simulate different expert behavior patterns, therefore their opinions can be changed throughout the process. Since it is possible to simulate different experts behaviors and analyses several consensus models. Then, it permits to evaluate if a model in a specific problem is more

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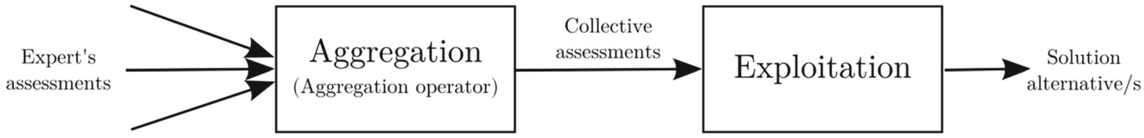

Fig. 1 Selection process for the resolution of GDM problems

appropriate than other in the same conditions or determine what the weaknesses of each one are.

After applying AFRYCA 1.X in several GDM situations it was detected that the framework presented some limitations. The main detected limitations were: (1) outdated technology and dependencies with other software tools; (2) complex structure that causes high complexity to add new models and difficulty introducing parameters of the different consensus models; (3) impossibility to modify several parameters values by the user; (4) inability to check the CRP evolution and (5) inability to analyze consensus models across process because of the rounds results were not saved. Due to these limitations it has been necessary to create a new version for working out them and extend the functionality of AFRYCA 1.X. The new AFRYCA 2.0 features which are presented in this paper are: (1) migrating to the 4.X new branch Eclipse RCP, e4; (2) new simpler structure; (3) new behavior configurations; (4) model parametrization by restrictions and relations between parameters; (5) built-in support for the behavior development; (6) built-in support for the multi-dimensional scaling (MDS); (7) new consensus models incorporated; (8) a new behavior simulation for experts; (9) CRP step-by-step visualization and (10) metrics support, including several experimental predefined metrics.

To show how the new functionalities incorporated in AFRYCA 2.0 can be really helpful for the analysis of CRPs, an experimental study is also conducted, using different consensus models and different behavior patterns. Furthermore, the CRPs evolution will be shown by means of a visual tool, together their performance according to several metrics.

This paper is organized as follows: in Sect. 2 concepts about GDM and CRP included in AFRYCA 2.0 are briefly revised. Section 3 introduces the new characteristics of AFRYCA 2.0 by making a comparison with the AFRYCA 1.X version. Section 4 shows an experimental study that illustrates the new functionality of AFRYCA 2.0. Finally, some conclusions and future works are drawn in Sect. 5.

2 Background

This section makes a brief review on GDM and CRP concepts. Furthermore, the taxonomy of consensus models presented in [7] is also revised.

2.1 Group decision-making problems

A GDM problem can be formally defined as a decision situation where [2]:

- A group $E = \{e_1, \ldots, e_m\}$ of m individuals or experts.
- A set $X = \{x_1, \ldots, x_n\}$ of n alternatives or possible solutions.
- The experts try to achieve a common solution.

In GDM, each expert $e_i \in E$ expresses his/her opinions over different alternatives by means of a preference structure. One of the most common preference structures in GDM is the so-called fuzzy preference relation [8]. A fuzzy preference relation associated to expert e_i it is noted as $P_i = (p_i^{lk})_{n \times m}$ and can be represented, for X finite, as an $n \times n$ matrix as follows:

$$P_i = \begin{pmatrix} - & \ldots & p_i^{ln} \\ \vdots & \ddots & \vdots \\ p_i^{n1} & \ldots & - \end{pmatrix}$$

where each assessment, $p_i^{lk} = \mu_{P_i}(x_l, x_k) \in [0, 1]$ represents the preference degree of e_i over x_l regarding $x_k, l, k \in \{1, \ldots, n\}, l \neq k$, such that:

- $p_i^{lk} > 0.5$ the preference of the expert e_i over x_l is greater than x_k.
- $p_i^{lk} < 0.5$ the preference of the expert e_i over x_k is greater than x_l.
- $p_i^{lk} = 0.5$ there is no preference of the expert e_i between x_k y x_l.

The elements that complete the diagonal of the matrix, p_i^{ll}, there are no computing and they are noted as "–".

The solution for a GDM problem can be obtained by a selection process, applying either a direct approach or an indirect approach [3]. In a *direct approach*, the solution is directly obtained from the individual preferences of the experts, without a global opinion first. On the other hand, in an *indirect approach*, a global opinion or *collective preference*, P_c, is determined a priori from individual opinions.

Regardless of the approach considered, the classical alternative selection process for achieving a solution to

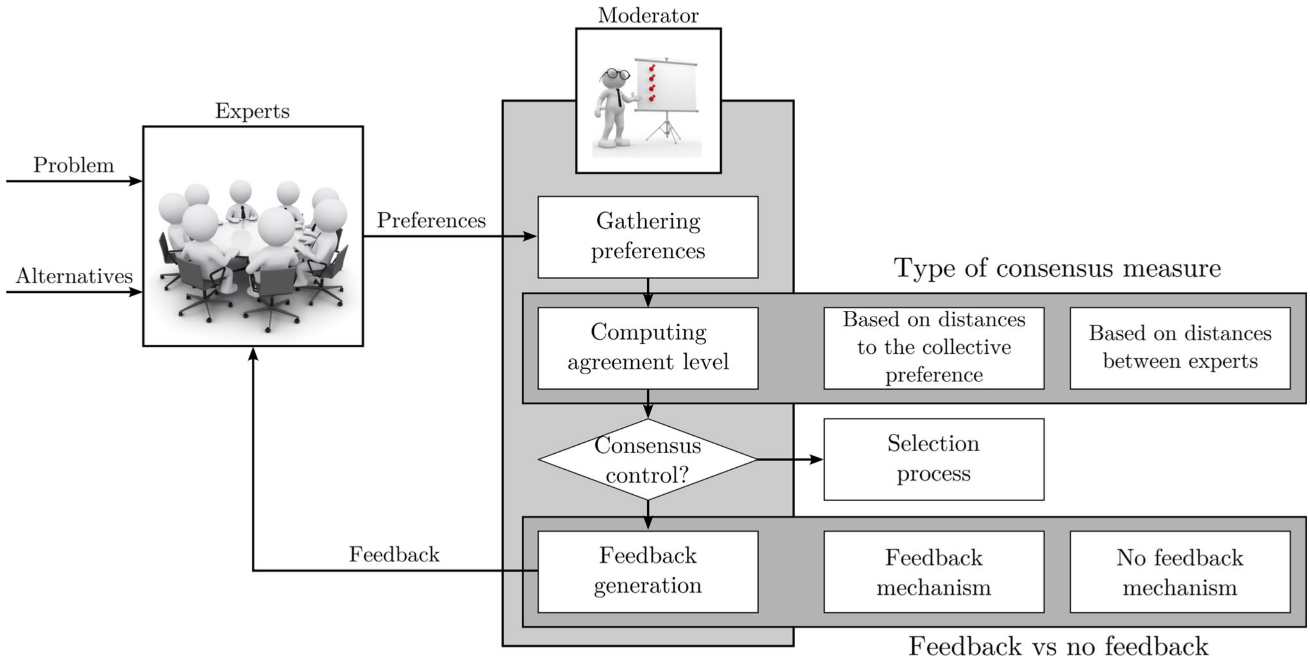

Fig. 2 General CRP scheme

GDM problem is composed of two phases [4] (see Fig. 1):

1. *Aggregation* the preferences of experts are combined by using an aggregation operator.
2. *Exploitation* an alternative or subset of them are obtained as the solution to the problem, by means of a selection criterion.

2.2 Consensus reaching process

The selection process does not take into account that the best solution can cause conflicts in the group because of several experts do not agree with the solution. Hence, problems may appear occasionally, for instance, that several experts think either that their opinions have not been taken into account, or do not take responsibility for the decision [5,9].

To resolve the previous inconveniences, CRPs were introduced as an additional phase in the GDM problem resolution.

What is consensus? The concept has been interpreted from different points of view, from total agreement (unanimity), which is hardly reachable in practice [10], to other definitions more flexible. In [5], Saint and Lawson defined consensus as “a state of mutual agreement among members of a group, where all legitimate concerns of individuals have been addressed to the satisfaction of the group”. Kacprzyk et al. introduced in [1] the notion of *soft consensus* based on the concept of fuzzy majority, which states that consensus exists when “most of the important individuals agree as to (their testimonies concerning) almost all of relevant opinions”.

The CRPs are iterative and dynamic, in which experts modify their initial preferences, in order to make their preferences closer to each other and achieve a high agreement level after several rounds of discussion [5]. Such process is often coordinated by a human figure known as *moderator*, who is responsible for supervising and guiding the discussion between experts. In Fig. 2 is shown a general scheme of CRPs as it is defined in [7]. The main phases in the process are:

1. *Gathering preferences* Each expert gives his opinion over the alternatives.
2. *Computing agreement level* The consensus degree of the group is worked out. It is possible to use different consensus measures [7], through aggregation operators and calculating distances between preferences.
3. *Consensus control* The consensus degree is compared with a threshold level of agreement, defined a priori, if the level of consensus desired has been achieved, the group moves onto the selection process, otherwise, it is necessary to carry out another round of discussion. The number of rounds allowed will be limited too.
4. *Feedback generation* A procedure is applied in order to increase the level of agreement in the following round of the CRP. Traditionally such a procedure has consisted of applying a *feedback generation process*, in which the moderator identified the assessments of experts which are farthest from consensus and advises them to modify such assessments [5,11]. Many existing consensus models incorporate feedback mechanisms [12–15]; how-

ever some other proposed models do not incorporate such mechanisms and instead they implement approaches that update information automatically [16–18].

2.3 A taxonomy of consensus approaches

A large number of consensus models have been proposed during recent decades [6,19–25].

As it is seen there are many consensus models, each one with different characteristic and performance. For this reason, a taxonomy that provided an overview and categorization of some existing consensus models for GDM was proposed in [7]. The classification (see Fig. 3) considered two different criteria, if the consensus model uses *feedback mechanism* or not and what kind of consensus measure is used. From these two criteria, the taxonomy was based on two axes, so that they are combined into four different quadrants:

- Q_1 Consensus models with feedback mechanism and a consensus measure based on computing distances to the collective preference.
- Q_2 Consensus models with feedback mechanism and a consensus measure based on computing distances to the collective preference.
- Q_3 Consensus models without a feedback mechanism and with a consensus measure based on computing distances to the collective preference.
- Q_4 Consensus models without a feedback mechanism and with a consensus measure based on computing pairwise similarities.

By employing this taxonomy, it is possible to categorize a consensus model based on its characteristics. The classification of models allows to estimate their behavior and facilitates conduct comparative studies between them.

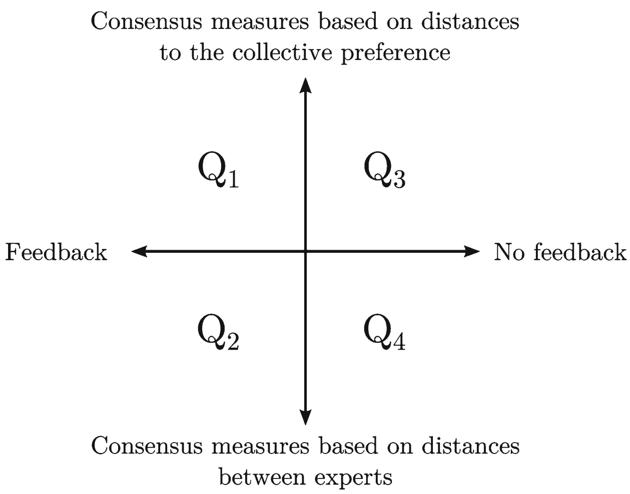

Fig. 3 A taxonomy of approaches for consensus reaching

3 AFRYCA 2.0: new functionalities

This paper aims at introducing a new version of AFRYCA, called AFRYCA 2.0, that extends previous version. For the sake of clarity, the description of AFRYCA 2.0 will be based on a comparison with previous version. Hence, first, AFRYCA 1.X architecture and functionality will be outlined. After that, AFRYCA 2.0 will be presented from the perspective of the new functionalities and improvements made in this version compared to the previous one.

3.1 AFRYCA 1.X

AFRYCA is an acronym for A FRamework for the analYsis of Consensus Approaches. It is a component-based application which was developed by using Eclipse Rich Client Platform (Eclipse RCP) [26], a platform to build and deploy desktop rich client applications easy to maintain and extend.

In its initial version, it was composed of five types of components (see Fig. 4):

1. *Graphical user interface* (*GUI*) Components that allow to interact with the framework.
2. *Preference generator* Components that generate experts' preferences based on a specific preference structure and information domain. This version is able to generate consistent reciprocal fuzzy preference relations [27].
3. *Preference visualization* Components that provides a graphical 2-D representation with the positions of experts' preferences and the group. For this, an MDS data visualization technique is used to visualize data [28].
4. *Behavior simulation* Components that simulate experts' behavior regarding the advice received. AFRYCA 1.X included only one behavior, the *standard* behavior that consist of following feedback suggestion.

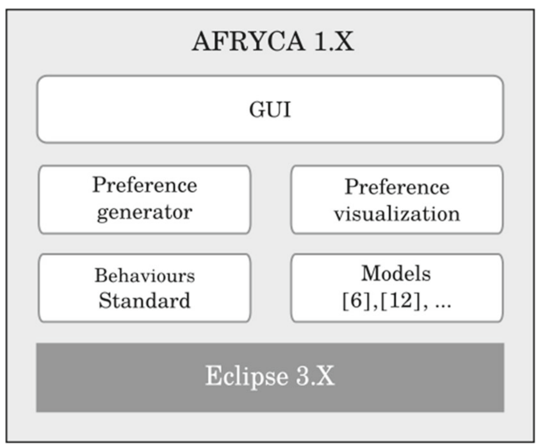

Fig. 4 AFRYCA 1.X architecture

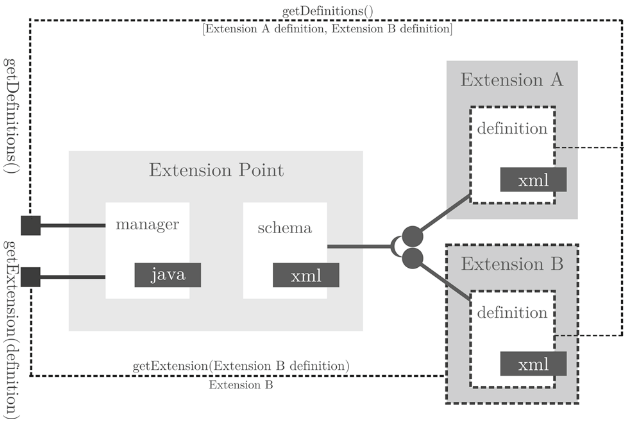

Fig. 5 Extension point scheme

5. *Consensus models* Components that implement consensus models proposed in the literature. Each component corresponds to a consensus model and it includes the different phases and parameters considered in such a model. AFRYCA 1.X came with six consensus model components [14,29–33].

Accompanying these components, AFRYCA 1.X defines several extension points, basic elements in a component-based RCP application. These elements aim to avoid tight coupling between components. When it is desirable for the functionality of a given component to be extended or customized, this component will declare an extension point. An extension point defines a contract to which extensions must conform. Components that extend or customize the functionality provided by the component must implement that contract in their extension. The general scheme of an extension point can be seen in Fig. 5.

Even though the framework fulfilled the objectives proposed before to its development, its use revealed several limitations mentioned in the introduction. These limitations consist of:

1. *Technology and dependencies* AFRYCA 1.X has been developed using 3.X branch Eclipse RCP hence, it does not take advantage of the new existing technologies. Furthermore, AFRYCA 1.X uses an external software to carry out a variety of statistical techniques and this causes difficulties in managing development aspects.
2. *Complex structure* Several elements of AFRYCA 1.X present a convoluted structure by definition. Elements such as extension points are defined in a complex way and it makes difficult, for instance, to add new consensus models to the framework. Set parameters' values in the consensus models defined in AFRYCA 1.X is a hard and sometimes confusing task too. A consensus model does not define any information about its parameters in its extension thus, the users do not know how many parameters have consensus models, which are the restrictions of these parameters and what the order is in which they must introduce them.
3. *Behavior configuration* AFRYCA 1.X does not allow to modify several parameters' values, for instance, in experts' behavior patterns and in values associated with the probability distributions used in the behavior.
4. *Evolution of CRPs* AFRYCA 1.X only shows the visualization of the final CRP state by means of MDS. Therefore, it is not possible to see the CRP evolution in each discussion round.
5. *Analysis of CRPs* The results obtained for each round in a consensus model are not saved in AFRYCA 1.X. Hence, it is not possible to evaluate the performance of a CRP.

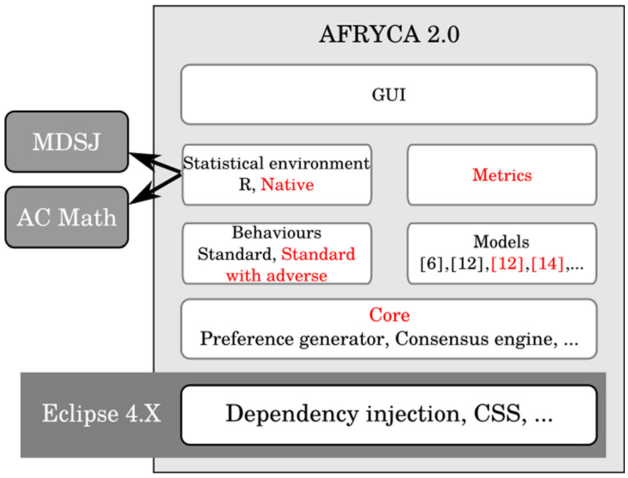

Fig. 6 AFRYCA 2.0 architecture

AFRYCA 2.0 aims at overcoming such limitations.

3.2 AFRYCA 2.0

This subsection introduces the new features incorporated in AFRYCA 2.0, overcoming the limitations previously mentioned. Section 3.2.1 focuses on the new AFRYCA 2.0 technology and its independence of other software tools. Section 3.2.2 presents the new AFRYCA simpler structure that allows to include new models in a simple way. Section 3.2.3 is focus on the flexibility to configure the different experts' behavior patterns. Section 3.2.4 presents the possibility in AFRYCA 2.0 to check the CRPs evolution thanks to the visualization of each discussion by means of MDS and last, Sect. 3.2.5, shows how AFRYCA 2.0 can analyze the performance of CRPs through the use of metrics.

Taking into account these modifications, AFRYCA 2.0 uses more than 40 components, which are grouped in six basics types (see Fig. 6): (1) GUI, (2) statistical environments; (3) metrics; (4) behaviors; (5) models and (6) core.

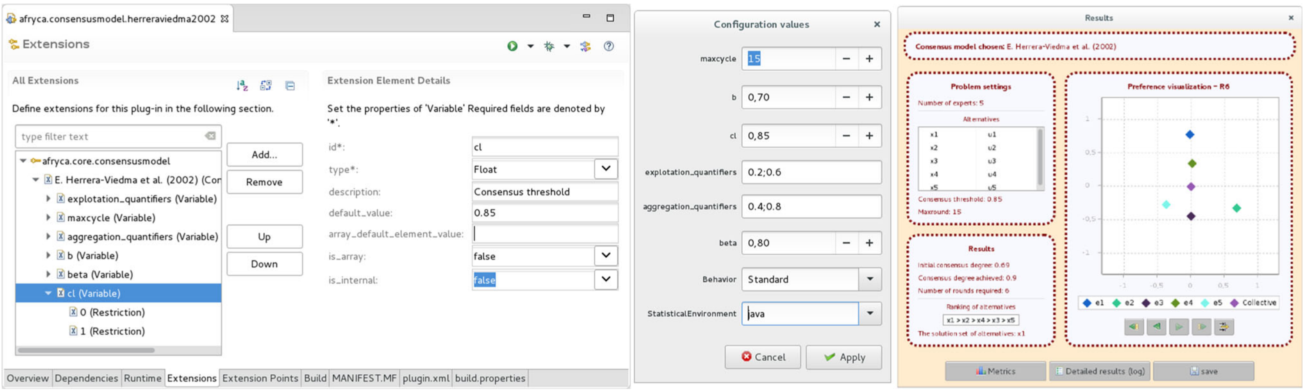

Fig. 7 Parametrization of models

3.2.1 Migration and independence

AFRYCA 1.X was developed using 3.X branch Eclipse RCP. That entails to greater stability, documentation and compatibility with third components, but it is detrimental to the maintenance and extension of the developed applications, which cannot benefit from new technologies and software paradigms.

Aware of it, in AFRYCA 2.0 has been made the migration of the framework to the 4.X new branch Eclipse RCP, e4, which is adapted to new tendencies in the applications develop under components architecture, such as dependency injection, declarative services, application model design or development of graphical interfaces with style sheets.

One of the most important features in AFRYCA is its integration with software R.[1] It facilitates the employment of a wide variety of statistical techniques in an easy way. This greater potency and flexibility that R provides can be an inconvenient because it makes it difficult to manage development aspects such as its migration to other platforms, the development environment configuration or debugging.

For this reason, in AFRYCA 2.0 all functionality that, until now, was executed through R has been implemented natively. For that, it has been developed a statistical environment able to carry out MDS [28] of the preferences and the simulation of pattern behaviors by means of probability distributions, functions that were carried out using R. Thus, the framework is free of the statistical environment which can be selected in runtime.

To develop the native statistical environment in AFRYCA 2.0, two free libraries have been used: (1) *MDSJ* [34], a statistical library which implements the most commonly used MDS algorithms and (2) *Apache Commons Math* [35], a statistical library which allows to use more than 30 different probability distributions.

3.2.2 A simpler AFRYCA structure: adding new models

Since its inception, AFRYCA aims to be a tool focused on the research, focusing its design phase in simplifying the development of new models and tools for analysis. Thus, AFRYCA 2.0 incorporates a new mechanism for development of consensus models that aims to facilitate their definition.

At this moment, to develop in AFRYCA 2.0 a consensus model, it is possible to define easily all variables and parameters which will be used in it, as well as its natures, default values, restrictions and relations between them (see Fig. 7). Because of this, researchers are free of the necessity of checking that all values used to execute their code are valid. This also avoids appearance of errors and reduces the amount of code lines necessary to develop a model.

Several consensus models were included in AFRYCA 1.X [14,29–33]. In AFRYCA 2.0, it has been incremented the amount of consensus models included by default with two new consensus models with feedback mechanism called in the framework as *Palomares2014* [36] and *Quesada2015* [37].

In [36], Palomares et al. have introduced a consensus model that incorporates a novel mechanism based on computing with words and fuzzy set theory to assign weights to experts based on their behavior at each round of the consensus process. Each expert's behavior is evaluated based on the amount of received feedback that they apply in favor of consensus, and assigns them an importance weight accordingly, which is taken into account when computing the group preference.

In [37], Quesada et al. have presented a consensus model that extends the above described, by introducing an approach based on uninorm aggregation operators to manage the behavior of experts in the consensus process. Due to the full reinforcement property of uninorm operators, they allow to weigh experts based not only on their behavior at the current round, but also on their previous behavior since the beginning of the discussion. Furthermore, this approach reinforces positively or negatively the weight of experts with a repeated

[1] https://www.r-project.org/.

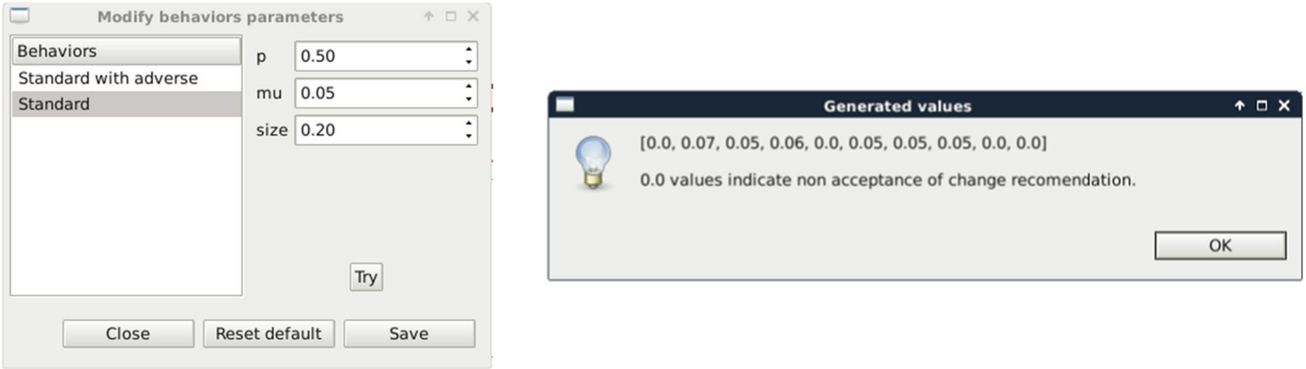

Fig. 8 Wizard for the configuration and simulation of the behaviors

good or bad behavior, respectively, in several consensus rounds.

To illustrate the performance of both models, both will be used in the experimental study conducted in Sect. 4.

3.2.3 Behavior configuration

The functioning of the behaviors is more flexible in AFRYCA 2.0 thanks to the new wizard developed, which permits to establish values associated with the probability distributions used in the behavior as well as to do configuration testing (see Fig. 8).

Furthermore, adding new behaviors patterns is easier thanks to the restructuring of AFRYCA. As a result, a new behavior pattern called *standard with adverse* has been incorporated, which permits to simulate a set of experts reticent to accept recommendations. This behavior complements the existing *standard* behavior, which simulates a set of experts receptive to accept recommendations.

To carry through such behaviors, two different aspects have to be taken into account:

1. *Behavior of experts regarding advice* The amount of advice on assessments that accepts an expert based on the total of advice that have been suggested to him/her. This characteristic is simulated in AFRYCA 2.0 by means of binomial probability distribution, whose parameter values can be fixed by the user (probability of success in binomial distribution). Specifically, in the standard behavior, the probability of accepting a change recommendation, p, has been modeled using this probability distribution and it is associated with this parameter. Additionally, in the standard with adverse behavior, the probability of adverse behavior when receives an advise, c, has also been modeled in this way.
2. *Degree of change* The degree of change that an expert might apply to an assessment, if he/she has accepted to do a modification. This feature is modeled in AFRYCA 2.0 with a negative binomial probability distribution, so the values generated by the probability distribution, represent the degree of change applied to the assessment. As in the previous point, the framework allows to configure the parameters of the probability distribution to modify the behavior of the experts. In particular, it is possible to fix the average of the distribution, μ, as well as its size, *size*, values that represent the average degree of change on an assessment and the range of possible degrees of change around the average, respectively.

Table 1 Behaviors default values

	Standard	Standard with adverse
p	0.5	0.5
c	–	0.25
μ	0.05	0.05
Size	0.2	0.2

AFRYCA 2.0 behavior default values are shown in Table 1.

3.2.4 Evolution of CRPs

Another significant change is established in the preferences visualization in the CRP. In AFRYCA 1.X, it was just possible to visualize the state of the preferences at the end of the process, in AFRYCA 2.0, that happens round by round.

The experimental study carried out in Sect. 4 will be conducted in several simulations using both behaviors to analyze their differences.

3.2.5 Analysis of CRPs: metrics

So far, despite the existence of multiple consensus models, there is no systematic way to choose the most suitable one for each GDM problem. Hence in order to analyze such consensus models and the CRPs performance, metrics are essential, since they might allow to study different aspects of the CRPs

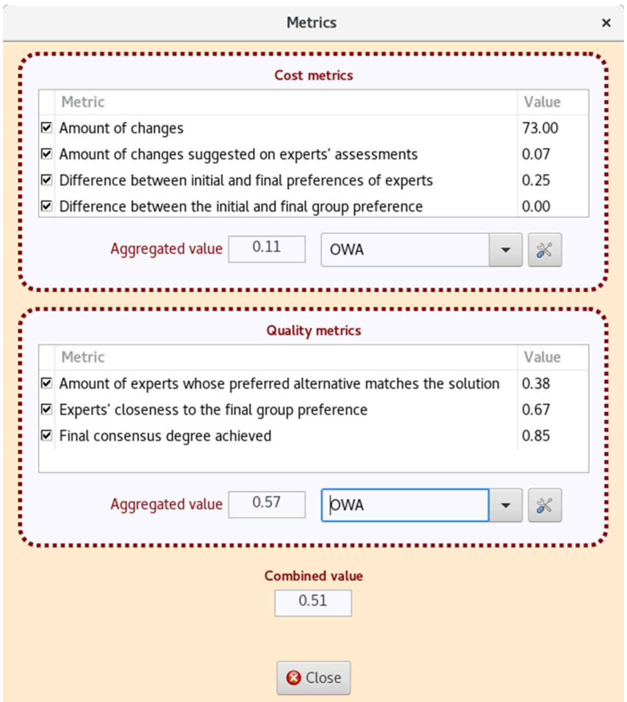

Fig. 9 Metrics dialog

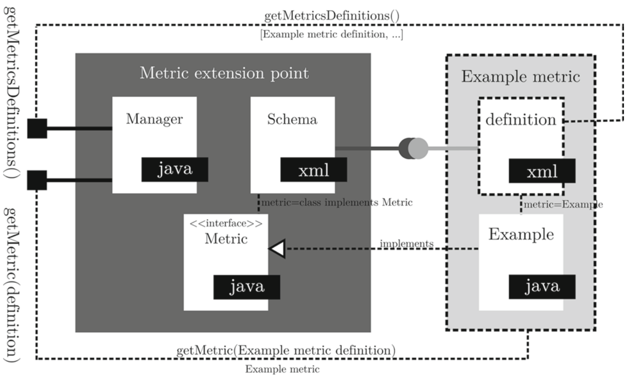

Fig. 10 Metric extension point

and check the CRP performance. Because of this importance, AFRYCA 2.0 has evolved and the framework uses a new model development mechanism and it defines at the same time an extended programming interface that supports the storage of the results obtained in each discussion. Thus, it is possible to implement analyze metrics, that can be embedded in the framework for its use. The metrics environment simplifies the operation with the stored results and offers an option for working with them together. Making use of it, an experimental set of metrics has been defined which is available in the new version (see Fig. 9).

AFRYCA 2.0 has implemented simple metrics such as, *amount of changes, difference between initial and final preferences of experts, expert's closeness to the final group preference*, etc. But it also allows to add metrics in an easy way. Solely by defining one component for each metric and using an extension point already defined in AFRYCA 2.0, it is possible to create all kinds of metrics (see Fig. 10). As a result, AFRYCA 2.0 is not limited in the creation of new metrics and, thus, it will be able to perform more and more complete CRPs analyses.

Section 4.3 illustrates how a metric is developed and used in AFRYCA 2.0.

4 Experimental study

In [7] was carried out an experimental study in which several CRPs were simulated in different GDM problems using AFRYCA, in order to illustrate its purpose. In this section we use that study to show how, the new features added in AFRYCA 2.0, can be truly helpful in the analysis of consensus models.

The experimental study conducted in [7] supposed a company divided into four departments of eight employees each one: *Technical Department*, *Human Resources Department*, *Marketing Department* and *Sales Department*. Each department plans to celebrate a Christmas dinner separately, hence for each group is given a GDM problem. The four possible alternatives for all of them are the following: *Restaurant Thamesis*, *Catalina Castle*, *Restaurant La Zaga* and *Juleca Complex*. In the study, all preferences are expressed as fuzzy preference relations, having been available the data sets for public access in AFRYCA website.[2] To find a satisfactory solution for each department, a CRP will be carried out in each GDM problem with which it seeks to achieve a minimum level of agreement of $\mu = 0.85$, regardless of the number of consensus rounds necessary to perform.

Our study will be focused on the GDM problem defined for the *Technical Department*, for which eight CRPs have been simulated using different consensus models as well as different experts' behaviors:

S_{1-2} The consensus models without a feedback mechanism proposed by (S_1) Wu et al. [32] and (S_2) Xu et al. [33].

S_{3-5} The consensus models with feedback mechanism proposed by (S_3) Herrera-Viedma et al. [14], (S_4) Palomares et al. [36] and (S_5) Quesada et al. [37] using the standard behavior.

S_{6-8} The consensus models used in (S_6) S_3, (S_7) S_4 and (S_8) S_5 using the standard with adverse behavior.

For each simulation performed, behaviors have been configured with the values shown in Table 1. The consensus

[2] http://sinbad2.ujaen.es/afryca.

Table 2 Consensus models parameters

Wu et al. [32]	Xu et al. [33]	Herrera-Viedma et al. [14]	Palomares et al. [36]	Quesada et al. [37]
$\beta = 0.8$	$\alpha = 0.2$	$\beta = 0.8$	$\beta = 0.6$	$\beta = 0.6$
$\overline{CI} = 0.15$	$\gamma = 0.15$	Aggregation quantifier $= F_{\text{most}}$	$\gamma = 0.2$	$\gamma = 0.2$
$w_i = \frac{1}{8}, i = 1, \ldots, 8$	$w_i = \frac{1}{8}, i = 1, \ldots, 8$	Exploitation quantifier $= F_{\text{as many as possible}}$	$\epsilon = 0.05$	$\epsilon = 0.05$
			$h_{\text{start}} = 2$	$h_{\text{start}} = 2$
			Increment $= 0.1$	Increment $= 0.1$
				$\eta = 0.5$
				$G = 0.5$

Table 3 CRP simulations results

	Initial consensus degree	Final consensus degree	Number of rounds	Ranking	Solution set of alternatives
S_1	0.698 (0.302)	0.858 (0.142)	10	$x_1 \succ x_3 \succ x_2 \succ x_4$	x_1
S_2	0.844 (0.156)	0.893 (0.107)	3	$x_1 \succ x_3 \succ x_2 \succ x_4$	x_1
S_3	0.75	0.9	3	$x_1 \succ x_3 \succ x_2 \succ x_4$	x_1
S_4	0.77	0.855	11	$x_1 \succ x_3 \succ x_2 \succ x_4$	x_1
S_5	0.77	0.855	10	$x_1 \succ x_3 \succ x_2 \succ x_4$	x_1
S_6	0.75	0.88	4	$x_1 \succ x_3 \succ x_2 \succ x_4$	x_1
S_7	0.77	0.858	19	$x_1 \succ x_3 \succ x_2 \succ x_4$	x_1
S_8	0.77	0.853	18	$x_1 \succ x_3 \succ x_2 \succ x_4$	x_1

models have been configured with the values shown in Table 2.

4.1 Results

The CRP simulations performed with each consensus model have yielded the values summarized in Table 3. To facilitate the analysis, given the fact that Wu et al. model measures consensus using Individual Consensus Indices $\text{ICI}(P_i) = d(P_i, P_c)$ for each $e_i \in E$ [32], and Xu et al. model using Group Consensus Index (GCI) [33], the consensus degrees shown for S_1 and S_2 are given by $1 - \max_i \text{ICI}(P_i)$ and $1 - \text{GCI}$, respectively, as it was done in [7].

To analyze how evolution of consensus has been during the process for each simulation, we use one of the novelties of AFRYCA 2.0 which was presented in Sect. 3.2, the step-to-step MDS visualization of preferences. Figures 11, 12, 13, 14, 15, 16, 17 and 18 show the step-by-step MDS visualization of all simulations.

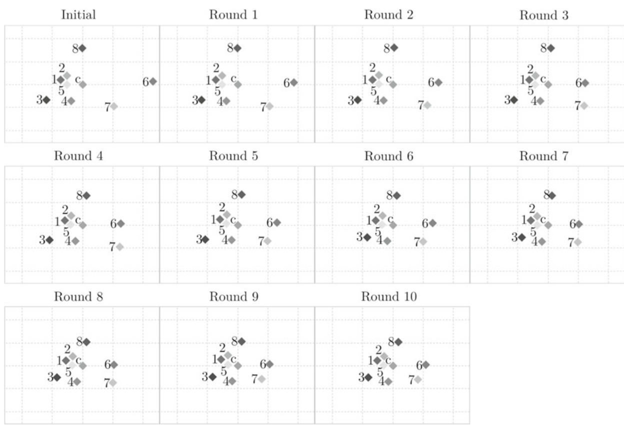

Fig. 11 MDS visualization of CRP using Wu et al. model [32]

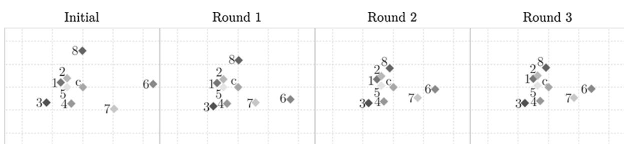

Fig. 12 MDS visualization of CRP using Xu et al. model [33]

4.2 Discussion of the experimental study

From the analysis of the obtained information, several aspects can be highlighted. It should be noted that the results obtained are conditioned by the configuration established for each simulation, so that in our analysis we do not focus on establishing which consensus model is better, but on what seems to be the behavior of the consensus models according to their typology and how, the new AFRYCA 2.0 available behavior pattern, consensus models and functionalities described in Sect. 3.2, allow us to analyze new aspects of CRPs.

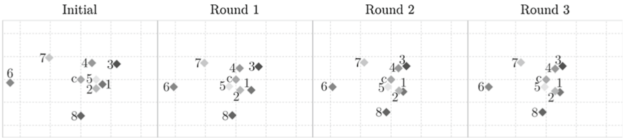

Fig. 13 MDS visualization of CRP using Herrera-Viedma et al. model [14] and standard behavior

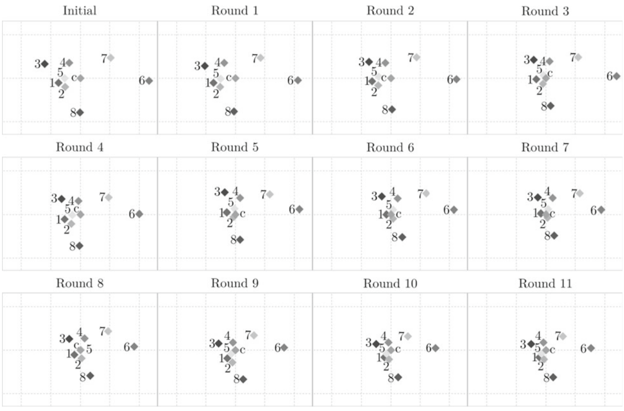

Fig. 14 MDS visualization of CRP using Palomares et al. model [36] and standard behavior

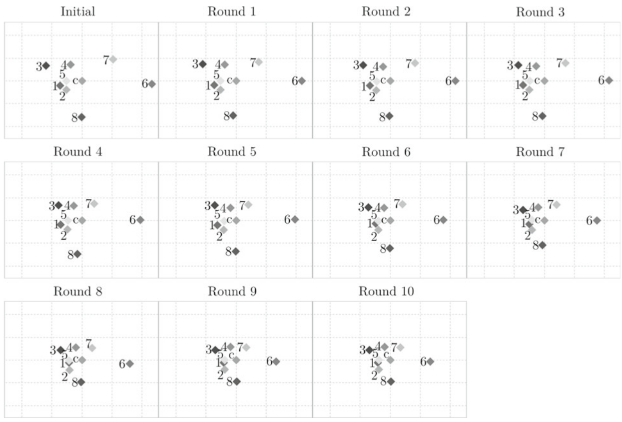

Fig. 15 MDS visualization of CRP using Quesada et al. model [37] and standard behavior

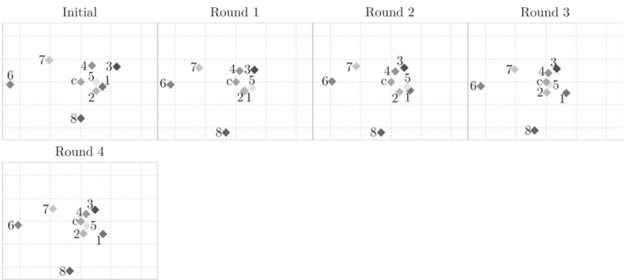

Fig. 16 MDS visualization of CRP using Herrera-Viedma et al. model [14] and standard with adverse behavior

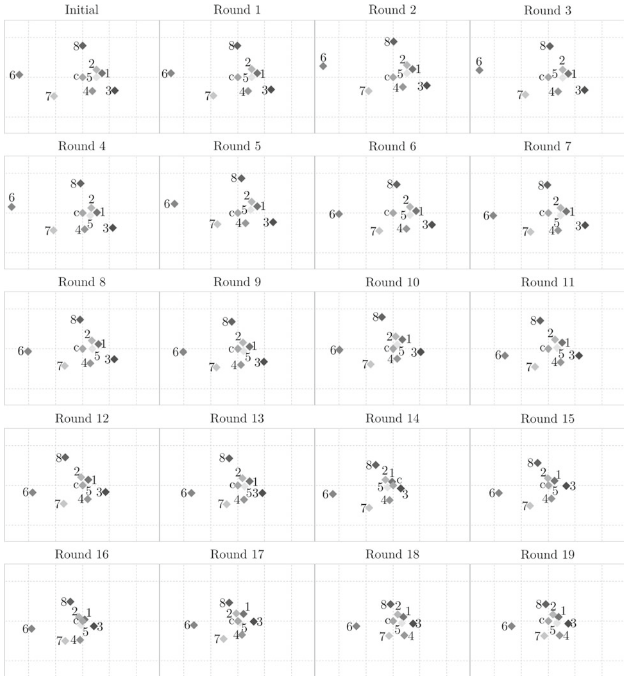

Fig. 17 MDS visualization of CRP using Palomares et al. model [36] and standard with adverse behavior

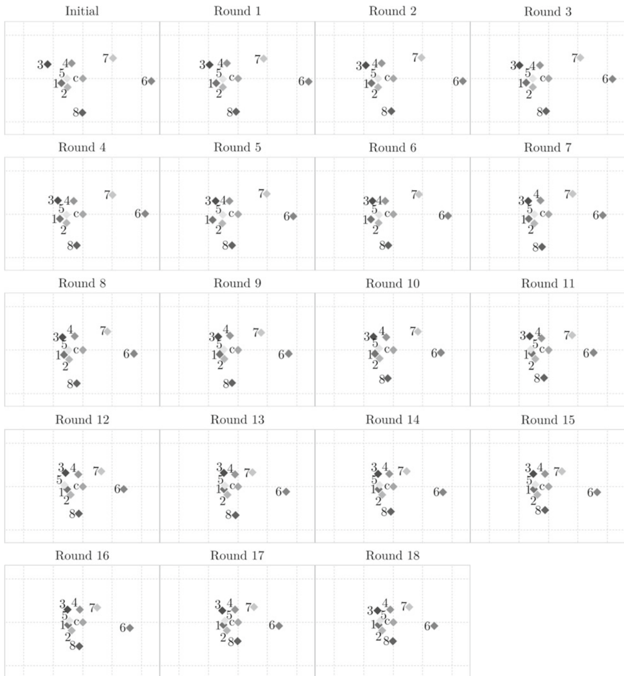

Fig. 18 MDS visualization of CRP using Quesada et al. model [37] and standard with adverse behavior

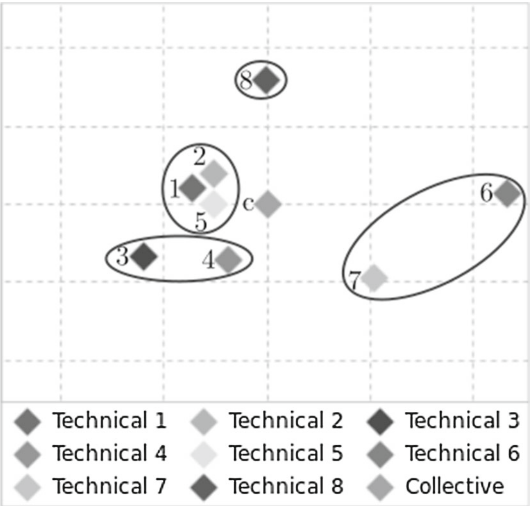

Fig. 19 A possible clustering of the experts

- All the simulations carried out return the same ranking of alternatives, that is, $x_1 \succ x_3 \succ x_2 \succ s_4$, where x_1 is always the best alternative and x_4 is always the worst of them. A priori, this suggests that the initial degree of disagreement among experts is not very large, which is reflected in the initial degree of consensus calculated in each simulation, which are in the range of [0.698 (S_1), 0.844 (S_2)].
- In our simulations, the number of consensus rounds required to reach the established consensus degree threshold appears to be unaffected whether a model with feedback mechanism is used or not. In the simulations S_{1-2}, in which consensus models without feedback are used, 10 (S_1) and 3 (S_2) consensus rounds having been carried out, and in the simulations S_{3-5} in which consensus models with feedback and a standard behavior are used, 3, 11 and 10 consensus rounds are carried out, values similar to the previous ones.
- Adverse behavior of experts seems to increase the number of consensus rounds needed to reach the established consensus threshold. Thus, it can be seen that more consensus rounds have been carried out in S_6 than in S_3 (4 vs. 3, +1 or 33.3% more), in S_7 than in S_4 (19 vs. 11, +8 or 72% more) and in S_8 than in S_5 (18 vs. 10, +8 or 80% more).
- The behavior used also seems to affect the degree of final consensus degree achieved, which is similar or lower when the adverse behavior is used: 0.88 versus 0.9 in S_6 and S_3, respectively (−0.02 or 2.22% less), 0.858 versus 0.855 in S_7 and S_4, respectively (+0.003 or 0.35% more), and 0.853 versus 0.855 in S_8 and S_5, respectively (−0.002 or 0.23% less).

Thanks to the step-by-step MDS of experts' preferences made by AFRYCA 2.0, it is possible to analyze aspects of

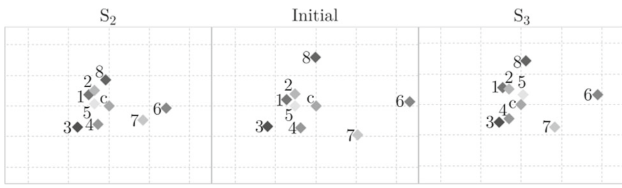

Fig. 20 Difference between final preferences in S_2 and S_3

the simulations that could not be analyzed with the previous version. Thus, it is possible to discover groups of experts with similar preferences in a GDM problem (see Fig. 19), to detect which experts disagree with the collective or to analyze how the CRP converges.

For example, in Fig. 20, in which can be seen the initial preferences and the final preferences in the simulations S_2 and S_3, it can be seen how, although an equal number of rounds have been carried out in both simulations and the final consensus degree reached has been very close (0.893 in S_2 and 0.9 in S_3), S_3 seems to be more respectful with initial individual experts' preferences than S_2, which may impact on the degree of satisfaction of experts with the CRP as well as with the final solution.

4.3 Metrics environment

Thanks to the metrics environment developed in AFRYCA 2.0, it is now possible to define our own metrics to analyze, from a quantitative point of view, specific aspects of CRPs. This allows to carry out comparative studies between different consensus models, which makes it easier to determinate the most appropriate model for a given GDM problem.

In order to demonstrate the power and utility of the metrics environment, we have defined an experimental metric that has been used to analyze the results obtained in the study. It should be noted that the proposed metric only responds to illustrative reasons, so we have tried to define a metric whose code and results are easy to understand.

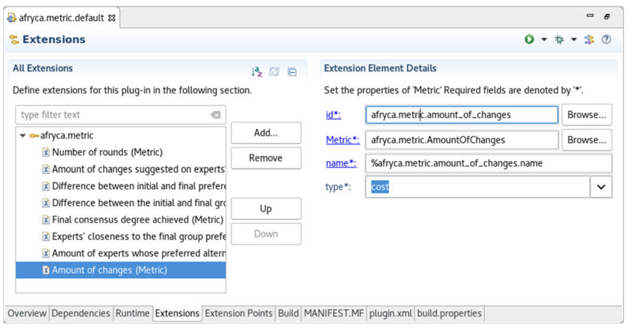

Fig. 21 Amount of changes metric extension

The defined metric has been called $M_{\text{amount of changes}}$, and as its name suggests, it measures the number of changes made by a consensus model in a CRP.

As it is defined in Sect. 3.2.5, a metric is defined in AFRYCA by defining an extension that extends the extension point for metrics (see Fig. 21) and implementing a Java class that implements the interface afryca.metric.Metric (see Listing 1).

Listing 1 Amount of changes in metric implementation

```
package afryca.metric;

import java.util.*;
import afryca.consensusmodel.definition.*;
import afryca.fpr.FPR;
import afryca.metric.Metric;

public class AmountOfChanges implements Metric {
 public Double execute(
     List<Map<ERoundResult, Object>> rounds,
     Map<EResultElements, Object> result) {
   Double changes = Double.NaN;
   if (!rounds.isEmpty()) {
     changes = 0d;
     FPR[] pre, pos;
     for (Map<ERoundResult, Object> r : rounds) {
       pre = (FPR[]) r.get(ERoundResult.
                              pre_preferences);
       pos = (FPR[]) r.get(ERoundResult.
                              pos_preferences);
       changes += countRoundChanges(pre, pos);
     }
   }
    return changes;
 }

 private static Integer countRoundChanges(
     final FPR[] pre,
     final FPR[] pos) {
   int result = 0;
   // Last is collective
   int elements = pre.length - 1;
   for (int i = 0; i < elements; i++) {
     result += countChanges(pre[i], pos[i]);
   }
   return result;
 }

 private static Integer countChanges(
     final FPR pre,
     final FPR pos) {
   int result = 0;
   int alternatives = pre.getNumberOfAlternatives();
     for (int r = 0; r < (alternatives - 1); r++) {
       for (int c = (r + 1); c < alternatives; c++) {
         if (pre.getValue(r, c).floatValue() !=
             pos.getValue(r, c).floatValue()) {
           result++;
         }
       }
     }
   return result;
 }
}
```

Using the defined metric, the values shown in Table 4 have been obtained.

The values returned by the metric allow to establish new conclusions of the results of the simulations:

– The simulation S_2 is the one that makes the least number of changes in experts' preferences.

Table 4 Amount of changes metric

	$M_{\text{Amount of changes}}$
S_1	51
S_2	6
S_3	34
S_4	38
S_5	39
S_6	85
S_7	94
S_8	73

– All the simulations carried out using consensus models with feedback mechanism and the standard behavior, that is, S_{3-5}, have made a similar number of changes in experts' preferences. This result is of particular interest considering that the number of consensus rounds conducted in S_3 (3) is significantly lower than the rounds conducted in the other two simulations (11 in S_4 and 10 in S_5).
– Similarly to the previous point, in the simulations in which standard with adverse behavior has been used, that is, S_{6-8}, the number of changes made in each case are much more similar than they might seem from the number of rounds performed in the simulation.

The results obtained allow us to demonstrate how the range of functionalities offered by AFRYCA 2.0 facilitates the study and analysis of CRPs in GDM problems, making it a much more powerful and versatile framework.

5 Conclusions

In GDM problems it is convenient that all experts reach a consensus before making a decision; thus, it is necessary to apply consensus reaching processes in their resolution. The large number of existing consensus models, each one with different characteristics, makes the selection of the most suitable one for a determined problem complex.

In this paper, a new version of AFRYCA has been presented, which allows to analyze distinct consensus models by means of a simulated environment. An experimental study has also been conducted to illustrate the usefulness of the new version. This latest version of the framework assumes a complete internal restructuring, adapting its architecture to the new software development patterns and facilitating its extension and modification. Furthermore, AFRYCA 2.0 includes several new features that focus on simplifying its use and increasing its capabilities, providing a more powerful and versatile analysis framework.

AFRYCA 2.0 opens a large range of possibilities for future works, considering, immediately, (i) to perform comparative studies about the different models implemented in distinct environments; (ii) CRP simulations in which several experts behaviors are possible and (iii) definition of consensus-based metrics that allow performance analysis of consensus models.

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4.10. APOLLO: Una Herramienta de Soporte para la Toma de Decisión Multi-criterio Difusa en Política Climática

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Research Article

APOLLO: A Fuzzy Multi-criteria Group Decision-Making Tool in Support of Climate Policy

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ABSTRACT

Multi-criteria decision-making is a daily process in everyday life, in which different alternatives are evaluated over a set of conflicting criteria. Decision-making is becoming increasingly complex, and the apparition of uncertainty and vagueness is inevitable, especially when related to sustainability issues. To model such lack of information, decision makers often use linguistic information to express their opinions, closer to their way of thinking, giving place to linguistic decision-making. However, the participation of multiple experts usually involves disagreements within the group, leading to unreliable solutions. To assist in decision-making and reduce such complexities, A grouP decisiOn fuzzy TOoL in support of cLimate change pOlicy making (APOLLO), a fuzzy decision support system, is introduced to deal with such problems in climate change and policy. The tool implements a framework for group decision-making, using 2-tuple Technique for Order Preference by Similarity to Ideal Solution (TOPSIS), coupled with a new consensus measuring model to increase robustness of selected solutions. The operation of the software tool is showcased in a real case carried out in Austria, where stakeholders were asked to assess the risks embedded in pathways for decarbonizing the country's iron and steel sector. Results indicate that a coherent strategy addressing funding and competition issues is necessary, with experts displaying a consensus level of 85% in that these risks are the most threatening for the transition.

1. INTRODUCTION

Decision-making (DM) problems range from the most common situations in human beings' daily lives (e.g., what film to see at the cinema) to much more complex ones that may affect larger social units, including communities (a new policy to reduce pollution in a city center), nations (a financial incentive to boost technological innovation), regions (sectoral coverage of the European Union's Emissions Trading System), or the globe (effort sharing in mitigating climate change). A DM problem always comprises a set of alternatives or possible solutions for the problem, and often a group of experts with different attitudes, who evaluate these alternatives in order to collectively select the "best" one. More often than not, the evaluation of the alternatives is based on several criteria, leading to *multi-criteria decision-making* (MCDM) [1,2].

However, in many MCDM problems, complexity significantly increases, with conflicts emerging among alternatives' performances across the evaluation criteria and reaching one optimal solution not being a straightforward process [3]. Furthermore, combined with the lack of information related to the alternatives, this complexity often implies the apparition of uncertainty. In this situation, modeling uncertainty is not a trivial task, since experts are usually unable to express it by using exclusively discrete assessments. To overcome the latter limitation, *linguistic variables* [4] have been used successfully [5]. By means of such variables, experts can express their opinions by using linguistic terms, such as *good* or *very bad, high*, or *insignificant*, etc., which are closer to their way of thinking. Under these conditions, MCDM becomes *linguistic decision-making* (LDM) [6].

The classical resolution scheme for MCDM problems considers only the aggregation of the experts' opinions over the alternative actions in order to obtain a ranking of these actions and select the best one [7]. This could often lead to situations where the possible disagreements that may emerge in the group are ignored or not reflected in the aggregate preferential model [8]. Consequently, some experts might not agree with the solution achieved and feel outside of the decision process. To increase the robustness of the chosen solutions, a consensus level of the experts can be measured [9] to identify sources of proximity and disagreement.

Nowadays, many of the most important real-world MCDM problems are related to sustainability issues [10]. The effects of global environmental change are becoming increasingly obvious and its impacts on our societies, economies, and environment, today and in the near future, constitute one of the main concerns worldwide. This is why nations have long set out to address this challenge (e.g.,

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the Kyoto Protocol and, recently, the Paris Agreement), in a globally coordinated and cooperative manner [11].

The enormous complexity of problems associated with climate change and action, especially in the context of an all-inclusive, participatory, and transparent dialogue, based on the principles of Talanoa [12], makes experts often come up with a series of assumptions that fail to reflect the real-world constraints, in order to reduce such complexity. MCDM has long been used to address challenges and resolve problems associated with environmental, energy, and climate policy [13]. Respectively, decision support systems, i.e., software tools used to support decisions, judgements, and courses of action, have recently been developed, featuring the capacity to solve climate change-related MCDM problems from the perspective of multiple stakeholders (e.g., Nikas *et al.* [14]; Jeong [15]), without however aiming to improve consensus.

In this direction, this research aims to make an important qualitative contribution within the climate change policy research area by presenting a new fuzzy decision support system, *A grouP decisiOn fuzzy TOoL in support of cLimate change pOlicy making* (APOLLO). The main aim of APOLLO is to facilitate a consensus measuring process of a group of individuals toward reaching the best decision for an MCDM problem related to climate change and policy issues. Additionally, the software has the ability to analyze the conflicts (or disagreements) that emerge among the experts. Furthermore, in order to validate it and showcase its usefulness, APOLLO is presented and stress-tested in a real-world case study that was carried out in Austria, in the context of assessing the risks embedded in pathways for decarbonizing the country's iron and steel sector.

From a methodological point of view, this paper seeks to contribute to the literature by establishing a new decision support system that focuses on dealing with problems related to climate change adaptation and mitigation policy issues. It takes into account the challenges of engaging with multiple actors from various stakeholder groups and thereby increases ownership of decisions, while introducing a new consensus analysis method, drawing from the literature.

The rest of the paper is organized as follows: Section 2 reviews several basic concepts toward facilitating the understanding of proposed method and tool. Section 3 introduces both the resolution process and the architecture of APOLLO. Section 4 describes the real-world case study on evaluating risks associated with the decarbonization of Austria's iron and steel sector, showcasing the performance of the APOLLO decision support tool. Finally, in Section 5 some conclusions and prospects of our research are drawn.

2. METHODS AND TOOLS

This section describes the proposed methods and tools that will be implemented in APOLLO. First, the choice of linguistic variables is facilitated through the review of LDM and the presentation of the 2-tuple linguistic model. Second, the 2-tuple Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) model that APOLLO uses to solve group DM problems is described. Finally, the new consensus measuring framework is introduced.

2.1. Linguistic Decision-Making

Human beings are continuously faced with decision problems; what to eat, what mobile phone to buy, or what shoes to wear today are common examples of this type of problems. As the problematic, along with the impacts of a decision to address it, shifts from individuals to larger social units (e.g., policymaking), the decision-making process requires ownership of a collectively acceptable solution and therefore entails the engagement of more than one decision maker. Formally, in these cases, the DM problem is formed by a set of experts, $E = \{e_1, \dots, e_k\}$, who evaluate different alternatives, $A = \{a_1, \dots, a_m\}$, and choose the best one(s) as solution(s) to the problem, by evaluating them against a set of different conflicting criteria, $C = \{c_1, \dots, c_n\}$ [14].

As complexity of a DM problem increases, with decision makers not knowing all of the information required to make a decision about the problem, uncertainty and vagueness are present. Under these circumstances, the classical probabilistic models cannot be used to obtain a solution and a different approach to deal with these problems is necessary. The *fuzzy linguistic approach* and *fuzzy variables* [4] have been widely used in the DM area in order to model the inherent uncertainty that appears in many decision situations, giving place to LDM [6]. In an LDM problem, the group of engaged individuals provide their opinions by using linguistic expressions, which are considered closer to the way in which human beings express their ideas.

Due to experts using linguistic expressions to give their opinions, it is essential to carry out computations with such linguistic information in order to provide consistent solutions for the LDM problems. Furthermore, these results should also be represented linguistically to promote understanding from the decision makers' point of view. The *Computing with Words* (CWWs) methodology [16–18] tries to mimic the reasoning process of human beings, by obtaining linguistic outputs from the linguistic inputs provided by the stakeholders. Many DM methods follow this methodology to solve an LDM problem. In this research, we focus on an extension of the TOPSIS, based on the 2-tuple linguistic model.

2.2. The 2-Tuple Linguistic Model

The 2-tuple linguistic computational model [19] is a symbolic model that was introduced as an improvement of other linguistic modeling approaches [20]. It carries out linguistic computational processes in an easy and comprehensive manner, without losing information, using a continuous linguistic domain, and outputs results that are expressed in the same linguistic domain [14].

To represent linguistic information, the 2-tuple model uses a pair of values that is called linguistic 2-tuple (s, a), where s is a linguistic term and a is a numeric value representing a symbolic translation.

Let $S = \left\{s_0, \dots, s_g\right\}$ be a linguistic term set and $\beta \in \left[0, g\right]$ be the result of a symbolic aggregation operation, where $g + 1$ is the cardinality of S. Let $i = round(\beta)$ and $\alpha = \beta - i$ be two values, such that $i\epsilon\,[-0.5, 0, 5)$; then α is called a symbolic translation. The symbolic translation of a linguistic term s_i is a numerical value within $[-0.5, 0, 5)$ indicating the difference of the information between the calculated value $\beta \in [0, g]$, and its closest element within $\{s_0, \dots, s_g\}$ indicating the content of the closest linguistic term $S\,(i = round\,(\beta))$.

In essence, the 2-tuple linguistic representation model extends the use of indexes modifying the fuzzy linguistic approach, by adding a symbolic translation that represents the linguistic information by means of a linguistic 2-tuple.

$$a = \begin{cases} [-0.5, 0.5)\,, if s_i \in \{s_1, s_2, \dots, s_{g-1}\} \\ [0, 0.5)\,, if\, s_i = s_0 \\ [-0.5, 0)\,, if\, s_i = s_g \end{cases} \quad (1)$$

Finally, for a linguistic term set $S = \{S_0, \dots, s_g\}$ and a value supporting the result of a symbolic aggregation operation $\beta \epsilon [0, g]$, the 2-tuple expressing the equivalent information to β is calculated:

$$\Delta : [0, g] \to S \times (-0.5, 0.5)$$

$$\Delta\,(\beta) = (s_i, \alpha)\,,\ with \begin{cases} s_i\, i = round(\beta) \\ \alpha = \beta - i\, \alpha \epsilon\, [-0.5, 0, 5) \end{cases} \quad (2)$$

Evidently, the conversion of a linguistic term into a linguistic 2-tuple consists of adding a value 0 as symbolic translation: $s_i \,\epsilon\, S \Rightarrow (s_i, 0)$.

2.3. The 2-Tuple TOPSIS Model

TOPSIS [21] is an MCDM method based on the idea that the best alternative is the closest to a positive ideal solution and the farthest from a negative ideal solution. Initially, TOPSIS was proposed as an MCDM method that can deal with numerical assessments and has been found to be relevant in the climate policy domain [14]; but, as already discussed, uncertainty often appears in many DM problems and, consequently, the need for linguistic information emerges. Several fuzzy TOPSIS methods have been proposed both in the broader literature as well as in climate policy support research [13].

Here, we build on the 2-tuple TOPSIS approach introduced in Ref. [22], which makes use of the 2-tuple linguistic model [23] and a new distance function that allows to obtain more precise and interpretable results than other models. However, instead of aggregating the initial input from the stakeholders using average values and then perform the 2-tuple TOPSIS, we follow the methodology established by Krohling and Campanharo [24] where the fuzzy TOPSIS was used in the experts' preference to create a global model and then another round of fuzzy TOPSIS was performed to acquire the global solution with the experts' individual solutions acting as the criteria. Nikas *et al.* [14] expanded the concept of using a double round of TOPSIS in group DM by using behavioral instead of fuzzy TOPSIS. The 2-tuple TOPSIS method to be used on this study consists of the following steps:

i. Defining a weight vector $U_t = \left(u_j^t\right)_{1*n}^T$, where $u_j^t \in U$ is the linguistic preference by stakeholder e_t for criterion c_j and U is a linguistic term set, with $U = \{u_1, u_2, \dots, u_p\}$ transformed into a 2-tuple linguistic decision matrix $U_t = \left(u_j^t, 0\right)_{1*n}^T$.

ii. Calculating the normalized 2-tuple weight vector $U_t^N = \left(\overline{u}_j^t, \overline{\beta}_j^t\right)_{1*n}^T$ for each stakeholder e_t as

$$\left(\overline{u}_j^t, \overline{\beta}_j^t\right) = \Delta_u \left(\frac{\Delta_u^{-1}\left(u_j^t, 0\right)}{T_U - 1} \right), \quad (3)$$

$j = 1, 2, \dots, n$ and T_U is the cardinal of set U.

Normalizing with the cardinal of the linguistic scale instead of the maximum value, as suggested in the original method, is preferred to avoid exaggerating the differences between the responses.

iii. Defining the decision matrix $X_t = \left(r_{ij}^t\right)_{m*n}$, where $\left(r_{ij}^t\right) \in S$ is the linguistic value preference provided by stakeholder e_t for alternative a_i over criterion c_j, and S is the linguistic term set, with $S = \{s_1, s_2, \dots, s_t\}$ transformed into a 2-tuple linguistic decision matrix $X_t = \left(r_{ij}^t, 0\right)_{m*n}$.

iv. Calculating the weighted decision matrix $\overline{X}_t = \left(\overline{r}_{ij}^t, \overline{a}_{ij}^t\right)_{m*n}$ for each stakeholder e_t, with

$$\left(\overline{r}_{ij}^t, \overline{a}_{ij}^t\right) = \Delta_S \left(\Delta_u^{-1}\left(\overline{u}_j^t, \overline{\beta}_j^t\right) . \Delta_S^{-1}\left(r_{ij}^t, 0\right) \right), \quad (4)$$
$$i = 1, 2, \dots, m, j = 1, 2, \dots, n.$$

v. Calculating the positive and negative ideal solutions for each stakeholder e_t as: $\left(r^{t,+}, \alpha^{t,+}\right) = \left\{\left(r_1^{t,+}, \alpha_1^{t,+}\right), \left(r_2^{t,+}, \alpha_2^{t,+}\right), \dots, \left(r_n^{t,+}, \alpha_n^{t,+}\right)\right\}$ and $\left(r^{t,-}, \alpha^{t,-}\right) = \left\{\left(r_1^{t,-}, \alpha_1^{t,-}\right), \left(r_2^{t,-}, \alpha_2^{t,-}\right), \dots, \left(r_n^{t,-}, \alpha_n^{t,-}\right)\right\}$, where $\left(r_j^{t,+}, \alpha_j^{t,+}\right) = \max_i \left\{\left(\overline{r}_{ij}^t, \overline{a}_{ij}^t\right) | c_j \in B\right\}$ *or* $\min_i \left\{\left(\overline{r}_{ij}^t, \overline{a}_{ij}^t\right) | c_j \in B'\right\}$ and $\left(r_j^{t,-}, \alpha_j^{t,-}\right) = \min_i \left\{\left(\overline{r}_{ij}^t, \overline{a}_{ij}^t\right) | c_j \in B\right\}$ *or* $\max_i \left\{\left(\overline{r}_{ij}^t, \overline{a}_{ij}^t\right) | c_j \in B'\right\}$, where $i = 1, 2, \dots, m, j = 1, 2, \dots, n$ and where B and B' are the benefit and cost criteria sets respectively.

vi. Determining the distance of each alternative from the positive and negative ideal solutions for each stakeholder e_t as

$$\left(\xi_i^{t,+}, \eta_i^{t,+}\right) = \Delta_{S'} \left(\frac{1}{n} \sum_{j=1}^{n} \frac{(T_{S'} - 1)}{(T_S - 1)} \cdot \left(\left| \Delta_S^{-1}\left(\overline{r}_{ij}^t, \overline{a}_{ij}^t\right) - \left(r_j^{t,+}, \alpha_j^{t,+}\right) \right| \right) \right) \quad (5)$$

and

$$\left(\xi_i^{t,-}, \eta_i^{t,-}\right) = \Delta_{S'} \left(\frac{1}{n} \sum_{j=1}^{n} \frac{(T_{S'} - 1)}{(T_S - 1)} \cdot \left(\left| \Delta_S^{-1}\left(\overline{r}_{ij}^t, \overline{a}_{ij}^t\right) - \left(r_j^{t,-}, \alpha_j^{t,-}\right) \right| \right) \right) \quad (6)$$

where $S' = \{s'_1, s'_2, \dots, s'_{t'}\}$ is the linguistic term set for the distances, T_S. and $T_{S'}$ the cardinals of sets S and S' respectively.

vii. Calculating the relative closeness degree of each alternative from the positive ideal solution for each stakeholder e_t as

$$\left(\xi_i^t,\eta_i^t\right)=\Delta_{S'}\left(\left(\frac{\Delta_{S'}^{-1}\left(\xi_i^{t,-},\eta_i^{t,-}\right)}{\Delta_{S'}^{-1}\left(\xi_i^{t,+},\eta_i^{t,+}\right)+\Delta_{S'}^{-1}\left(\xi_i^{t,-},\eta_i^{t,-}\right)}\right)\cdot(T_S-1)\right), \quad (7)$$
$$i=1,2,\dots,m \text{ and } T_S \text{ the cardinal of set } S.$$

the current form the results are expressed in the linguistic scale S used by the stakeholders to increase interpretability. The results could have been displayed in the scale S' which was defined explicitly to express distances; however, presenting the results in the new terms, despite being considered more appropriate, might confuse the stakeholders.

viii. Computing the collective 2-tuple linguistic decision matrix $X=(\tilde{r}_{it},\tilde{\alpha}_{it})_{m*k}$, where $(\tilde{r}_{it},\tilde{\alpha}_{it})=\left(\xi_i^t,\eta_i^t\right)$, $i=1,2,\dots,m$, $t=1,2,\dots,k$. In this step the stakeholders are considered equally weighted. By adjusting steps 1–4, the new matrix X could be calculated to also include weights for the expert.

ix. Calculating the positive and negative ideal collective as $\left(r^+,\alpha^+\right)=\left\{\left(r_1^+,\alpha_1^+\right),\left(r_2^+,\alpha_2^+\right),\dots,\left(r_k^+,\alpha_k^+\right)\right\}$ and $\left(r^-,\alpha^-\right)=\left\{\left(r_1^-,\alpha_1^-\right),\left(r_2^-,\alpha_2^-\right),\dots,\left(r_k^-,\alpha_k^-\right)\right\}$, where $\left(r_t^+,\alpha_t^+\right)=\max_i\left\{(\tilde{r}_{it},\tilde{\alpha}_{it})\,|c_j\in B\right\}$ or $\min_i\left\{(\tilde{r}_{it},\tilde{\alpha}_{it})\,|c_j\in B'\right\}$ and $\left(r_t^-,\alpha_t^-\right)=\min_i\left\{(\tilde{r}_{it},\tilde{\alpha}_{it})\,|c_j\in B\right\}$ or $\max_i\left\{(\tilde{r}_{it},\tilde{\alpha}_{it})\,|c_j\in B'\right\}$, where $i=1,2,\dots,m$, $t=1,2,\dots,k$ and B and B' are the benefit and cost criteria sets respectively.

x. Determining the distance of each alternative form the positive and negative ideal solutions for each stakeholder t as $\left(\xi_i^+,\eta_i^+\right)=\Delta_{S'}\left(\frac{1}{k}\sum_{t=1}^{k}\frac{(T_{S'}-1)}{(T_S-1)}\cdot\left(\,|\Delta_S^{-1}(\tilde{r}_{it},\tilde{\alpha}_{it})-(r_t^+,\alpha_t^+)|\,\right)\right)$ and $\left(\xi_i^-,\eta_i^-\right)=\Delta_{S'}\left(\frac{1}{k}\sum_{t=1}^{k}\frac{(T_{S'}-1)}{(T_S-1)}\cdot\left(\,|\Delta_S^{-1}(\tilde{r}_{it},\tilde{\alpha}_{it})-(r_t^-,\alpha_t^-)|\,\right)\right)$, where $S'=\left\{s_1',s_2',\dots,s_{t'}'\right\}$ is the linguistic term set for the distances, T_S and $T_{S'}$ the cardinals of sets S and S' respectively.

xi. Finally, calculating the relative closeness degree of each alternative from the positive ideal solution as

$$\left(\xi_i,\eta_i\right)=\Delta_{S'}\left(\left(\frac{\Delta_{S'}^{-1}\left(\xi_i^-,\eta_i^-\right)}{\Delta_{S'}^{-1}\left(\left(\xi_i^+,\eta_i^+\right)\right)+\Delta_{S'}^{-1}\left(\xi_i^-,\eta_i^-\right)}\right)\cdot(T_S-1)\right),$$
$$i=1,2,\dots,m \text{ and } T_S \text{ is the cardinal of set } S. \quad (8)$$

The results could have been displayed in the distance scale S', but instead they are converted to the scale the stakeholders provided their answers in for clarity of results, needed in the next steps.

2.4. Consensus Measuring

MCDM methods allow to obtain a solution for a DM problem. In certain occasions, however, the solutions obtained do not satisfy all of the engaged stakeholders participating in the decision-making process. For this reason, Ref. [25] suggests measuring a realistic and "human-consistent" degree of consensus to calculate these differences, softening the concept of complete agreement by introducing the "soft" consensus degree [1,26,27]. Kuncheva [28] identifies five metrics for consensus measuring based on comparisons between the experts' evaluations, which capture either common ground among the answers or sources of disagreement [9]. Many studies used such metrics to extract consensus level information from comparing the experts' preference data [29,30]. However, Herrera-Viedma *et al.* [31] argue that these methods can withhold information or underestimate consensus, since different evaluations may lead to similar solutions. To avoid this bias, they propose an alternative approach, which is based on comparing the rankings of the experts' assessments with a global solution instead of each other's preferences. Boroushaki and Malczewski [32] adapted the model to integrate geographical information systems with MCDA, while Ben-Arieh and Chen [33] also considered the degree of importance of each expert.

However, in this approach, alternatives with similar evaluations in the global solution may result in huge differences in the rankings, which will subsequently lead to exaggerations of dissimilarity, if only the rankings are taken into account. Here, we build on Ref. [31] by applying a consensus measuring model that is similarly based on the comparison of a global solution with the experts' assessments but takes advantage of the 2-tuple TOPSIS evaluations provided by the distance function instead of the rankings. The model is described below:

i. The dissimilarity of each expert for each alternative $p_i\left(x_j\right)$ is calculated by comparing the distance between the result of the 2-tuple TOPSIS of that alternative in the experts' individual solution and in the collective one as follows:

$$p_i\left(x_j\right)=p\left(R^i,R^c\right)\left(x_j\right)=\left(\frac{\left|R_j^c-R_j^i\right|}{T-1}\right)^b\in[0,1],\ b\geq 0 \quad (9)$$

where i stands for each expert, j stands for each alternative, b can be in the range of $(0,1)$ to control the rigorousness of the model, R_j^c is the result of the 2-tuple TOPSIS of the alternative j in the group solution, R_j^i is the result of the 2-tuple TOPSIS of the alternative j in expert's i solution, and T is the cardinal of the linguistic term set, used to normalize the dissimilarity values. With this approach, the evaluation of the group solution and the expert is compared for each alternative instead of the positions in the ranking, enabling us to capture the full information provided by the stakeholders.

ii. Next, we calculate the consensus degree of all experts on each alternative x_j using the following expression:

$$C\left(x_j\right)=1-\sum_{i=1}^{m}\frac{p_i\left(x_j\right)}{m} \quad (10)$$

where m stands for the total number of experts.

iii. Finally, we calculate the consensus measure over the set of alternatives, called C_X:

$$C_X = \frac{\sum_{j=1}^{k} C\left(x_j\right) * R_j^C}{\sum_{j=1}^{k} R_j^C} \tag{11}$$

where k is the total number of alternatives. In the original model, the aggregation of the consensus degree of each alternative into the final consensus measure was performed by using the S-OWA OR LIKE operator [34]. Through this process the set of alternatives was split in a set of solutions and a set of remaining alternatives, where the former is given an increased weight, leading to the dependence of the consensus measure on the choice of the OWA operator. To avoid this issue, in our approach, the aggregation is performed through a weighted average formula, where the evaluation of the 2-tuple TOPSIS of the global solution for each alternative is used as the weight of the consensus degree over this alternative.

iv. Applying a similar approach with the consensus measure, the proximity of i-th expert to the global solution can be calculated:

$$P_X^i = \frac{\sum_{j=1}^{k} \left(1 - p_i\left(x_j\right)\right) * R_j^C}{\sum_{j=1}^{k} R_j^C} \tag{12}$$

3. APOLLO

This section introduces a fuzzy MCDM group decision tool, APOLLO, to solve multicriteria problems under uncertainty, related to climate change and policy. First, we discuss the different steps that describe the resolution scheme of the introduced software, and then we present its architecture.

3.1. Resolution Scheme

APOLLO has been developed with the aim of solving LDM problems related to climate change issues, fully aligned with policy developments, such as the Paris Agreement and the Talanoa dialogue, as well as with emerging scientific paradigms in support of these developments (e.g., Doukas *et al.* [11]; Weitzel *et al.* [35]). Furthermore, due to the complexity and importance usually linked to these kinds of problems and in the aim of maximizing governance (of science, risks, and policy), our goal is to also provide solutions in which the majority of stakeholders (and stakeholder groups) participating in the decision process agree with one another. Hence, it is necessary to propose a specific LDM solving process that, on one hand, allows using MCDM methods in order to provide solutions for the decision problem and, on the other hand, guarantees that such solutions satisfy the largest part of the group of engaged individuals as much as possible, mitigating potential disagreements. APOLLO's resolution scheme is composed by different steps that are described in the following subsections (see Figure 1).

3.1.1. Problem definition (Framework)

This step allows defining the MCDM problem. Stakeholders, criteria, alternatives, and the expression domains that the stakeholders use to provide their preferences. In this application, we consider that stakeholders use linguistic expressions in order to facilitate the preference elicitation process, thus the expression domains are represented by fuzzy linguistic term sets, the label numbers of which can be selected by the user/analyst.

3.1.2. Knowledge domain assignment (Knowledge)

Although linguistic expression domains are created in the previous step, it is essential to match these domains to each participating stakeholder. In doing so, several linguistic scales can be defined, each one tailored to the knowledge/preference of each engaged decision maker.

3.1.3. Preference elicitation (Gathering)

At this stage, stakeholders provide their assessments by using linguistic expressions. In this version, stakeholders may use expressions represented by single linguistic terms, such as *Good, Bad, High*, or *Very Low*.

3.1.4. Multi-criteria solution (Rating)

This phase carries out the resolution of the MCDM problem. This version of APOLLO uses the 2-tuple TOPSIS method to solve the defined MCDM problem by following the steps introduced in Section 2.3.

3.1.5. Consensus measuring

This step allows us to measure the consensus and proximity level of the solution found in the previous stage. APOLLO calculates consensus based on the model presented in Section 2.4

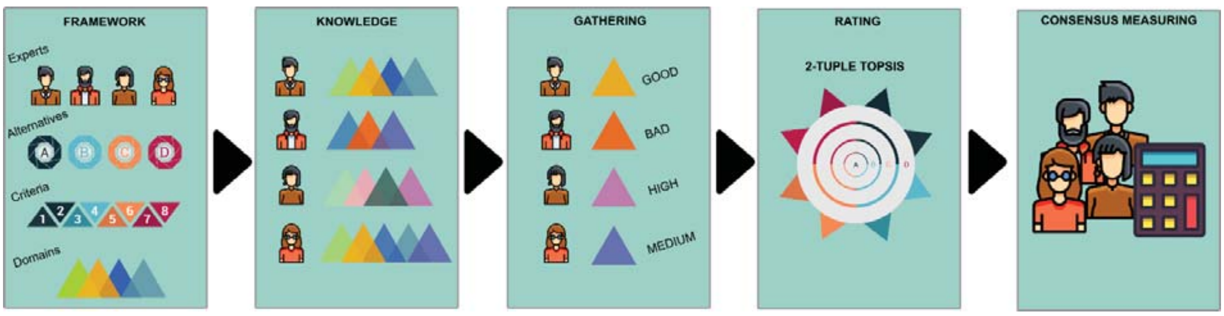

Figure 1 A group decision fuzzy tool in support of climate change policy making's (APOLLO) resolution scheme.

If desired, a consensus reaching process (CRP) [36,37] can be applied to bring the experts' assessments closer with one another and achieve an acceptable level of agreement in the group (consensus control) [38]. The initial experts' preferences would then be modified through iterative rounds and used to obtain a consensual solution for the problem (feedback process), to conclude the CRP cycle [39].

3.2. Architecture

APOLLO has been developed using an Eclipse Rich Client Platform (RCP) developed by IBM and created for building desktop applications with richer functionality. The main advantage of this technology is the capability to extend, modify, and reuse the applications easily in different operative systems thanks to the components-based architecture. Components or also so-called plugins are small pieces of software interconnected with each other that compose the whole RCP application. The use of plugins allows connecting them to other RCP applications and increase their functionality without the need to have a full understanding of how the application works. APOLLO is composed by several plugins classified into different categories:

- *User interface:* the plugins which belong to this category are used to visualize the user interface of the application (buttons, plots, etc.).
- *MCDM:* the plugins included in this category represent all the information related with the MCDM problems and their resolution. Here we can find plugins to represent the different elements of the problems, for instance, experts, alternatives, criteria, or expressions domains. In addition, the MCDM models to solve the problem are also classified in this category. For this version of APOLLO, the 2-tuple TOPSIS is the selected MCDM method but others can be added.
- *Consensus:* APOLLO solves MCDM problems by using MCDM methods but also incorporates plugins that measure the consensus level. In this way, the selection of the best alternatives is accompanied by a consensus level to obtain a more robust solution.

The APOLLO's architecture is represented in Figure 2.

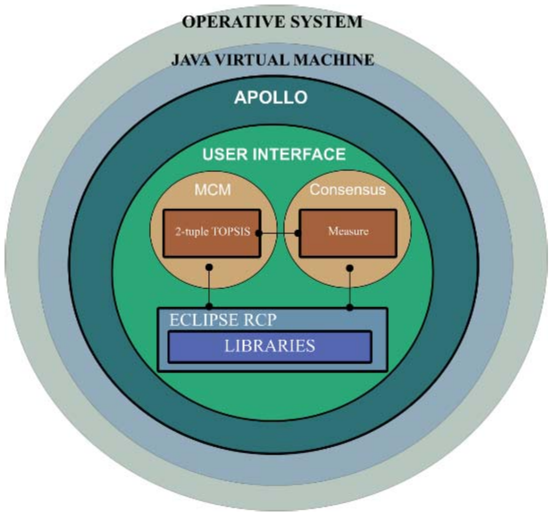

Figure 2 | A group decision fuzzy tool in support of climate change policy making's (APOLLO) architecture.

4. CASE STUDY

In order to show the usefulness of APOLLO, we use it to solve a real MCDM problem related to the decarbonization of iron and steel production in Austria.

4.1. Background Information

Iron and steel is considered an energy-intensive industry [40], accounting for 4%–7% of the industrial CO_2 emissions in the EU [41], while in 2017 contributed almost 16% of the industrial and 1.5% of the total GHG emissions [42]. In Austria, these shares are even higher, with iron and steel producing 65% of the industrial and 14% of the total GHG emissions in 2017, according to the UNFCCC Inventory, highlighting the importance of decarbonization of the sector as part of the country's emissions mitigation targets. As seen in Figure 3, the emissions of the sector do not only represent a high share, but they steadily increased through time, despite the fluctuation of the total emissions and the obvious decrease from the 2005 level, even rebounding from the decrease caused by the economic crisis in 2008.

Part of the intensity of the iron and steel industry can be attributed to technological reasons for the production process. The dominant process for primary production is the energy-intensive Blast Furnace/Basic Oxygen Furnace route (BF-BOF), where iron ores are reduced to iron, using coke as a reducing agent [43]. The secondary steelmaking process is the Electric Arc Furnace (EAF) route which produces steel from recycled scrap, requiring a third of the energy needed in the BF/BOF route [44].

In Austria the majority of iron and steel produced is based on the BF/BOF route [45]. The dominance of BF-BOF compared to other European regions makes Austria one of the most sensitive countries to CO_2 prices in the EU [46]. Therefore, radical innovations need to be implemented in the sector to be able to adapt to deep decarbonization strategies [47], since simple solutions like the Best Available Techniques have limitations [48]. Such cutting-edge technologies include hydrogen-based production that could drastically reduce emissions intertwined with renewable energy production [49]. However, actors are usually skeptical about large-scale transitions out of fear of the cost and risk associated with the adoption of radical innovations [50]. These fears need to be considered during the development of policies, since actively engaging stakeholders in the process could provide valuable insights on their point of view towards a "greener" industry [51]. This background constitutes the motivation of our study, showcasing why the Austrian iron and steel sector was selected as a case study.

4.2. Alternatives and Evaluation Criteria

In order to facilitate the transition pathway of the Austrian iron and steel industry, risks associated with this transition are prioritized through the engagement of stakeholders in an iterative co-creative

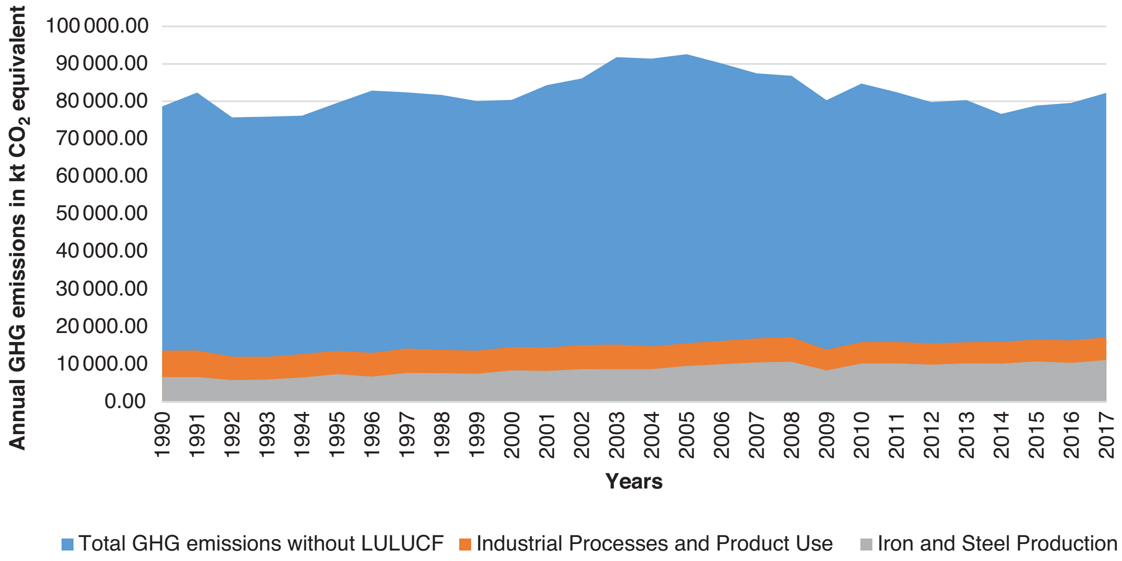

Figure 3 Total, Industrial and Iron and Steel greenhouse gas (GHG) emissions in Austria. Source: UNFCCC [42], own elaboration.

process that will provide insights into what key actors of the system fear the most.

In our study, we focus on risks that cut across a number of dimensions, such as energy infrastructure, the political and institutional status, environmental issues related to end-use acceptance, financial markets, and technological innovation (Table 1), adapting from the clustering of risks performed in Bachner *et al.* [51] and Wolkinger *et al.* [52]

Most risks are intertwined with the need to achieve wide-scale diffusion of centralized and decentralized renewable energy sources, in order to support green hydrogen production to be used in industry. This is evident in the infrastructure cluster, where the challenges posed to the stability of the grid due to the increase of renewable generation [53] and storage limitations are analyzed. Importance is also given to the institutional level to manage policy-related risks and financially support technological innovation that will pave the way for a just transition that will be acceptable by the society despite lock-ins in the dominant regimes [54]. The list of risks is not exhaustive, given the multiplicity of the various risks that can hinder the envisaged transition pathway, but was considered by the stakeholders to be representative of the risks decelerating the energy transition.

The identified risks are evaluated against four criteria: (a) their likelihood to manifest; (b) the level of the perceived impact that they can have on the climate mitigation policy framework; (c) lack of state/societal capacity to mitigate them; and (d) level of concern.

4.3. Stakeholder Input

Based on the stakeholder dialogue format described in Ref. [51], ten stakeholders (E1...E10) from the Austrian iron and steel sector were engaged in the process through bilateral interviews and workshops.

Initially the stakeholders were asked to assess the importance of the four evaluation criteria, using a 5-term linguistic scale {None (N), Low (L), Medium (M), High (H), Extreme (E)}. The evaluations are presented in Figure 4.

Despite significant variance in the responses, the majority of the stakeholders consider the level of concern over each risk to be an important evaluation factor, with six of them weighting concern with extreme importance, two with high importance and only two considered it of low importance.

In the next step, stakeholders were asked to evaluate each alternative/risk against these criteria answering to the questions in Table 2.

The responses of the stakeholders are then converted in the same scale used for the weights, {None, Low, Medium, High, Extreme}, while the answers for Criteria 3 are appropriately adjusted to reflect the lack of capacity.

Based on the adjusted answers, the distribution of the assessments for each term of the linguistic scale is presented in Figure 5. Most of the experts' answers are concentrated around medium and neighboring terms.

However, the experts seemed more reluctant to use the higher scales, since "none" received almost double the answers of "extreme," while "low" received a higher number of responses than "high." This indicates that the experts showcased a moderate behavior being less willing to use stricter terms.

4.4. Results

4.4.1. Experts' individual solutions

After initial assessments, the 2-tuple TOPSIS model described in Section 2.3 is applied to the answers of each expert independently, in order to calculate the rank and the score of each alternative. In Table 3, the assessments and results of 2-tuple TOPSIS are presented for Expert 1; a similar process is followed for the rest of the experts.

In Figure 6, the results of the 2-tuple TOPSIS for each expert are presented. Despite general similarities among the results, significant differences between individual choices exist. For example, Expert 4 considers alternative R22 "Lock-ins due to capacity mechanisms" to be the most important risk with an evaluation of (Extreme, −0.19),

Table 1 Risk classification and evaluation criteria.

Group	Alternatives	Evaluation Criteria
Energy infrastructure	R1. Lack of transparency R2. Grid Instability R3. Lack of storage capacity R4. Complicated investment procedures	C1. Likelihood to manifest C2. Impact on policy C3. Lack of mitigation capacity C4. Level of concern
Environmental/acceptability	R5. Social injustices R6. Insufficient consideration of lifestyles R7. Resource consumption overlooked R8. Social resistance against investments R9. Lack of investment framework	
Political/institutional framework	R10. Non-evidence-based regulatory framework R11. Short-sighted energy/climate planning R12. Market distortions R13. Lack of political leadership R14. Fluctuation of CO_2 prices	
Financial	R15. Non-coordination at the EU level R16. Uneven distribution of transition costs R17. Non-engaging/unstable markets R18. Narrow consideration of competition R19. Price risks due to new technologies	
Innovation and technology	R20. Limited funding capacity R21. Bad timing of new industry technologies R22. Technological lock-ins in iron and steel R23. Little integration across multiple sectors R24. Lack of information flows R25. Imperfect picture of transition	

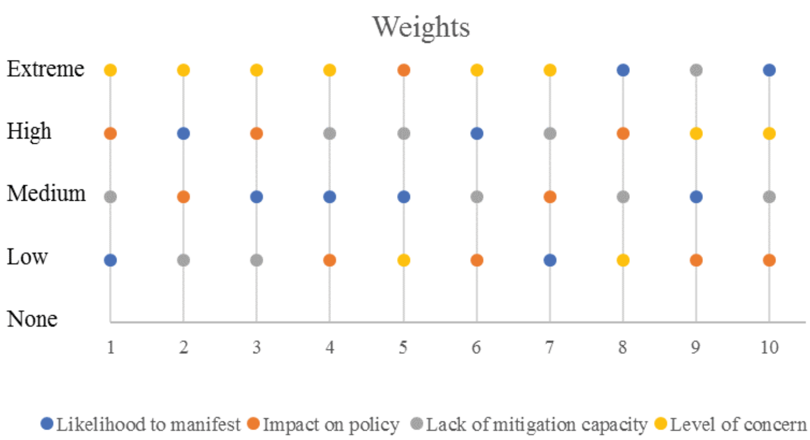

Figure 4 Criteria weights assigned by the ten stakeholders.

Table 2 Questions asked to the stakeholders for the evaluation of each risk against the four criteria.

Evaluation Criteria	Question	Linguistic Scale of the Answers
C1. Likelihood to manifest	What is the likelihood for the following risks to occur?	{Very unlikely, Unlikely, As likely as not, Likely, Very Likely}
C2. Impact on policy	If the following risks were to occur, what would be the extent of their impact?	{Limited, Considerable, Great, Extreme, Catastrophic}
C3. Lack of mitigation capacity	If the following risk were to occur, how would you estimate the capacity of relevant actors to mitigate them?	{None, Low, Medium, High, Extreme}
C4. Level of concern	How worried are you about following risks?	{Not worried, A little worried, Somewhat worried, Very worried, Extremely worried}

while Expert 3 considers it to be the risk with the lowest importance and a score of (Low, −0.45). These differences illustrate that the stakeholder pool is well diversified, mitigating possible biases in the collective solution.

4.4.2. Collective solution

The results for each individual expert are used to create the new matrix to be used to calculate the collective solution of the group. In

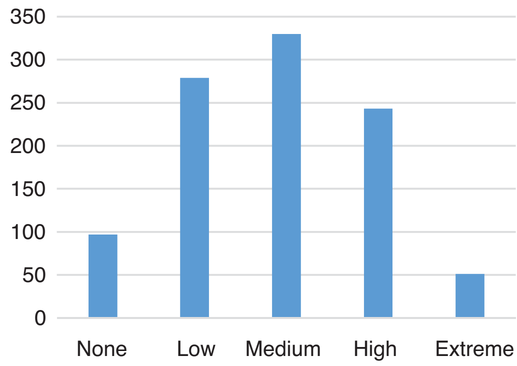

Figure 5 Distribution of the experts' assessments for each linguistic term.

Table 3 Assessments and results for Expert 1.

	C1	C2	C3	C4	Results
Weights	L	H	M	E	
R1	M	L	L	L	0.92
R2	E	H	L	M	2.77
R3	H	M	N	M	1.85
R4	H	H	M	M	2.92
R5	H	L	N	L	0.77
R6	H	L	N	L	0.77
R7	M	L	L	L	0.92
R8	H	H	L	M	2.62
R9	E	H	L	H	3.38
R10	M	M	M	M	2.31
R11	H	L	M	M	2.00
R12	L	N	L	L	0.31
R13	H	M	L	H	2.77
R14	H	M	M	M	2.46
R15	H	H	L	H	3.23
R16	H	M	M	M	2.46
R17	H	M	M	M	2.46
R18	H	M	H	M	2.77
R19	H	L	L	M	1.69
R20	M	M	M	M	2.31
R21	H	H	L	M	2.62
R22	M	L	L	L	0.92
R23	H	M	L	H	2.77
R24	E	H	M	M	3.08
R25	H	M	M	M	2.46

that case, the experts will play the role of equally weighted criteria. The 2-tuple TOPSIS is then run again to the new matrix (Table 4) to assess the importance of each alternative as a collective group.

The ranking of the alternatives according to the second 2-tuple TOPSIS are presented in Table 5. Out of 25 risks examined, 8 were evaluated in the scale of "High," the majority fluctuates around medium values, while only 3 received a "Low" score. Despite the moderate answers of the experts who avoided higher rates as discussed in Section 4.3, the percentage of high-importance risks indicate a broad concern of the stakeholders for the envisaged transition. Specifically, the risks with the higher importance with almost identical scores are the "Lack of investment framework" and the "Narrow consideration of competition." The performance of these risks indicative a request from the experts to the state to develop a coherent strategy that will address the high investments costs of the transition and deal with competitiveness issues especially from major exporting countries, like China, that can offer cheaper commodities due to lower energy efficiency investments [55] and the slower development of a universal carbon market [56]. This is further established by the high performance of the risk "Imperfect picture of transition," leading to the conclusion that the design of a clear transitional pathway that addresses the aforementioned concerns is vital

In Figure 7, the results are presented following the allocation of the risks to the groups described in Table 1.

Table 4 New decision matrix for the collective solution.

	E1	E2	E3	E4	E5	E6	E7	E8	E9	E10
R1	0.92	1.04	2.21	0.57	1.75	1.30	0.41	1.85	0.93	0.67
R2	2.77	2.22	1.24	0.38	2.00	0.86	1.66	1.13	2.53	2.13
R3	1.85	2.81	2.76	2.86	2.50	2.38	2.34	2.77	2.27	1.87
R4	2.92	2.52	0.55	1.14	2.75	1.41	2.34	1.44	0.80	3.07
R5	0.77	2.81	0.69	1.90	1.63	2.70	1.10	2.26	0.53	0.40
R6	0.77	3.41	3.03	1.52	2.75	2.81	1.24	3.18	1.60	2.93
R7	0.92	3.70	2.90	2.10	1.25	2.49	2.34	1.64	2.00	3.33
R8	2.62	2.22	3.45	1.52	2.38	2.49	2.34	2.05	1.47	2.80
R9	3.38	2.37	2.48	1.71	3.13	3.24	2.07	3.38	2.67	2.53
R10	2.31	1.48	1.66	0.76	1.50	1.19	1.66	1.13	1.33	2.53
R11	2.00	2.22	1.79	3.05	3.00	3.24	1.79	0.82	2.93	1.73
R12	0.31	1.63	1.38	0.95	0.88	2.70	1.38	2.46	2.13	2.67
R13	2.77	2.81	1.38	1.52	0.75	3.14	2.21	1.95	2.40	2.53
R14	2.46	0.59	3.03	2.29	1.25	2.70	1.79	1.85	1.60	3.47
R15	3.23	0.30	2.21	3.43	0.88	3.03	2.90	1.85	1.87	2.80
R16	2.46	2.67	0.97	0.19	1.13	3.03	1.52	2.67	2.40	2.53
R17	2.46	3.85	3.45	2.86	3.13	1.19	3.17	0.82	2.40	2.27
R18	2.77	3.26	2.76	3.43	2.00	3.14	2.48	1.74	2.40	2.80
R19	1.69	2.37	1.24	1.14	2.88	2.92	1.38	1.13	2.67	2.13
R20	2.31	1.48	2.21	2.10	1.88	2.16	1.38	2.15	2.13	1.20
R21	2.62	1.48	1.52	1.71	1.00	2.49	2.48	2.77	2.40	1.60
R22	0.92	2.96	0.55	3.81	0.75	1.41	1.79	2.26	1.07	1.87
R23	2.77	1.93	2.76	2.10	2.25	3.03	1.79	1.13	1.07	2.40
R24	3.08	1.78	2.21	1.52	3.25	3.24	2.90	1.74	1.20	2.40
R25	2.46	3.26	3.17	2.67	2.38	2.70	3.17	0.51	2.13	2.67

Table 5 Final ranking of risks based on the collective solution.

Ranking	Alternative	2-tuple TOPSIS Linguistic
1	R9	(High, 0.04)
2	R18	(High, 0.02)
3	R17	(High, −0.15)
4	R25	(High, −0.21)
5	R3	(High, −0.31)
6	R8	(High, −0.46)
7	R24	(High, −0.46)
8	R6	(High, −0.47)
9	R7	(Medium, 0.45)
10	R11	(Medium, 0.44)
11	R15	(Medium, 0.42)
12	R13	(Medium, 0.29)
13	R23	(Medium, 0.25)
14	R14	(Medium, 0.23)
15	R21s	(Medium, 0.09)
16	R16	(Medium, 0.02)
17	R19	(Medium, 0.02)
18	R20	(Medium, −0.05)
19	R4	(Medium, −0.06)
20	R22	(Medium, −0.27)
21	R2	(Medium, −0.34)
22	R12	(Medium, −0.40)
23	R10	(Low, 0.47)
24	R5	(Low, 0.37)
25	R1	(Low, −0.06)

TOPSIS, Technique for Order Preference by Similarity to Ideal Solution.

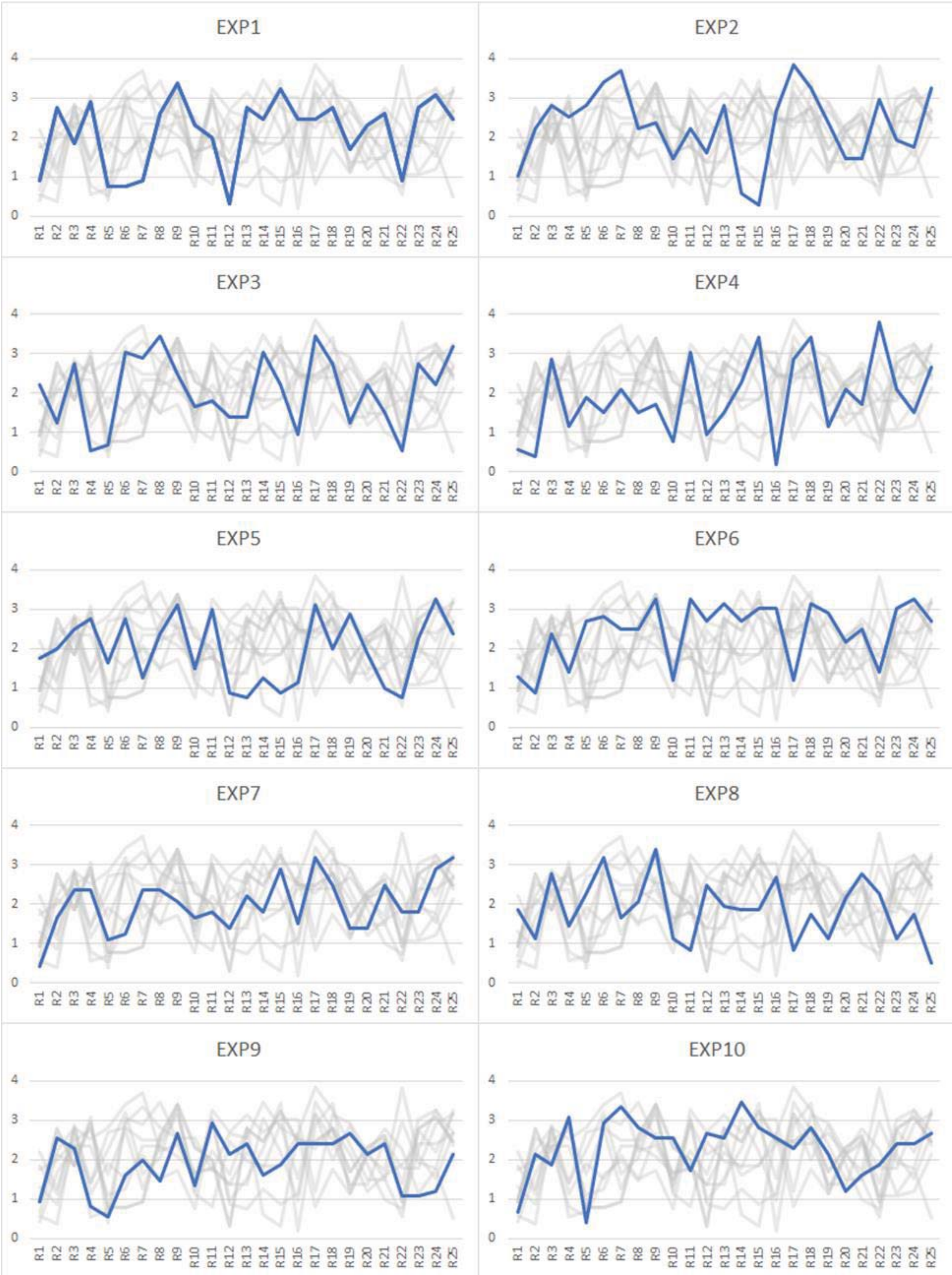

Figure 6 | Results of 2-tuple Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) for each individual expert.

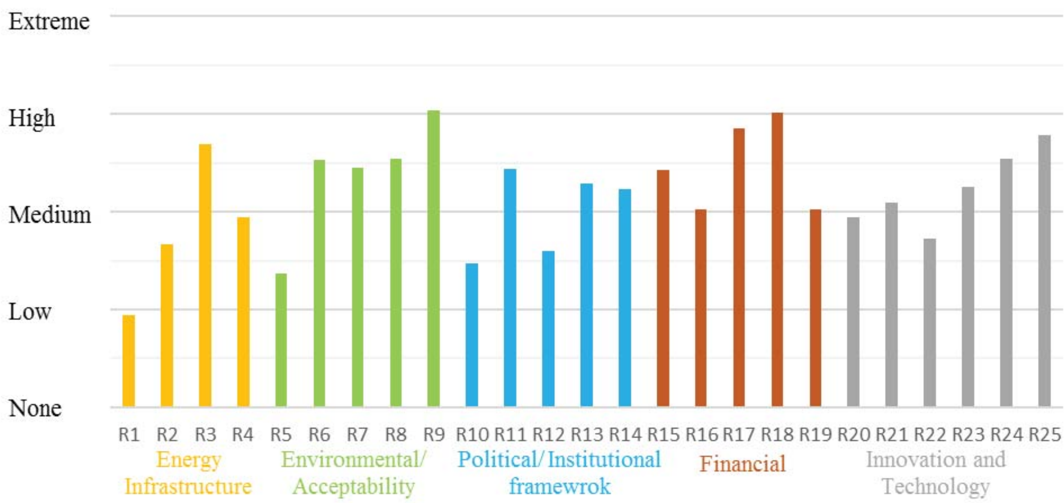

Figure 7 Clustered results of the collective solution.

From an infrastructural perspective, the "Lack of storage capacity" is considered the most important risk, since it is associated with the ability of the grid to maintain high shares of renewable energy productions. The procedures for investments in infrastructure and the stability of the grid perform slightly below medium, showcasing that, if the storage capacity is improved, the stakeholders are confident about the efficiency of the infrastructure economically and technologically. The transparency of the infrastructural procedures concerns stakeholders the least, not only in the same cluster, but over the complete set of alternatives, which indicates that, if the financial, technological, and social aspects of the transition are determined, it will be easier to adapt to procedural requirements.

In the environmental cluster, apart from the lack of an investment framework, consumption of resources received attention, since the activities of the iron and steel industry commence from the iron ores, as discussed in Section 4.1. Significant concern also exists over the behavior of the end-users both through "Insufficient consideration of lifestyles" and "Social resistance against investments." Interestingly, however, the risk of "Social Injustices" that could arise in a transition and affect the local communities received low importance, ending in the second to last place. Despite being concerned over the resistance they may face over the transition of the sector, stakeholders lack the understanding or the will to address the primal reasons that can cause resistance from the community. The importance of understanding the negative impacts, such as job losses, in the process of developing the plan requested by the stakeholders should be a key aspect of a "just transition" [57], built on a social dialogue that includes all interested parties [58].

Regarding the political/institutional framework, the balanced results indicate that there are some concerns over "Fluctuation of CO2 prices" and the "Lack of political leadership" that should not be neglected, but they do not raise immediate threats. On this cluster "Short-sighted energy/climate planning" seems to be the most important risk, with the stakeholders fearing that the current plans lack long-term vision. On the other hand, the stakeholders believe that "Market distortions" and the "Non-evidence-based regulatory framework" do not constitute significant risks, placing them in the lower positions of the ranking.

Having discussed the "Narrow consideration of competition" which has been identified by stakeholders as one of the top two risks, "Nonengaging/unstable markets" also received a comparably high score, establishing the financial cluster as an important factor of the risks associated with the transition. Industries like iron and steel that provide supplies for other major industries are bound to the stability of these markets and especially their reluctancy of adopting cleaner solutions [59]. This is associated with the "Price risks due to new technologies," since low-carbon products may cause higher prices, which may lead to "Uneven distribution of transition costs," two risks that both received medium importance. Financial coordination among the EU countries is also an aspect identified as fairly important by the stakeholders to outbalance the competitive advantage of countries like China, as previously discussed.

In the innovation and technology group, we discussed the importance of developing a clear picture of the envisaged transition, with the clustered results also indicating this picture should incorporate effective information flow channels. In the innovation system of iron and steel, these networks will allow cooperation in the distribution of knowledge and implementation of innovative projects [60]. "Technological lock-ins in iron and steel," "Limited funding capacity," "Bad timing of new industry technologies," and "Little integration across multiple sectors" are risks of medium importance that should be taken into account, as part of this broader strategy.

4.4.3. Consensus level

To calculate the consensus level of the experts compared to the global solution we use the methodology proposed in Section 2.4 and then compare the results with the original method proposed by Ref. [31]. The results are shown in Table 6 and Figure 8, where for both models a value of $b = 1$ was used, since only one round of stakeholder engagement took place so there was no need to add rigorousness on the assessments. Specifically, for the methodology of Ref. [31] the OWA operator was set to $\beta = 0.8$ in the middle of the proposed interval for the variable, the ties in the rankings were not broken, while it is presumed that the set of solutions consist of the alternative ranked first.

Table 6 Consensus measure and proximity levels of individual solutions compared to the collective.

		Herrera-Viedma *et al.* [31]	Proposed Methodology
Proximity level	*EXP1*	94.4	84.4
	EXP2	58.3	83.1
	EXP3	66.3	85.6
	EXP4	55.9	82.8
	EXP5	91.5	84.3
	EXP6	94.9	86.3
	EXP7	60.1	89.1
	EXP8	92.9	79.4
	EXP9	90.8	84.6
	EXP10	64.7	86.9
Consensus measure		**77.0**	**84.6**

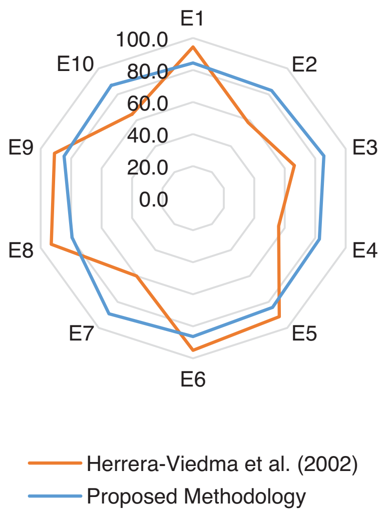

Figure 8 Proximity level of each expert.

From the results, it is showcased that the proposed consensus model is less rigorous than the initial methodology both in terms of the total consensus level and the variance of the proximity of each expert.

Our method results to a consensus level of 84.6% compared to the 77% level of the initial model. The main reason for this difference derives from the way Herrera-Viedma *et al.* calculate the dissimilarity, which is based on the position in the rankings of the collective and the experts' solutions, whereas in the proposed method the scores of TOPSIS are used. In this case study, many alternatives where concentrated around the "medium" scale. For that reason, calculating dissimilarity simply based on the position can exaggerate the existing differences. For example, as we can see on Table 5, the positions from 6 to 11 in the collective solution are separated by only a 0.12 difference in the five-term scale. Therefore, no strong preference can be deduced, rather than merely a tendency. However, the 5-place distance between the rankings of these risks in a total of 25 alternatives can strongly increase the dissimilarity level. This exaggeration is mitigated in the proposed methodology, since the 2-tuple TOPSIS results are used, taking into consideration the exact distance in the assessment of the individual expert and the collective solution, thus using all the available information to calculate the consensus level.

For the proximity levels of each expert to the collective solution, the results show less variance in the proposed methodology compared to the initial partially due to the exaggeration explained above, but also because of the choice and use of the OWA operator, a bias already recognized by Herrera-Viedma *et al.* Specifically, by using the value of $\beta = 0.8$ the alternatives that are considered part of the solution set are given a dominant weight compared to the rest. In this case, we considered the set of solutions to include only the first alternative in the collective solution. However, the argument about the bias can be valid even if more alternatives were included in the solution set, since the exaggeration would simply include a limited number of alternatives rather than the complete set. In MCDM methods based on ranking in the energy sector, valuable insights can be gained even by examining the patterns on the last places [61]. To limit the dependence on the first alternatives, we used the scores of 2-tuple TOPSIS as the weights of the distances placing more importance on the risks ranked higher, while not undercalculating the outputs from the lower positions. For example, Expert 8 performed poorly on the majority of the alternatives both based on our methodology and the calculation of the differences excluding the top alternative for the methodology of Herrera-Viedma *et al.* However, because the expert ranked "R9" first similarly with the collective solution, he received a very high proximity level on the latter method, whereas in our case matching the first solution managed to keep them to adequate proximity levels around 80%, but they were also punished for their failure to assess the rest of the alternatives appropriately, receiving a smaller percentage than the rest of the experts. The opposite phenomenon was observed in the case of Expert 7 who performed very well in most alternatives, but miscalculated the first alternative by rating it with a medium score, leading to an exaggeration of their consensus level by the method of Herrera-Viedma *et al.*

As seen in Figure 5 the experts collectively showed a moderate behavior toward lower grades. For example, Expert 9's solutions show a low deviation with most of them slightly fluctuating around medium values, as seen in Figure 6. However, both consensus models gave the expert a high consensus percentage due to the fact that many alternatives in the collective solutions were also rated around medium. Both models need to consider this bias toward moderate behavior and not punish experts that are more willing to use the full extent of the linguistic scale to better express the existing differences among the alternatives.

5. CONCLUSIONS

In this research, APOLLO, a fuzzy decision support tool is presented to deal with MCDM problems in climate change and policy issues. Stakeholder engagement processes are enabled by using linguistic variables which are more similar to the way experts think. Therefore, it is easier for them to provide the initial feedback and understand the final results derived by the tool. On the first stage, APOLLO uses an adaptation of the 2-tuple TOPSIS [22] to analyze the initial assessments and calculate the ranking and the evaluations of the alternatives for each expert independently. These evaluations are then used as input of the next 2-tuple TOPSIS calculation to find the collective solution of the group of experts [23].

However, the assessments of the experts may include significant dissimilarities, which threaten the acceptance of the final solution. To increase robustness of the solution, APOLLO incorporates a new consensus measuring model that builds on Ref. [31]. The contribution of the model lies on the fact that it uses the 2-tuple TOPSIS evaluations to weight the distances between the experts and the collective solution. From that perspective, each alternative is given the necessary importance for the calculation of consensus and proximity, limiting rigorous assessments.

The added value of APOLLO lies in it constituting a complete tool to perform risk assessments and solve broader problems of DM related to sustainability and decarbonization policies, as its features are tailored to the specificities of the domain (in terms of types of alternatives and criteria, need for large number of stakeholders, and requirements for socially just action driven by consensus). The tool provides robust solutions through measuring consensus among experts, and results that are comprehensible to all audiences and thus all stakeholder groups, making it easier for them to trust the analysis and convert findings into concrete actions.

The tool and the proposed framework are used in an Austrian case study, where stakeholders evaluate the importance of potential risks threatening the low-carbon transition of the iron and steel industry.

We showcase that despite the generally moderate initial answers provided by the stakeholders, many risks received a final evaluation of "high" based on the 5-scale term used for in linguistic model. This indicates that there is a broad concern over the sustainable transition of the sector. Experts agreed with a consensus level of 85% that the most important risks threatening the transition refer to the "Lack of investment framework" and the "Narrow consideration of competition," closely followed by the "Nonengaging/unstable markets" and the "Imperfect picture of the transition." These results can be interpreted as a plea from the experts to policymakers to create a coherent and clear transformational strategy that provides financial resources toward low-carbon technologies that are associated with increased shares of RES production, while also dealing with competition from emerging powerhouses. Regarding the system's ability to manage the high penetration of RES, storage capacity is another risk evaluated as important from the experts.

In our study, a key limitation was that the experts evaluated the alternatives only once, which eliminated the possibility to perform a complete CRP, by providing them with feedback to alter their initial assessments. Therefore, APOLLO can be enhanced to incorporate a CRP cycle [39], which can be tested in a multiple-round stakeholder engagement case study to achieve a higher level of agreement in the group [38]. As part of consensus measuring, the moderate behavior should be formally examined, since current models may punish an expert that deviates from median values. However, such an expert can provide insights along the entire scale used, instead of fluctuating around median values. APOLLO can also be coupled with evolutionary approaches like the "multi-level perspective" [62] to create a holistic framework that captures both the qualitative aspects of innovation in a transition and quantitative multi-criteria risk assessment.

CONFLICTS OF INTEREST

The authors have declared no conflicts of interest.

AUTHORS' CONTRIBUTIONS

Conceptualization, Alexandros Nikas, Konstantinos Koasidis, Haris Doukas; research literature, Alexandros Nikas, Konstantinos Koasidis, Álvaro Labella; methodology, Konstantinos Koasidis, Alexandros Nikas, Álvaro Labella; software development, Álvaro Labella; data curation, Konstantinos Koasidis, Apostolos Arsenopoulos, formal analysis, Konstantinos Koasidis, Álvaro Labella; writing—original draft preparation, Konstantinos Koasidis, Alexandros Nikas, Álvaro Labella; writing—review and editing, Alexandros Nikas, Konstantinos Koasidis, Haris Doukas; funding acquisition, Haris Doukas. All authors have read and agreed to the published version of the manuscript.

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Capítulo 5

Conclusiones y Trabajos Futuros

Para concluir esta memoria expondremos las diferentes conclusiones extraídas en el transcurso de esta investigación, junto con los posibles trabajos futuros que pueden abordarse a partir de los resultados obtenidos.

5.1. Conclusiones

La creciente complejidad en problemas de *Toma de Decisión en Grupo* (TDG) da lugar a la aparición de incertidumbre, que dificulta significativamente la labor de los expertos a la hora de expresar sus opiniones. El modelado lingüístico de las preferencias permite representar dicha incertidumbre de una manera satisfactoria, dando lugar a la *Toma de Decisión Lingüística*. Sin embargo, a pesar de las bondades que presentan los modelos de representación lingüísticos actuales enfocados a problemas de TDG, éstos también presentan diferentes limitaciones en términos de interpretabilidad y/o precisión.

El primer objetivo de esta memoria de investigación consistía en proponer un modelo de representación lingüístico que permitiera superar dichas limitaciones en TDG en contextos lingüísticos. Para lo cual, se ha propuesto el uso de Expresiones Lingüísticas Comparativas con Translación Simbólica (ELICIT), que han permitido mejorar la flexibilidad de las expresiones de los expertos, al ser similares al modelo cognitivo de los seres humanos. Además, su modelo computacional, junto con los operadores de agregación propuestos basados en él, han facilitado la obtención de resultados interpretables y más precisos que en los modelos existentes en la literatura.

Otro de los objetivos marcados era mejorar los *Procesos de Alcance de Consenso* (PAC) en contextos de incertidumbre. En primer lugar, se llevó a cabo un estudio sobre el uso de modelos de consenso clásicos, orientados a problemas de TDG con pocos expertos, en problemas de TDG a gran escala. Este estudio mostró que los modelos clásicos tienen importantes limitaciones y no son capaces de adaptarse a las peculiares características de este tipo de problemas. Por lo que, en primer lugar, se diseñó un modelo de consenso para problemas de TDG a gran escala capaz de afrontar el reto de la escalabilidad mediante la generación de subgrupos de expertos y que propone una nueva medida de cohesión para estos últimos que mejora la convergencia de las opiniones hacia el acuerdo. Posteriormente se han propuesto dos modelos de consenso lingüísticos con expresiones lingüísticas comparativas y ELICIT, que facilitan el modelado lingüístico de las preferencias de los expertos y mejoran la interpretabilidad y precisión de los resultados.

El siguiente objetivo de esta memoria, se basaba en la definición de una métrica que permitiese evaluar el desempeño de distintos modelos de consenso. Por lo que, en primer lugar, se definió el concepto de *modelo integral de mínimo coste*, un modelo basado en programación lineal capaz de obtener la solución óptima de un PAC. A partir de dicho concepto, se desarrolló una métrica de coste que permite evaluar de una forma objetiva el desempeño de distintos PAC sobre un problema de TDG.

Los modelos y herramientas definidas anteriormente se han aplicado a problemas de decisión del mundo real, tal y como indicaba el cuarto objetivo de esta investigación, habiéndose obtenido resultados satisfactorios que proporcionan nuevas soluciones a problemas de TDG que antes no podían ser obtenidas y que mejoran los resultados de modelos del estado del arte.

Finalmente, se han implementado dos sistemas de soporte a la decisión para automatizar y facilitar la resolución de PAC y problemas de TDG anteriores, tal y como pretendía el último objetivo de nuestra investigación. El primer sistema implementado es AFRYCA 2.0, un sistema de carácter general que permite la simulación y análisis de PAC para TDG de cualquier tipo. El segundo sistema, APOLLO, es un sistema de ayuda a problemas de TDG enfocados a políticas climáticas, siendo único en su campo de acuerdo a sus características y funcionalidad.

Por tanto, cabe destacar que se han alcanzado todos los objetivos definidos al inicio de esta memoria de investigación proporcionando herramientas, modelos y resultados que mejoran el estado del arte anterior a nuestra investigación y abren la posibilidad a nuevas investigaciones como las descritas en la siguiente sección.

5.2. Trabajos Futuros

A partir de los resultados obtenidos en esta investigación, se pueden definir posibles trabajos que continúen con la investigación realizada a lo largo de esta tesis doctoral. Estos trabajos futuros son:

- Extender el modelo computacional ligado al modelo de representación lingüístico ELICIT mediante la definición de nuevos operadores de agregación.

- Estudiar nuevos tipos de problemas de decisión, como los de clasificación (sorting), cuyo objetivo es proporcionar una clasificación de las alternativas en diferentes clases y proponer nuevas metodologías que permitan su resolución y aplicarlas a problemas del mundo real.

- Proponer nuevos modelos de consenso que se enfrenten a diferentes retos relacionados con los problemas de toma de decisión actuales como la polarización de opiniones o las opiniones minoritarias.
- Desarrollar nuevas métricas para modelos de consenso que modelen las preferencias de los expertos mediante información lingüística.
- Incrementar la funcionalidad del sistema de ayuda a la decisión AFRYCA 2.0 con la inclusión de nuevos modelos de consenso, tipos de comportamiento y otras nuevas funcionalidades.
- Aumentar la funcionalidad del software APOLLO mediante la inclusión de nuevos modelos de decisión y consenso y posteriormente usarlo en la resolución de distintos problemas relacionados con el cambio climático del mundo real.
- Realizar el proceso de registro de APOLLO para que sea reconocida su autoría.

5.3. Publicaciones Adicionales

En el desarrollo de esta investigación se han presentado otras publicaciones que no han sido recogidas en esta memoria y que enumeramos a continuación:

- En revistas internacionales
  - R. M. Rodríguez, A. Labella, B. Dutta y L. Martínez. Comprehensive minimum cost models for large scale group decision making with consistent fuzzy preference relations. Knowledge-Based Systems, 106780, 2021.
  - A. Labella, A. Ishizaka y L. Martínez. Consensual Group-AHPSort: Applying consensus to GAHPSort in sustainable development and industrial engineering. Computers & Industrial Engineering, 152, 107013, 2021.
  - A. L. Moreno-Albarracín, A. Licerán-Gutierrez, C. Ortega-Rodríguez, A. Labella y R. M. Rodríguez. Measuring What Is Not Seen—Transparency and Good Governance Nonprofit Indicators to Overcome the Limitations of Accounting Models. Sustainability, 12(18): 7275, 2020.
  - A. Labella, J. C. Rodríguez-Cohard, J. D, Sánchez-Martínez y L. Martínez. An AHPSort II Based Analysis of the Inequality Reduction within European Union. Mathematics, 8(4): 646, 2020.

- A. Labella, R. M. Rodríguez y L. Martínez. Extending the linguistic decision suite FLINTSTONES to deal with comparative linguistic expressions with symbolic translation information. Journal of Intelligent & Fuzzy Systems, 39: 6245–6258, 2020.
- L. Wang, A. Labella, R. M. Rodríguez, Y. M. Ming Wang y L. Martínez. Managing non-homogeneous information and experts' psychological behavior in group emergency decision making. Symmetry, 9(10): 234, 2017.

- En congresos internacionales

  - A. Labella, R. M. Rodríguez y L. Martínez. Green Supplier Selection by means of a Decision Making Method based on ELICIT Information. En 14th International Conference FLINS Conference, 750-758, World Scientific, 2020.
  - A. Labella, D. Uztürk, R. M. Rodríguez, G. Büyüközkan y L. Martinez. Product development partner selection based on ELICIT information, En 14th International Conference FLINS Conference, 767-775, World Scientific, 2020.
  - A. Nikas, A. Arsenopoulos, H. Doukas y A. Labella. Prioritisation of risks associated with decarbonisation pathways for the Austrian iron and steel sector using 2-tuple TOPSIS, En 14th International Conference FLINS Conference, 776-783, World Scientific, 2020.
  - A. L. Moreno-Albarracín, C. Ortega-Rodríguez, A. Licerán-Gutiérrez, A. Labella y Luis Martínez. How are donations managed? A proposal of transparency measurement for non-profit organizations in a Spanish setting. En XIX Encuentro Internacional de la Asociación Española de Profesores Universitarios de Contabilidad, 2020.
  - A. Labella, H. Liu, R. M. Rodríguez, L. Martínez. Comprehensive Minimum Cost Models Based on Consensus Measures. En The international virtual workshop of business analytics EUREKA 2019, 2019.
  - A. Labella, A. Ishizaka y L. Martínez. Consensual Group-AHPSort. En 30th European Conference on Operational Research, 2019
  - A. Labella, R. M. Rodríguez y L. Martínez. A Comparative Performance Analysis of Consensus Models Based on a Minimum Cost Metric. En International Conference on Intelligent and Fuzzy Systems, 1506-1514. Springer, Cham, 2020.
  - A. Labella, R. M. Rodríguez y L. Martínez. A Novel Linguistic Cohesion Measure for Weighting Experts' Subgroups in Large-Scale Group Decision Making Methods. En 2019 IEEE 14th International Conference on Intelligent Systems and Knowledge Engineering (ISKE), 9-15. IEEE, 2019.

- A. Labella, R. M. Rodríguez y L. Martínez. An Adaptive Consensus Reaching Process Dealing with Comparative Linguistic Expressions in Large-scale Group Decision Making. En 11th Conference of the European Society for Fuzzy Logic and Technology (EUSFLAT 2019), 170-177. Atlantis Press, 2019.
- A. Labella y L. Martínez. FLINTSTONES 2.0 an Open and Comprehensive Fuzzy Tool for Multi-criteria Decision Analysis. En International Conference on Intelligent and Fuzzy Systems, 762-769. Springer, Cham, 2019.
- D. Uztürk, A. Labella, G. Büyüközkan y L. Martinez. Fuzzy linguistic integrated methodology for sustainable hospital building design. En International Conference on Intelligent and Fuzzy Systems, 1180-1188. Springer, Cham, 2019.
- A. Labella, R. M. Rodríguez, G. De Tré y L. Martínez. A Cohesion Measure for Improving the Weighting of Experts' subgroups in Large-scale Group Decision Making Clustering Methods. En 2019 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE), 1-6. IEEE, 2019.
- R. Yera, A. Labella, J. Castro y L. Martínez. On group recommendation supported by a minimum cost consensus model. En Data Science and Knowledge Engineering for Sensing Decision Support: Proceedings of the 13th International FLINS Conference (FLINS 2018), 11: 227, 2018.
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- A. Labella, L. Martínez y R. M. Rodríguez. Can classical consensus models deal with large scale group decision making?. En 2017 12th International Conference on Intelligent Systems and Knowledge Engineering (ISKE), 1-7. IEEE, 2017.
- L. Wang, A. Labella, R. M. Rodríguez y Luis Martínez. An emergency group decision making method based on prospect theory. En 14th Information Systems for Crisis Response and Management (ISCRAM), 2017.

- Congresos nacionales

  - A. Labella, R. M. Rodríguez y L. Martínez. Uso de expresiones lingüísticas comparativas en AFRYCA 3.0. En XVIII Conferencia de la Asociación Española para la Inteligencia Artificial (CAEPIA 2018) 23-26 de octubre de 2018 Granada, España (pp. 341-346). Asociación Española para la Inteligencia Artificial (AEPIA), 2018.

- R. M. Rodríguez, Á. Labella y L. Martínez. Un modelo de consenso para toma de decisiones en grupo a gran escala usando conjuntos difusos dudosos. En XVIII Conferencia de la Asociación Española para la Inteligencia Artificial (CAEPIA 2018) 23-26 de octubre de 2018 Granada, España (pp. 316-321). Asociación Española para la Inteligencia Artificial (AEPIA).
- A. Labella, F. J. Estrella y L. Martínez. AFRYCA 2.0: Análisis de Procesos de Alcance de Consenso. En XVII Conferencia de la Asociación Española para la Inteligencia Artificial (CAEPIA 2016) 14-15 de septiembre de 2016 Salamanca, España (pp. 573-582). Asociación Española para la Inteligencia Artificial (AEPIA), 2016.
- F. J. Estrella, R. M. Rodríguez, A. Labella y L. Martínez. Un Sistema Basado en FLINTSTONES para Procesos de Selección Mediante un Modelo Difuso TOPSIS. En XVI Conferencia de la Asociación Española para la Inteligencia Artificial (CAEPIA 2015) 9-12 de noviembre de 2015 Albacete, España (pp. 491-500). Asociación Española para la Inteligencia Artificial (AEPIA), 2015.

5.4. Reconocimientos

Cabe destacar que algunos de los trabajos elaborados a lo largo de esta tesis doctoral recibieron un reconocimiento por parte de la comunidad científica. En particular, dos presentados en congresos internacionales, el primero titulado “FLINTSTONES 2.0 an Open and Comprehensive Fuzzy Tool for Multi-criteria Decision Analysis” expuesto en la *International Conference on Intelligent and Fuzzy Systems* en Estambul, Turquía, que recibió el premio *Best student paper award*. Por otro lado, el artículo titulado “A Novel Linguistic Cohesion Measure for Weighting Experts’ Subgroups in Large-Scale Group Decision Making Methods”, presentado en la *IEEE 14th International Conference on Intelligent Systems and Knowledge Engineering*, en Dalian, China, recibió el premio *Best paper award*. Además, el software desarrollado en esta tesis doctoral, AFRYCA 2.0, recibió en 2017 el premio en la modalidad *Mejor aplicación* en los *III Premios Ada Lovelace en Tecnologías de la Información y la Comunicación de la Universidad de Jaén*. Todos los diplomas asociados a los anteriores reconocimientos se incluyen a continuación.

Best Student Paper Award

Dear

Álvaro Labella & Luis Martinez,

INFUS community would like to thank you for taking part at International Conference on Intelligent and Fuzzy Systems organized by Industrial Engineering Department of Istanbul Technical University in July 23-25, 2019 at Istanbul, Turkey.

Your research,

"FLINTSTONES 2.0 an Open and Comprehensive Fuzzy Tool for Multi-Criteria Decision Analysis"

has been selected to be the best student paper presented at INFUS 2019.

INFUS
International Conference on
Intelligent and Fuzzy Systems

Istanbul Technical University
Faculty of Management
Industrial Engineering Department

İTÜ

Prof.Dr.Cengiz Kahraman
Conference
Chair

IEEE 14th International Conference on Intelligent Systems and Knowledge Engineering
(ISKE2019)

BEST PAPER AWARD

Presented to:

Alvaro Labella, Rosa M. Rodr´ıguez, Luis Mart´ınez

Paper Title:

A Novel Linguistic Cohesion Measure for Weighting Experts' subgroups in Large-scale Group Decision Making Methods (SS14: Fuzzy decision making)

Date: November 14-16, 2019
Location: Dalian, China

Li Zou

Li Zou
Chair of International Program Committee
of ISKE2019

Jie Lu
Chair of ISKE2019 Steering Committee

Universidad de Jaén

CEATIC

CENTRO DE ESTUDIOS AVANZADOS EN TECNOLOGÍAS DE LA INFORMACIÓN Y LA COMUNICACIÓN

D. ÁLVARO LABELLA ROMERO

con DNI Nº: 77348227D

Ha quedado como ganador en los III Premios Ada Lovelace en Tecnologías de la Información y la Comunicación de la Universidad de Jaén, en la modalidad a la mejor aplicación, por su propuesta "AFRYCA: A Framework for the analysis of Consensus Approaches", organizados por el CEATIC , y para que conste a los efectos oportunos, se expide el presente:

CERTIFICADO GANADOR

III Premios Ada Lovelace en Tecnologías de la Información y la Comunicación de la Universidad de Jaén

Jaén, 8 de Noviembre de 2017

Director del Centro de Estudios Avanzados en Tecnologías de la Información y la Comunicación

D. Luis Alfonso Ureña López

5.5. Estancias y Colaboraciones

En el transcurso de la tesis doctoral se han llevado a cabo varias estancias y colaboraciones en el extranjero, con el objetivo de mejorar la formación investigadora del doctorando a través del conocimiento y la experiencia de personas expertas en la temática.

Gracias a las becas EDUJA 2017 que ofrece la Universidad de Jaén, ha sido posible realizar una estancia de 3 meses en la School of Electrical and Computer Engineering en la National Technical University (NTUA), Atenas, colaborando de forma estrecha con el Prof. Haris Doukas y su grupo en el desarrollo de un proyecto Europeo de investigación.

Además, a través de una beca de Formación Predoctoral (FPI) otorgada por parte del Ministerio de Ciencia, Innovación y Universidades, se realizó otra estancia con una duración de 2 meses en la Universidad de Portsmouth, Reino Unido, junto al Prof. Alessio Ishizaka.

Apéndice A

English Summary

This Appendix covers an English summary of the thesis entitled, *Complex Modelling of Linguistic Information for Decision Making Problems under Uncertainty*, which is written in English language, as partial requirement for obtaining the International Ph.D Award.

Firstly, a brief introduction to the research topic and a motivation for the research conducted is shown. The objectives established in such a research are then exposed and the structure of chapters that compose this research memory is described. After that, a summary of the research proposals of this memory is presented. Finally, some conclusions, future works and publications related to this research are pointed out.

A.1. Motivation

In our daily life, we are used to face multiple situations of *Decision Making* (DM), everyday we need to choose for instance the clothes we are going to wear or what we are going to have for breakfast. Formally, DM is defined as a cognitive process, in which through different mental and reasoning processes an expert selects, among multiple alternatives or possible solutions, the best one [73]. In certain complex situations, it is very common that the resolution of a DM problem is not carried out only by one person, but by a set of experts with different points of view and knowledge, giving rise to what is known as *Group Decision Making* (GDM) [34, 62, 66, 76, 80].

The participation of several experts in the resolution of GDM problems inevitably implies the appearance of polarization, conflicts and disagreements among the experts when choosing the solution to the problem [16, 17]. Classical resolution schemes for GDM problems did not take this aspect into account, so it was possible to obtain a solution in which not all experts agreed, because they may feel ignored and out of the decision process [7, 61]. For this reason, prior to the process of selecting the best alternative, a *Consensus Reaching Process* (CRP) is included, in which experts discuss and modify their initial opinions with the objective of reaching a solution that satisfies as many experts as possible. Although, CRPs are key to obtain agreed solutions in GDM problems, there is such a large number of consensus models proposed in the literature [34, 35, 56, 83], that it is often really complex to determine which model best fits a given decision problem [49]. The lack of metrics that allow evaluating the performance of these models on a GDM problem is presented as one of the main limitations within this field of DM.

On the other hand, most real world GDM problems and their corresponding CRPs are defined in contexts that are constantly changing, leading to a lack of information and the emergence of uncertainty. Therefore, not all DM problems are as simple as those mentioned above, many present uncertainty whose nature is not probabilistic and which are called *GDM under uncertainty* problems [33]. In this type of problems, experts may find difficulties to express their knowledge appropriately, so they prefer to use linguistic expressions closer to their way of thinking. Under these circumstances, *fuzzy logic* [88], *fuzzy linguistic approach* [89] and other *soft computing* tools have been used with great success in modelling uncertainty in GDM problems using linguistic variables, giving rise to *Linguistic Decision Making* (LDM).

The use of linguistic expressions to model expert judgements in LDM problems implies the need of performing operations on linguistic information. There are numerous methodologies to perform these operations, but within the field of fuzzy logic, the *Computing with Words* (CW) approach stands out [20, 41, 84, 90]. Through this methodology, computations are performed on words or phrases given in a natural or artificial language, rather than numerical values, mimicking the human beings' reasoning process. A key premise of this methodology is that the input information must be linguistic and, once is manipulated, the results must be also linguistically expressed to ensure comprehension (see Fig. A.1).

Figure A.1: Computing with Words general scheme.

Nowadays, there are many computational models applied to LDM problems that follow a CW approach and allow modelling experts' opinions using linguistic information [59]. One of the most prominent is the *2-tuple linguistic model* [39] which, thanks to the use of symbolic translation, allows to carry out operations on a continuous domain with high accuracy. However, a 2-tuple linguistic value is composed of only one linguistic term, which may be insufficient in problems with high complexity where experts may hesitate and are not able to decide on a single linguistic term. To overcome this limitation, other approaches have defined processes to elaborate more complex linguistic expressions that allow modelling the experts' doubt, such as the *hesitant fuzzy linguistic terms set* (HFLTS) [57], the *comparative linguistic expressions* (CLEs) [58], etc. However, these new proposals still present several limitations in terms of expressiveness and/or precision that are summarized below:

1. *Modelling uncertainty*: some proposals [39, 78] are not able to represent expert's doubt in GDM problem, limiting the linguistic representation to a single term, insufficient considering that GDM problems are becoming more and more complex and doubt in experts' opinions more common.

2. *Modelling linguistic expressions*: although some proposals model experts' opinions using linguistic expressions that are more complex than a single linguistic term [19, 71], often these expressions are far from the way humans express themselves and are difficult to understand and unusable in practice.

3. *Precision and interpretability*: it is common in many approaches [45, 75] to transform the input linguistic information into numerical values, which implies loss of information and precision in the results. Moreover, such results are represented by non-linguistic structures that are difficult for experts to interpret, infringing the main feature of the CW approach.

As mentioned above, there are a large number of decision and consensus models proposed in the literature, each with its own characteristics, advantages and disadvantages. However, these models are often not simple to understand, most are algorithms composed of multiple steps or based on mathematical models such as linear programming [69]. Considering the high complexity faced by experts when solving a GDM problem and even more so under conditions of uncertainty where information is vague and imprecise, it is unthinkable that they also have to invest their time in understanding, analyzing and manually applying these models, further increasing such complexity and keeping in mind that, often, certain decision situations are taken under pressure and require a quick solution. Therefore, the development of *Decision Support Systems* (DSS) that facilitate the experts' task, and thus the resolution of DM problems in any context, is key. In spite of this, there is a great lack of software tools focused on this objective and, the existing ones, present limitations such as the impossibility to deal with DM problems under uncertainty [26], an insufficient battery of available decision models [18], or the inability to solve problems applying a CW approach [25].

The main limitations in the current linguistic models for solving LDM problems and their CRPs and the lack of software tools to manage such problems led us at the beginning of this research to formulate the following hypothesis:

1. *The definition of a new and better methodological framework with models, methodologies and tools based on soft computing for the fuzzy modelling of uncertainty that, by means of complex linguistic models for GDM processes under uncertainty and CRPs, will allow overcoming different challenges imposed by the new circumstances and trends in which these problems have to be developed and that cannot be solved at present.*

2. *The definition of a metric for CRPs will facilitate a better evaluation of the performance of the different current and new CRPs.*

3. *The application of a novel methodological framework in new CRPs and GDM models. In addition, its integration in a software system, which will produce a major progress in real-world CRPs and GDM by facilitating automatic problem solving and supporting decision makers with understandable and appropriate tools.*

A.2. Objectives

Taking into account the limitations previously exposed in the current LDM models and the starting hypothesis, our goal in this doctoral thesis is focused on the investigation and definition of DM models and complex linguistic CRPs, which allow overcoming these limitations. Based on this, we propose the following objectives:

1. *Defining a methodological framework for modelling and treatment of uncertainty in GDM and its CRPs using complex linguistic expressions*, which allows to model in an appropriate way the experts' opinions and to obtain easily interpretable and accurate results.

2. *Defining new complex linguistic consensus models for GDM problems under uncertainty* that overcome the limitations of the existing proposals in the specialized literature, improving the detection of dissent in the group and the recommendations for the experts and, in this way, increasing the consensus among them in the shortest possible time.

3. *Elaborating metrics for consensus processes that establish performance benchmarks in the scope of consensus* and thus analyse and select the best CRP to apply in each GDM problem.

4. *Study different GDM problems and CRPs in the real world*, identifying their main characteristics and the challenges they pose in order to analyse and select the resolution approach that provides the best possible solution.

5. *Support for the above problems* by developing DSS that help experts handle the increasing complexity inherent in GDM problems.

A.3. Structure

This doctoral thesis, in accordance with the provisions of Article 25, point 2, of the current regulations of the Doctoral Studies at the University of Jaén (RD. 99/2011), will be composed of a series of articles published by the PhD student, whose purpose is based on achieving the objectives set out in the previous section. Specifically, this research memory is composed of ten articles, nine of them published in international journals indexed in the Journal Citation Reports (JCR) database, and the one published in a international journal indexed by Scopus.

The research memory is divided into the following chapters:

1. **Chapter 2**: basic concepts related to the topic of the doctoral thesis are reviewed. We will introduce the fuzzy logic theory and the fuzzy linguistic approach. Subsequently, we will focus on concepts related to GDM, LDM and the CW approach. Furthermore, we will analyse the advantages and limitations of existing linguistic decision models focusing mainly on the 2-tuple linguistic model, HFLTSs and CLEs. Finally, we will discuss the need of CRPs to reach consensus solutions.

2. **Chapter 3**: it will summarize the main proposals that make up this research memory, highlighting the results obtained and the conclusions drawn in each of them.

3. **Chapter 4**: the ten articles previously mentioned compose this section.

4. **Chapter 5**: finally, the main conclusions obtained throughout the development of the doctoral thesis are drawn and possible future works are outlined.

Finally, this memory concludes with a bibliographic compilation of the most relevant articles related to this doctoral thesis.

A.4. Background

In this chapter, we will briefly summarize the theoretical concepts and background related to the research presented in this memory. Initially, we will introduce basic concepts about fuzzy logic and the fuzzy linguistic approach. We will go then deeper into the definition of decision making under uncertainty and we will analyse some of the most important proposals that allow modelling such uncertainty by means of linguistic expressions. Finally, we will describe the consensus reaching processes in decision making.

A.4.1. Fuzzy Logic and Fuzzy Linguistic Approach

L. Zadeh introduced the *Fuzzy Logic Theory* [88] for the purpose of modelling uncertainty or imprecision. For this purpose, he extended the definition of classical set to the *fuzzy set*, where the set boundaries are not strictly defined. On the one hand, a classical set is marked by a strict dichotomy constraint, i.e., an object may or may not belong to a set. This binary classification can be defined by the following characteristic function:

Definition 1 *Let A a set in a discourse universe X, the characteristic function associated to A, $A(x), x \in X$, is defined as follows:*

$$A(x) = \begin{cases} 1, & if\ x \in A \\ 0, & if\ x \notin A. \end{cases}$$

According to the Definition 1, the membership or not of an object x to the set A is defined by a function $A : X \to \{0, 1\}$ whose value is 1 when the object belongs to the set and 0 otherwise. The definition of a fuzzy set relaxes the restriction of the characteristic function of a classical set, allowing intermediate values to be obtained. In a fuzzy set, the characteristic function is called *membership function*:

Definition 2 *[88] A fuzzy set $\tilde{A}$ over X is defined by a membership function that transforms the elements belonging to the discourse universe X in the interval* $[0, 1]$.

$$\mu_{\tilde{A}} : X \longrightarrow [0, 1]$$

Therefore, a fuzzy set $\tilde{A}$ in X can be represented by a set of ordered pairs formed by an element $x \in X$ and its membership degree $\mu_{\tilde{A}}(x)$:

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) / \ x \in X, \mu_{\tilde{A}}(x) \in [0, 1]\}$$

The membership function of a fuzzy set is more complex than the characteristic function of a classical set, since it allows to obtain a membership value between 0 and 1, the closer to 1, the higher the degree of membership. Therefore, it is necessary to define functions describing the membership of a fuzzy set. Although fuzzy sets can be represented by many types of functions, as long as they satisfy the condition $\mu_{\tilde{A}} \in [0, 1]$, parametric functions are the most commonly used. Within this family of functions, the most common are the triangular and trapezoidal type (see Fig. A.2), whose membership functions are defined below:

- *Triangular membership function*:

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & \text{if} \quad x \leq a \\ \dfrac{x-a}{b-a}, & \text{if} \quad x \in (a, b] \\ \dfrac{c-x}{c-b}, & \text{if} \quad x \in (b, c) \\ 0, & \text{if} \quad x \geq c \end{cases}$$

- *Trapezoidal membership function*:

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & \text{if} \quad x \leq a \\ \dfrac{x-a}{b-a}, & \text{if} \quad x \in (a, b] \\ 1 & \text{if} \quad x \in (b, c] \\ \dfrac{d-x}{d-c}, & \text{if} \quad x \in (c, d) \\ 0, & \text{if} \quad x \geq d \end{cases}$$

Fuzzy Logic plays a fundamental role dealing with most real world decision problems, which are usually defined under a context of uncertainty and lack of information. The key question is how to model such uncertainty in a simple and human interpretable way, the answer

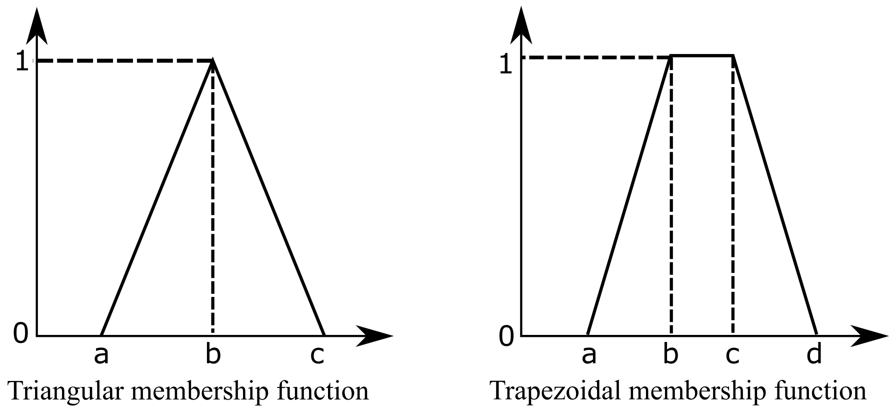

Figure A.2: Parametric functions.

has already been addressed with great success through *linguistic modelling* [38]. Linguistic modelling of uncertainty allows to use natural language words such as *high*, *easy* or *comfortable* to evaluate qualitative aspects of a problem that have to do with perceptions or feelings. There are multiple approaches for modelling linguistic information [42, 43, 89] but in this research, the *fuzzy linguistic approach* has been used.

The fuzzy linguistic approach is based on Fuzzy Set Theory and allows modelling linguistic information by means of the concept of *linguistic variable* defined by L. Zadeh [89]. In the words of L. Zadeh, a linguistic variable is a "*variable whose values are not numbers but words or phrases in a natural or artificial language*". The formal definition of a linguistic variable is presented below:

Definition 3 *[89] A linguistic variable is composed of a quintuple (H, T(H), U, G, M), in which H represents the name of the variable, T(H) a set of linguistic terms of H, where each value is a fuzzy variable denoted as X and varying over the universe of discourse U, G is a syntactic rule for generating the names of the values of H and M is a semantic rule that associates meaning M(X) to each element of H, which is a fuzzy set of U.*

In summary, a linguistic variable is mainly made up of a syntactic value or label (a word belonging to a set of linguistic terms) and a semantic value represented by a given fuzzy set in a universe of discourse.

In Fig. A.3 we can see an example of a set of linguistic terms. From this set of terms, a person could express knowledge using any of the linguistic descriptors that compose the set, in this case, *Horrible*, *Very bad*, *Bad*, *Average*, *Good*, *Very good* or *Excellent*. We can also see that the semantics of the variables are represented by fuzzy triangular membership functions, although other types of functions could be used, such as the trapezoidal ones mentioned above.

The number of terms of the set of linguistic terms (also so-called *cardinality*) is an important aspect to consider how defining a set of linguistic terms. This decision will depend on the degree of knowledge that is intended to be expressed, a set with few terms implies lack of

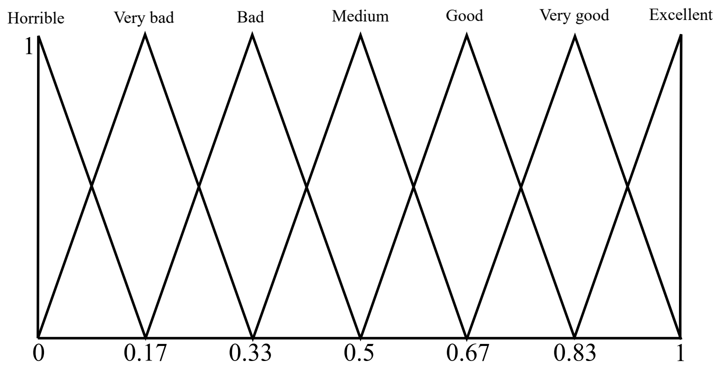

Figure A.3: Linguistic terms set.

knowledge and, in turn, loss of expressiveness, on the contrary, a set with a high cardinality, presents a higher granularity of uncertainty, which is adequate when the degree of knowledge is high. The most common values of cardinality are usually odd values such as 5, 7 or 9 [44], in which the middle term represents a value of approximately 0.5 and the rest of the terms are distributed around this one [6].

On the other hand, fuzzy modelling of linguistic information has not been limited exclusively to the use of single linguistic terms. The need for more complex and flexible linguistic expressions to appropriately represent people's knowledge, has given rise to several proposals based on the fuzzy linguistic approach. This doctoral thesis introduces in Section 4.1 a publication which reviews some of such extensions.

A.4.2. Decision Making under Uncertainty

Decision Making (DM) is a daily activity in human beings' life that involves selecting, among a set of possible alternatives, the best one as a solution to a decision problem. Some DM problems are so simple that they can be solved in a short period of time and by a single person. However, other DM problems turn out to be much more complex and require the participation of several experts with different points of view and knowledge [28, 46, 52, 77, 82] giving rise to the *Group Decision Making* (GDM). Formally, a GDM problem consists of a finite set of experts $E = \{e_1, e_2, \ldots, e_m\}$ whose main task is to select one or several alternatives within a finite set of possible options $X = \{x_1, x_2, \ldots, x_n\}$ as solution(s) to the decision problem. In multiple problems, alternatives are evaluated from a finite set of attributes or criteria $C = \{c_1, c_2, \ldots, c_s\}$, resulting in the *Multi-criteria Decision Making* [27, 60, 87].

The classical resolution scheme of a GDM problem is composed of two phases (see Fig. A.4):

1. *Aggregation*: the individual experts' opinions on each alternative and criterion are aggregated using an appropriate aggregation operator. In this way, a collective value is obtained for each alternative of the problem.

2. *Exploitation*: the collective values obtained in the previous phase are ordered by means

of selection functions that allow selecting the best alternative(s) as the solution to the problem.

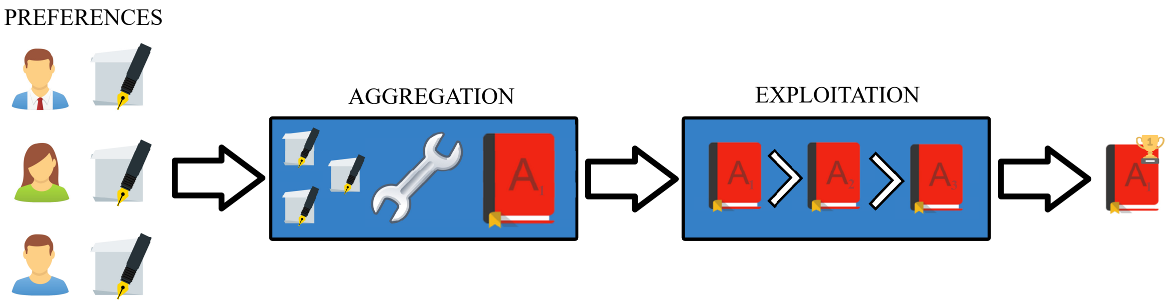

Figure A.4: General resolution scheme for a GDM problem.

In the real world, human beings face DM problems conditioned by the lack of information and the inevitable apparition of uncertainty, since it is almost impossible to know all states of the nature of the problem. The modelling of such uncertainty by means of linguistic information has offered excellent results [21, 40], giving rise to the *Linguistic Decision Making* (LDM) problems. In this type of problems, the fuzzy linguistic approach is presented as one of the most widely used approaches to model experts' preferences by means of linguistic expressions (see Section A.4.1).

The resolution scheme for a LDM problem varies slightly from the classical one, incorporating two additional phases [21] (see Fig. A.5):

1. *Selection of the set of linguistic terms and their semantics*: the set of linguistic terms that the experts will use to appropriately express their knowledge about the set of alternatives is defined.

2. *Selection of an aggregation operator for linguistic information*: the opinions provided by the experts through linguistic expressions are aggregated by means of a linguistic operator, obtaining a collective value for each alternative.

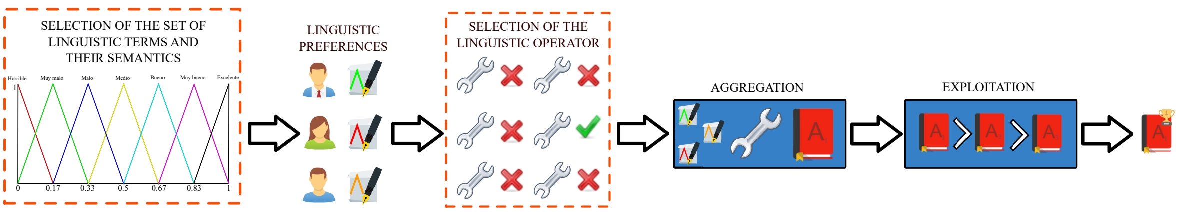

Figure A.5: General resolution scheme for LDM problems.

The resolution scheme presented in Fig. A.5 evidences the need to perform operations with linguistic information to find the solution to a LDM problem. In this sense, the *Computing* *with Words* (CW) approach [42, 90] mimics the reasoning process of the human beings, generating linguistic results from also linguistic premises. According to the definition provided by L. Zadeh, CW is "*a methodology in which words are used instead of numbers to calculate, reason and make decisions*". The CW methodology has been successfully applied to carry out computational processes in several fields such as LDM problems [12, 20], machine learning [48] or database [86].

In this research memory, we will focus on the computational processes carried out through CW in LDM problems, in which this methodology has been intensively applied [12, 20, 39, 40] and which has given rise to different CW schemes [67, 84, 85]. However, they all emphasize the need for accurate and understandable linguistic results. R.R. Yager introduced a CW scheme consisting of two main processes, *translation* and *retranslation* (depicted in Fig. A.6). The former consists of translating the input linguistic information into a fuzzy logic-based format that can be manipulated by a machine. The second one is responsible for retranslating the manipulated information back to a linguistic format that is easy to interpret by humans.

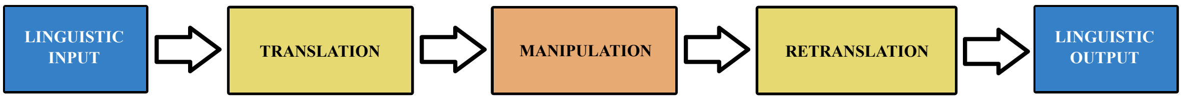

Figure A.6: CW scheme proposed by R. R. Yager.

A.4.3. Linguistic Computational Models

As mentioned in the previous section, modelling uncertainty by means of linguistic information involves carrying out CW processes. Based on this, a large number of computational linguistic models that carry out operations with linguistic information have been proposed. In this section, we will briefly review the most relevant models related to the research developed in this memory. These same models, along with others, are reviewed together with their respective computational models in greater depth in one of the articles included in Chapter 4, Section 4.1.

A.4.3.1. The 2-tuple Linguistic Model

The *2-tuple linguistic model* [39], based on the fuzzy linguistic approach, is one of the most widely used linguistic representation models in LDM. The main features of this model are the high interpretability and precision of the results. The first one is achieved thanks to the development of CW processes that allow obtaining results represented in a linguistic way. The second is determined by the representation in a continuous domain of the linguistic values, which allows obtaining accurate results without loss of information.

One of the most important concepts presented in the 2-tuple linguistic model is the *symbolic translation*, a numeric value representing the shift of the membership function of a linguistic label. Formally, the linguistic information in the 2-tuple linguistic model is represented from a pair of values (s_i, α) in which s_i is a linguistic label belonging to a set of linguistic terms $S = \{s_0, s_1, \dots, s_g\}$ and the symbolic translation value $\alpha \in [-0{,}5, 0{,}5)$, which represents the shift of the membership function of the s_i term (see Fig A.7). The value of α is defined as:

$$\alpha = \begin{cases} [-0{,}5, 0{,}5) & if\ s_i \in \{s_1, s_2, \dots, s_{g-1}\} \\ [0, 0{,}5) & if\ s_i = s_0 \\ [-0{,}5, 0] & if\ s_i = s_g \end{cases}$$

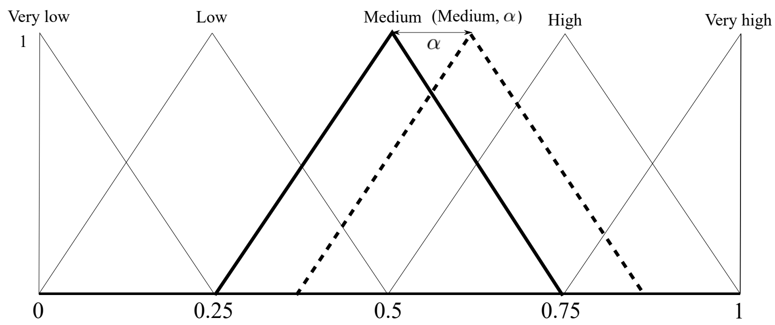

Figure A.7: Symbolic translation.

A 2-tuple linguistic value $(s_i, \alpha) \in \overline{S}$, where $\overline{S}$ is the set of 2-tuple linguistic values associated with S defined as $\overline{S} = S \times [-0{,}5, 0{,}5)$, can also be represented by an equivalent numerical value $\beta \in [0, g]$:

Proposition 1 *[39] Let $S = \{s_0, \dots, s_g\}$ a linguistic terms set and $(s_i, \alpha) \in \overline{S}$ a 2-tuple linguistic value. There is a function, Δ^{-1} so:*

$$\Delta^{-1} : \overline{S} \to [0, g]$$
$$\Delta_S^{-1}(s_i, \alpha) = \alpha + i = \beta$$

In turn, a numerical value $\beta \in [0, g]$ can be transformed to its respective 2-tuple linguistic value as follows:

Definition 4 *[39] Let $S = \{s_0, \dots, s_g\}$ be a linguistic terms set and $\overline{S}$ a set of 2-tuple linguistic values associated with S defined as $\overline{S} = S \times [-0{,}5, 0{,}5)$. The function $\Delta_S : [0, g] \to \overline{S}$ is given by:*

$$\Delta_S(\beta) = (s_i, \alpha),\ with\ \begin{cases} i = round(\beta) \\ \alpha = \beta - i \end{cases}$$

where $round(\cdot)$ assigns to β the closest integer number $i \in \{0, \dots, g\}$.

The 2-tuple linguistic model was defined together with a computational model that can be consulted in further detail in references [23, 39].

A.4.3.2. Hesitant Fuzzy Linguistic Terms Set

The 2-tuple linguistic model presents very remarkable advantages in terms of accuracy and interpretability. However, 2-tuple linguistic values are represented by a single linguistic term, which may be insufficient in situations where experts hesitate between several linguistic terms when expressing their opinions. With the aim of overcoming this limitation and modelling experts' hesitation, the *Hesitant Fuzzy Linguistic Terms Set* (HFLTS) was defined [57].

Definition 5 *[57] Let S a linguistic term set, $S = \{s_0, \dots, s_g\}$, a HFLTS H_S is defined as a ordered finite subset of consecutive linguistic terms belonging to S.*

$$H_S = \{s_i, s_{i+1}, \dots, s_j\}$$

To clarify this concept, let see an example:

Example 1 *Let us suppose a linguistic term set S = {Very insecure, Insecure, Medium, Secure, Very secure}, some illustrative HFLTS may be:*

$$H_S^1 = \{Insecure, Medium\}$$
$$H_S^2 = \{Medium, Secure, Very\ secure\}$$
$$H_S^3 = \{Secure, Very\ secure\}$$

A.4.3.3. Comparative Linguistic Expressions

HFLTSs allow experts to express their opinions through several linguistic terms in situations of doubt where they are not clear about which one to choose. However, these are quite far from the way human beings elicit their opinions. Therefore, it is evident the need to create more complex linguistic expressions that allow modelling the experts' doubt with a structure similar to the expressions used by human beings to express their knowledge. With this aim, Rodriguez et al. [58] defined a new type of more expressive and complex linguistic expressions so-called *Comparative Linguistic Expressions* (CLEs).

CLEs are based on HFLTSs, but they are generated using a context-free grammar, which allows modelling expressions closer to the language used by experts in LDM problems. Rodriguez et al. introduced the following context-free grammar to generate CLEs [58]:

Definition 6 *[58] Let G_H a context-free grammar and $S = \{s_0, \dots, s_g\}$ a linguistic terms set. The elements of $G_H = (V_N, V_T, I, P)$ are defined as:*

$V_N = \{(primary\ term),\ (composite\ term),$

$(unary\ relation),\ (binary\ relation),\ (conjunction)\}$

$V_T = \{at\ least,\ at\ most,\ lower\ than,\ greater\ than,\ between,\ and,\ s_0,\ s_1, \dots, s_g\}$

$I \in V_N$

The production rules are defined in an extended Backus-Naur form:

$$P = \{I ::= \langle primary\ term\rangle | \langle composite\ term\rangle$$
$$\langle composite\ term\rangle ::= \langle unary\ relation\rangle\langle primary\ term\rangle |$$
$$\langle binary\ relation\rangle\langle primary\ term\rangle\langle conjunction\rangle\langle primary\ term\rangle$$
$$\langle primary\ term\rangle ::= s_0|s_1|\dots|s_g$$
$$\langle unary\ relation\rangle ::= at\ least|at\ most|lower\ than|greater\ than$$
$$\langle binary\ relation\rangle ::= between$$
$$\langle conjunction\rangle ::= and\}$$

Example 2 *Let us suppose a linguistic terms set $S = \{$Very uncomfortable, Uncomfortable, Normal, Comfortable, Very comfortable$\}$ and the context-free grammar G_H shown in Definition 6, some illustrative CLEs may be:*

$$ELC_1 = at\ least\ Comfortable$$
$$ELC_2 = at\ most\ Normal$$
$$ELC_3 = lower\ than\ Comfortable$$
$$ELC_4 = greater\ than\ Normal$$
$$ELC_5 = between\ Comfortable\ and\ Very\ comfortable$$
$$ELC_6 = Normal$$

A.4.4. Consensus Reaching Processes

In section A.4.2 we have seen different resolution schemes for GDM and LDM problems (see Figs. A.3 and A.4). In both schemes, it can be seen that the experts' opinions are directly aggregated, ignoring the possible disagreements that may exist between them. The main consequence of this omission is that some experts may not agree with the solution obtained, feeling ignored and questioning the confidence in the decision process. Nowadays, consensual decisions are really valued in different areas of society [16, 49, 82], so it seems evident the need to add a *Consensus Reaching Process* (CRP) in the resolution scheme of a GDM problem before the selection of the best alternative.

Before defining in detail what a CRP is, we will explain the meaning of *consensus*. The concept of consensus can generate some controversy, since there are multiple points of view on its meaning. Some strict approaches define consensus as *unanimity* or total agreement, which can hardly be achieved in practice [32]. Other approaches are more flexible, such as the view of Kacprzyk, who proposed the concept of *soft consensus* [29, 30], an approach based on the *fuzzy majority* that establishes consensus in a group when "*the majority of most relevant experts agree on almost all relevant options*". In this research, we will consider Kacprzyk's view of soft consensus.

A CRP is an iterative and dynamic process in which experts discuss, review and modify their initial opinions with the aim of bringing positions closer together and reaching a consensual solution in a given number of debate rounds. This process is usually guided by a *moderator*, a person in charge of identifying those experts who are furthest away from the global opinion of the group and suggesting the necessary changes in their opinions to avoid deadlocks in the decision process. Generally, a CRP consists of the following phases (represented in Fig. A.8):

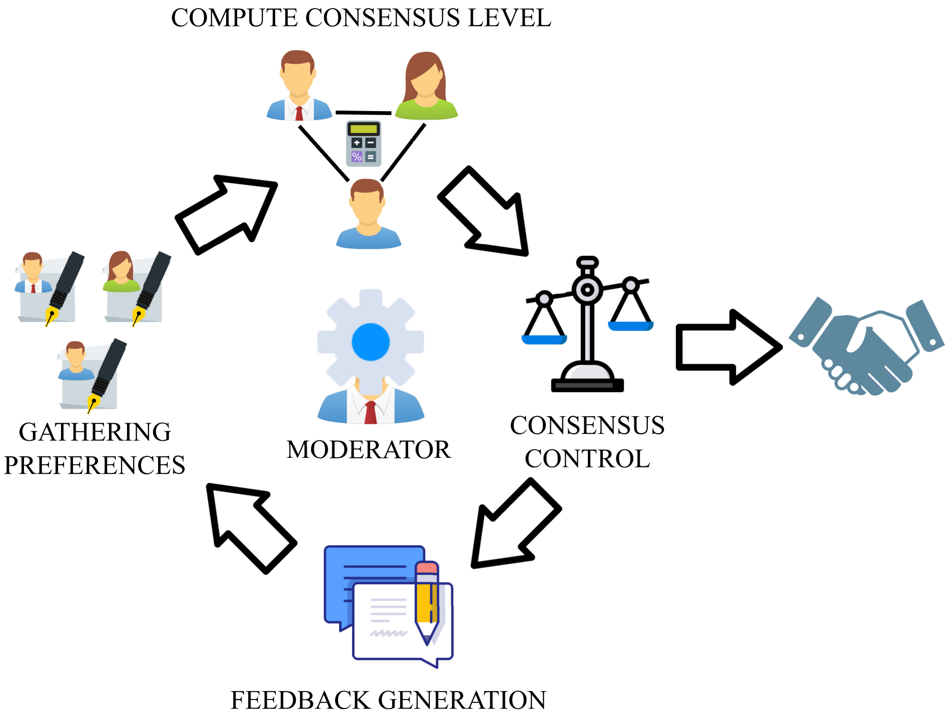

Figure A.8: General scheme of a CRP.

1. *Gathering preferences*: the preferences provided by the experts on the alternatives are collected.

2. *Compute consensus level*: the current level of consensus in the experts' group is calculated through a consensus measure. There are two basic types of consensus measures [49]:

   - *Consensus measure based on the distance to the collective opinion of the group*: the collective opinion of the group is calculated by aggregating the individual opinions of the experts. Subsequently, the distance between the collective opinion and the individual opinion of each expert is computed.
   - *Consensus measure based on the distance between experts' opinions*: the similarity of opinions is calculated for each pair of experts. Subsequently, the similarity values are aggregated to obtain the consensus value in the group.

3. *Consensus control*: it is checked whether the current consensus of the group has reached a minimum required and predefined consensus level. If it has been reached, the CRP finishes and the process of selecting the best alternative would start, otherwise another round of discussion is necessary. The maximum number of discussion rounds is also established a priori in order to avoid endless processes. If the maximum number of rounds is reached but not the minimum consensus required, the process will end without reaching an agreement.

4. *Feedback generation* if no agreement is reached in the current round, the moderator identifies the major points of dissent in the group and advises the experts to change certain opinions in order to increase the level of consensus in the group. There are consensus models that eliminate the role of the moderator and carry out the changes in opinions automatically without the need to involve the experts. These models are often used as a tool to support real-world CRPs.

There is a huge amount of consensus models proposed in the literature [35, 56, 83], some of the most relevant were reviewed in the development of this doctoral thesis in the article included in Section 4.4. This fact led Palomares et al. [49] to introduce a taxonomy of consensus models for GDM problems, which classifies models on the basis of two basic aspects (see Fig. A.9):

- *With or without feedback generation*: models are classified depending on whether or not they incorporate a feedback mechanism.

- *Consensus measure*: models are classified depending on the measure they use to compute consensus, either based on the distance to the collective opinion or based on the distance between the individual opinions of the experts.

Even the taxonomy proposed by Palomares et al. allows us to clearly categorize the consensus models based on their main characteristics, the large number of proposals makes it very difficult to select the most appropriate consensus model for a given GDM problem. This problematic is approached in this research from different points of view. On the one hand, Section 4.8 includes an article which presents a consensus metric to evaluate the performance of a consensus model applied to a GDM problem. On the other hand, the development of a decision support system that allows us to carry out simulations of different consensus models and determine which one best fits the needs of the problem. Also in Chapter 4, Section 4.9, an article presents a software focused on CRPs support in GDM.

A.5. Discussion of Results

This chapter will summarize the proposals that shape this research memory together with the results and conclusions obtained from them. This chapter is structured in three main goals

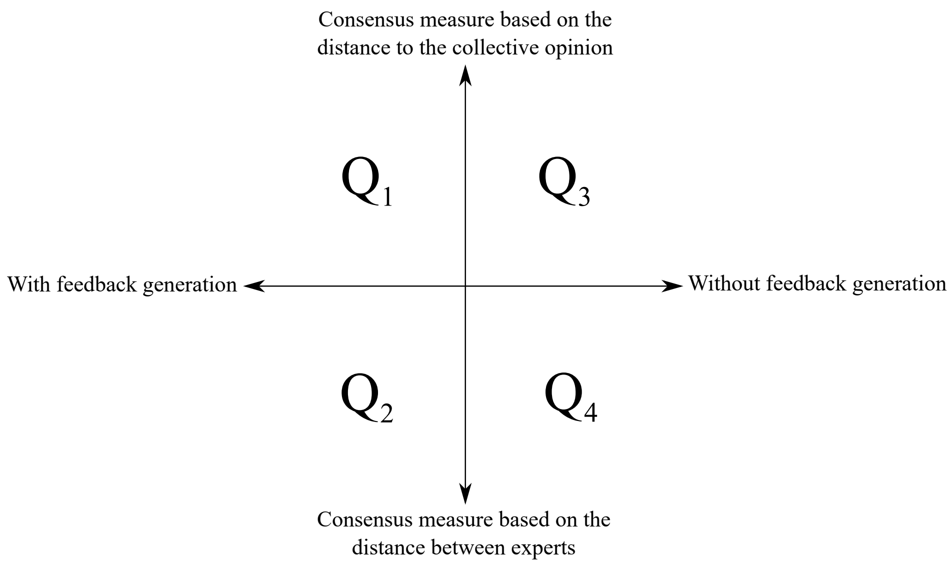

Figure A.9: Consensus models taxonomy.

that are divided into different specific objectives:

1. *Modelling and Processing of Linguistic Information using Complex Linguistic Expressions.* This proposal is divided into three objectives:

   - *Overview of Existing Proposals on Preference Modelling using Linguistic Expressions in Decision Making.*
   - *Definition of Comparative Linguistic Expressions with Symbolic Translation in Decision Making.*
   - *Aggregation Operators for Comparative Linguistic Expressions with Symbolic Translation.*

2. *Consensus Reaching Processes in Group Decision Making.* This proposal is divided into five objectives:

   - *Comparative Study of Classical Consensus Models in Large-Scale Group Decision Making Problems.*
   - *Large scale Consensus Reaching Process Managing Group hesitation.*
   - *Consensus Reaching Process with Comparative Linguistic Expressions.*
   - *Consensus Reaching Process with Comparative Linguistic Expressions with Symbolic Translation.*
   - *A Cost Consensus Metric for Consensus Reaching Processes based on a Comprehensive Minimum Cost Model.*

3. *Support for Group Decision Making Problems and Consensus Reaching Processes.* This proposal is divided into two objectives:

   - *Software for the Analysis of Consensus Reaching Processes: AFRYCA 2.0.*
   - *Software for the Support of Decision Problems based on Climate Policy: APOLLO.*

A.5.1. Modelling and Processing of Linguistic Information using Complex Linguistic Expressions

This goal's section begins by analysing the advantages and disadvantages of the main proposals in the literature based on the linguistic modelling of experts' preferences in LDM problems. Then, based on these advantages and disadvantages, a new linguistic representation model based on ELICIT (Comparative Linguistic Expressions with Symbolic Translation) is proposed, which overcomes the limitations of the existing models. Finally, any linguistic representation model must have an associated computational model that allows carrying out operations with linguistic information. For this purpose, the information aggregation process is key, so we propose different aggregation operators to deal with ELICIT information.

A.5.1.1. Overview of Existing Proposals on Preference Modelling using Linguistic Expressions in Decision Making

In this objective, we review the main proposals based on the fuzzy linguistic approach for preferences modeling using complex linguistic expressions in DM problems under uncertainty [11, 37, 57, 58, 64, 70, 71]. From the analysis of these proposals, we extract their main advantages and disadvantages and a clear view of what are the main aspects to be improved in preferences modelling using linguistic expressions. Some of these aspects are summarized below:

- Although some proposals are quite flexible generating linguistic expressions [11, 37], they do not define a formal process for their generation or are far from the usual human language. On the other hand, those expressions that are closer to the common language of human beings, present less flexible expressions [58]. Therefore, a key aspect would be to elaborate linguistic expressions that are closer to the thinking of human beings and that in turn are more flexible.

- The modelling of uncertainty in DM problems is usually established by applying a single technique. However, this may not be realistic, considering the multiple approaches that can be applied to solve a problem. Therefore, it would be worth studying the modelling of uncertainty by combining several approaches simultaneously, taking advantage of the benefits of each one of them.

- The analysed proposals provide a unique meaning to the linguistic expressions they generate. However, it is obvious that a linguistic expression can have different meanings depending on the person. Consequently, it would be interesting to develop representation mechanisms for linguistic expressions that consider this aspect.

The article associated with this review is included in Section 4.1. It should be noted that this article is, according to the InCites Essential Science Indicators published by Clarivate Analytics, highly cited (Highly Cited Paper).

A.5.1.2. Definition of Comparative Linguistic Expressions with Symbolic Translation in Decision Making

This objective starts from the conclusions drawn of the review presented in the previous section, which highlight the need to define a new linguistic representation model that overcomes the limitations existing in other models published in the literature. These limitations are mainly encompassed in two basic aspects, the precision in the processes carried out with linguistic expressions and their interpretability. The previous review analyzes two proposals that present good characteristics in relation to these two aspects, although separately. On the one hand, the 2-tuple linguistic representation model (see Section A.4.3.1), carries out precise CW processes thanks to the use of the concept of symbolic translation. However, these expressions are formed by a single linguistic term, insufficient in situations in which experts hesitate among several linguistic terms. Such a limitation is overcome by the CLEs based on the HFLTSs (see Section A.4.3.3), which allow modeling the experts' hesitation as well as providing a rich linguistic representation close to the human way of thinking. Although multiple models have employed CLEs [9, 45, 54], they all present drawbacks from different points of view. Most of these proposals, transform the CLEs to carry out the computational processes, losing information in that process and, consequently, also the main characteristic of these expressions, their interpretability.

The previous statements evidence the limitations of both the 2-tuple linguistic representation model and the CLEs, but also their benefits, which lead us to think about a combined use of both proposals which could offer excellent results in the modelling of linguistic information. Other proposals have already combined to a lesser or greater extent concepts related to CLEs, HFLTSs and the 2-tuple linguistic model [1, 63, 74, 95], although none of them in a fully satisfactory way. For this reason, we propose a new linguistic representation model that combines the expressiveness of the CLEs and the precision of the 2-tuple linguistic model. This linguistic representation model represents linguistic information by Comparative Linguistic Expressions with Symbolic Translation (ELICIT), CLEs extended to a continuous domain by using symbolic translation. The proposed expressions are generated through a context-free grammar whose terms are formed by 2-tuple linguistic values instead of single linguistic terms.

Together with the ELICIT linguistic representation model, a CW approach for ELICIT information is proposed that allows, starting from linguistic premises represented by CLEs and ELICIT information, to carry out accurate CW processes based on fuzzy operations [53] and obtain understandable results represented also by ELICIT information. To carry out operations with ELICIT information in CW processes, a computational model with basic operations such as negation, comparison between ELICIT or aggregation operators is also defined.

Finally, the benefits of the new linguistic model are demonstrated by solving a LDM problem and comparing it with other previous linguistic models. The results obtained show that the ELICIT representation model provides a more accurate, interpretable and reliable solution than other approaches.

The article associated with this part is included in Section 4.2.

A.5.1.3. Aggregation Operators for Comparative Linguistic Expressions with Symbolic Translation

The previous proposal allows us to model linguistic preferences using ELICIT information and to carry out CW processes for LDM problems. A key stage in the resolution of a LDM problem is the aggregation phase using linguistic aggregation operators, in which the individual experts' opinions on the alternatives are combined based on different criteria or attributes, obtaining a global value for each alternative (see Section A.4.2). Sometimes, the attributes that compose a LDM problem are related to each other and it is necessary to model such interaction in order to correctly carry out the aggregation process and obtain a reliable solution. However, in the previous work, no aggregation operator was proposed to consider the interrelationship between the criteria, neither it take into account the individual importance of the attributes in the aggregation process, which is key in many decision processes.

Taking into account the lack of proposals, in this work we aim to define new linguistic aggregation operators for ELICIT information, which consider different patterns of relationship between attributes and their importance in the aggregation process. These operators are based on the Bonferroni mean and its variants [5, 13, 14], capable of capturing different types of relationships between the aggregated attributes. In total, three new aggregation operators are proposed, the first one focused on ELICIT expressions whose interrelation is homogeneous or, in other words, each input expression has relation with the rest. The second approach is based on an aggregation operator that deals with ELICIT expressions with a heterogeneous interrelation, that is, certain expressions may or may not have a relation with the rest. Finally, the third aggregation operator deals with the partitioned interrelation of ELICIT expressions, where the input expressions are divided into sets formed by ELICIT expressions with an interrelation between them, but not between expressions from different sets.

Finally, the proposed ELICIT aggregation operators are applied in solving a LDM problem to show their performance and are compared with other aggregation operators that do not consider the interrelation between the attributes of the problem. As a conclusion, we appreciate that the ranking obtained through aggregation operators that consider the interrelationship between attributes is totally different from that provided by operators that do not consider this aspect, demonstrating the need and importance of considering the relationship between attributes in a LDM problem.

The article associated with this part is included in Section 4.3.

A.5.2. Consensus Reaching Processes in Group Decision Making

This goal studies the main challenges that exist nowadays in CRPs for GDM problems, taking into account aspects such as the increase in the number of experts involved in the decision process, and analyses whether some of the most widely used consensus models in the literature can meet these new challenges. Subsequently, different CRP proposals able to deal with real world GDM problems are presented. Finally, a consensus metric is proposed to evaluate the performance of CRPs and to determine which one is more suitable for a specific GDM problem.

A.5.2.1. Comparative Study of Classical Consensus Models in Large-Scale Group Decision Making Problems

In this objective, we study and analyse the new challenges related to GDM problems and their CRPs due to the expansion of technological paradigms in our society, such as social networks or Big Data, which have given way to new GDM problems in which the number of people involved can be hundreds or thousands. In this type of problems, CRPs are even more necessary if possible, since a large number of experts implies in turn the inevitable appearance of a large number of conflicts, the need to deal with multiple points of view and behaviours, the detection of coalition between groups, etc.

The classical consensus models presented in the specialized literature deal with GDM problems in which the number of experts is small, which leads us to ask an obvious question, are consensus models focused on GDM problems with few experts suitable to deal with problems in which the number of experts is much higher? To answer this question, this paper reviews the most influential consensus models in the literature oriented to small groups of experts and, due to the large number of existing proposals, a selection of them is made. In order to make this selection as representative as possible, we use the taxonomy proposed by Palomares et al. [49] (see Section A.4.4), choosing representative consensus models for each of the categories defined in that taxonomy [10, 24, 79, 81, 92].

Once the consensus models have been chosen, the next step is to analyse their performance using a GDM problem with a significant number of experts under different decision scenarios. Specifically, we define three possible scenarios based on the possible experts' behaviour: (i) all the experts accept the recommendations provided by the model, (ii) 80 % of the experts accept the recommendations and the remaining 20 % reject them, and finally (iii) 70 % accept the recommendations, 20 % reject them and the remaining 10 % have a defensive attitude that aims to sabotage the consensus in the group. The simulation of the consensus models on the different scenarios is carried out thanks to the software AFRYCA 2.0, a decision support system for CRPs that has been developed during this doctoral thesis and that will be introduced in the Section A.5.3.1.

The simulations allow us to draw valuable conclusions. Classical consensus models that do not use a feedback mechanism are not affected by the behaviour of experts, since their participation in the CRP is not required, which apparently makes them suitable in problems with large groups. However, this feature together with their mathematical nature could be their main limitations, since, on the one hand, the experts might not trust the solution obtained since they are removed from the CRP and, on the other hand, the mathematical model might not find a feasible solution. Because of this, classical models that incorporate a feedback mechanism could be considered as the most suitable for solving this type of problem. However, classical models assume a collaborative behaviour of the experts; if this behaviour does not occur, deadlocks could happen and the desired consensus could never be reached. Therefore, it is clear that classical consensus models cannot cope with GDM problems with large groups of experts, so it is necessary to develop new proposals to meet the challenges that this type of problems propose.

The article associated with this part is included in Section 4.4.

A.5.2.2. Large scale Consensus Reaching Process Managing Group hesitation

In the previous work, the need to develop new consensus models that are able to cope with the new challenges presented by current GDM problems was highlighted. One of these challenges is scalability, which appears in GDM problems involving a large number of experts and, consequently, the simultaneous processing of a large amount of information. On the other hand, it is logical to think that problems with large groups of experts are implicitly associated with high complexity and, therefore, uncertainty and lack of information, which may cause experts to hesitate when expressing their preferences. Based on these assumptions, this paper presents a new consensus model focused on problems with large scale groups that overcomes the scalability problems and models expert hesitation.

To mitigate the scalability problem, the proposed consensus model applies a *clustering process* based on the Fuzzy C-means algorithm [4] that classifies experts into different subgroups based on the similarity between their opinions. Therefore, those experts whose opinions are similar will be part of the same subgroup. In this way, information processing is not applied to the whole set of experts, but to the different subgroups. A key aspect of any clustering technique is the assignment of weights to the subgroups. These weights will determine the influence of the subgroup on the CRP, the higher the weight, the greater its influence on the

CRP and on the solution of the problem. Usually, the weighting of subgroups is based solely on their size, the greater the number of members the group has, the greater the associated weight. However, the fact that a subgroup is formed by experts with similar opinions does not guarantee that there are no disagreements within it. For this reason, this proposal includes a mechanism to calculate the importance of subgroups based on two aspects: the size of the subgroup and its cohesion. In this way, two subgroups with the same number of members but different cohesion will be weighted differently, giving priority to those with greater cohesion.

The consensus model also includes an adaptive feedback process. Depending on the current level of global consensus, recommendations are applied to a group or individual experts. This distinction is established on the basis of a pre-established consensus threshold value. If the current consensus is below the threshold, it is considered that the group of experts is still far from reaching the desired consensus and that a significant change in the experts' preferences is necessary, so the subgroups formed by experts whose opinions are further away from the remaining ones are detected and all the experts forming the subgroup are recommended to change their preferences. If, on the other hand, the current consensus is greater than the set threshold, it means that the subgroup is close to reaching the desired consensus and that it is not necessary to make many changes in the preferences, so the experts whose opinions are farther away from the majority are detected individually, and those opinions are the only ones recommended to be modified.

It should be noted that the experts' preferences are modelled by means of hesitant fuzzy sets (HFS) [68]. These sets are an extension of the fuzzy sets, which allow assigning various degrees of membership of an element to a fuzzy set and thus, modelling the experts' hesitation and preserving as much information as possible in the computational processes carried out in the proposed CRP.

Finally, the new proposal is applied to the resolution of a large scale GDM problem and a comparative analysis with different consensus models published in the literature is carried out. The results obtained from the AFRYCA 2.0 software (see Section A.5.3.1) show that the consensus model is able to deal with GDM problems with large groups, reaching the desired consensus in a few rounds of discussion. Moreover, the comparative analysis shows that the consensus reached by the proposal is higher than that reached by other consensus models and needs fewer rounds to reach it.

The article associated with this part is included in Section 4.5.

A.5.2.3. Consensus Reaching Process with Comparative Linguistic Expressions

Nowadays, GDM problems and their CRPs are becoming increasingly complex and difficult to solve, so the apparition of uncertainty and doubt in experts' opinions is quite common. Most of consensus models presented in the literature [8, 72, 94], model such uncertainty by simple linguistic terms, which are not expressive enough to model experts' hesitation. In order to cover this lack of proposals, this paper presents a consensus model that models experts' preferences by means of CLEs, rich linguistic expressions that allow representing doubt in experts' opinions (see Section A.4.3.3).

This proposal employs the fuzzy representation of CLEs by making use of the concept of *fuzzy envelope* [36], a function that allows transforming the HFLTS associated with an CLE into a fuzzy number. In this way, it is possible to carry out the CRP computations accurately and without loss of information.

The consensus model also proposes a feedback mechanism. This mechanism is based on the computation of the proximity between the individual opinions of the experts and the collective opinion of the group. If the collective consensus on some of the alternatives is below the desired consensus threshold, it is recommended to modify the opinions on these alternatives. The experts who should carry out these modifications will be those whose opinions on these alternatives are furthest from the group's opinion. Once the experts and dissenting alternatives have been identified, it is necessary to define how the recommendations will be carried out. Contrary to other proposals, this work applies the changes directly on the CLEs initially provided by the experts, facilitating the interpretability of the results.

The good performance of the proposed consensus model is tested by solving a GDM problem. The use of CLEs and their fuzzy representation, together with the formalization of a feedback process applied directly on the CLEs initially provided by the experts, makes it possible to solve the problem in very few rounds of discussion. These characteristics make the proposal superior to other consensus models presented in the literature, as demonstrated in the comparative analysis carried out. Again it should be noted that the resolution of the GDM problem and the comparative analysis with other consensus models is carried out using AFRYCA 2.0, presented in Section A.5.3.1.

However, this work also has an important limitation, because experts express their opinions from a discrete expression domain, limited by the finite set of linguistic terms that experts can use to express their opinions. Therefore, changes on preferences are limited to the granularity of the set of linguistic terms, which could harm the CRP.

The article associated with this part is included in Section 4.6.

A.5.2.4. Consensus Reaching Process with Comparative Linguistic Expressions with Symbolic Translation

The previous work evidenced the lack of consensus models that were capable of modelling the uncertainty and experts' hesitation in the GDM problems and their CRPs. Therefore, a consensus model was proposed to model the experts' preferences by means of CLEs and carried out a feedback mechanism that is directly applied on these expressions. However, the proposal presented an important drawback, as these recommendations were limited by the discrete expression domain that experts use to express their opinions. This limitation could be an obstacle to reaching the desired consensus. This paper aims to overcome this limitation.

The new consensus model replaces the CLEs by ELICIT information (see Section A.5.1.2), which allows to maintain the interpretability of the CLEs and to use linguistic expressions defined under a continuous domain of expression and therefore not limited exclusively to the finite set of terms that form such domain. The computational processes carried out in the consensus model are performed accurately and without loss of information, thanks to the use of the fuzzy representation of ELICIT expressions.

This proposal also includes a feedback process. In this case, the alternatives in which there is the greatest dissent within the group are identified. If the collective consensus on an alternative is below the desired consensus threshold, it is necessary to recommend to certain experts to modify their opinions on that alternative. The experts who should modify their preferences are those whose opinions on the dissenting alternatives are furthest from the group's opinion. Once the experts and the alternatives have been identified, it is necessary to define the preference recommendation. The proposal includes an adaptive process that identifies whether the change to be applied should be more or less drastic, a key aspect of our contribution since, contrary to other existing proposals, the ELICIT information allows modifying the experts' preferences in a continuous domain. While other consensus models apply the change in experts' preferences only on linguistic terms belonging to a predefined linguistic set, our proposal can use the concept of symbolic translation of ELICIT information to apply changes in the continuous values that exist between linguistic terms. This helps to generate more accurate recommendations, avoiding excessive modifications in preferences that may cause a deadlock in the consensus process.

To evaluate the performance of the proposal, we proceed to the resolution of a GDM problem and a subsequent comparative analysis with the proposal presented in the previous section, due to the similarity between both. The simulation carried out using AFRYCA 2.0 shows excellent results. On the one hand, the consensus model is able to solve the problem just in a few rounds of discussion and reaching a high level of consensus. This is achieved thanks to the use of ELICIT information, which allows us to carry out fuzzy operations that avoid the loss of information in the resolution process and generate recommendations in the right measure, avoiding excessive changes in preferences that negatively influence the group's agreement. In addition, the changes are applied on the ELICIT expressions, which facilitates their interpretation by the experts. The comparative analysis also shows a better performance with respect to the previous proposal, since the latter is not able to reach the desired consensus in the maximum number of predefined discussion rounds.

The article associated with this part is included in Section 4.7.

A.5.2.5. A Cost Consensus Metric for Consensus Reaching Processes based on a Comprehensive Minimum Cost Model

As we have seen in previous works, CRPs are of great importance within GDM since, on many occasions, certain decision problems require a consensual solution. For this reason, a large number of consensus models have been proposed in the literature. The number of consensus models is such that it is really complex to decide which one to use in solving a GDM problem and, in addition, there is no objective measure to evaluate the good or bad performance of a CRP and facilitate such a decision. This work aims to overcome this limitation by defining a metric to evaluate the performance of consensus models.

The initially proposed metric is based on *minimum cost consensus* (MCC) models [2, 3, 93]. These models define consensus as the minimum distance between individual expert opinions and the collective opinion and seek to minimize the cost of modifying these opinions using a linear function. Therefore, they are able to obtain a consensus solution by modifying as little as possible the initial opinions of the experts based on a predefined maximum distance threshold value between the experts' opinions and the collective opinion. The smaller the value of this threshold, the smaller the distance between the experts' opinions and the collective one and, consequently, the higher the level of agreement reached in the group. However, small distances between the individual expert opinions and the collective opinion do not always guarantee reaching the desired level of agreement within the group.

To solve the above problem, this paper proposes new MCC models that include an additional constraint related to the calculation of the consensus within the group of experts, which we will name *comprehensive minimum cost consensus* (CMCC) models. In this way, it is guaranteed that, in case a feasible solution is found, it will meet the consensus needs required by the problem. In total, four CMCC models are proposed based on two aspects. The first aspect is related to the measure of consensus used to calculate the consensus within the group, which can be based on the similarity between the experts' opinion and the collective opinion or based on the similarity between experts. The second aspect is related to the type of preference structure that experts use to express their opinions. In this case, we consider two possible structures, those formed by utility vectors [65] or by fuzzy preference relations[47].

The next step is to define the metric for consensus models. The metric, called *consensus cost metric*, could use any of the four models described above, the selection depends on the characteristics of the consensus model to be analysed. Once the CMCC model is selected, it will provide, if it exists, the optimal solution to the problem, which is the one with the lowest cost or that requires the smallest change in the experts' preferences based on the consensus and distance conditions set for the decision problem. Subsequently, this optimal solution is compared, by means of the metric, with the solution provided by the analysed consensus model, computing the distance between both solutions and returning a value between -1 and 1. If the resulting value is 0, the analysed consensus model provides the same solution as the CMCC, i.e., the optimal solution. If the result is 1, it means that the consensus model provides a solution in which the experts' preferences have not been modified at any time. For intermediate values, the closer to one, the worse the solution proposed by the consensus model. For negative values, -1 represents the worst possible solution due to excess cost, i.e., the experts' preferences have been excessively modified. For intermediate values, the more negative the value, the worse the solution.

Finally, to show the usefulness of the metric, a number of representative consensus models are selected in order to evaluate their performance on a GDM problem. The simulations carried out by AFRYCA 2.0 demonstrate that the new metric can be effectively used to make comparisons between CRPs, as it allows to detect anomalous situations in their performance that cannot be detected with other criteria.

The article associated with this part is included in Section 4.8.

A.5.3. Support for Group Decision Making Problems and Consensus Reaching Processes

Real-world DM problems are becoming increasingly complex due to the continuous development of society. Experts often have to deal with decision problems that involve uncertainty and lack of information and sometimes also demand solutions in a short period of time. Under these conditions, experts are under high-pressure situations that can directly affect their behaviour and negatively influence the decision process. Decision support systems are created with the aim of supporting experts and facilitating their work in decision making.

In this part, we will show two decision support systems that were developed in this doctoral thesis. First, we will introduce *AFRYCA 2.0*, an improved version of the framework for the analysis of CRPs proposed by Palomares et al. [49]. This new version of the software includes new consensus models and features that allow the treatment of a larger number of GDM problems among other advantages that will be developed in detail in the next section. We will also present the decision support system APOLLO the acronym for "A grouP decision fuzzy TOoL in support of cLimate change pOlicy making", which allows solving GDM problems related to climate change policies.

A.5.3.1. Software for the Analysis of Consensus Reaching Processes: AFRYCA 2.0

CRPs are essential when consensus solutions are required in GDM problems. There are multiple consensus models proposed in the literature that simulate these CRPs and can be used as a support tool for the experts in solving the problem. However, most of these models have a complex algorithmic structure consisting of different steps and experts cannot spend the already scarce time they have in determining which consensus model to use and manually carry out all the computations related to the performance of the model. With this in mind, AFRYCA [49] (A FRamework for the analYsis of Consensus Approaches), a decision support system that includes different consensus models to simulate a real CRP, was initially designed. Its main objectives are (i) to discover the advantages and disadvantages of consensus models, (ii) to determine the most suitable model for a specific GDM problem and (iii) to carry out comparisons between the different models.

The use of AFRYCA in multiple GDM situations revealed certain shortcomings in the software, such as outdated technology, a complex structure that makes it difficult to include new consensus models and their parameters, the impossibility for the user to modify several relevant simulation parameters, lack of information on the simulation results, and the inability to analyse the models in more detail. With all these limitations in mind, an improved version of the software, AFRYCA 2.0, is presented in this paper.

AFRYCA 2.0 has the following advantages over its predecessor:

- *Migration and independence*: AFRYCA 2.0 is developed under the new Eclipse RCP 4.0 branch, which includes several new features at the technological level such as dependency injection, declarative services, model application, etc. In addition, the first version of AFRYCA made use of external libraries for certain functionalities such as the use of the statistical programming language RCP [55], which made its migration to other platforms difficult. In the version 2.0, the language is incorporated natively, so the statistical environment can be selected at runtime.

- *New consensus models*: AFRYCA 2.0 incorporates a new and simpler mechanism for adding new consensus models to the software. It is now possible to define all the parameters associated with the model and apply a series of constraints and relationships between them, saving users from having to manually check if all the values are correct. In addition, two new consensus models have been included in the software [50, 51].

- *Behaviours configuration*: AFRYCA 2.0 provides greater flexibility in the configuration of the simulation of experts' behaviour, making it possible to model the probability distribution associated with them. The mechanism to include new behaviours has also been facilitated and a new one called "standard with adverse" has been included, which allows to simulate experts reluctant to accept the recommendations.

- *Evolution of CRPs*: the first version of AFRYCA visualized the state of the experts' preferences at the end of the CRP. AFRYCA 2.0, however, shows such a visualization for each of the discussion rounds that have been necessary in the course of the CRP.

- *Metrics*: AFRYCA 2.0 includes several metrics that allow studying different aspects of the consensus models and analysing their performance.

This paper also includes an experimental study in which several consensus simulations are carried out on different GDM problems with the aim of showing the new features and advantages of AFRYCA 2.0.

The article associated with this part is included in Section 4.9.

A.5.3.2. Software for the Support of Decision Problems based on Climate Policy: APOLLO

Nowadays, many of most important GDM problems are related to sustainability issues. The effects of climate change are increasingly evident and its impact on our society, economy and environment today and in the future is one of our main concerns. This challenge has been addressed through different climate policies, however, its enormous complexity means that experts have to assess the risks of applying different policies in a given geographical area, leaving themselves open to a series of assumptions that do not reflect real-world constraints. This work focuses on reducing this complexity by developing a climate policy-focused decision support system called APOLLO.

The main objective of APOLLO is to facilitate the consensus process of a group of experts to reach the best possible solution to a GDM problem related to climate change and climate issues. To this end, APOLLO presents a resolution scheme divided into several steps:

1. *Problem definition*: this step defines the GDM problem and all the elements related to it, the alternatives, the criteria to evaluate them, experts or expression domains. Specifically, APOLLO focuses on the modelling of preferences by means of linguistic information with the objective of facilitating the task of the experts.

2. *Assignment of expression domains*: in this phase the expression domains created in the first stage are assigned to the different experts. In this way, the experts can use the expression domain with which they feel more comfortable when expressing their knowledge.

3. *Consensus*: climate policies affect society, so consensual solutions are much more highly valued. APOLLO measures the level of consensus in the group of experts, carrying out a CRP, if it is necessary, with the aim of getting the experts to modify their initial preferences and increase the agreement among them.

4. *Rating*: finally, at this stage APOLLO carries out the resolution of the GDM problem by means of the linguistic method 2-tuple TOPSIS [62] providing a ranking of the alternatives based on the consensus opinions of the experts.

The performance of APOLLO is tested by solving a real case study related to the decarbonization of iron and steel production in Austria. The case study aims to facilitate the transition path of the Austrian steel industry by prioritizing the risks associated with this transition by engaging stakeholders in a process that will provide information on what key players in the system fear the most. A total of twenty five possible risks are considered and evaluated on the basis of four different criteria. APOLLO makes it possible to detect disagreements within the group of experts, to simulate a CRP to support the experts in modifying their preferences and reaching a higher level of agreement and, finally, to provide a ranking of the different risks. All this keeping in mind the fact that the software always makes use of linguistic information, which facilitates the understanding of the results by the experts.

The article associated with this part is included in Section 4.10.

A.6. Conclusions and Future Works

To conclude this research memory, we will present the different conclusions drawn in this research, together with the possible future works that can be addressed on the basis of the results obtained.

A.6.1. Conclusions

The increasing complexity of *Group Decision Making* (GDM) problems leads to the emergence of uncertainty, which makes more difficult for experts to express their opinions. Linguistic modelling of preferences allows to represent this uncertainty in a satisfactory way, giving rise to *Linguistic Decision Making*. However, despite the benefits of current linguistic representation models focused on GDM problems, they also have different limitations in terms of interpretability and/or accuracy.

The first objective of this research memory was to propose a linguistic representation model to overcome these limitations in GDM under linguistic contexts. For this purpose, we have proposed the use of Comparative Linguistic Expressions with Symbolic Translation (ELICIT), which have improved the flexibility of the experts' expressions, as they are similar to the human beings' cognitive model. Moreover, its computational model, together with the proposed aggregation operators based on it, have made it easier to obtain interpretable and more accurate results than existing models in the literature.

Another aim was to improve *Consensus Reaching Processes* (CRPs) in contexts of uncertainty. First, a study was carried out on the use of classical consensus models, focused on GDM problems with few experts, in large scale GDM problems. This study proved that classical models have important limitations and are not able to adapt to the peculiar characteristics of this type of problems. Therefore, first, a consensus model for large scale GDM problems was designed, capable of meeting the challenge of scalability by generating experts' subgroups and proposing a new cohesion measure for the latter that improves the convergence of opinions towards agreement. Subsequently, two linguistic consensus models with comparative linguistic expressions and ELICIT have been proposed, which facilitate the linguistic modelling of experts' preferences and improve the interpretability and accuracy of the results.

The next aim of this memory was based on the definition of a metric to evaluate the performance of different consensus models. Therefore, first of all, the concept of *comprehensive minimum cost model* was defined, a linear programming model capable of obtaining the optimal solution of a CRP. From this concept, a cost metric was developed that allows to evaluate the performance of different CRPs on a GDM problem in an objective way.

The models and tools defined above have been applied to real-world decision problems, as indicated in the fourth objective of this research. Satisfactory results have been obtained, providing new solutions to GDM problems that could not be obtained before and improving the results of state-of-the-art models.

Finally, as a consequence of the previous objective, two decision support systems have been implemented to automate and facilitate the resolution of previous CRPs and GDM problems, as the last objective of our research was intended. The first decision system implemented is AFRYCA 2.0, a general framework that allows the simulation and analysis of CRPs for GDM problems of any type. The second software tool, APOLLO, is a decision support system for GDM problems focused on climate policies, being unique in its field according to its characteristics and functionality.

Therefore, it should be noted that all the objectives defined at the beginning of this research memory have been achieved, providing tools, models and results that improve the state of the art prior to our research and open the possibility for new research such as it is described in the following section.

A.6.2. Future Works

Based on the results obtained, it is possible to define possible works that continue with the research carried out throughout this doctoral thesis and presented in this research memory. These future works are:

- To extend the computational model associated to the ELICIT linguistic representation model by defining new aggregation operators.
- To study new types of decision problems, such as classification problems, which aim to provide a classification of alternatives into different classes and propose new methodologies that allow their resolution and application to real world problems.
- To propose new consensus models that face different challenges related to current decision making problems such as polarization of opinions or minority opinions.
- To develop new metrics for consensus models that model experts' preferences using linguistic information.
- To increase the functionality of the decision support system AFRYCA 2.0 by including new consensus models, behaviour types, and other new functionality.

- To increase the functionality of APOLLO by including new decision and consensus models and subsequently use it in solving various real-world climate change related problems.
- To perform the registration process of APOLLO so that its authorship is recognized.

A.6.3. Additional Publications

In the development of this research, other publications have been presented that have not been included in this research memory, which are listed below:

- In International Journals
  - R. M. Rodríguez, A. Labella, B. Dutta y L. Martínez. Comprehensive minimum cost models for large scale group decision making with consistent fuzzy preference relations. Knowledge-Based Systems, 106780, 2021.
  - A. Labella, A. Ishizaka y L. Martínez. Consensual Group-AHPSort: Applying consensus to GAHPSort in sustainable development and industrial engineering. Computers & Industrial Engineering, 152, 107013, 2021.
  - A. L. Moreno-Albarracín, A. Licerán-Gutierrez, C. Ortega-Rodríguez, A. Labella y R. M. Rodríguez. Measuring What Is Not Seen—Transparency and Good Governance Nonprofit Indicators to Overcome the Limitations of Accounting Models. Sustainability, 12(18): 7275, 2020.
  - A. Labella, J. C. Rodríguez-Cohard, J. D, Sánchez-Martínez y L. Martínez. An AHPSort II Based Analysis of the Inequality Reduction within European Union. Mathematics, 8(4): 646, 2020.
  - A. Labella, R. M. Rodríguez y L. Martínez. Extending the linguistic decision suite FLINTSTONES to deal with comparative linguistic expressions with symbolic translation information. Journal of Intelligent & Fuzzy Systems, 39: 6245–6258, 2020.
  - L. Wang, A. Labella, R. M. Rodríguez, Y. M. Ming Wang y L. Martínez. Managing non-homogeneous information and experts' psychological behavior in group emergency decision making. Symmetry, 9(10): 234, 2017.
- In International Conferences
  - A. Labella, R. M. Rodríguez y L. Martínez. Green Supplier Selection by means of a Decision Making Method based on ELICIT Information. In 14th International Conference FLINS Conference, 750-758, World Scientific, 2020.
  - A. Labella, D. Uztürk, R. M. Rodríguez, G. Büyüközkan y L. Martinez. Product development partner selection based on ELICIT information, In 14th International Conference FLINS Conference, 767-775, World Scientific, 2020.

- A. Nikas, A. Arsenopoulos, H. Doukas y A. Labella. Prioritisation of risks associated with decarbonisation pathways for the Austrian iron and steel sector using 2-tuple TOPSIS, In 14th International Conference FLINS Conference, 776-783, World Scientific, 2020.
- A. L. Moreno-Albarracín, C. Ortega-Rodríguez, A. Licerán-Gutiérrez, A. Labella y Luis Martínez. How are donations managed? A proposal of transparency measurement for non-profit organizations in a Spanish setting. In XIX International Meeting of the Spanish Association of University Professors of Accountancy, 2020.
- A. Labella, H. Liu, R. M. Rodríguez, L. Martínez. Comprehensive Minimum Cost Models Based on Consensus Measures. In The International Virtual Workshop of Business Analytics EUREKA 2019, 2019.
- A. Labella, A. Ishizaka y L. Martínez. Consensual Group-AHPSort. In 30th European Conference on Operational Research, 2019
- A. Labella, R. M. Rodríguez y Lui Martínez. A Comparative Performance Analysis of Consensus Models Based on a Minimum Cost Metric. In International Conference on Intelligent and Fuzzy Systems, 1506-1514. Springer, Cham, 2020.
- A. Labella, R. M. Rodríguez y L. Martínez. A Novel Linguistic Cohesion Measure for Weighting Experts' Subgroups in Large-Scale Group Decision Making Methods. In 2019 IEEE 14th International Conference on Intelligent Systems and Knowledge Engineering (ISKE), 9-15. IEEE, 2019.
- A. Labella, R. M. Rodríguez y L. Martínez. An Adaptive Consensus Reaching Process Dealing with Comparative Linguistic Expressions in Large-scale Group Decision Making. In 11th Conference of the European Society for Fuzzy Logic and Technology (EUSFLAT 2019), 170-177. Atlantis Press, 2019.
- A. Labella y L. Martínez. FLINTSTONES 2.0 an Open and Comprehensive Fuzzy Tool for Multi-criteria Decision Analysis. In International Conference on Intelligent and Fuzzy Systems, 762-769. Springer, Cham, 2019.
- D. Uztürk, A. Labella, G. Büyüközkan y L. Martinez. Fuzzy linguistic integrated methodology for sustainable hospital building design. In International Conference on Intelligent and Fuzzy Systems, 1180-1188. Springer, Cham, 2019.
- A. Labella, R. M. Rodríguez, G. De Tré y L. Martínez. A Cohesion Measure for Improving the Weighting of Experts' subgroups in Large-scale Group Decision Making Clustering Methods. In 2019 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE), 1-6. IEEE, 2019.
- R. Yera, A. Labella, J. Castro y L. Martínez. On group recommendation supported by a minimum cost consensus model. In Data Science and Knowledge Engineering for Sensing Decision Support: Proceedings of the 13th International FLINS Conference (FLINS 2018), 11: 227, 2018.

- A. Labella y L. Martínez. A new visualization for preferences evolution in group decision making. In Data Science And Knowledge Engineering For Sensing Decision Support-Proceedings Of The 13th International FLINS Conference, 11: 235. World Scientific, 2018.
- A. Labella y L. Martínez. AFRYCA 3.0: An improved framework for consensus analysis in group decision making. In International Conference on Intelligent Decision Technologies, 76-86. Springer, Cham, 2018.
- A. Labella, L. Martínez y R. M. Rodríguez. Can classical consensus models deal with large scale group decision making?. In 2017 12th International Conference on Intelligent Systems and Knowledge Engineering (ISKE), 1-7. IEEE, 2017.
- L. Wang, A. Labella, R. M. Rodríguez y Luis Martínez. An emergency group decision making method based on prospect theory. In 14th Information Systems for Crisis Response and Management (ISCRAM), 2017.

- In National Conferences

  - A. Labella, R. M. Rodríguez y L. Martínez. Uso de expresiones lingüísticas comparativas en AFRYCA 3.0. In Conference of the Spanish Association for Artificial Intelligence (CAEPIA 2018) 23-26 October 2018 Granada, Spain (pp. 341-346). Association for Artificial Intelligence (AEPIA), 2018.
  - R. M. Rodríguez, A. Labella y L. Martínez. Un modelo de consenso para toma de decisiones en grupo a gran escala usando conjuntos difusos dudosos. In XVIII Conference of the Spanish Association for Artificial Intelligence (CAEPIA 2018) 23-26 October 2018 Granada, Spain (pp. 316-321). Association for Artificial Intelligence (AEPIA).
  - A. Labella, F. J. Estrella y L. Martínez. AFRYCA 2.0: Análisis de Procesos de Alcance de Consenso. In XVII Conference of the Spanish Association for Artificial Intelligence (CAEPIA 2016) 14-15 September 2016 Salamanca, Spain (pp. 573-582). Association for Artificial Intelligence (AEPIA), 2016.
  - F. J. Estrella, R. M. Rodríguez, A. Labella y L. Martínez. Un Sistema Basado en FLINTSTONES para Procesos de Selección Mediante un Modelo Difuso TOPSIS. In XVI Association for Artificial Intelligence (CAEPIA 2015) 9-12 November 2015 Albacete, Spain (pp. 491-500). Association for Artificial Intelligence (AEPIA), 2015.

A.6.4. Awards Received

It is worth mentioning that some of the published works along this doctoral thesis were awarded. In particular, two papers presented in international conferences, the first one, entitled “FLINTSTONES 2.0 an Open and Comprehensive Fuzzy Tool for Multi-criteria Decision Analysis” presented at the *International Conference on Intelligent and Fuzzy Systems* in Istanbul, Turkey, which received the *Best student paper award*. The second one, entitled "A Novel Linguistic Cohesion Measure for Weighting Experts' Subgroups in Large-Scale Group Decision Making Methods", presented at the *IEEE 14th International Conference on Intelligent Systems and Knowledge Engineering*, in Dalian, China, received the *Best paper award*. In addition, the software developed in this doctoral thesis, AFRYCA 2.0, received in 2017 the award in the modality "Best applicationïn the *III Ada Lovelace Awards in Information and Communication Technologies of the University of Jaén*. All the diplomas associated with the above awards are included below.

Best Student Paper Award

Dear

Álvaro Labella & Luis Martinez,

INFUS community would like to thank you for taking part at International Conference on Intelligent and Fuzzy Systems organized by Industrial Engineering Department of Istanbul Technical University in July 23-25, 2019 at Istanbul, Turkey.

Your research,

"FLINTSTONES 2.0 an Open and Comprehensive Fuzzy Tool for Multi-Criteria Decision Analysis"

has been selected to be the best student paper presented at INFUS 2019.

INFUS
International Conference on
Intelligent and Fuzzy Systems

Istanbul Technical University
Faculty of Management
Industrial Engineering Department

Prof.Dr.Cengiz Kahraman
Conference
Chair

IEEE 14th International Conference on Intelligent Systems and Knowledge Engineering (ISKE2019)

BEST PAPER AWARD

Presented to:

Alvaro Labella, Rosa M. Rodr´ıguez, Luis Mart´ınez

Paper Title:

A Novel Linguistic Cohesion Measure for Weighting Experts' subgroups in Large-scale Group Decision Making Methods (SS14: Fuzzy decision making)

Date: November 14-16, 2019
Location: Dalian, China

Li Zou

Li Zou
Chair of International Program Committee of ISKE2019

Jie Lu
Chair of ISKE2019 Steering Committee

Universidad de Jaén

CEATIC

CENTRO DE ESTUDIOS AVANZADOS EN TECNOLOGÍAS DE LA INFORMACIÓN Y LA COMUNICACIÓN

D. ÁLVARO LABELLA ROMERO

con DNI Nº: 77348227D

Ha quedado como ganador en los III Premios Ada Lovelace en Tecnologías de la Información y la Comunicación de la Universidad de Jaén, en la modalidad a la mejor aplicación, por su propuesta "AFRYCA: A Framework for the analysis of Consensus Approaches", organizados por el CEATIC , y para que conste a los efectos oportunos, se expide el presente:

CERTIFICADO GANADOR

III Premios Ada Lovelace en Tecnologías de la Información y la Comunicación de la Universidad de Jaén

Jaén, 8 de Noviembre de 2017

Director del Centro de Estudios Avanzados en Tecnologías de la Información y la Comunicación

D. Luis Alfonso Ureña López

A.6.5. Stays and Collaborations

During the development of this doctoral thesis, several stays and collaborations abroad have been carried out, with the aim of improving the research training of the PhD student through the knowledge and experience of experts in the topic.

Thanks to the EDUJA 2017 fellowship granted by the University of Jaén, it has been possible to carry out a stay for 3 months at the School of Electrical and Computer Engineering at the National Technical University (NTUA), Athens, collaborating closely with the Prof. Haris Doukas in the development of a European research project.

In addition, the pre-doc fellowship for the training of doctors (FPI) granted by the Spanish Ministry of Science, Innovation and Universities, gave me the opportunity to carry out another stay for 2 months at the University of Portsmouth, UK, with the Prof. Alessio Ishizaka.

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