UNIVERSITY OF JAÉN
School of Engineering and Computing
Computer Science Department

Linguistic Decision Making Based On Hybrid Mathematic Models

Thesis Memory Presented by

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yaya liu

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Supervisors

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The Thesis entile *Linguistic decision making based on hybrid mathematic models*, presented by Dª Yaya Liu to obtain the PhD degree in Computer Science, has been carried out in the Computer Science Department of the University of Jaén with the supervisors Dr. Luis Martínez López and Dr. Rosa Mª Rodríguez Domínguez. To be evaluated, this research memory is presented as a set of published articles, according to Article 23, point 3, Regulation of Doctoral Studies of the University of Jaén, approved in February 2012.

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Contents

Chapter 1

Introduction

1.1 Motivation

Classical mathematic models fail to deal with daily life decision making (DM) problems that contain uncertain information. Although several mathematic models, such as fuzzy set theory, rough set theory and vague set theory, have been proposed and extensively applied in DM problems under uncertainty, they suffer from one common limitation when they are used in an isolated way, that is, lack of parameterization tools. This limitation indicates that individually these models are unable to consider alternatives from different parameters aspects. Molodstov [50] proposed a model called *soft set*, that successfully overcomes this limitation. Afterwards, to enhance the ability of soft sets dealing with different kinds of uncertainties, various generalization models of soft sets have been proposed by combining soft sets with other models.

Popular hybrid soft sets can be divided into two main categories: (i) hybrid models obtained from combination of fuzzy sets (and generalization models of fuzzy sets) with soft sets; and (ii) hybrid models obtained from combination of rough sets (and generalization models of rough sets) with soft sets. Fuzzy soft sets belong to the first category, while rough soft sets and soft rough sets belong to the second category. These hybrid models of soft sets are typical and simple, therefore, researchers have proposed more complex hybrid models to generalize the previous ones. For instance, intuitionistic fuzzy soft sets [36] and interval-valued intuitionistic fuzzy soft sets [28] could be viewed as extended models of fuzzy soft sets. Jiang et al. [27] and Zhang et al. [85] extended Feng et al.'s DM approach based on fuzzy soft sets [14] to come up with an intuitionistic fuzzy soft sets based DM approach and an interval-valued intuitionistic fuzzy soft set based DM approach, respectively.

The two most popular fuzzy soft set based DM approaches are: i) the *fuzzy choice value based approach* [30] and ii) the *score based approach* [61]. Up to present, there are still arguments about which one is more reasonable [14, 30], and both of these approaches still have some limitations. For instance, the score based approach proposed by Roy and Maji in [61] requests a large amount of computations when parameters are added or deleted during the DM process, which causes some drawbacks to those problems that deal with dynamic information. The research on rough soft sets based DM and group decision making (GDM) approaches is still in an initial stage. Besides, there is not methods apply assessments on alternatives provided by decision makers to make the decision based on the combination of rough sets and soft sets yet, the previous ones request an optimal decision made by each decision maker before a GDM process is carried out. Based on the above concern, we think it is necessary to carry out a systematic research that aims at the improvement of DM approaches based on fuzzy soft sets and rough soft sets in order to overcome such limitations.

Although various algorithms based on soft rough set models and their fuzzy extensions have been proposed to solve DM problems [81, 82, 83], no researches have been carried out systematically on the inner-relationships among these models yet. In order to make more flexible the application of various soft rough sets models in DM and make it more convenient the selection of suitable models in practical DM circumstances according to properties of different models, it is necessary to carry out a research on their inner-relationships.

In some real world DM problems, decision makers may use linguistic information rather than crisp values to provide their assessments over alternatives. Among various hybrid soft set models, linguistic value soft set is the only one that could be used to deal with linguistic information under the framework of soft sets, however if this model is applied, it requests that decision makers always provide their assessments by means of a single linguistic term, which might be hard, since decision makers may hesitate among several linguistic terms and the use of only one linguistic term would not be enough to reflect their knowledge in a proper way. Therefore, it seems convenient to define new models of soft sets able to deal with not only with single linguistic terms, but also with linguistic expressions.

To deal with linguistic information in DM problems, the fuzzy linguistic approach has been successfully applied [78]. It models the uncertainty by linguistic variables rather than numerical values. In fuzzy linguistic approach, words mean different things to different people, therefore a fuzzy set is adopted to capture uncertainty contained in a word. However, most linguistic models [39, 40] only use single and simple linguistic terms to express preferences of decision makers, which fails to reflect decision makers' real opinion in context with a high level of uncertainty. To overcome this limitation, recently a model called hesitate fuzzy linguistic term set (HFLTS) has been introduced [58]. A context-free grammar was proposed to generate comparative linguistic expressions (CLEs), that are close to cognition of human-being. By using a transformation function, CLEs can be easily translated into HFLTSs [58]. The application of CLEs based on HFLTSs allows experts to provide more complex, elaborated and flexible assessments than single linguistic terms. Therefore, it seems worthy to research the use of CLEs to define new soft sets models able to deal with linguistic expressions.

Within our research in hybridizing soft sets and linguistic expressions, and the need of computing with such expressions, it was noted that in a similar way to linguistic terms in computing with words (CW), the comparative linguistic expressions (CLEs) also mean different things to different people. Hence, to deal with the uncertainty contained in a CLE, suitable representation model for HFLTSs should be constructed. So far, the representation models for HFLTSs are based on linguistic intervals [58] or type-1 fuzzy sets [33]. None of these models take into account the hesitancy and fuzzy uncertainty contained in HFLTSs, which might cause loss of information when CLEs are applied in DM. Therefore, it is necessary to construct new representation models for HFLTSs which can reflect and deal with linguistic uncertainties in a more comprehensive way.

1.2 Objectives

Based on the motivation and considerations raised in the previous section, the purpose of this research is focused on the improvement of application methodologies of hybrid mathematic models in DM, especially in linguistic DM.

Based on this purpose, we set the following objectives:

1. To make a comparative study on the existing DM approaches based on fuzzy soft sets and rough soft sets, point out their limitations and analyze the reasons why they have such limitations. Afterwards, to present new methodologies to overcome these limitations. It is also important to explore new approaches based on these hybrid models to meet different demands of applications.

2. To make a comparative study on the existing soft rough set models as well as their fuzzy extensions, studying the inner-relationships among different models and point out the potential use of these relationships in DM. To explore new ways to combing soft set theory and rough set theory and propose new soft rough set models. To study the application of the new constructed soft rough set models in DM.

3. To define a new hybrid soft set model able to deal with CLEs to improve the

elicitation of linguistic information. To construct some novel algorithms based on the new constructed model to solve DM and GDM problems. Afterwards, to examine the performance of the proposed algorithm in GDM by comparing it with existing algorithms based on other hybrid soft sets.

4. To construct a new fuzzy representation model for CLEs, such a model can be used to reflect and deal with linguistic uncertainties contained in CLEs. Since CLEs could be transformed into HFLTSs, the new representation model in form of type-2 fuzzy sets should reflect and deal with both fuzzy uncertainties and hesitancy contained in HFLTSs. Furthermore, to examine the performance of the new representation model for CLEs in linguistic DM, it should be compared with other existing representation models.

1.3 Structure

To achieve the objectives set out in the previous section, taking into account the article 23, point 3, of the current regulations for Doctoral Studies at the University of Jaén, in accordance with the program established in the RD 99/2011, this research memory will be presented as a set of articles published by the Phd student.

Two articles have been published in International journals indexed by JCR database, produced by ISI. And other two articles have been submitted to two International journals indexed by JCR. In summary, the report is composed of a total of four articles which have been published or submitted in prestigious International journals.

Next, we make a brief description of the structure of this research memory:

- Chapter 2: It revises some theoretical concepts that are used in our proposals to achieve the objects presented: the notions of soft sets, fuzzy soft sets, rough soft sets and several other hybrid soft sets models; the concepts of fuzzy linguistic approach, HFLTSs and comparative linguistic expressions generated by a context-free grammar.

- Chapter 3: It introduces in short the proposals of the published or on-going articles that form the research memory. For each article, a brief discussion of the obtained results is presented.

- Chapter 4: This chapter acts as the core of the doctoral thesis, which contains the publications obtained as result of the research. For each publication, the quality indexes where the proposals have been published are indicated.

- Chapter 5: It points out the final conclusions drawn from this research, and discusses some future works as the development of the current research.

Chapter 2

Basic Concepts and Background

2.1 Decision making under uncertainty

In this section, the concept of decision making (DM) under uncertainty will be reviewed. Some mathematic models, soft sets and hybrid soft sets, which could be applied to solve DM problems with uncertain information are revised. Afterwards, several approaches based on hybrid soft set which are applied to DM problems with vague and uncertain information are briefly discussed.

2.1.1 Decision making under uncertainty and difficulties

DM problems appear frequently in many daily life fields of human being, such as in political agencies [19], engineering [20], business [56], governmental [64], social and economic sciences [71], etc. A DM problem consists of several alternatives and a decision maker who has to make a choice to obtain the best one(s) as solution of the problem. Classical DM problems contain the following elements:

- objectives to be reached;
- several alternatives to be chosen for reaching the objectives;
- a decision context to formulate the decision problem;
- a function to determine the utility values of alternatives under the decision context.

DM problems are classified by means of the decision context as follows: (i) under certain environment; (ii) under risk environment; (iii) under uncertainty environment. In the current research, we focus on the DM problems under uncertain environment, in which it is hard to have probabilistic knowledge about alternatives.

Increasing complexity of the socio-economic environment makes that real-life DM problems are full of uncertainty and imprecise information. In such situations, classical mathematics cannot handle this type of uncertainty, because they require all mathematical notions to be exact, otherwise, precise reasoning would be impossible.

From the mathematic modeling point of view, the main difficulties for dealing with DM problems under uncertainty are:

1. Lack of mathematic models to deal with uncertain information.

   Fuzzy set theory, fuzzy linguistic approach, rough set theory and soft set theory have been proved as effective approaches to deal with different types of uncertainties. The applications of these models are limited since each of them usually only deals with a single type of uncertainty, however in DM problems different types of uncertainties might exist at the same time. To face this situation, it is necessary the construction of new hybrid models taking the advantages of each specific model and dealing with more complex DM situations.

2. Lack of methodologies to apply mathematic models to solve DM problems.

   Although some hybrid models such as fuzzy rough soft sets and rough fuzzy soft sets have been proposed by researchers, DM algorithms or DM approaches based on these models still need to be explored. The existing DM approaches by using hybrid soft sets are few, and the application scope of each approach is usually very limited, since it can only be used to deal with specific situations, therefore new approaches need to be developed to meet the demands of various DM situations.

2.1.2 Soft sets and hybrid soft sets: mathematic models for dealing with uncertainty

Soft sets and hybrid soft sets are useful mathematic models for dealing with uncertainty. In this section, we make a brief review on the notion of soft sets and introduce a list with some popular hybrid soft sets models.

2.1.2.1 Soft sets

Let U be the universe and E the set of all possible parameters under consideration with respect to U. (U, E) is called a soft space. Usually, parameters are attributes, characteristics, or properties of objects in U. A soft set is defined as follows:

Definition 1 *[50] A pair (F, A) is called a soft set over U, where $A \subseteq E$ and F is a mapping given by $F : A \to P(U)$.*

A soft set over U is a parameterized family of subsets of U. For $e \in A$, $F(e)$ may be considered as the set of e-approximate elements of the soft set (F, A).

2.1.2.2 Hybrid soft sets.

To enhance the ability of soft sets in dealing with uncertainty, it has been studied the combination of soft sets with other models. Most of the hybrid soft sets models come from the combination of soft set theory with fuzzy set theory [77] and rough set theory [55]. Several popular hybrid soft set models are revised.

1. **Fuzzy soft sets.**

   Maji et al. [37] initiated the study on hybrid structures involving both fuzzy sets and soft sets and introduced the notion of fuzzy soft sets:

   Definition 2 *[37] Let (U, E) be a soft space. A pair (F, A) is called a fuzzy soft set over U, where $A \subseteq E$ and F is a mapping given by $F : A \to F(U)$.*

   Fuzzy soft sets are a fuzzy generalization of soft sets. Compared to soft sets, in fuzzy soft sets, fuzzy sets on the universe U are used as substitutes for the crisp subsets of U. Therefore, every soft set could be considered as a fuzzy soft set.

2. **Rough soft sets**

   Considering the approximations of soft sets in a Pawlak approximation space, Feng et al. [15] introduced the notion of rough soft sets.

   Definition 3 *[15] Let (U, R) be a Pawlak approximation space and $\mathfrak{S} = (F, A)$ be a soft set over U. The lower and upper rough approximations of $\mathfrak{S} = (F, A)$ with respect to (U, R) are denoted by $\underline{Apr}_R(\mathfrak{S}) = (\underline{F}_R, A)$ and $\overline{Apr}_R(\mathfrak{S}) = (\overline{F}_R, A)$, which are soft sets over U with the set-valued mappings given by*

   $\underline{F}_R(e) = \underline{Apr}_R(F(e))$,

   $\overline{F}_R(e) = \overline{Apr}_R(F(e))$,

   where $e \in A$. The operators $\underline{Apr}_R$ and $\overline{Apr}_R$ are called the lower and upper rough approximation operators on soft sets. If $\underline{Apr}_R(\mathfrak{S}) = \overline{Apr}_R(\mathfrak{S})$ the soft set $\mathfrak{S}$ is said to be definable; otherwise $\mathfrak{S}$ is called a rough soft set.

3. **Soft rough sets**

   In [15], Feng et al. initiated the notion of soft rough sets (F-soft rough sets).

Definition 4 *[15] Let $S = (f, A)$ be a soft set over U. The pair $P = (U, S)$ is called a soft approximation space. Based on P, the following two operations are defined:*

$$\underline{apr}_P(X) = \{u \in U; \exists a \in A(u \in f(a) \subseteq X)\} \tag{2.1}$$

$$\overline{apr}_P(X) = \{u \in U; \exists a \in A(u \in f(a), f(a) \cap X \neq \emptyset)\} \tag{2.2}$$

assigning to every subset $X \subseteq U$ two sets $\underline{apr}_P(X)$ and $\overline{apr}_P(X)$ called the F-lower and F-upper soft rough approximations of X in S, respectively. If $\underline{apr}_P(X) = \overline{apr}_P(X)$, X is said to be F-soft definable in P; otherwise X is called a F-soft rough set.

Compared to rough set theory, in a F-soft rough set, a soft set instead of an equivalence relation is used to granulate the universe of discourse.

4. **Modified soft rough sets**

Shabir et al. [63] noted that if $S = (f, A)$ is not a full soft set, then there exists $x \in U$ such that $x \in Neg_P(X) = U - \overline{apr}_P(X)$ for all $X \subseteq U$. Thus $X \subseteq \overline{apr}_P(X)$ and some basic properties of rough sets do not hold in general. Based on these observations, Shabir et al. [63] proposed the notion of modified soft rough sets (MSR sets).

Definition 5 *[63] Let (f, A) be a soft set over U and $\varphi : U \to P(A)$ be a map defined as $\varphi(x) = \{a \in A; x \in f(a)\}$. The pair (U, φ) is then called MSR-approximation space and for any $X \subseteq U$, the lower MSR approximation and upper MSR approximation of X are denoted by $\underline{X}_\varphi$ and $\overline{X}_\varphi$ respectively, which are defined as*

$$\underline{X}_\varphi = \{x \in U; \forall y \in X^c(\varphi(x) \neq \varphi(y))\} \tag{2.3}$$

$$\overline{X}_\varphi = \{x \in U; \exists y \in X(\varphi(x) = \varphi(y))\tag{2.4}$$

If $\underline{X}_\varphi = \overline{X}_\varphi$, then X is said to be MSR definable, otherwise X is said to be a MSR set.

5. **Soft rough fuzzy sets**

Based on the combination of rough, fuzzy and soft sets, Feng et al.[15], Meng et al. [49] and Zhan et al. [83] proposed different notions of soft rough approximation operators on fuzzy sets and presented three different soft rough fuzzy set models. To facilitate the discussion, we denote them as F-soft rough fuzzy set, M-soft rough fuzzy set and Z-soft rough fuzzy set respectively. These models are briefly reviewed as below.

Let $S = (f, A)$ be a full soft set over U and $P = (U, S)$ be a soft approximation space. For a fuzzy set $\mu \in F(U)$:

Definition 6 *[49] The lower F-soft rough approximation $\underline{sap}_P(\mu)$ and upper F-soft rough approximation $\overline{sap}_P(\mu)$ of μ are fuzzy sets in U given by:*

$$\underline{sap}_P(\mu)(x) = \wedge\{\mu(y); \exists a \in A(\{x, y\} \subseteq f(a))\} \tag{2.5}$$

$$\overline{sap}_P(\mu)(x) = \vee\{\mu(y); \exists a \in A(\{x, y\} \subseteq f(a))\} \tag{2.6}$$

for all $x \in U$. If $\underline{sap}_P(\mu) = \overline{sap}_P(\mu)$, μ is said to be F-soft definable; otherwise μ is called a F-soft rough fuzzy set.

Definition 7 *[15] The lower M-soft rough approximation $\underline{sap}'_P(\mu)$ and upper M-soft rough approximation $\overline{sap}'_P(\mu)$ of μ are fuzzy sets in U given by:*

$$\underline{sap}'_P(\mu)(x) = \vee_{x\in f(a)} \wedge_{y\in f(a)} \mu(y) \tag{2.7}$$

$$\overline{sap}'_P(\mu)(x) = \wedge_{x\in f(a)} \vee_{y\in f(a)} \mu(y) \tag{2.8}$$

for all $x \in U$. If $\underline{sap}'_P(\mu) = \overline{sap}'_P(\mu)$, μ is said to be M-soft definable; otherwise μ is called a M-soft rough fuzzy set.

Definition 8 *[83] The Z-lower soft rough approximation $\underline{\mu}_\varphi$ and Z-upper soft rough approximation $\overline{\mu}_\varphi$ of μ are fuzzy sets in U given by:*

$$\underline{\mu}_\varphi(x) = \wedge\{\mu(y); y \in U \wedge \varphi(x) = \varphi(y)\} \tag{2.9}$$

$$\overline{\mu}_\varphi(x) = \vee\{\mu(y); y \in U \wedge \varphi(x) = \varphi(y)\} \tag{2.10}$$

for all $x \in U$. If $\underline{\mu}_\varphi = \overline{\mu}_\varphi$, μ is said to be Z-soft definable; otherwise μ is called a Z-soft rough fuzzy set.

2.1.3 Uncertain decision making based on hybrid soft sets

We only provide a brief review on DM based on fuzzy soft sets and rough soft sets, since these two models are simple and very popular and our research presented in Section 4.1 is closely related to them. The application of these two models have potential to be extended to more complex models and situations.

Up to present, there still exist arguments and limitations for the existing algorithms based on fuzzy soft sets and rough soft sets, and the application methodologies are far away to meet different demands of applications:

1. In terms of fuzzy soft sets based DM methods, there exist two popular approaches: (i) the *score based approach* [61], and (ii) the *fuzzy choice value based approach* [30].

   - In the *score based approach*, a comparison matrix is constructed and through computing the row sum and column sum of the comparison matrix the scores could be obtained. The final decision is to choose the alternative with maximum score.
   - In the *fuzzy choice value based approach*, the sum of all the membership values of alternatives with respect to all parameters are computed as the fuzzy choice value, and the decision is made by selecting the alternative with maximum fuzzy choice value.

   There has been a fierce argue on which of these two approaches is more reasonable. In [14], it is proposed an adjustable approach by using level soft sets. Since choice values of alternatives in the soft sets are applied to make the decision, their adjustable approach could be viewed as the improvement of the fuzzy choice values based approach.

2. In terms of rough soft sets based DM approaches, we have only noticed two algorithms provided by Ma et al. [35]. However, such algorithms can only solve specific DM problems and could only be viewed as the initial attempt for application.

2.2 Linguistic preference modeling for decision making under uncertainty

In this section, we make a brief review about fuzzy linguistic approach, computing with words (CW), elicitation of comparative linguistic expressions (CLEs) based on hesitant fuzzy linguistic term sets (HFLTSs), and several representation models for HFLTSs.

2.2.1 Fuzzy linguistic approach

Fuzzy linguistic approach has been widely applied to model linguistic preferences in DM. In fuzzy linguistic approach [78], linguistic information is represented by linguistic variables. A linguistic variable is described as "a variable whose values are not numbers but words or sentences in a natural or artificial language" [78], and it could be formally defined as follows:

Definition 9 *[79] A linguistic variable is characterized by a quintuple* $(H, T(H), U, G, M)$ *in which* H *is the name of the variable;* $T(H)$ *is the term set of* H*, i.e., the set of names of linguistic values of* H*, with each value being a fuzzy variable that is denoted by* X *and ranging across a universe of discourse* U*, which is associated with the base variable* u*,* G *is a syntactic rule (which usually takes the form of a grammar) for the generation of the names of values of* H*; and* M *is the semantic rule for associating its meaning with each* H*,* $M(X)$*, which is a fuzzy subset of* U*.*

To deal with linguistic variables, it is necessary to select suitable descriptors for linguistic terms, and define the appropriate semantics (see Fig. 2.1). The linguistic descriptors of terms could be obtained by using an ordered structure approach [23, 75], they can also be obtained by using a context-free grammar, in which the linguistic descriptors are sentences generated by a context-free grammar G [5, 6, 79]. Accordingly, the semantics of terms could be accomplished based on an ordered structure of the linguistic term set, or by using membership functions of primary linguistic terms and a semantic rule to provide membership functions of non-primary linguistic terms [58].

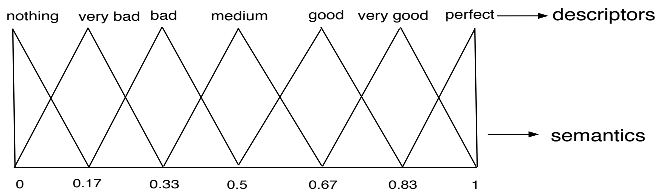


Figure 2.1: Descriptors and semantics of a linguistic term set

The use of linguistic information implies the necessity of operating with linguistic variables. CW is defined as a methodology for reasoning, computing and decision making using natural language information [48]. The general scheme for CW is shown by Fig. 2.2.

Figure 2.2: Computing with words scheme [12, 21]

The application of linguistic information in recommender systems [3, 43, 57], social choice [18], data mining [26], and many other practical fields would not be possible without carrying out CW processes. Tools such as probability [31, 32], Fuzzy Logic [80], and Fuzzy Linguistic Approach [78] grounds the basis for different computational models for CW.

2.2.2 Elicitation of comparative linguistic expressions based on HFLTSs

In real world linguistic DM problems, sometimes it is difficult for decision makers to use single linguistic terms to elicit their assessments on alternatives, since they may hesitate among several linguistic terms at the same time when they are under time pressure, lack of confidence or consider other uncertain issues.

In order to model this hesitant situations, Rodríguez et al. introduced the concept of HFLTS [58].

Definition 10 *[58] Let $S = \{s_0, \ldots, s_g\}$ be a linguistic term set, and a $HFLTS$ H_S, is defined as an ordered finite subset of consecutive linguistic terms of S, $H_S = \{s_i, s_{i+1}, \ldots, s_j\}$ such that $s_k \in S$, $k \in \{i, \ldots, j\}$.*

Although HFLTSs could be used to deal with linguistic preferences in hesitant situations, they are not similar to the way of thinking and reasoning in real world problems. Therefore, in [59] it was proposed the use of a context-free grammar to generate comparative linguistic expressions (CLEs) close to the natural language used by human beings in real world.

Definition 11 *[59] Let G_H be a context-free grammar and $S = \{s_0, \ldots, s_g\}$ be a linguistic term set. The elements of $G_H = (V_N, V_T, I, P)$ are defined as follows:*
$V_N = \{\langle primary\ term\rangle, \langle composite\ term\rangle, \langle unary\ relation\rangle, \langle binary\ relation\rangle, \langle conjunction\rangle\}$.
$V_T = \{at\ most, at\ least, between, and, s_0, \ldots, s_g\}$.
$I \in V_N$.
$P = \{I ::= \langle primary\ term\rangle | \langle composite\ term\rangle$
$\langle composite\ term\rangle ::= \langle unary\ relation\rangle\langle primary\ term\rangle | \langle binary\ relation\rangle\langle primary\ term\rangle\langle conjunction\rangle\langle primary\ term\rangle$
$\langle primary\ term\rangle ::= s_0|s_1|\ldots|s_g$
$\langle unary\ relation\rangle ::= at\ most|at\ least$
$\langle binary\ relation\rangle ::= between$
$\langle conjunction\rangle ::= and\}$.

Different CLEs generated by the context-free grammar G_H can be transformed into HFLTSs by a transformation function. In this way, CLEs could be semantically represented by HFLTSs, and operations on CLEs could be realized through carrying out operations on HFLTSs.

Definition 12 *[58] A transformation function E_{G_H}, which is able to transform a CLE, ll into a $HFLTS$, H_S, where S is the linguistic term set used by E_{G_H}, is defined as:*

$$E_{G_H} : ll \rightarrow H_S \tag{2.11}$$

Based on E_{G_H}, CLEs generated by G_H can be transformed into $HFLTSs$ in different ways according to their meaning:
$E_{G_H}(s_i) = \{s_i | s_i \in S\}$,
$E_{G_H}(\text{at most } s_i) = \{s_j | s_j \leq s_i \text{ and } s_j \in S\}$,
$E_{G_H}(\text{at least } s_i) = \{s_j | s_j \geqslant s_i \text{ and } s_j \in S\}$,
$E_{G_H}(\text{between } s_i \text{ and } s_j) = \{s_k | s_i \leq s_k \leq s_j \text{ and } s_k \in S\}$.

2.2.3 Representation of comparative linguistic expressions

To facilitate the computations with CLEs based on HFLTSs, a representation model for HFLTSs in form of linguistic interval was proposed.

Definition 13 *[58] The envelope of an HFLTS H_S, denoted by $env(H_S)$, is a linguistic interval whose limits are obtained by:*
$env(H_S) = [H_{S^-}, H_{S^+}]$, $H_{S^-} \leq H_{S^+}$
where $H_{S^-} = min\{s_i\}$ and $H_{S^+} = max\{s_i\}$ for $s_i \in H_S$, $\forall i$.

The envelope in form of linguistic intervals loses the fuzzy representation of linguistic information, therefore Liu et al. [33] proposed another kind of envelope in form of trapezoidal fuzzy numbers (TFNs), called type-1 fuzzy envelope. A scheme to obtain fuzzy envelopes of HFLTSs has been provided in [33] by using OWA operators [16] and the semantics of linguistic terms in a linguistic term set. The scheme is briefly summarized as follows:

- Obtain elements to aggregate.

  Each linguistic term of the HFLTS s_k is presented as $A^k = T(a_L^k, a_M^k, a_R^k)$, and it is logical to use the set of all points of all membership functions of linguistic terms as the elements to aggregate, for computing the fuzzy envelope of $H_S = \{s_i, s_{i+1}, \dots, s_j\}$.

  $T = \{a_L^i, a_M^i, a_R^i, a_M^{i+1}, a_L^{i+2}, a_R^{i+1}, \dots, a_L^j, a_R^{j-1}, a_M^j, a_R^j\}$.

  Since $a_R^{k-1} = a_M^k = a_L^{k+1}$, $k = 1, 2, \dots, g-1$, the elements to aggregate could be deduced to

  $T = \{a_L^i, a_M^i, a_M^{i+1}, \dots, a_M^j, a_R^j\}$.

- Compute the parameters of fuzzy envelopes in form of TFNs.

  For any HFLTS, H_S, the fuzzy envelopes for H_S is a TFN, $F_{H_S} = T(a, b, c, d)$, where the parameters a, b, c, d are obtained from aggregation of elements in $T = \{a_L^i, a_M^i, a_M^{i+1}, \ldots, a_M^j, a_R^j\}$. The aggregation operators are OWA operators reviewed in the next step.

- Compute the OWA operators.

  The approach in [16] is adopted to compute the OWA operators, i.e., W^1 and W^2 are chosen as the associated weights to aggregate elements in $T = \{a_L^i, a_M^i, a_M^{i+1}, \ldots, a_M^j, a_R^j\}$ according to the CLE:

  Definition 14 *[16] Let $\alpha \in [0, 1]$,*
  the first type of OWA weights $W^1 = (w_1^1, w_2^1, \ldots, w_n^1)$ is defined as
  $w_1^1 = \alpha$, $w_2^1 = \alpha(1-\alpha)$, $w_3^1 = \alpha(1-\alpha)^2$, $\ldots$, $w_{n-1}^1 = \alpha(1-\alpha)^n - 2$, $w_n^1 = (1-\alpha)^{n-1}$;

  the second type of OWA weights $W^2 = (w_1^2, w_2^2, \ldots, w_n^2)$ is defined as
  $w_1^2 = \alpha^{n-1}$, $w_2^2 = (1-\alpha)\alpha^{n-2}$, $w_3^2 = (1-\alpha)\alpha^{n-3}$, $\ldots$, $w_{n-1}^2 = (1-\alpha)\alpha$, $w_n^2 = 1-\alpha$.

- Obtain the fuzzy envelope.

  Fuzzy envelope F_{H_S} for H_S is obtained as a TFN $T(a, b, c, d)$.

2.3 Group decision making and consensus reaching process

In this section, we first review the concepts of group decision making and consensus reaching process, then we discuss the limitations of group decision making approaches based on hybrid soft set models.

2.3.1 Group decision making

When more than one decision maker takes part in and takes responsibility for the decision result in a DM problem, it is called a group decision making (GDM) problem. More decision makers may make the decision result more reliable, at the same time, it also may increase the difficulties to make the decision considering factors such as the time consumption. A GDM problem is formally characterized by the following elements [29]:

- A common problem to be solved.

- A set of alternatives or a set of possible solutions denoted by X to the problem.

$$X = \{x_1, \ldots, x_n\} \ (n \geqslant 2) \tag{2.12}$$

- A set of decision makers, denoted by E, who express their preferences on alternatives and try to obtain a common solution to the problem.

$$E = \{e_1, \ldots, e_m\} \ (m \geqslant 2) \tag{2.13}$$

Each decision maker provides his/her opinion over alternatives by using a preference structure. Some popular structures in GDM problems under uncertainty are: utility vectors [7], fuzzy preference relation [51, 54, 68], and preference ordering [67]. Different information domains to provide preferences are allowed in GDM problems. Some frequently utilized in GDM under uncertainty are: numerical domain [7, 51, 84], interval-valued domain [17, 73], and linguistic domain [11, 22, 25, 38, 41].

There are two kinds of approaches for solving a GDM problem: a direct approach or an indirect approach [24]. In the direct approach, a solution could be directly obtained from the individual preferences of decision makers, it is not necessary to obtain a social opinion in front. In the indirect approach, a social opinion or a collective preference need to be computed first, and afterwards the collective preference is utilized to obtain the final solution.

The classical selection process to obtain the solution of a GDM problem consists of two phases [60], as shown in Fig. 2.3.

- Aggregation phase: in this phase, the preferences provided by decision makers are combined using aggregation operators to obtain a collective preference.

- Exploitation phase: in this phase, one or a subset of alternatives will be selected by using a selection criterion as the solution for the problem.

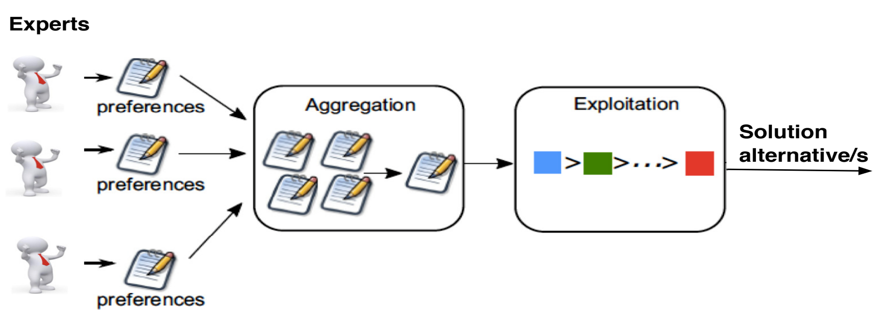


Figure 2.3: Selection process for the solution of a GDM problem

2.3.2 Consensus reaching process in group decision making

When the solution of a GDM problem is directed obtained by the selection process, a desired agreement level among decision makers is difficult to be guaranteed, which may lead to a solution that is not accepted by decision makers who feel that their individual opinions have not been taken into consideration [8]. Since a high level of acceptance of the whole group is critical in real-life GDM problems, it is necessary to consider a phase called "consensus" for GDM problems. A consensus reaching process (CRP) is a dynamic and iterative process consisting of several rounds of discussion, in which decision makers modify their initial opinions to make themselves closer to the majority of the group and to ensure a desired group agreement before making the decision [8, 62].

CRP is usually coordinated by a moderator, who takes responsibility for supervising and guiding the discussion amongst decision makers [8, 62]. A general CRP scheme (see Fig. 2.4) consists of four main phases:

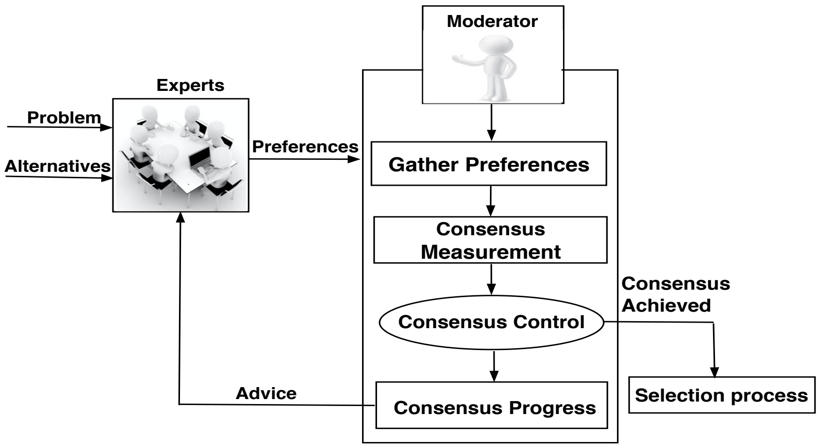


Figure 2.4: A general CRP scheme

1. Gathering preferences

   The preferences of decision makers are collected in this phase.

2. Consensus measurement

   The moderator makes use of individual preferences of decision makers to estimate a group agreement level by consensus measures. Based on the type of information fusion procedures, the existing consensus measures could be classified into two categories [52]: 1) consensus measures based on distances from individual to collective preference; 2) consensus measures based on distances between preferences of different pairs of experts.

3. Consensus control

   The consensus degree obtained previously is compared with a *consensus threshold* $\mu \in [0, 1]$, which indicates the minimum value of acceptable agreement. If the consensus degree exceeds the consensus threshold, μ, means that the desired consensus has been achieved, and therefore, the group moves into the selection process; otherwise, another discussion round should be carried out. A value $maxrounds \in N$, which indicates the maximum number of allowed rounds will be set a prior in order to prevent a never ending process.

4. Consensus progress

   If the current consensus degree is not enough, a procedure should be adopted to increase the level of agreement throughout the discussion rounds. The procedure can also be classified into two categories [53]:

   - Traditionally, such a procedure incorporates a feedback generation process, in which the moderator identifies the farthest assessments from consensus in the current round, and then some advise are generated to modify decision maker's assessments to get closer to the rest of the group and increase the consensus degree [42, 62].
   - Some other consensus models employ a procedure without a feedback generation process, in which assessments of decision makers are updated automatically to increase the consensus degree [4, 72, 84]. In this model, decision makers only need to provide initial preference information, since it is no necessary for them to be involved in the following rounds.

2.3.3 Hybrid soft sets based group decision making and their limitations

Several GDM approaches by using hybrid soft sets have been proposed by researchers. Based on the revisions of linguistic preference modeling, GDM, and CRP, group decision making approaches based on hybrid soft set models present several limitations that could be listed as follows:

1. A priori optimal group decisions.

   Current GDM algorithms based on soft rough sets request that each decision maker provides their optimal choices before making the group decision. That is, each decision maker has to make his/her own decision before the GDM process carries out. This request is very strict, usually can not be carried out in real life decision making, thus the applications for the existing GDM algorithms are limited.

2. Lack of models and approaches to deal with linguistic information under the framework of hybrid soft sets.

   It has been mentioned before that the use of linguistic information is very common in GDM problems. Although hybrid soft sets have been applied in GDM, they can not be applied when decision makers provide linguistic information. As far as we know, there is only one hybrid soft set model called *linguistic value soft sets* [65] that is able to deal with linguistic preferences.

3. No consensus reaching process have been considered.

   As it was mentioned in the above section, a CRP is very important for ensuring a decision accepted by the whole group. However, no consensus models have been defined either applied to GDM problems by using hybrid soft set models.

Chapter 3

Research Results

This chapter presents a summary of the main proposals considered in this research memory. Research findings and research results will be briefly discussed for each proposal. It is structured in four proposals which are related with the objectives described in the introduction chapter:

1. Improving DM approaches based on fuzzy soft sets and rough soft sets.
2. A comparative study of some soft rough sets.
3. Hesitant linguistic expression soft sets: Application to group decision making.
4. Type-2 fuzzy envelope for HFLTSs and its application to multi-criteria decision making.

3.1 Improving decision making approaches based on fuzzy soft sets and rough soft sets

In order to achieve the first objective mentioned in Section 1.2, in this proposal, we analyze the limitations of existing approaches based on fuzzy soft sets and rough soft sets. Afterwards, we provide some methods to improve them, and we propose some new algorithms to enrich the methodologies.

3.1.1 Improving decision making approaches based on fuzzy soft sets

There has been an argument on DM approaches based on fuzzy soft sets about which one "*the score based method*" or "*the fuzzy choice value based method*" is more reasonable. In this proposal, we point out that it is hard to determine which one is more reasonable without setting a certain circumstance, since their decision criteria are different. We state that "the score based method" is used to select the objects that cooperate with more attributes in quantity, whereas "the fuzzy choice value based method" is used to select the objects that cooperate with more attributes in quality. Both approaches have some limitations, and there should be proposed new methods to overcome such limitations.

The main limitation for fuzzy choice value based method is that the direct addition of all membership degrees with respect to attributes is not always reasonable. We summarize the methodologies overcoming this limitation to avoid the unreasonable direct addition of membership values:

(i) We can weight the parameters according to advices provided by relevant experts, and the fuzzy choice value could be submitted by using the aggregated result obtained by OWA operators [74]. The approaches for synthesizing or redefining fuzzy choice value is not unique, the use of OWA operators is only an illustration.

(ii) We can use the adjustable approach proposed in [14] to avoid direct addition. In their approach, through using a threshold value, fuzzy soft sets will be transformed into soft sets, then DM problems based on fuzzy soft sets will become a problem based on soft sets. Therefore, choice value instead of fuzzy choice value will be adopted to evaluate alternatives. If this approach is applied, we state that the most important task is to select a suitable threshold value according to practical circumstances.

The score based method has two main limitations: (i) when parameters need to be updated during the DM process, each entry in the comparison matrix have to be recomputed, which result in a large amount of computation. (ii) there exists DM problems which can not be successfully dealt with by using the score based method [14]. Thus, we provide some new methods which could be used to overcome these limitations:

1. To improve the score based method, we introduce a new concept so-called D-Score and a mathematic tool D-score table. It is proved that the results of a DM problem are the same by using the score based method and the proposed approach with D-Score table. However, the amount of computations obviously decreases when the parameters are added/deleted in the DM problem. Therefore, the proposed approach is able to overcome the limitation (i) of the score based method because it does not repeat computation when the information is updated. Consequently, the new approach based on the D-score table could be considered as an improvement.

2. A novel adjustable DM approach based on fuzzy soft set is introduced by using thresholds when comparing two membership degrees to obtain different kinds of scores for alternatives. Benefitting from Feng's idea of introducing thresholds [14], a comparison threshold will be taken into consideration when comparing the membership degrees of two objects with respect to a common parameter. In this way, if the exceed degree of one membership degree over another is not less than the comparison threshold, we say that the object relatively possesses that parameter. With the introduction of threshold values, the scores of alternatives will be different from the scores by using the score based approach, and different optimal decision will be obtained by setting different comparison thresholds, which makes this approach adjustable. This new adjustable approach could be used to solve problems which cannot be solved by using the score based approach.

3.1.2 Improving decision making approaches based on rough soft sets

After studying the existing DM approaches based on rough soft sets, we have found two main limitations that should be overcome:

(i) The application scopes of existing DM approaches based on rough soft sets are very limited, difficult to meet various demands of DM.

(ii) So far, there is not any application of rough soft set in GDM.

Taking into account these two limitations, we introduce new approaches:

1. Two new DM approaches based on rough soft set are introduced. One selects the optimal choice whose upper approximations cover all alternatives while the lower approximations cover a specific number of alternatives. Another approach determines the best choice by selecting attributes whose upper approximations cover the most number of alternatives. Several examples are provided to illustrate the feasibility of both approaches. Different selection mechanisms are proposed to enrich the methodologies for applying rough soft sets in DM.

2. A GDM approach based on rough soft set which successfully solve problems when the initial evaluation information provided by experts are their assessments on alternatives from different parameters aspects. It has been analyzed that most of existing GDM approaches based on soft rough sets have a strong requirement, because each decision maker has to make optimal choices before

a GDM process. Our proposed GDM approach based on rough soft set successfully gets rid of this strong requirement, since the group decision results could be obtained based on the assessments provided by decision makers over alternatives. It is noticed that our approach is the first attempt in applying rough soft set in GDM.

The article associated to this proposal is the following one:

Y Liu, K Qin, L Martínez. Improving decision making approaches based on fuzzy soft sets and rough soft sets. Applied Soft Computing, 2018, 65: 320-332.

3.2 A comparative study of some soft rough sets

In this section, hybrid soft sets models constructed by combining soft sets and rough sets, as well as fuzzy extension models of rough sets, are collectively refer to as soft rough sets. To achieve the second objective of this research, this section is devoted to two main directions:

1) The discussion on relationships among various existing soft rough sets;

2) The introduction of a novel soft rough set model and a decision making approach based on such a model.

3.2.1 The relationships among various soft rough sets

The combination of soft set, fuzzy set and rough set is one of the most important issues in the development of soft set theory, since it can enhance the ability of soft sets dealing with multiple types of uncertainty. We notice that the relationships among different soft rough sets have not been systematically studied. In this proposal, we study the relationships among various soft rough sets, and our research result could be briefly summarized as follows:

1. The relationship between F-soft rough approximations (see page 12, Def. 4) and MSR approximations (see page 13, Def. 5):
   $\overline{X}_{\varphi} \subseteq \overline{apr}_P(X)$, $\overline{apr}_P(X) \subseteq \overline{X}_{\varphi}$, $\underline{X}_{\varphi} \subseteq \underline{apr}_P(X)$, if some specific conditions hold, respectively.

2. The relationship between F-soft rough sets (see page 12, Def. 4) and Pawlak's rough sets [55]:
   F-soft rough sets in (U, S) could be identified with Pawlak's rough sets in (U, R_S), when the underlying soft set is a partition soft set.

3. The relationship between MSR approximations (see page 13, Def. 5) and Pawlak's rough approximations [55]:
   MSR approximation operator is a kind of Pawlak rough approximation operator.

4. The relationship between Z-lower, Z-upper soft rough approximation operators (see page 14, Def. 8) and Dubois and Prade's lower and upper rough fuzzy approximation operators [13]:
   Z-lower and Z-upper soft rough approximation operators are equivalent to Dubois and Prade's lower and upper rough fuzzy approximation operators.

5. The relationship between the (classical) rough fuzzy sets [13] and M-soft rough fuzzy sets (see page 14, Def. 7):
   The (classical) rough fuzzy sets in Pawlak approximation space (U, R) and M-soft rough fuzzy sets in soft approximation space (U, S) are equivalent when the underlying soft set S is a partition soft set.

6. The relationship among Z-soft rough approximation operators (see page 14, Def. 8) and M-soft rough approximation operators (see page 14, Def. 7) and F-soft rough approximation operators (see page 13, Def. 6) on fuzzy set μ:
   $\underline{sap}_P(\mu) \subseteq \underline{sap}'_P(\mu) \subseteq \underline{\mu}_\varphi \subseteq \mu \subseteq \overline{\mu}_\varphi \subseteq \overline{sap}'_P(\mu) \subseteq \overline{sap}_P(\mu)$.

7. The relationship between soft fuzzy rough approximation [49] and Dubois and Prade's fuzzy rough approximation [13]:
   The soft fuzzy rough approximation is a kind of Dubois and Prade's fuzzy rough approximation [13].

8. The relationship between F-soft rough set (see page 12, Def. 4) and soft rough soft set (proposed by us):
   Soft rough soft set is an extension of F-soft rough set.

3.2.2 A novel soft rough set and decision making

By extending the notion of F-soft rough set [15], we define a new soft rough set model, called soft rough soft set, and then study its application in decision making.

1. By using a soft set as the elementary knowledge to compute the approximations of another soft set, we originally define a new hybrid soft set model called *soft rough soft set*. Compared to F-soft rough sets, soft rough soft sets introduce parameter tools to the universe description and a soft set (instead of a subset of the universe) is approximated. Compared to rough soft set [15], a soft set instead of an equivalence relation has been adopted in soft rough soft sets to

compute the approximations of soft set. The parameter tool is necessary not only for the knowledge representation, but also for the universe description. In soft rough soft sets, parameterized tools have been used in both aspects.

2. A multi-group decision making approach based on soft rough soft sets has been provided to illustrate the application of the proposed model. This approach is designed to deal with problems in which two groups of experts take part in the decision. The decision principle could be briefly described as follows:

   Assessments of each expert in group A are performed as a soft set, the group assessments of A is a soft set obtained by using the intersection operation of soft sets. The best alternatives selected by another specialists of the group B also form a soft set. B is supposed to be more reliable than A. Through computing the lower and upper soft rough approximations of the soft set related to B in the soft approximation space, consisted of the soft set related to A, and the alternative which occupies most number of benefit properties and may be important for both groups, will be chosen. The proposed approach takes full advantage of information provided by two independent groups.

The article associated to the second objective of this research is the following one:

Y Liu, L Martínez, K Qin. A Comparative Study of Some Soft Rough Sets. Symmetry, 2017, 9(11): 252.

3.3 Hesitant linguistic expression soft sets: Application to group decision making.

Although several hybrid soft set models have been constructed by combing soft sets with other mathematic models for dealing with uncertainty, as far as we know, there is only one model called linguistic value soft set that combines soft set theory with linguistic information which allows decision makers to provide assessments with single linguistic terms. However, linguistic terms may be too strict in hesitant situations and decision makers might hesitate among several terms. In such situations, they may prefer to use linguistic expressions rather than linguistic terms. The use of CLEs based on HFLTSs could be suitable to describe hesitant linguistic information. Nevertheless, linguistic value soft sets fail to deal with CLEs or any other complex linguistic expression, therefore, more flexible and practical models need to be constructed in order to deal with CLEs under the framework of soft sets.

In this proposal, first a novel hybrid soft set model called hesitant linguistic expression soft set (HLE soft set) is defined, which is able to deal with CLEs. Second, a decision making approach based on HLE soft set is introduced and an illustrative example is shown. Third, a novel GDM model to reach consensus is proposed. The performance of the new GDM approach is examined by comparing it with an existing GDM approach [65], through handling a GDM problem by using these two approaches, respectively. In this way, we achieve the third objective mentioned in Section 1.2.

3.3.1 Hesitant linguistic expression soft set

Soft set theory provides a framework for considering assessments from different parameters aspects. In this proposal, to improve the ability and flexibility of linguistic value soft set ecilitating linguistic expressions more complex than single linguistic terms, we do further research on the combination of soft set theory and HFLTSs to introduce the notion of hesitant linguistic expression soft set (HLE soft set), which makes possible to evaluate alternatives from different parameters aspects by using CLEs.

Let U be the universe set and E be related parameters. Let $S = \{s_0, s_1, \ldots, s_g\}$ be a linguistic term set and $\mathcal{P}(U)$ be the power set of all CLEs built from S for the universe U. A pair (F^{cle}, E) is called a HLE soft set over U, where F^{cle} is a mapping from a parameter set E to the power set of all CLEs built from S for U, i.e., $F^{cle} : E \to \mathcal{P}(U)$.

According to the context-free grammar, CLEs contains linguistic terms, which means that a HLE soft set will degenerate to a linguistic value soft set when all CLEs degenerate to single linguistic terms. However, it improves the linguistic value soft set by allowing assessments on alternatives with respect to parameters presented in form of both linguistic terms and CLEs.

In order to carry out computations on HLE soft sets and solve DM problems dealing with HLE soft sets, it is necessary to define suitable syntax and semantics for the linguistic terms in the linguistic term set S. In this proposal:

1. The semantic of linguistic terms in S is defined by means of triangular membership functions. CLEs in HLE soft sets are semantically represented by the fuzzy envelope of HFLTSs.

2. The syntax of linguistic terms in S describes the satisfactory degree of alternatives with respect to parameters. To consider extreme situations when alternatives absolutely satisfy parameters or not satisfy parameters at all, we adopt two single terms, “none” and “absolute” with semantics $T(0, 0, 0, 0)$ and $T(1, 1, 1, 1)$, as the smallest and the largest linguistic terms in S.

3. To carry out the CW process with CLEs in HLE soft sets, fuzzy envelopes for HFLTSs in form of trapezoidal fuzzy numbers (TFNs) will be used. However,

the methodology for computing type-1 fuzzy envelopes [33] is adjusted when the linguistic terms "none" and "absolute" are considered.

As it is mentioned above, CLEs could be transformed into HFLTSs by using a transformation function (see Def. 11), the fuzzy envelope for the HFLTSs (in form of TFNs) could be computed by means of the adjustment of the approach in [33]. On the basis of a principle that "the larger the fuzzy envelope of a HFLTS is, the larger the corresponds CLE should be", we use a ranking approach for CLEs based on a ranking approach for TFNs [1]. Based on this ranking approach for CLEs, operations on HLE soft sets are studied, including the complement, relative complement, extended intersection, restricted intersection, restricted union, and extended intersection of HLE soft sets.

3.3.2 Decision making based on HLE soft sets

A DM approach based on HLE soft set is proposed to deal with multi-criteria DM problems. To solve a DM problem by using the proposed method/approach, first of all, the assessments on all alternatives provided by decision makers should be CLEs, and all assessments form a HLE soft set. To carry out the computation on a HLE soft set, all CLEs in the HLE soft sets need to be transformed into HFLTSs. Afterwards, the type-1 fuzzy envelopes of all HFLTSs are computed. The following step is to compute the magnitudes of all fuzzy envelopes. Finally, the scores of alternatives are computed based on the magnitudes values, and the alternative with the maximum score is selected.

This approach extends the DM approach based on fuzzy soft set proposed in [2]. The novelty of our approach is that we use the magnitude [1] of type-1 fuzzy envelope of the HFLTSs transformed from CLEs to compute the score of each alternative. The advantage of this approach is the use of all linguistic information provided by decision makers from different parameters aspects to make the final decision.

An example is provided to illustrate the feasibility of the proposed DM algorithm. As far as we know, there are few algorithms that could be used to solve linguistic DM problems under the background of soft sets, the proposed approach promote the application of soft set theory with linguistic information.

3.3.3 Group decision making based on HLE soft sets

A GDM approach based on HLE soft set is proposed to deal with multi-criteria GDM problems. To facilitate the computation during the GDM process, a new operator called CLE-OWA operator is defined to aggregate CLEs. A CLE-OWA operator could be viewed as a special ordinary OWA operator [28] in which the weights are linguistic terms while the aggregation objects are CLEs. By using CLE-OWA operator, we obtain a collective HLE soft set which reflects the opinion of the group.

The parameterization tool of soft set theory makes HLE soft sets more comprehensive by using linguistic information with regards to different parameters, it is more difficult to deal with the CRP because there will be a larger amount of data. To deal with this issue, a consensus model (see Fig. 3.1) based on HLE soft set is proposed.

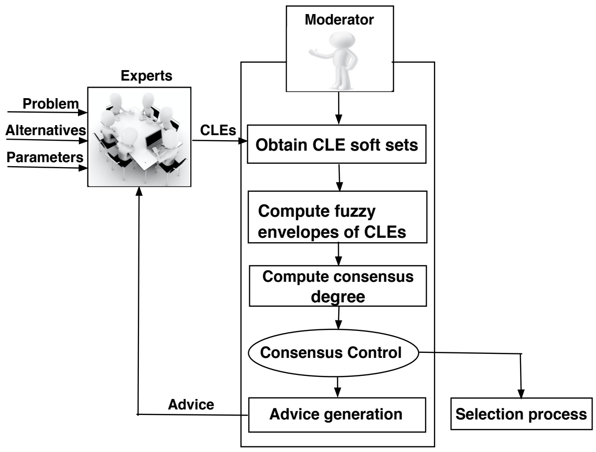


Figure 3.1: A consensus model based on HLE soft set

This new GDM approach based on HLE soft sets can be briefly introduced as follows:

1. Collect all assessments upon alternatives with respect to parameters, and the assessments provided by each decision maker form a HLE soft set. In this way, several HLE soft sets can be obtained.

2. A CRP is carried out to ensure that the decision result is accepted by the majority of decision makers.

3. A collected HLE soft set (group opinion) is computed by using the CLE-OWA operator to aggregate CLEs.

4. Based on the collected HLE soft set, the GDM problem becomes a DM problem. The DM approach mentioned in Section 3.4.2 is applied, and the alternative(s) with maximum score is selected.

Considering that there is only one GDM method proposed by Sun et al. [65] which deals with linguistic assessments of decision makers under the framework of soft set theory based on linguistic value soft set, we provide a simple GDM problem to make a comparison between the two approaches and show some advantages of our proposal. After carrying out different experiments with these two approaches, we can point out the following advantages:

1. When parameters considered by all decision makers are the same, the decision can not be made by using Sun et al.'s approach. However, the same problem can be solved by using our GDM approach.

2. Compared with Sun et al.'s approach, a CRP has been considered in our GDM approach, which makes the final decision closer to the opinion of the majority, and ensures a more reasonable decision result.

3. Sun et al.'s approach is constructed based on linguistic value soft sets, whereas our GDM approach is constructed based on HLE soft sets. It depends on different hybrid soft set models they adopt, Sun et al.'s approach can only deal with decision makers' assessments in form of linguistic terms, whereas our GDM approach can deal with not only linguistic terms but also with CLEs.

Based on the above analysis, we conclude that our GDM proposal goes beyond Sun et al.'s, since the former can be used to solve GDM problems that can not be handled by the latter.

The article related to this approach is:

Y Liu, R M. Rodríguez, J C R. Alcantud, K Qin and L Martínez. Hesitant based HLE soft sets: Application to group decision making, Computers & Industrial Engineering, submitted.

3.4 Type-2 fuzzy envelope for HFLTSs and its application to multi-criteria decision making

Decision makers may prefer to provide assessments on alternatives by using linguistic terms rather than crisp values. A lot of factors, such as lack of confidence, time pressure, may cause decision makers hesitate among several linguistic terms at the same time when they are requested to give their evaluations. To deal with hesitant situations, different elaborated models [34, 66, 69] have been proposed to provide more flexible and richer expressions than single linguistic terms. However, none of these models are close to the way of thinking and reasoning of human being or they do not formalize the generation of such linguistic expressions. As is mentioned in

Section 3.3, the concept of CLEs based on HFLTSs could be a suitable choice to facilitate the elicitation of linguistic information and model hesitant situations. CLEs are generated formally by using a context-free grammar, and are easily transformed and semantically represented by means of HFLTSs, which allows experts to elicit several linguistic values for a linguistic variable. Therefore, CLEs are convenient because they are similar to the expressions used by decision makers in the daily life.

The use of linguistic information implies CW processes [40, 44, 76]. Existing representations for HFLTSs, called envelopes, are proposed in form of linguistic intervals or type-1 fuzzy sets. However, it should be noticed that CLEs mean different things to different people because there exists uncertainties contained in CLEs. None of the two kinds of envelopes for HFLTSs [33, 58] can reflect or describe the uncertainty contained in CLEs, which motivates us to make further research on the construction of a new fuzzy envelope for HFLTSs based on type-2 fuzzy sets. It will be tested in comparison with other previous representations.

3.4.1 An approach to construct a type-2 fuzzy envelope for HFLTS

With this proposal, we achieve the fourth objective mentioned in Section 1.2 which consists of defining a new fuzzy representation model for CLEs that can reflect the linguistic uncertainty contained in CLEs. This approach is divided into three-steps (see Fig. 3.2).

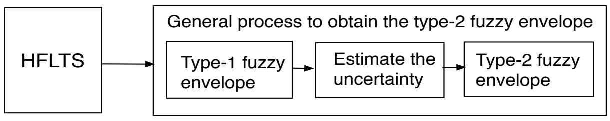


Figure 3.2: A three-steps process to construct the type-2 fuzzy envelope for HFLTS

1. Compute type-1 fuzzy envelope.

   In the first step, the type-1 fuzzy envelope is obtained by using the method proposed in [33].

2. Compute the uncertainty contained in HFLTSs.

   Recently, a measure called comprehensive entropy for HFLTSs has been introduced [70] to evaluate both fuzzy uncertainty and hesitancy of HFLTSs. Compared to other measures to evaluate uncertainties of HFLTSs, the comprehensive entropy takes full use of uncertain information and therefore it is good enough to capture the uncertainties contained in CLEs. In this proposal, comprehensive entropy will be used in the second step of the general process of type-2 fuzzy envelope.

The fuzzy uncertainty of linguistic terms have been studied by using fuzzy entropy measure in [70]. The larger fuzzy entropy of a linguistic term is, the more fuzzy the linguistic term is. In this research, we introduce an example to show that the hesitancy among linguistic terms more fuzzy will lead to more uncertainty compared with hesitancy among linguistic terms less fuzzy. To perform different treatments for hesitations contained in different HFLTSs by determining the importance degrees of hesitancy according to the specific characteristics of HFLTSs, we use a function $\beta(H_S)$ to control the importance of hesitancy, and compute the comprehensive entropy $E_c(H_S)$ for HFLTSs by using the following equation:

$$E_c(H_S) = \frac{E_f(H_S) + \beta(H_S)E_h(H_S)}{1 + \beta(H_S)E_h(H_S)} \tag{3.1}$$

where $E_f(H_S)$ and $E_h(H_S)$ are the fuzzy and hesitant entropy of H_S, respectively. The function $\beta(H_S)$ represents the importance degree of the hesitancy when evaluating the overall uncertainty contained in H_S. The larger $\beta(H_S)$, the greater the value of overall uncertainty $E_c(H_S)$, because more hesitancy will be taken into account when the overall uncertainty contained in H_S is computed.

Several principles to determine $\beta(H_S)$ have been provided considering two main factors: the number of linguistic terms and the positions of linguistic terms contained in the $HFLTS$, H_S. These principles ensures that:

(i) when all linguistic terms are contained in H_S, the importance level of hesitancy reaches the highest;

(ii) when only one single term is contained in H_S, the importance level of hesitancy reaches the lowest;

(iii) the importance of hesitancy increases when the level of hesitancy contained in H_S increases;

(iv) the changing quantity of importance of hesitancy is positively correlated to the fuzzy degree of the linguistic term added/deleted of a given HFLTSs;

(v) when both the fuzzy uncertainty and the hesitancy are equal for two different HFLTSs, the importance of hesitancy should be the same.

According to the context-free grammar revised in Def. 9, there are three kinds of CLEs: "at least s_i", "at most s_i" and "between s_i and s_j", where s_i and

s_j are linguistic terms in the linguistic term set S. For these three different cases, we provide different equations to compute $\beta(H_S)$, the equations are listed below:

$$\beta(E_{G_H}(\text{at least } s_i)) = \frac{1}{2}cos\frac{\pi}{g}i + \frac{1}{2}, i \in [0, g]. \quad (3.2)$$

$$\beta(E_{G_H}(\text{at most } s_i)) = \frac{1}{2}sin(\frac{\pi}{g}i - \frac{\pi}{2}) + \frac{1}{2}, i \in [0, g]. \quad (3.3)$$

$$\beta(E_{G_H}(\text{between } s_i \text{ and } s_j)) = \frac{1}{2}cos\frac{\pi}{g}i + \frac{1}{2}sin(\frac{\pi}{g}j - \frac{\pi}{2}), \quad i, j \in [0, g]. \quad (3.4)$$

These three equations are proved to satisfy the principles set a prior. However, it should be noticed that the way for computing $\beta(H_S)$ is not unique, the equations proposed are a possible choice, but any others that satisfy the principles could be used.

3. Construct a type-2 fuzzy envelope for HFLTSs.

   A new representation model for HFLTSs, called type-2 fuzzy envelope will be constructed by using type-2 fuzzy sets [46, 47]. To simplify the computation in the initial exploration of type-2 fuzzy envelope, interval type-2 fuzzy sets (IT2 FSs) [45] are applied in our proposal, which means that a secondary grade of 1 will be put at all points of $FOU(\tilde{A})$. An IT2 FS can be uniquely determined by its FOU, if the LMF and UMF are determined, the FOU could be uniquely determined, as well as the IT2 FS.

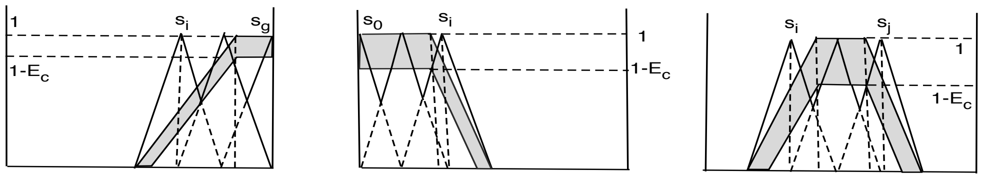
   Therefore, we construct the type-2 fuzzy envelope $\tilde{F}_{H_S}$ of H_S. To do so, we use a type-1 fuzzy envelope F_{H_S} as the upper membership function, and the lower membership function of $\tilde{F}_{H_S}$ is presented as

$$\underline{\mu}_{\tilde{F}_{H_S}}(x) = max\{0, F_{H_S}(x) - E_c(H_S)\}, \ \forall x \in X$$

   The uncertainty contained in H_S can be approximately reflected by the width of FOU, and the type-2 fuzzy envelope can be presented as an IT2 FS $\tilde{F}_{H_S} = 1/FOU(\tilde{F}_{H_S})$ with

$$FOU(\tilde{F}_{H_S}) = \{(x, u) : x \in X, u \in [max\{0, F_{H_S}(x) - E_c(H_S)\}, F_{H_S}(x)]\}. \quad (3.5)$$

   Footprints of type-2 fuzzy envelope for three types of CLEs are shown by Fig. 3.3.



$FOU(\tilde{F}_{E_{G_H}(\text{at least } s_i)})$ $FOU(\tilde{F}_{E_{G_H}(\text{at most } s_i)})$ $FOU(\tilde{F}_{E_{G_H}(\text{between } s_i \text{ and } s_j)})$

Figure 3.3: Footprints of type-2 fuzzy envelopes for CLEs

In summary, in order to construct suitable representation models for CLEs based on HFLTSs: first, we compute the comprehensive entropy for HFLTSs. The comprehensive entropy synthesizes the fuzzy uncertainty and hesitancy for HFLTSs. Afterwards, the comprehensive entropy for HFLTSs is used to measure the uncertainties contained in CLEs when we construct the type-2 fuzzy envelopes for HFLTSs (that are obtained from CLEs). The approach to construct type-2 fuzzy envelope for HFLTSs in form of IT2 FSs uses the type-1 fuzzy envelope as the lower approximation membership function of the footprint and the comprehensive entropy as the width of the footprint. Finally, the type-2 fuzzy envelopes for HFLTSs are obtained as the type-2 fuzzy representations for CLEs, which successfully reflects the uncertainties contained in CLEs.

3.4.2 Comparisons between type-1 and type-2 fuzzy envelopes in decision making.

The second part of the fourth objective mentioned in Section 1.2 was to examine the performance of the new representation model with existing ones. To do so, a comparison on the application of type-1 and type-2 fuzzy envelope of HFLTSs has been carried out by considering a multi-criteria DM problem. The problem has been handled by using type-1 fuzzy TOPSIS [9] and type-2 fuzzy TOPSIS [10] methods with type-1 and type-2 fuzzy envelopes, respectively. It is shown by experiments that the decision result by using type-2 fuzzy envelope is consistent with the decision result by using type-1 fuzzy envelope. However, in situations in which two alternatives cannot be distinguished by using type-1 fuzzy envelope, the use of type-2 fuzzy envelope provides a more precise result.

The reason why the use of type-2 fuzzy envelope provides more precise decision result has been analyzed: compared with type-1 fuzzy envelope, the construction of type-2 fuzzy envelope takes into account both the fuzzy uncertainty and hesitancy contained in the linguistic expressions. The comparison between the use of type-1 TOPSIS method with type-1 fuzzy envelopes for HFLTSs, and the use of type-2 fuzzy

TOPSIS with type-2 fuzzy envelopes successfully reduces the loss of information caused by computations on CLEs during the DM process, therefore it achieves more accurate decision result.

The article related to this proposal is the following one:

Y Liu, R M. Rodríguez, H Hagras, H Liu, K Qin, and L Martínez. Type-2 fuzzy envelope of hesitant fuzzy linguistic term set: a new representation model of comparative linguistic expressions, IEEE Transactions on Fuzzy Systems, submitted.

Chapter 4

Publications

4.1 Improving decision making approaches based on fuzzy soft sets and rough soft sets

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Improving decision making approaches based on fuzzy soft sets and rough soft sets

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ABSTRACT

Hybrid soft sets, such as fuzzy soft sets and rough soft sets, have been extensively applied to decision making. However in both cases, there is still a necessity of providing improvements on approaches to obtain better decision results in different situations. In this paper several proposals for decision making are provided based on both hybrid soft sets. For fuzzy soft sets, a computational tool called D-score table is introduced to improve the decision process of a classical approach and its convenience has been proved when attributes change across the decision process. In addition, a novel adjustable approach based on decision rules is introduced. Regarding rough soft sets, several new decision algorithms to meet different decision makers' requirements are introduced together a multi-criteria group decision making approach. Several practical examples are developed to show the validity of such proposals.

1. Introduction

Classical mathematical tools, which require all notions to be exact, usually fail to handle the uncertainty, imprecision and vagueness in a wide variety of practical fields. Although theories such as fuzzy set theory [1], rough set theory [2], intuitionistic fuzzy set theory [3] and vague set theory [4] have been proved useful mathematical approaches in modeling these uncertainties, all of them have a common limitation—the inadequacy of the parameterization tool. In 1999, soft set theory was put forward by Molodstsov [5] as a new mathematic tool for dealing with uncertainty, which is free from the above mentioned limitation. Afterwards, the generalized models of soft sets (hybrid soft sets) come forth rapidly and there has been an increasing interest in the practical applications of hybrid soft set theories, especially with regard to their applications in decision making [6–17].

The popular hybrid soft set models contain two main categories: (1) The combination of soft set theory with fuzzy set theory and the generalized models of fuzzy set theory [18–24]; (2) The combination of soft set theory with rough set theory and the generalized models of rough set theory [25–28]. As two representative hybrid soft set models in these two different categories, fuzzy soft sets [18] and rough soft sets [26] are interconnected [29]. All decision making methods based on fuzzy soft sets or rough soft sets have the potential to be extended to deal with more complex hybrid soft set models situations. For instance, Jiang et al. [30] and Zhang et al. [31] extended Feng et al.'s decision making approach based on fuzzy soft sets in [32] to come up with an intuitionistic fuzzy soft sets based decision making approach and an interval-valued intuitionistic fuzzy soft set based decision making approach, respectively.

In terms of fuzzy soft set based decision making methods, Roy and Maji [33] provided a novel method (the score based method) for decision making based on fuzzy soft sets, which builds upon concepts such as the comparison table and the scores of objects. However, no researchers have paid attention to the improvement of the score based method in order to overcome its own limitations and make it fit for more practical situations until now, although its reasonability has already been verified [32]. With the development of information technology in modern society, the practical information updates rapidly as time goes by, adding new data and removing old data. In this paper, we will improve the score based method by introducing a new mathematic tool called D-Score table and then successfully make it more convenient to obtain the decision result when parameters should be added/deleted in decision making problems, this improvement will be useful for practical problems solving which contains updating information. Furthermore, we propose a new approach to fuzzy soft set based decision making by introducing comparison thresholds when comparing two membership values to obtain different kinds of scores for

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objects. After choosing different comparison thresholds, we will construct different level D-Score tables and then obtain different optimal decision sets, which makes the new approach adjustable. In this way, the new approach can be successfully used to deal with some problems which cannot be solved by the initial score based method.

In terms of rough soft sets based decision making approaches, the researches are so far few. Recently, Ma et al. [34] introduced some initial algorithms for decision making based on rough soft sets and an algorithm for group decision making based on MSR-set [35]. However, the algorithms proposed by Ma et al. [34] are far from enough to meet various practical demands. Furthermore, the group decision making method based on rough soft sets has not been studied yet. In the present work, with the construction of rough soft sets and a fuzzy soft set, we determine the weights of experts by using the similarity measure of soft sets and then provide a new approach based on rough soft sets to solve the group decision making problem. As an important hybrid soft set model generated with rough set theory, rough soft sets has huge potential to be used in dealing with practical problems that contain uncertainty, and it is a promising topic to find out more decision making approaches based on rough soft sets to meet the different demand of decision.

Although some researchers have systematically discussed the decision making approaches based on fuzzy soft sets and rough soft sets recently [34,36], they concentrated on proposals review or revision, rather than improving existing approaches or providing new approaches to meet various application demands. There still exist arguments on the fuzzy soft sets based decision making approaches [32,37], and it can be said that the research of rough soft sets based decision making approaches is still in an initial stage, that is the reason why it necessary to carry out a research focuses on the improvement of decision making approaches based on fuzzy soft sets and rough soft sets. In the current work, the limitations of some popular existing proposals will be systematically discussed and afterwards several solutions will be provided. All the improved proposals or new approaches provided in the current research have the potential to be extended to more complex hybrid models situations.

The present paper is organized as follows: Some basic notions on soft sets, fuzzy soft sets and rough soft sets are reviewed in Section 2. In Section 3, we recall an existing argument of fuzzy soft sets based decision making approaches, provide our opinion on this argument, afterwards present an improvement of the score based method. On the basis of this improved score based method, a new adjustable decision making approach based on fuzzy soft sets is proposed. In Section 4, we discuss the limitation of existing rough soft set based decision making methods and the necessary to enrich the approaches, afterwards two algorithms are provided to conquer these limitations and to meet various practical demands. It is worth noticing that we originally apply rough soft set as a tool to deal with group decision making problems, which successfully solve the problems according to assessments on alternatives provided by decision makers, rather than according to specific decision results made by separate decision makers which have been adopted in some other existing approaches. Finally, conclusions are given in Section 5.

2. Preliminaries

In this section we briefly recall some concepts that will be useful in subsequent discussions.

Let U be the initial universe of objects and E be the set of attributes related to objects in U. Both U and E are assumed to be nonempty finite sets. Let $P(U)$ be the power set of U and $A \subseteq E$.

Definition 1. [5]: A pair (F, A) is called a soft set over U, where F is a mapping given by $F: A \longrightarrow P(U)$.

For any attribute $e \in A$, $F(e) \subseteq U$ may be considered as the set of e-approximate elements of the soft set (F, A). In other words, the soft set is not a kind of set in the ordinary sense, but a attributeized family of subsets of U. We denote by (U, E) the set of all soft sets over U.

For two soft sets (F, A) and (G, B) over a common universe U, we say that (F, A) is a soft subset of (G, B) if

(1) $A \subseteq B$;
(2) $\forall e \in A$, $F(e) \subseteq G(e)$.

This relationship is denoted by $(F, A) \subseteq (G, B)$. (G, B) is said to be a soft super set of (F, A), if (F, A) is a soft subset of (G, B).

Definition 2. [38]: A mapping $S: (U, E) \times (U, E) \rightarrow [0, 1]$ is said to be a similarity measure if the following axioms hold for arbitrary $(F, A), (G, B) \in (U, E)$:

(1) $0 \leq S((F, A), (G, B)) \leq 1$;
(2) $S((F, A), (F, A)) = 1$;
(3) $S((F, A), (G, B)) = S((G, B), (F, A))$;
(4) If $(F, A) \subseteq (G, B) \subseteq (H, C)$, then $S((F, A), (H, C)) \leq S((F, A), (G, B))$, $S((F, A), (H, C)) \leq S((G, B), (H, C))$.

The theory of fuzzy sets, first introduced by Zadeh [1] in 1965, provides an appropriate framework for representing and processing vague concepts by allowing partial memberships. A fuzzy set F in the universe U is defined as $F = \{(x, \mu_F(x))/x \in U, \mu_F(x) \in [0, 1]\}$. μ_F is called the membership function of F and $\mu_F(x)$ indicates the membership degree of x to F. The family of all fuzzy sets on U is denoted by $F(U)$.

In 2001, Maji et al. [18] initiated the study on hybrid structures involving both fuzzy sets and soft sets. They introduced the notion of fuzzy soft sets, which can be seen as a fuzzy generalization of crisp soft sets.

Definition 3. [18]: A pair (F, A) is called a fuzzy soft set over U, where $A \subseteq E$ and F is a mapping given by $F: A \longrightarrow F(U)$.

For any attribute $e \in A$, $F(e)$ is a fuzzy subset of U and it is called fuzzy value set of attribute e. If for any attribute $e \in A$, $F(e)$ is a crisp subset of U, then the fuzzy soft set (F, A) degenerated to the standard soft set. Let us denote $\mu_{F(e)}(x)$ the membership degree that object x holds attribute e where $x \in U$ and $e \in A$. Then $F(e)$ can be written as $F(e) = \{ < x, \mu_{F(e)}(x) > | x \in U\}$.

For two fuzzy soft sets (F, A) and (G, B) over a common universe U, we say that (F, A) is a fuzzy soft subset of (G, B) if

(1) $A \subseteq B$;
(2) $\forall e \in A$, $F(e)$ is a fuzzy subset of $G(e)$, that is $\mu_{F(e)}(x) \leq \mu_{G(e)}(x)$ for all $x \in U$.

This relationship is denoted by $(F, A) \subseteq (G, B)$. (F, A) and (G, B) are said to be fuzzy soft equal if and only if $(F, A) \subseteq (G, B)$ and $(F, A) \supseteq (G, B)$, We write $(F, A) = (G, B)$.

A fuzzy soft set (F, A) over U is said to be null fuzzy soft set, if for $\forall e \in A$, we have $F(e) = \emptyset$; A fuzzy soft set (F, A) over U is said to be absolute fuzzy soft set, if for $\forall e \in A$, we have $F(e) = U$.

The rough set theory proposed by Pawlak [2] provides a systematic approach for dealing with vague concepts caused by indiscernibility in situation with insufficient and incomplete information.

Definition 4. [2]: Let R be an equivalence relation on the universe U. (U, R) is called a Pawlak approximation space. For any $X \subseteq U$, the lower approximation $\underline{Apr}_R(X)$ and the upper $\bar{Apr}_R(X)$ of X are defined as:

$\underline{Apr}_R(X) = \{x \in U : [x]_R \subseteq X\}$,

$\bar{Apr}_R(X) = \{x \in U : [x]_R \cap X \neq \emptyset\}$.

A subset $X \subseteq U$ is called definable if $\underline{Apr}_R(X) = \bar{Apr}_R(X)$; otherwise, X is said to be a rough set.

Considering the lower and upper approximations of a soft set in a Pawlak approximation space, Feng et al. [26] introduced the concept of rough soft sets.

Definition 5. [26]: Let (U, R) be a Pawlak approximation space and $\mathfrak{S} = (F, A)$ be a soft set over U. The lower and upper rough approximations of $\mathfrak{S} = (F, A)$ with respect to (U, R) are denoted by $\underline{Apr}_R(\mathfrak{S}) = (\underline{F}_R, A)$ and $\bar{Apr}_R(\mathfrak{S}) = (\bar{F}_R, A)$, which are soft sets over U with the set-valued mappings given by

$\underline{F}_R(e) = \underline{Apr}_R(F(e))$,

$\bar{F}_R(e) = \bar{Apr}_R(F(e))$, where $e \in A$. The operators $\underline{Apr}_R$ and $\bar{Apr}_R$ are called the lower and upper rough approximation operators on soft sets. If $\underline{Apr}_R(\mathfrak{S}) = \bar{Apr}_R(\mathfrak{S})$ the soft set $\mathfrak{S}$ is said to be definable; otherwise $\mathfrak{S}$ is called a rough soft set.

3. Improving decision making approaches based on fuzzy soft sets

In this section, first we will state our opinion on an existing argument upon two popular decision making approaches based on fuzzy soft sets. Afterwards, we summarize the limitations of these two approaches and furthermore provide solutions. Especially, some improvement approaches are proposed to overcome the limitations of "the score based method" (Roy-Maji method) [33].

3.1. An argument on fuzzy soft sets based decision making

Which fuzzy soft sets based decision making method is more reasonable: "the score based method" or "the fuzzy choice value based method"? There has been a strong argument on this question. As follows we will present our opinion based on a brief list of these arguments and summarize the main limitations of both approaches.

By introducing the concept of comparison table and a measure called the score of object, Roy and Maji [33] introduced an original decision making method as below (see Algorithm 1).

Algorithm 1.

[Step 1.] Input the fuzzy soft sets (F, A), (G, B) and (H, C).
[Step 2.] Input the attribute set P as observed by the observer.
[Step 3.] Compute the corresponding resultant fuzzy soft set (S, P) from the fuzzy soft sets (F, A), (G, B), (H, C) and place it in tabular form.
[Step 4.] Construct the comparison table of the fuzzy soft set (S, P) and compute r_i and t_i for o_i, $\forall i$.
[Step 5.] Compute the score $s_i = r_i - t_i$ of o_i, $\forall i$.
[Step 6.] The decision is o_k if $s_k = max_i s_i$.
[Step 7.] If k has more than one value then any one of o_k may be chosen.

The comparison table is a square table in which both rows and columns are labelled by the objects $o_1, o_2, \ldots, o_n$, and the entry c_{ij} indicates the number of attributes for which the membership value of o_i exceeds or equals to the membership value of o_j. Clearly, $0 \leq c_{ij} \leq m$ and $c_{ii} = m(\forall i, j)$, where m is the number of attributes.

The row-sum r_i of object o_i is computed by

$$r_i = \sum_{j=1}^{n} c_{ij} \tag{1}$$

Table 1
Tabular representation of fuzzy soft set (S, P).

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	Choice value(c_i)
o_1	0.3	0.1	0.4	0.4	0.1	0.1	0.5	$c_1 = 1.9$
o_2	0.3	0.3	0.5	0.1	0.3	0.1	0.5	$c_2 = 2.1$
o_3	0.4	0.3	0.5	0.1	0.3	0.1	0.6	$c_3 = 2.3$
o_4	0.7	0.4	0.2	0.1	0.2	0.1	0.3	$c_4 = 2.0$
o_5	0.2	0.5	0.2	0.3	0.5	0.5	0.4	$c_5 = 2.6$
o_6	0.3	0.5	0.2	0.2	0.4	0.3	0.3	$c_6 = 2.2$

Table 2
Comparison table of the fuzzy soft set (S, P).

	o_1	o_2	o_3	o_4	o_5	o_6
o_1	7	4	2	4	4	4
o_2	6	7	5	5	3	3
o_3	6	7	7	5	3	3
o_4	4	4	4	7	2	3
o_5	3	4	4	6	7	6
o_6	4	5	4	6	3	7

Table 3
Score table of the fuzzy soft set (S, P).

	Row-sum(r_i)	Column-sum(t_i)	Score(s_i)
o_1	25	30	−5
o_2	29	31	−2
o_3	31	26	5
o_4	24	33	−9
o_5	30	22	8
o_6	29	26	3

The column-sum t_j of object o_j is computed by

$$t_j = \sum_{i=1}^{n} c_{ij} \tag{2}$$

Finally, the score s_i of object o_i is defined as

$$s_i = r_i - t_i \tag{3}$$

When dealing with decision making problems by Algorithm 1, the objects with the maximum score computed from the comparison table will be regarded as the optimal decision, so this method is called "the score based method" in this paper. Here is an example to illustrate it:

Example 1. [33]: Let $U = \{o_1, o_2, \ldots, o_6\}$ be the universe of objects. The tabular representation of the fuzzy soft set (S, P) (with choice values) is given by Table 1. The comparison table of (S, P) is shown by Table 2. Then we obtain Table 3, namely the score table of (S, P), by computing the row-sum r_i, column-sum t_i and the score s_i for each object o_i. From Table 3, it is clear that the optimal decision is o_5 since it has the maximum score $s_5 = 8$.

In [37], Kong et al. argued that "the score based method" was incorrect since the decision result obtained by using "the score based method" is not always the object with the maximum choice value. Besides, they revised Algorithm 1 from Step 4 by redesigning c_{ij} and r_i as follows.

$$c_{ij} = \sum_{k=1}^{m} (f_{ik} - f_{jk}) \tag{4}$$

$$r_i = \sum_{j=1}^{m} c_{ij} \tag{5}$$

where f_{ik} is the membership value of object o_i for the kth attribute, m is the number of attributes. The decision set obtained by Kong et al.'s revised algorithm is o_k if $r_k = max_i r_i$.

In [32], Feng et al. deduced Kong's c_{ij} as follows

$$c_{ij} = \sum_{k=1}^{m} f_{ik} - \sum_{k=1}^{m} f_{jk} = c_i - c_j \tag{6}$$

where c_i is the sum total of all membership values of object o_i with respect to different attributes, which is called the fuzzy choice value of object o_i.

The object with the maximum fuzzy choice value, instead of the object with the maximum score, will be selected as the optimal decision by Kong's redesigned algorithm [32]. Hence, Kong et al.'s algorithm [37] can be called "the fuzzy choice value based method". However, Feng et al. [32] argued that the direct addition of all the membership values with respect to different attributes in a fuzzy soft set is not always reasonable, it no longer represents the number of (good) attributes possessed by an object in decision making. Furthermore, they provided an example without setting a real-life circumstance to argue that the score based method is more suitable than the fuzzy choice value based method.

Actually, it is hard to determine whether "the score based method" or "the fuzzy choice value based method" is more reasonable without setting a certain circumstance, since their decision criteria are different. "the score based method" is to used to select the objects which cooperate with more attributes in quantity, whereas "the fuzzy choice value based method" is used to select the objects which cooperate with more attributes in quality. Both approaches have some limitations, and there should be proposed new methods to overcome these limitations.

1. For fuzzy choice value based method, the main limitation is that the direct addition of all membership values with respect to attributes is not always reasonable. To overcome this limitation:
   - Method 1: Firstly, if the direct addition of all membership values is not reasonable in some cases, we can consider other synthesized measures to construct the fuzzy choice value by the membership values with respect to every attribute. For instance, weight the attributes with the help of relevant experts and then use the OWA operator to compute the fuzzy choice value of each object.
   - Method 2: When it is hard to determine the weights for different attributes in some specific cases, Feng et al.'s adjustable approach in [32], i.e. translating a fuzzy soft set into a soft set by using threshold values is another choice to make the decision result more reasonable. By selecting certain thresholds and using corresponding decision rules, a fuzzy soft set will be translated into a crisp soft set, then choice value of objects in a soft set, instead of fuzzy choice value of objects in a fuzzy soft set, will be used to measure objects. If this approach is taken, selecting the most suitable threshold values according to practical circumstances becomes the most important task.
2. For the score based method, two main limitations of Algorithm 1 can be listed as below. To overcome these limitations, we will provide an improved algorithm and some new algorithms in Sections 3.2 and 3.3, respectively.
   - Limitation 1: During the process of decision making, sometimes new attributes need be added if the existing attributes are not enough to embody the character of objects. On the contrary, some attribute need to be deleted if these attributes are proven to be ineffective to the decision result. According to the calculation mechanism for scores of objects in Algorithm 1, a new comparison table has to be conducted when a set of attributes need to be added/deleted, which indicates a large amount of recalculations should be involved in order to obtain a new solution set.

Table 4
Tabular representation of the fuzzy soft set (F_1, E_1) with choice values and scores.

	e_1	e_2	e_3	e_4	Choice value(c_i)	Score(s_i)
o_1	0.92	0.88	0.08	0.12	$c_1 = 2.0$	$s_1 = 0$
o_2	0.82	0.60	0.18	0.40	$c_2 = 2.0$	$s_2 = 0$
o_3	0.24	0.46	0.83	0.47	$c_3 = 2.0$	$s_3 = 0$
o_4	0.12	0.40	0.96	0.52	$c_4 = 2.0$	$s_4 = 0$

   - Limitation 2 [32]: There exist some fuzzy soft set based decision problems in which Algorithm 1 cannot be successfully used to reach an optimal decision.

Example 2. [32]: Let (F_1, E_1) be a fuzzy soft set and Table 4 be its tabular representation. From Table 4, it is clear that all these objects have the same score (i.e., $s_1 = s_2 = s_3 = s_4 = 0$) and the same fuzzy choice value (i.e., $c_1 = c_2 = c_3 = c_4 = 2.0$). By using both the score based method and the fuzzy choice value based method we could hardly arrive at the final optimal decision, since any one of them could be selected as the optimal candidate.

3.2. An improved method of "the score based method"

Based on the limitations analysis of "the score based method" in Section 3.1, in this subsection, by introducing a tool called D-Score table, we will provide an equivalence approach of Algorithm 1 which successfully overcomes Limitation 1 of "the score based method".

Definition 6. Let $U = \{o_1, o_2, \ldots, o_n\}$ be the universe and $A = \{e_1, e_2, \ldots, e_m\}$ be the attribute set. The D-Score of object o_i on e_l is denoted by $S(o_i)(e_l)$ and defined by

$$S(o_i)(e_l) = R(o_i)(e_l) - T(o_i)(e_l) \tag{7}$$

where $R(o_i)(e_l) = |\{o_j \in U | \mu_{F(e_l)}(o_i) \geq \mu_{F(e_l)}(o_j)\}|$, $T(o_i)(e_l) = |\{o_j \in U | \mu_{F(e_l)}(o_j) \geq \mu_{F(e_l)}(o_i)\}|$.

The D-Score of object o_i is denoted by S_i and defined as

$$S_i = \sum_{l=1}^{m} S(o_i)(e_l). \tag{8}$$

The D-Score table is a table in which rows are labelled by the attributes $e_1, e_2, \ldots, e_m$ and columns are labelled by the objects $o_1, o_2, \ldots, o_n$. The entry corresponding to attribute e_l and object o_i is denoted by $S(o_i)(e_l)$.

An algorithm based on the D-Score table of a fuzzy soft set is given (see Algorithm 2).

Algorithm 2.

[Step 1.] Input a fuzzy soft set (F, A).
[Step 2.] Present the D-Score table for (F, A) and compute the D-Score S_i of o_i, $\forall i$.
[Step 3.] The optimal decision is to select o_j if $S_j = max_i S_i$.
[Step 4.] If j has more than one value, then any o_j can be chosen as the decision result.

Theorem 1. *Let (F, A) be a fuzzy soft set on U. For any $o_i \in U$, calculate its score s_i by Algorithm 1 and its D-Score S_i by Algorithm 2, then we have $s_i = S_i$.*

Proof. Since $\sum_{j=1}^{n} c_{ij} = \sum_{l=1}^{m} R(o_i)(e_l)$ and $\sum_{j=1}^{n} c_{ji} = \sum_{l=1}^{m} T(o_i)(e_l)$, we obtain $s_i = r_i - t_i = \sum_{j=1}^{n} c_{ij} - \sum_{j=1}^{n} c_{ji} = \sum_{l=1}^{m} R(o_i)(e_l) - \sum_{l=1}^{m} T(o_i)(e_l) = \sum_{l=1}^{m} (R(o_i)(e_l) - T(o_i)(e_l)) = \sum_{l=1}^{m} S(o_i)(e_l) = S_i$.

Example 3. Consider the fuzzy soft set (S, P) in Example 1, the D-Score table of (S, P) is presented as Table 5. For any $o_i \in \{o_1, o_2, \ldots, o_6\}$, its score s_i in Table 3 is equal to its D-Score S_i in Table 5.

By Theorem 1, we know that for any object $o_i \in U$, its D-Score S_i obtained by Algorithm 2 is always the same as its score s_i obtained

Table 5
The D-Score table of (S, P) with D-Scores.

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	D-Score (S_i)
o_1	−1	−5	1	5	−5	−2	2	S_1 = −5
o_2	−1	−2	4	−3	0	−2	2	S_2 = −2
o_3	3	−2	4	−3	0	−2	5	S_3 = 5
o_4	5	1	−3	−3	−3	−2	−4	S_4 = −9
o_5	−5	4	−3	3	5	5	−1	S_5 = 8
o_6	−1	4	−3	1	3	3	−4	S_6 = 3

Table 6
The tabular representation of (G, P').

	e'_1	e'_2	e'_3
o_1	0.3	0.3	0.4
o_2	0.4	0.7	0.5
o_3	0.4	0.3	0.6
o_4	0.5	0.2	0.7
o_5	0.6	0.1	0.8
o_6	0.3	0.5	0.2

Table 7
D-Score table of $(H, P \cup P')$.

	S_i	e'_1	e'_2	e'_3	S'_i	$S_i + S'_i$
o_1	−5	−4	0	−3	−7	−12
o_2	−2	0	5	−1	4	2
o_3	5	0	0	1	1	6
o_4	−9	3	−3	3	3	−6
o_5	8	5	−5	5	5	13
o_6	3	−4	3	−5	−6	−3

by Algorithm 1, which indicates that the optimal decision sets obtained by Algorithms 1 and 2 are always the same, and thus Algorithms 1 and 2 are equivalent. The "score" and "D-Score" of an object will not be distinguished in the following discussion since they are always equal.

Suppose that (F, E_1) is an original fuzzy soft set and a new attribute set $E_2 = \{e'_1, e'_2, \ldots, e'_r\}$ should be added to E_1. If we use Algorithm 1, to obtain the scores of objects, we have to compute the new comparison table for the new fuzzy soft set $(H, E_1 \cup E_2)$. If we use Algorithm 2, after adding attributes, we only need to calculate the D-Score table of (G, E_2) to obtain the D-Score table for $(H, E_1 \cup E_2)$. In this way, we say although Algorithms 1 and 2 are equivalent, Algorithm 2 has the advantage that reduce the time consumption by avoid redundant computations of Algorithm 1 when attributes are added/deleted in a decision making problem. Adding attributes and deleting attributes are similar cases, so we only discuss the cases when attributes should be added.

Here is an example to illustrate the convenience of Algorithm 2 in avoiding redundant computations:

Example 4. Let (S, P) be the fuzzy soft set on U given by Table 1 in Example 1. Suppose that some new attributes $P' = \{e'_1, e'_2, e'_3\}$ should be added to P, let (G, P') be the corresponding fuzzy soft set which is shown by Table 6. If we use Algorithm 2, then we only need calculate the D-Score table for (G, P'). For an object o_i, its D-Score in $(H, P \cup P')$ is the sum of its D-Score in (S, P) and its D-Score in (G, P'), i.e., $S_i + S'_i$. The D-Score table of $(H, P \cup P')$ is shown by Table 7. In contrast, if we use Algorithm 1, we need recalculate all issues in the new comparison table of $(H, P \cup P')$, which is shown by Table 8.

Here is an example carried on data of moderate size to illustrate the advantage of Algorithm 2 in reducing time consumption:

Example 5. Suppose that there are n objects that are related with m attributes in the fuzzy soft set which we will apply to make the decision. By writing codes in C++, an experiment is performed on a PC Intel Core i-5 with 4 GB RAM and Windows 7 as operating system. This experiment is divide into two stages. In the first stage, the time consumption will be tested when the number of objects (n) keeps constant as 200 while the number of parameters "m" changes among 50, 100, 150, 200, 250, 300, 350, 400, 450, as is shown in Table 9. In the second stage, 10 new parameters and corresponds new data will be added to the initial data set, and the time consumption are shown in Table 10 for obtaining the final decision results on the basis of the median decision results obtained in stage 1. From Table 9 we can easily observe that based on the same scale of initial fuzzy soft sets, the time consumption for achieving decision results by using Algorithms 1 and 2 are very similar. However, Table 10 show that in the second stage when 10 new parameters needed to be considered, the second-stage time consumption increases with the number of the whole parameter sets increases if Algorithm 1 is applied, whereas the time consumption stay unchanged if Algorithm 2 is adopted. This example serves as a strong evidence for that algorithm 2 effectively decreases the time consumption when parameters are requested to be added to a decision making problem during the decision process.

Table 8
Comparison table of the fuzzy soft set $(H, P \cup P')$.

	o_1	o_2	o_3	o_4	o_5	o_6
o_1	10	4	3	5	5	6
o_2	9	10	7	6	4	6
o_3	9	9	10	6	4	5
o_4	6	6	6	10	3	5
o_5	5	6	6	8	10	8
o_6	6	5	5	7	4	10

Table 9
Time consumption in the first stage.

m	50	100	150	200	250	300	350	400	450
$A1$[1] (s)	0.020	0.040	0.060	0.081	0.100	0.120	0.140	0.160	0.180
$A2$ (s)	0.020	0.040	0.060	0.081	0.101	0.121	0.143	0.164	0.184

[1] In Tables 9 and 10, Algorithms 1 and 2 are denoted by A1 and A2, respectively. The time consumption is measured in seconds.

Table 10
Time consumption in the second stage.

m	60	110	160	210	260	310	360	410	460
$A1$ (s)	0.024	0.044	0.064	0.084	0.104	0.124	0.145	0.164	0.184
$A2$ (s)	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004

Lemma 1. *Let $x_1, x_2, x_3, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ be two number sequences, if $x_i \le x_j \Leftrightarrow y_i \le y_j$ (for $\forall i, j \in \{1, 2, 3, \ldots, n\}$), the two sequences are called the same ordered and this relationship is denoted by $q(x_1, x_2, x_3, \ldots, x_n) = q(y_1, y_2, \ldots, y_n)$.*

Theorem 2. *Let (F, A) and (F', A) be two fuzzy soft sets on the universe U. Suppose that o_i is an object in the universe U. In the fuzzy soft set (F, A), denote the D-Score of o_i on e_l by $S(o_i)(e_l)$ and calculate the D-Score of o_i by $S_i = \sum_{l=1}^{m} S(o_i)(e_l)$. In the fuzzy soft set (F', A), denote the D-Score of o_i on e_l by $S'(o_i)(e_l)$ and calculate the D-Score of o_i by $S'_i = \sum_{l=1}^{m} S'(o_i)(e_l)$. If $q(S(o_1)(e_l), S(o_2)(e_l), \ldots, S(o_n)(e_l)) = q(S'(o_1)(e_l), S'(o_2)(e_l), \ldots, S'(o_n)(e_l))$ for all $e_l \in A$, then we have $S_i = S'_i$.*

When dealing with a decision making problem by Algorithms 1 or 2, if the membership value of one object is larger than another object with respect to one attribute, then it is supposed that the former object relatively possesses that attribute. Under this supposition, how much the membership value of one object is larger than another has not been taken into consideration. That is, no matter the membership value of one object is larger than another by a little or by a lot, we all conclude the same when we computing D-

Table 11
Tabular representation of a fuzzy soft set (F, A) with D-Scores.

	e_1	e_2	e_3	e_4	D-Score(S_i)
o_1	0.8	h_{12}	0.1	0.1	S_1 =−6
o_2	0.5	0.7	0.5	0.3	S_2 = 0
o_3	0.3	0.6	0.8	0.9	S_3 = 2
o_4	0.2	0.9	h_{43}	0.8	S_4 = 4

Scores of objects. With respect to an attribute, as long as the order of the membership values of objects stay unchanged, the D-Scores of objects with respect to this attribute are determined. Hence, as long as the order of the membership values of objects with respect to every attribute stay unchanged, then the D-Scores of objects are determined, and thus the optimal decision set will stay unchanged.

Example 6. Let (F, A) be a fuzzy soft set and Table 11 be the tabular representation of it. When the value of h_{12} varies between [0, 0.6), the value of h_{43} varies between (0.8, 1], the order of membership values of objects with respect to every attribute stay unchanged, hence the D-Scores of objects are determined, i.e., S_1 =−6, S_2 = 0, S_3 = 2, S_4 = 4. By Algorithm 2, we obtain that the optimal candidate is o_4.

3.3. *An adjustable decision making approach based on fuzzy soft sets*

Benefitting from Feng's idea of introducing thresholds in [32], a comparison threshold will be taken into consideration when comparing the membership values of two objects with respect to a common attribute. In this way, a new approach will be provided on this basis of algorithm 2 (see Section 3.2). Only if the exceed degree of one membership value over another is not less than the comparison threshold, we say that the object relatively possesses that attribute. People can obtain different optimal decision set by setting different comparison threshold, which make this approach adjustable. This new adjustable approach can also be regarded as an improvement of the score based method in [33] since it follows the initial idea of "scores" of objects and successfully overcomes Limitation 2 of Algorithm 1.

3.3.1. *t-Level D-Score table of fuzzy soft sets.*

By introducing a measure called "t-level D-Score" of object and the new tool called t-level D-Score table, we present an adjustable approach to solve fuzzy soft set based decision making problems (see Algorithm 3).

Definition 7. Let $U = \{o_1, o_2, \ldots, o_n\}$ be the universe, E be the attribute set, $A \subseteq E$ and $A = \{e_1, e_2, \ldots, e_m\}$. Suppose that (F, A) is a fuzzy soft set over U. For $t \in [0, 1]$, the t-level D-Score of object o_i on e_l is denoted by $S(o_i)(e_l)_t$ and defined by

$$S(o_i)(e_l)_t = R(o_i)(e_l)_t - T(o_i)(e_l)_t, \tag{9}$$

where $R(o_i)(e_l)_t = |o_j \in U\backslash\{o_i\} : \mu_{F(e_l)}(o_i) - \mu_{F(e_l)}(o_j) \geq t|$ and $T(o_i)(e_l)_t = |o_j \in U\backslash\{o_i\} : \mu_{F(e_l)}(o_j) - \mu_{F(e_l)}(o_i) \geq t|$.

The t-level D-Score of object o_i is denoted by S_i^t and defined by

$$S_i^t = \sum_{l=1}^{m} S(o_i)(e_l)_t. \tag{10}$$

The t-level D-Score table is a square table in which rows are labelled by the attribute names $e_1, e_2, \ldots, e_m$, columns are labelled by the object names $o_1, o_2, \ldots, o_n$, and the entry corresponding to attribute e_l and object o_i is $S(o_i)(e_l)_t$. The t-level D-Score table can be regarded as an extension of the D-Score table for a fuzzy soft set, and $t \in [0, 1]$ can be viewed as a comparison threshold between membership values of two objects with regard to each attribute. For real-life application of fuzzy soft set based decision making, the threshold t can be chosen by decision makers according to their requirement.

3.3.2. *Level D-Score table with respect to a comparison threshold fuzzy set*

In the definition of t-level D-Score table, the level comparison threshold assigned to each attribute is always a constant value $t \in [0, 1]$. However, it may happen that the decision makers would like to impose different comparison thresholds on different attributes in some special decision making problems. To deal with such situations, we can use a function instead of a constant number as the comparison threshold.

Now we introduce a measure called the level D-Score with respect to λ, and the new tool called the level D-Score table with respect to a fuzzy set λ.

Definition 8. Let $U = \{o_1, o_2, \ldots, o_n\}$ be the universe, E be the attribute set, $A \subseteq E$ and $A = \{e_1, e_2, \ldots, e_m\}$. Let $\lambda : A \longrightarrow [0, 1]$ be a fuzzy set on A which is called a comparison threshold fuzzy set. The level D-Score of object o_i on e_l with respect to λ is denoted by $S(o_i)(e_l)_\lambda$ and defined by

$$S(o_i)(e_l)_\lambda = R(o_i)(e_l)_\lambda - T(o_i)(e_l)_\lambda \tag{11}$$

where $R(o_i)(e_l)_\lambda = |o_j \in U\backslash\{o_i\} : \mu_{F(e_l)}(o_i) - \mu_{F(e_l)}(o_j) \geq \lambda(e_l)|$ and $T(o_i)(e_l)_\lambda = |o_j \in U\backslash\{o_i\} : \mu_{F(e_l)}(o_j) - \mu_{F(e_l)}(o_i) \geq \lambda(e_l)|$. The level D-Score of object o_i with respect to λ is denoted by S_i^λ and defined by

$$S_i^\lambda = \sum_{l=1}^{m} S(o_i)(e_l)_\lambda. \tag{12}$$

The level D-Score table with respect to the fuzzy set λ is a square table in which rows are labelled by the attributes $e_1, e_2, \ldots, e_m$, columns are labelled by the objects $o_1, o_2, \ldots, o_n$, and the entry corresponding to attribute e_l and object o_i is $S(o_i)(e_l)_\lambda$.

The D-Score table with respect to a comparison threshold fuzzy set generalize the t-level D-Score table by substituting a fuzzy set $\lambda : A \longrightarrow [0, 1]$ for a constant $t \in [0, 1]$. Let $\hat{t}$ denote the constant fuzzy set on A given by $\hat{t}(e) = t$ for $e \in A$, then we immediately have $S(o_i)(e_l)_{\hat{t}} = S(o_i)(e_l)_t$, that is, the level D-Score table with respect to the constant fuzzy set $\hat{t}$ coincides with the t-level D-Score table.

Example 7 *(The mid-level-comparison threshold of a fuzzy soft set).* Let the universe $U = \{o_1, o_2, \ldots, o_n\}$, E be the attribute set, $A \subseteq E$ and $A = \{e_1, e_2, \ldots, e_m\}$. Based on the fuzzy soft set (F, A), we can define a fuzzy set $\lambda_F^{mid} : A \longrightarrow [0, 1]$ by

$$\lambda_F^{mid}(e_l) = \frac{1}{|U|}(\vee_{o_i \in U}\mu_{F(e_l)}(o_i) - \wedge_{o_i \in U}\mu_{F(e_l)}(o_i)),$$

for all $e_l \in A$. The fuzzy set λ_F^{mid} is called the mid-level-comparison threshold of fuzzy soft set (F, A). In addition, the level D-Score table with respect to the mid-level-comparison threshold fuzzy set λ_F^{mid} is called the mid-level D-Score table of fuzzy soft set (F, A). In what follows the mid-level-comparison rule will mean using the mid-level-comparison threshold and considering the mid-level D-Score table of a fuzzy soft set in decision making process.

For a concrete example of mid-level-comparison threshold fuzzy set and mid-level D-Score table, let us reconsider the fuzzy soft set (F_1, E_1) with its tabular representation given by Table 4. It is clear that the mid-level-comparison threshold of (F_1, E_1) is a fuzzy set

$$\lambda_{F_1}^{mid} = \{(e_1, 0.20), (e_2, 0.12), (e_3, 0.22), (e_4, 0.10)\}.$$

Example 8 *(The min-level-comparison threshold of a fuzzy soft set).* Let the universe $U = \{o_1, o_2, \ldots, o_n\}$, E be the attribute set, $A \subseteq E$ and

Table 12
The 0.15-level D-Score table of (F_1, E_1) with 0.15-level D-Scores.

	e_1	e_2	e_3	e_4	$S_i^{0.15}$
o_1	2	3	−2	−3	$S_1^{0.15}=0$
o_2	2	0	−2	1	$S_2^{0.15}=1$
o_3	−2	−1	2	1	$S_3^{0.15}=0$
o_4	−2	−2	2	1	$S_4^{0.15}=-1$

Table 13
The mid-level D-Score table of (F_1, E_1) with mid-level D-Scores.

	e_1	e_2	e_3	e_4	S_i^{mid}
o_1	2	3	−2	−3	$S_1^{mid}=0$
o_2	2	1	−2	0	$S_2^{mid}=1$
o_3	−2	−2	2	1	$S_3^{mid}=-1$
o_4	−2	−2	2	2	$S_4^{mid}=0$

$A = \{e_1, e_2, \ldots, e_m\}$. Based on the fuzzy soft set (F, A), we can define a fuzzy set $\lambda_F^{\min} : A \longrightarrow [0, 1]$ by

$$\lambda_F^{\min}(e_l) = \wedge_{\{o_i, o_j \in U\}} |\mu_{F(e_l)}(o_i) - \mu_{F(e_l)}(o_j)|,$$

for all $e_l \in A$. The fuzzy set $\lambda_F^{\min}$ is called the min-level-comparison threshold of the fuzzy soft set. In addition, the level D-Score table with respect to the min-level-comparison threshold fuzzy set $\lambda_F^{\min}$ is called the min-level D-Score table of fuzzy soft set (F, A). In what follows the min-level-comparison rule will mean using the min-level-comparison threshold and considering the min-level D-Score table of a fuzzy soft set in decision making process.

For a concrete example of min-level-comparison threshold fuzzy set and min-level D-Score table, let us reconsider the fuzzy soft set (F_1, E_1) with its tabular representation given by Table 4. It is clear that the min-level-comparison threshold of (F_1, E_1) is a fuzzy set

$$\lambda_{F_1}^{\min}(e_1) = |\mu_{F_1(e_1)}(o_1) - \mu_{F_1(e_1)}(o_2)| = 0.10,$$
$$\lambda_{F_1}^{\min}(e_2) = |\mu_{F_1(e_2)}(o_3) - \mu_{F_1(e_2)}(o_4)| = 0.06,$$
$$\lambda_{F_1}^{\min}(e_3) = |\mu_{F_1(e_3)}(o_1) - \mu_{F_1(e_3)}(o_2)| = 0.10,$$
$$\lambda_{F_1}^{\min}(e_4) = |\mu_{F_1(e_4)}(o_3) - \mu_{F_1(e_4)}(o_4)| = 0.05,$$
$$\lambda_{F_1}^{\min} = \{(e_1, 0.10), (e_2, 0.06), (e_3, 0.10), (e_4, 0.05)\}.$$

Example 9 *(The max-level-comparison threshold of a fuzzy soft set).* Let the universe $U = \{o_1, o_2, \ldots, o_n\}$, E be the attribute set, $A \subseteq E$ and $A = \{e_1, e_2, \ldots, e_m\}$. Based on the fuzzy soft set (F, A), we can define a fuzzy set $\lambda_F^{\max} : A \longrightarrow [0, 1]$ by

$$\lambda_F^{\max}(e_l) = \vee_{\{o_i, o_j \in U\}} (\mu_{F(e_l)}(o_i) - \mu_{F(e_l)}(o_j)),$$

for all $e_l \in A$. The fuzzy set $\lambda_F^{\max}$ is called the max-level-comparison threshold of the fuzzy soft set. In addition, the level D-Score table with respect to the max-level-comparison threshold fuzzy set $\lambda_F^{\max}$ is called the max-level D-Score table of fuzzy soft set (F, A). In what follows the max-level-comparison rule will mean using the max-level-comparison threshold and considering the max-level D-Score table of a fuzzy soft set in fuzzy soft set based decision making.

For a concrete example of max-level-comparison threshold fuzzy set and max-level D-Score table, let us reconsider the fuzzy soft set (F_1, E_1) with its tabular representation given by Table 4. It is clear that the max-level-comparison threshold of (F_1, E_1) is a fuzzy set

$$\lambda_{F_1}^{\max}(e_1) = \mu_{F_1(e_1)}(o_1) - \mu_{F_1(e_1)}(o_4) = 0.80,$$
$$\lambda_{F_1}^{\max}(e_2) = \mu_{F_1(e_2)}(o_1) - \mu_{F_1(e_2)}(o_4) = 0.48,$$
$$\lambda_{F_1}^{\max}(e_3) = \mu_{F_1(e_3)}(o_4) - \mu_{F_1(e_3)}(o_1) = 0.88,$$
$$\lambda_{F_1}^{\max}(e_4) = \mu_{F_1(e_4)}(o_4) - \mu_{F_1(e_4)}(o_1) = 0.40,$$
$$\lambda_{F_1}^{\max} = \{(e_1, 0.80), (e_2, 0.48), (e_3, 0.88), (e_4, 0.40)\}.$$

In the fuzzy soft set (F, A), the level D-Score of object o_i with respect to λ_F^{mid}, $\lambda_F^{\min}$ and $\lambda_F^{\max}$ are denoted by S_i^{mid}, $S_i^{\min}$ and $S_i^{\max}$, respectively. Now we present the level D-Scores based decision making approach as below (see Algorithm 3).

Algorithm 3.

[Step 1.] Input a fuzzy soft set (F, A).
[Step 2.] Input a comparison threshold fuzzy set $\lambda : A \longrightarrow [0, 1]$ (or give a comparison threshold value $t \in [0, 1]$; or choose the mid-level-comparison decision rule; or choose the min-level-comparison decision rule; or choose the max-level-comparison decision rule) for decision making.
[Step 3.] Present the level D-Score table with respect to fuzzy set λ of (F, A) and compute the level D-Score of o_i with respect to λ, i.e. S_i^{λ}, $\forall i$ (or present the t-level D-Score table of (F, A) and compute the t-level D-Score S_i^t of o_i, $\forall i$; or present the mid-level D-Score table of (F, A) and compute the mid-level D-Score S_i^{mid} of o_i, $\forall i$; or present the min-level D-Score table of (F, A) and compute the min-level D-Score $S_i^{\min}$ of o_i, $\forall i$; or present the max-level D-Score table of (F, A) and compute the max-level D-Score $S_i^{\max}$ of o_i, $\forall i$).
[Step 4.] The optimal decision, which is denoted by $D((F, A), \lambda)$, is to select o_j if $S_j^{\lambda} = \max_i S_i^{\lambda}$ (or denoted by $D((F, A), t)$ and select o_j if $S_j^t = \max_i S_i^t$; or denoted by $D((F, A), \lambda_F^{mid})$ and select o_j if $S_j^{mid} = \max_i S_i^{mid}$; or denoted by $D((F, A), \lambda_F^{\min})$ and select o_j if $S_j^{\min} = \max_i S_i^{\min}$; or denoted by $D((F, A), \lambda_F^{\max})$ and select o_j if $S_j^{\max} = \max_i S_i^{\max}$).
[Step 5.] If j has more than one value then any one of o_j may be chosen.

In the last step of Algorithm 3, one may go back to the second step and change the comparison threshold that he/she once used so as to adjust the final optimal decision, especially when there are too many "optimal choices" to be chosen.

When comparing membership values of objects with respect to different attributes to evaluate the level D-Scores of objects, by introducing the comparison thresholds, Algorithm 3 takes into account both the quality and the quantity of attributes each object occupies. In this way, some decision making problems which cannot be dealt with by using Algorithm 1 can be solved by using Algorithm 3. In other words, Algorithm 3 overcomes Limitation 2 of Algorithm 1. Here is an example to illustrate.

Example 10. It is clear the 0.15-level D-Score table of (F_1, E_1) in Example 2 is given by Table 12. From Table 12, the 0.15-level D-Scores of objects are: $S_1^{0.15} = 0$, $S_2^{0.15} = 1$, $S_3^{0.15} = 0$, $S_4^{0.15} = -1$. It indicates that when using a comparison threshold value $t = 0.15$, we can obtain that o_2 is the optimal candidate by Algorithm 3.

The mid-level D-Score table of (F_1, E_1) is given by Table 13. From Table 13, we obtain $S_1^{mid} = 0, S_2^{mid} = 1, S_3^{mid} = -1, S_4^{mid} = 0$. It follows that if the mid-level-comparison decision rule is chosen, we also obtain o_2 as the optimal candidate by Algorithm 3.

Theorem 3. *Assuming that an actual decision making context is reduced to a fuzzy soft set (F, A) on the universe U. Let $D((F, A), \lambda)$ be the optimal decision set got by Algorithm 3, where λ is a comparison threshold fuzzy set of (F, A). If $\lambda(e_l) > \lambda_F^{\max}(e_l)$ for $\forall e_l \in A$, then we have $D((F, A), \lambda) = U$.*

3.3.3. Weighted D-Score based decision making

Now we introduce the concepts of weighted D-Scores, weighted t-level D-Scores and weighted level D-Scores with respect to a fuzzy set, and pay attention to their applications in decision making problems based on weighted fuzzy soft set.

Definition 9. [32] Let E be a set of attributes and $A \subseteq E$. A weighted fuzzy soft set is a triple $\mathfrak{T} = \{F, A, w\}$ where (F, A) is a fuzzy soft set over U, and $w : A \longrightarrow [0, 1]$ is a weight function specifying the weight $w_l = w(e_l)$ for each attribute $e_l \in A$.

Definition 10. Let $U = \{o_1, o_2, \ldots, o_n\}$ be the universe, E be the attribute set, $A \subseteq E$ and $A = \{e_1, e_2, \ldots, e_m\}$. Let $\mathfrak{T} = \{F, A, w\}$ be a

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weighted fuzzy soft set where (F, A) is a fuzzy soft set over U and $w: A \longrightarrow [0, 1]$ is a weight function specifying the weight $w_l = w(e_l)$ for each attribute $e_l \in A$. The weighted D-Score of $o_i \in U$ is defined by

$$\tilde{S}_i = \sum_{l=1}^{m} w_l \times S(o_i)(e_l), \tag{13}$$

where $S(o_i)(e_l)$ is the D-Score of object o_i on e_l calculated by Eq. (7).

Definition 11. Let $U = \{o_1, o_2, \ldots, o_n\}$ be the universe, E be the attribute set, $A \subseteq E$ and $A = \{e_1, e_2, \ldots, e_m\}$. Let $\mathfrak{T} = \{F, A, w\}$ be a weighted fuzzy soft set where (F, A) is a fuzzy soft set over U and $w: A \longrightarrow [0, 1]$ is a weight function specifying the weight $w_l = w(e_l)$ for each attribute $e_l \in A$. For $t \in [0, 1]$, the weighted t-level D-Score of $o_i \in U$ is defined by

$$\tilde{S}_i^t = \sum_{l=1}^{m} \tilde{S}(o_i)(e_l)_t \tag{14}$$

where $\tilde{S}(o_i)(e_l)_t = w_l \times S(o_i)(e_l)_t$ and $S(o_i)(e_l)_t$ is the t-level D-Score of object o_i on e_l calculated by Eq. (9).

Definition 12. Let $U = \{o_1, o_2, \ldots, o_n\}$ be the universe, E be the attribute set, $A \subseteq E$ and $A = \{e_1, e_2, \ldots, e_m\}$. Let $\mathfrak{T} = \{F, A, w\}$ be a weighted fuzzy soft set where (F, A) is a fuzzy soft set over U and $w: A \longrightarrow [0, 1]$ is a weight function specifying the weight $w_l = w(e_l)$ for each attribute $e_l \in A$. Let $\lambda : A \longrightarrow [0, 1]$ be a fuzzy set on A which is called a comparison threshold fuzzy set. The weighted level D-Score of $o_i \in U$ with respect to λ is defined by

$$\tilde{S}_i^\lambda = \sum_{l=1}^{m} \tilde{S}(o_i)(e_l)_\lambda \tag{15}$$

where $\tilde{S}(o_i)(e_l)_\lambda = w_l \times S(o_i)(e_l)_\lambda$ and $S(o_i)(e_l)_\lambda$ is the level D-Score of object o_i on e_l with respect to fuzzy set λ calculated by Eq. (11).

For every $o_i \in U$, its level D-Score with respect to λ which is calculated by Eq. (12) (t-level D-Score calculated by Eq.(10)) can be regarded as its weighted level D-Score with respect to λ (weighted t-level D-Score) in which every attribute be of equal importance.

The weighted level D-Score table with respect to the fuzzy set λ (weighted t-level D-Score table) is a square table in which rows are labelled by the attributes $e_1, e_2, \ldots, e_m$, columns are labelled by the objects $o_1, o_2, \ldots, o_n$ of the universe, and the entry corresponding to attribute e_l and object o_i is $\tilde{S}(o_i)(e_l)_\lambda$ ($\tilde{S}(o_i)(e_l)_t$). The weighted level D-Score table with respect to the mid-level-comparison threshold fuzzy set λ_F^{mid}, the min-level-comparison threshold fuzzy set $\lambda_F^{\min}$ and the max-level-comparison threshold fuzzy set $\lambda_F^{\max}$ are called the weighted-mid-level, weighted-min-level and weighted-max-level D-Score table of the weighted fuzzy soft set (F, A, w), respectively. In addition, the weighted level D-Score of object $o_i \in U$ with respect to λ_F^{mid}, $\lambda_F^{\min}$ and $\lambda_F^{\max}$ in the weighted fuzzy soft set (F, A, w) are denoted by weighted-mid-level D-Score ($\tilde{S}_i^{mid}$), weighted-min-level D-Score ($\tilde{S}_i^{\min}$) and weighted-max-level D-Score ($\tilde{S}_i^{\max}$), respectively.

Let $\hat{0}$ denote the constant fuzzy set on A given by $\hat{0}(e_l) = 0$ for $\forall e_l \in A$. Then we immediately have $\tilde{S}_i^{\hat{0}} = \sum_{l=1}^{m} \tilde{S}(o_i)(e_l)_{\hat{0}} = \sum_{l=1}^{m} w_l \times S(o_i)(e_l)_{\hat{0}} = \sum_{l=1}^{m} w_l \times S(o_i)(e_l)$, which is the weighted D-Score of object $o_i \in U$.

Algorithm 4 improves Algorithm 3 to deal with the decision making problems in which weights of attributes are different based on weighted fuzzy soft set and the corresponding weighted level D-Scores of objects with respect to a fuzzy set λ (weighted t-level D-Scores). In Algorithm 4, we take the weights of attributes into consideration and compute the weighted level D-Scores instead of level D-Scores. Since λ_F^{mid}, $\lambda_F^{\min}$ and $\lambda_F^{\max}$ are actually special comparison threshold fuzzy sets for (F, A), we will not sketch them out and highlight them in Algorithm 4.

Table 14
Tabular representation of weighted fuzzy soft set (F_1, E_1, w).

	$e_1, w_1 = 0.8$	$e_2, w_2 = 0.4$	$e_3, w_3 = 0.5$	$e_4, w_4 = 0.3$
o_1	0.92	0.88	0.08	0.12
o_2	0.82	0.60	0.18	0.40
o_3	0.24	0.46	0.83	0.47
o_4	0.12	0.40	0.96	0.52

Table 15
The weighted-mid-level D-Score table of (F_1, E_1, w) with weighted-mid-level D-Scores.

	$e_1, w_1 = 0.8$	$e_2, w_2 = 0.4$	$e_3, w_3 = 0.5$	$e_4, w_4 = 0.3$	$\tilde{S}_i^{mid}$
o_1	1.6	1.2	−1	−0.9	$\tilde{S}_1^{mid} = 0.9$
o_2	1.6	0.4	−1	0	$\tilde{S}_2^{mid} = 1.0$
o_3	−1.6	−0.8	1	0.3	$\tilde{S}_3^{mid} = -1.1$
o_4	−1.6	−0.8	1	0.6	$\tilde{S}_4^{mid} = -0.8$

Algorithm 4.

[Step 1.] Input a weighted fuzzy soft set (F, A, w).
[Step 2.] Input a comparison threshold fuzzy set $\lambda : A \longrightarrow [0, 1]$ (or give a comparison threshold value $t \in [0, 1]$) for the weighted fuzzy soft set (F, A, w).
[Step 3.] Present the weighted D-Score table with respect to fuzzy set λ for the weighted fuzzy soft set (F, A, w) and compute $\tilde{S}_i^\lambda$ ($\tilde{S}_i^t$), which is the weighted level D-Score with respect to λ (weighted t-level D-Score) of o_i, $\forall i$.
[Step 4.] The optimal decision is to select o_j if $\tilde{S}_j^\lambda = \max_i \tilde{S}_i^\lambda$ (or $\tilde{S}_j^t = \max_i \tilde{S}_i^t$).
[Step 5.] If j has more than one value then any one of o_j may be chosen.

Similarly to Algorithm 3, if too many "optimal choices" are obtained by Algorithm 4, one can also go back to the second step and change the comparison threshold previously used so as to adjust the final optimal decision. The notion of weighted level D-Score provide a framework for solving decision making problems by score based method in which all the attributes may not be of equal importance.

Example 11. Suppose that there are four candidates who apply for a position in a work place, the set of candidates $U = \{o_1, o_2, o_3, o_4\}$ is characterized by a attribute set $E_1 = \{e_1, e_2, e_3, e_4\}$ which is 'e_1 = technical information' ($w_1 = 0.8$), 'e_2 = experience' ($w_2 = 0.4$), 'e_3 = training' ($w_3 = 0.5$), 'e_4 = appearance' ($w_4 = 0.3$), respectively. Thus the decision maker has a weight function $w: E_1 \longrightarrow [0, 1]$ and the fuzzy soft set (F_1, E_1) in Example 2 is changed into a weighted fuzzy soft set (F_1, E_1, w) with its tabular representation as shown in Table 14.

As an adjustable approach, the decision maker can select different comparison thresholds when dealing with the problem. If we use the mid-level threshold in this case, we obtain the weighted-mid-level D-Score table of (F_1, E_1, w) as Table 15, then the optimal decision is o_2 by Algorithm 4.

4. Improving decision making approaches based on rough soft sets

This section first discusses the limitations of existing decision making approaches based on rough soft sets, some new approaches will be then provided to overcome such limitations.

4.1. Limitations of decision making methods based on rough soft sets

There are two main limitations of the rough soft sets based decision making approaches:

- Limitation 1: The existing few decision making algorithms are far from enough to meet the various demands of applications.

Table 16
The tabular representation of $\underline{Apr}_R(\mathfrak{S})$.

	o_1	o_2	o_3	o_4	o_5	o_6	o_7	o_8
e_1	0	0	0	0	0	0	0	0
e_2	1	1	0	0	0	0	0	0
e_3	0	0	1	1	0	0	0	0
e_4	0	0	1	1	1	1	0	0
e_5	0	0	0	0	0	0	0	0

Table 17
The tabular representation of $\bar{Apr}_R(\mathfrak{S})$.

	o_1	o_2	o_3	o_4	o_5	o_6	o_7	o_8
e_1	1	1	1	1	0	0	0	0
e_2	1	1	0	0	1	1	0	0
e_3	0	0	1	1	1	1	0	0
e_4	0	0	1	1	1	1	0	0
e_5	0	0	1	1	1	1	1	1

Decision making approaches based on rough soft sets has not obtained enough attention by the researchers so far. Let us reconsider the two material selection algorithms based on rough soft sets proposed by Ma et al. in [34], one of the algorithms is used to catch the closest one in all of the materials, the other one is used to reach the most representative materials. The scope of application of these algorithms is limited. The research on rough soft sets in decision making calls for improvement by proposing more approaches to meet various practical demands.

- Limitation 2: The research on application of rough soft set in group decision making has not appeared yet.

The combination of rough set theory and soft set theory shows great potential in solving group decision making problems. However, when a group decision making problem is solved by using [26] and MSR-sets [35], every expert have to present his/her best choice alternatives. In other words, every expert has already made their own decision before carrying out the group decision making process. This strict requirement is hardly fulfilled in some real-life situations since expert may prefer provide only their assessment of alternatives/candidates in different aspects when they are short of knowledge, time or just lack of confidence. All of the researchers only concentrate on the application of soft rough set in group decision making so far, no attempt has been done in solving such problems by using rough soft set. So the problem arises that how can we make full use of the information in form of assessments on candidates with respect to different aspects provided by the decision makers during the decision process and deal with the group decision making problem by using rough soft set?

In the following parts, we will come up with two new decision making algorithms based on rough soft set which enriches the scopes of applications and conquers Limitation 1 to a certain extend. Afterwards, a group decision making algorithm based on rough soft set which successfully solve group decision making problems when the initial evaluation information provided by experts are their assessments on alternatives from different aspects, which overcomes Limitation 2 and fills the blank that this respect field is studied.

4.2. Decision making methods based on rough soft sets

In this part, some new methods will be provided by deciding a most perspective attribute or a target attribute set using the rough approximation operators on a given soft set $\mathfrak{S} = (F, A)$.

Let $U = \{o_1, o_2, \ldots, o_n\}$ be the universe of objects and E be a set of related attributes. Let $\mathfrak{S} = (F, A)$ be a soft set over U and $A = \{e_1, e_2, \ldots, e_m\} \subseteq E$. Let (U, R) be a Pawlak approximation space where R be an equivalence relation on U. For $X \subseteq U$, let $|X|$ denote the number of objects in X, let $|X|_R$ denote the number of classes in U contained in X, where the classification is determined by an equivalent relation R. Then a decision making algorithm based on rough soft sets can be presented as Algorithm 5.

Algorithm 5.

[Step 1.] Input Pawlak approximation space (U, R) and a soft set $\mathfrak{S} = (F, A)$ on U.
[Step 2.] Compute the lower and upper rough soft approximation operators $\underline{Apr}_R(\mathfrak{S})$ and $\bar{Apr}_R(\mathfrak{S})$ on S, respectively.
[Step 3.] Select a threshold λ, which satisfies the condition $\lambda \in [0, \frac{|\underline{F}_R(e_1) \cup \underline{F}_R(e_2) \ldots \underline{F}_R(e_m)|_R}{|U|_R}]$.
[Step 4.] For each attribute $e_i \in A$, calculate $\bar{F}_R(e_i)$. If there is an attribute $e_i \in A$, s.t. $\bar{F}_R(e_i) = U$, turn to Step 5; if else, turn to Step 6.
[Step 5.] Calculate

$$\frac{|\underline{F}_R(e_k)|_R}{|U|_R} = \max_{\{e_i : \bar{F}_R(e_i) = U\}} \frac{|\underline{F}_R(e_i)|_R}{|U|_R}.$$

If $\frac{|\underline{F}_R(e_k)|_R}{|U|_R} \geq \lambda$, then $\{e_k\}$ is the expected decision set; if else, turn to Step 6.
[Step 6.] For all two attributes $e_i, e_j \in A$, calculate $\bar{F}_R(e_i) \cup \bar{F}_R(e_j)$. If there are attributes $e_i, e_j \in A$, s.t. $\bar{F}_R(e_i) \cup \bar{F}_R(e_j) = U$, turn to Step 7; If else, turn to Step 8.
[Step 7.] Calculate

$$\frac{|\underline{F}_R(e_k) \cup \underline{F}_R(e_l)|_R}{|U|_R} = \max_{\{e_i, e_j \in A : \bar{F}_R(e_i) \cup \bar{F}_R(e_j) = U\}} \frac{|\underline{F}_R(e_i) \cup \underline{F}_R(e_j)|_R}{|U|_R}.$$

If $\frac{|\underline{F}_R(e_k) \cup \underline{F}_R(e_l)|_R}{|U|_R} \geq \lambda$, then $\{e_k, e_l\}$ is the expected decision set; if else, turn to Step 8.
[Step 8.] If there are q $(q < m)$ attributes $e'_{i_1}, e'_{i_2}, \ldots, e'_{i_q} \in A$ can be found satisfying $\bar{F}_R(e'_{i_1}) \cup \bar{F}_R(e'_{i_2}) \cup \ldots \bar{F}_R(e'_{i_q}) = U$, then calculate

$$\frac{|\underline{F}_R(e_{i_1}) \cup \underline{F}_R(e_{i_2}) \cup \ldots \underline{F}_R(e_{i_q})|_R}{|U|_R}$$

$$= \max_{\{e'_{i_1}, e'_{i_2}, \ldots, e'_{i_q} \in A : \bar{F}_R(e'_{i_1}) \cup \bar{F}_R(e'_{i_2}) \cup, \ldots, \bar{F}_R(e'_{i_q}) = U\}} \frac{|\underline{F}_R(e'_{i_1}) \cup \underline{F}_R(e'_{i_2}) \cup \ldots \underline{F}_R(e'_{i_q})|_R}{|U|_R},$$

where $e_{i_1}, e_{i_2}, \ldots, e_{i_q} \in A$, if $\frac{|\underline{F}_R(e_{i_1}) \cup \underline{F}_R(e_{i_2}) \cup \ldots \underline{F}_R(e_{i_q})|_R}{|U|_R} \geq \lambda$, then $\{e_{i_1}, e_{i_2}, \ldots, e_{i_q}\}$ is an expected decision set (it is worth noticing that the expected decision set may be not unique);
if else, we will check $q+1$ attributes, $q+2$ attributes, $\ldots$, $q+(m-q)$ attributes, until we find the expected decision set.

The primary motivation for designing Algorithm 5 is to select the parameters whose upper approximation cover all the objects when their lower approximation cover a specified number of object classes in U. This selection mechanism can be used in many practical situations. Here is an example to illustrate:

Example 12. Suppose that a company decides to set up a working group for the expansion of business. To choose suitable members for the working group, they make a survey of the candidates on some professional skills (referred to as o_1, o_2, o_3, o_4, o_5, o_6, o_7 and o_8). Suppose that there are five candidates $A = \{e_1, e_2, e_3, e_4, e_5\}$ and each candidate have one or more skills: $F(e_1) = \{o_1, o_4\}$, $F(e_2) = \{o_1, o_2, o_6\}$, $F(e_3) = \{o_3, o_4, o_5\}$, $F(e_4) = \{o_3, o_4, o_5, o_6\}$ and $F(e_5) = \{o_3, o_5, o_8\}$. In this case, $(o_1, o_2) \in R$, $(o_3, o_4) \in R$, $(o_5, o_6) \in R$, $(o_7, o_8) \in R$ (R represents the equivalent relationship amongst skills). Now the skills are divide into four classes/types in U.

- If a candidate is good at one skill, he/she can be expected to handle its equivalent skills quicker (in a relative short time).
- If a candidate already handles all the skills in one class before the selection, he/she can be regarded as an expert in this type of skill.

The tabular representations of soft sets $\underline{Apr}_R(\mathfrak{S})$ and $\bar{Apr}_R(\mathfrak{S})$ are obtained as Tables 16 and 17, respectively.

Since $\frac{|\underline{F}_R(e_1) \cup \underline{F}_R(e_2) \ldots \underline{F}_R(e_5)|_R}{|U|_R} = \frac{3}{4}$, in this case we can set $\lambda = \frac{1}{2} \in [0, \frac{3}{4}]$.

The calculation process is given as below:

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1. For any attribute $e_i \in A$, it is obvious that $\bar{F}(e_i) \neq U$. Then we skip to two attributes situation;
2. For two attributes: $\bar{F}_R(e_1) \cup \bar{F}_R(e_5) = U$, $\bar{F}_R(e_2) \cup \bar{F}_R(e_5) = U$, it is easy to obtain $\frac{|\underline{F}_R(e_1) \cup \underline{F}_R(e_5)|_R}{|U|_R} = 0$, $\frac{|\underline{F}_R(e_2) \cup \underline{F}_R(e_5)|_R}{|U|_R} = \frac{1}{4}$. Since $\max\{0, \frac{1}{4}\} < \frac{1}{2}$, we skip to three attributes situation;
3. For three attributes:
   $\bar{F}_R(e_1) \cup \bar{F}_R(e_5) \cup \bar{F}_R(e_2) = U$, $\frac{|\underline{F}_R(e_1) \cup \underline{F}_R(e_5) \cup \underline{F}_R(e_2)|_R}{|U|_R} = \frac{|\{o_1, o_2\}|_R}{|U|_R} = \frac{1}{4}$.
   $\bar{F}_R(e_1) \cup \bar{F}_R(e_5) \cup \bar{F}_R(e_3) = U$, $\frac{|\underline{F}_R(e_1) \cup \underline{F}_R(e_5) \cup \underline{F}_R(e_3)|_R}{|U|_R} = \frac{|\{o_3, o_4\}|_R}{|U|_R} = \frac{1}{4}$.
   $\bar{F}_R(e_1) \cup \bar{F}_R(e_5) \cup \bar{F}_R(e_4) = U$, $\frac{|\underline{F}_R(e_1) \cup \underline{F}_R(e_5) \cup \underline{F}_R(e_4)|_R}{|U|_R} = \frac{|\{o_3, o_4, o_5, o_6\}|_R}{|U|_R} = \frac{1}{2}$.
   $\bar{F}_R(e_2) \cup \bar{F}_R(e_5) \cup \bar{F}_R(e_3) = U$, $\frac{|\underline{F}_R(e_2) \cup \underline{F}_R(e_5) \cup \underline{F}_R(e_3)|_R}{|U|_R} = \frac{|\{o_1, o_2, o_3, o_4\}|_R}{|U|_R} = \frac{1}{2}$.
   $\bar{F}_R(e_2) \cup \bar{F}_R(e_5) \cup \bar{F}_R(e_4) = U$, $\frac{|\underline{F}_R(e_2) \cup \underline{F}_R(e_5) \cup \underline{F}_R(e_4)|_R}{|U|_R} = \frac{|\{o_1, o_2, o_3, o_4, o_5, o_6\}|_R}{|U|_R} = \frac{3}{4}$.
   Here, $\frac{|\underline{F}_R(e_2) \cup \underline{F}_R(e_5) \cup \underline{F}_R(e_4)|_R}{|U|_R} = \max\{\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \frac{3}{4}\} = \frac{3}{4} > \frac{1}{2} = \lambda$, so we obtain that $\{e_2, e_4, e_5\}$ is the decision set.

1. If there is a skill which is not handled by anyone in the decision set, several of its equivalent skills must be handled by some memberships in the decision set. Thus, members in the decision set can be expected to handle all of the professional skills quicker (in a relative short time).
2. For at least two classes of skills, there are experts in the decision set. More classes of experts make the whole work more efficient.
3. Along with ensuring conditions 1–2, the members in the decision set will be the least, which contributes to save labor cost.
4. Along with ensuring conditions 1–3, with the same numbers of members, the number of skill classes with experts will be the most in the decision result set. Different types of experts make different types of work more efficient.

As following, another rough soft set based algorithm is proposed by selecting only one attribute whose upper approximation cover the most objects in the universe set.

Algorithm 6.

[Step 1.] Input Pawlak approximation space (U, R) and a soft set $\mathfrak{S} = (F, A)$ on U.
[Step 2.] Compute the upper rough soft approximation operator on $\mathfrak{S}$, i.e. $\overline{Apr}_R(\mathfrak{S})$.
[Step 3.] The optimal decision result is to select e_j if $|\bar{F}_R(e_j)| = \vee_{i \in \{1,2,\ldots,m\}} |\bar{F}_R(e_i)|$.
[Step 4.] If j has more than one value then any one of e_j may be chosen.

Example 13. Let us reconsider the decision making problem in Example 12. From Table 16, we can easily obtain that $|\bar{F}_R(e_4)| = \vee_{i \in \{1,2,\ldots,5\}} |\bar{F}_R(e_i)|$, so e_4 is the optimal decision by Algorithm 6. It is worth noticing that, by Algorithm 6, the optimal decision result is the candidate who has potential to handle most professional skills quicker (in a relative short time).

Remark 1. By Algorithms 5 and 6, we provide two different attempts of using rough soft sets to solve decision making problems. Different selection mechanisms make the methods have different scopes of application.

4.3. A group decision making method based on rough soft sets

Feng [39] and Zhan et al. [40] put forth approaches for group decision making problems based on soft rough sets [26] and MSR-sets [35], respectively. Benefitting from their ideas, now we will introduce an group decision making approach based on rough soft sets.

Assume that we have an expert group $G = \{T_1, T_2, \ldots, T_p\}$ consisting of p specialists to evaluate all the candidates $A = \{e_1, e_2, \ldots, e_m\}$. For each candidate, every specialist will be asked to provide an evaluation on him/her as aspect to all skills in $U = \{o_1, o_2, \ldots, o_n\}$ and will be requested to give judgement if a candidate is good at these skill or not. In this way, the judgements on all candidates with respect to all skills provided by every expert form a soft set. It is assumed that there exist some equivalent relationships between different skills. With these equivalent relationships, we can compute the rough approximations of the soft sets, the upper rough approximation of the soft set represents the low-confidence assessments provided by this expert while the lower approximation of the soft set represents the high-confidence assessments. The main character of our group decision making is that decision makers only need to provide their initial assessments of candidates (attributes) with respect to different aspects (objects), according to their knowledge/cognition of the problem, it is not necessary to provide their optimal alternatives before the group decision making process.

The evaluation result of each expert T_q ($q \in \{1, 2, \ldots, p\}$) can be described as an evaluation soft set $\mathfrak{T}_q = (F_{T_q}, A)$ over U, where $F_{T_q} : A \to P(U)$. Using rough approximations on soft set $\mathfrak{T}_q$, we can obtain two corresponding soft sets $\overline{Apr}_R(\mathfrak{T}_q) = (\overline{F_{T_q}}_R, A)$ and $\underline{Apr}_R(\mathfrak{T}_q) = (\underline{F_{T_q}}_R, A)$ over U, where $\overline{F_{T_q}}_R : A \to P(U)$ and $\underline{F_{T_q}}_R : A \to P(U)$.

We give a weighting vector $W' = (\eta_1, \eta_2, \ldots, \eta_p)$ such that $\eta_1 + \eta_2 + \ldots + \eta_p = 1$, where η_q ($q = 1, 2, \ldots, p$) represents the weight of expert T_q ($q = 1, 2, \ldots, p$) and can be calculated by:

$$\eta_q = \frac{S(\overline{Apr}_R(\mathfrak{T}_q), \underline{Apr}_R(\mathfrak{T}_q))}{\sum_{t=1}^{p} S(\overline{Apr}_R(\mathfrak{T}_t), \underline{Apr}_R(\mathfrak{T}_t))} \tag{16}$$

where $S(\overline{Apr}_R(\mathfrak{T}_q), \underline{Apr}_R(\mathfrak{T}_q))$ is the similarity between soft sets $\overline{Apr}_R(\mathfrak{T}_q) = (\overline{F_{T_q}}_R, A)$ and $\underline{Apr}_R(\mathfrak{T}_q) = (\underline{F_{T_q}}_R, A)$. (There are a lot of formulas can be used to calculate the similarity between two softs sets.)

The weight vector $W' = (\eta_1, \eta_2, \ldots, \eta_p)$ indicates different importance degree of different experts. It is noticed that people can use a lot of ways to determine the weights of experts. If there are enough additional information in evaluating the experts, the weights of experts can even been directly specified. Here we originally apply the similarity measures between soft sets to determine the weights of experts in a group decision making problem when the weights are not specified in advance. As is shown above, the opinion of each expert is represented by a soft set. We believe that the more similar the upper approximation and the lower approximation of one's opinion (the soft set) are, the more stable and reliable his/her opinion is, and thus the larger his/her weight should be.

Then the evaluation result of the whole expert group G could be formulated in terms of fuzzy sets:

$$\mu_{\mathfrak{T}'} : A \to [0, 1], e_i \mapsto \mu_{\mathfrak{T}'}(e_i) = (\frac{1}{n}) \sum_{q \in \{1,2,\ldots,p\}} \eta_q \times |F_{T_q}(e_i)|$$

where $i = 1, 2, \ldots, n$. Similarly, we can obtain two other fuzzy sets $\mu_{\underline{\mathfrak{T}}'}$ and $\mu_{\bar{\mathfrak{T}}'}$ in U, which are respectively given by

$\mu_{\underline{\mathfrak{T}}'} : A \to [0, 1], e_i \mapsto \mu_{\underline{\mathfrak{T}}'}(e_i) = (\frac{1}{n}) \sum_{q \in \{1,2,\ldots,p\}} \eta_q \times |\underline{F_{T_q}}_R(e_i)|$

$\mu_{\bar{\mathfrak{T}}'} : A \to [0, 1], \quad e_i \mapsto \mu_{\bar{\mathfrak{T}}'}(e_i) = (\frac{1}{n}) \sum_{q \in \{1,2,\ldots,p\}} \eta_q \times |\overline{F_{T_q}}_R(e_i)|$

where $i = 1, 2, \ldots, n$.

Then, we can construct a fuzzy soft sets to gather together the above fuzzy evaluation results. Let $C = \{L, M, H\}$ be a set of attributes, where L, M and H represent three kinds of confidence, respectively. Then we can define a fuzzy soft set $\mathfrak{F} = (G, C)$ over U, where $G : C \to F(U)$ is given by $G(L) = \mu_{\bar{\mathfrak{T}}'}$, $G(M) = \mu_{\mathfrak{T}'}$ and $G(H) = \mu_{\underline{\mathfrak{T}}'}$.

Now we give a weighting vector $W = (w_L, w_M, w_H)$ such that $w_L + w_M + w_H = 1$, we define

$$\nu(e_k) = (w_L) \times G(L)(e_k) + (w_M) \times G(M)(e_k) + (w_H) \times G(H)(e_k) \quad (17)$$

which is called the weighted evaluation value of the candidate $e_k \in A$. Finally we can select the attribute e_j such that $\nu(e_j) = \max(\nu(e_k))$ $(k = 1, 2, \ldots, m.)$ as the most preferred candidate. Now we present the decision making method based on rough soft sets by Algorithm 7.

Algorithm 7.

[Step 1.] Input Pawlak approximation space (U, R) and soft sets $\mathfrak{T}_1 = (F_{T_1}, A)$, $\mathfrak{T}_2 = (F_{T_2}, A), \ldots, \mathfrak{T}_p = (F_{T_p}, A)$ on U.
[Step 2.] For $\forall q \in \{1, 2, \ldots, p\}$, compute the lower and upper rough approximations on soft set $\mathfrak{T}_q$, i.e., $\bar{Apr}_R(\mathfrak{T}_q) = (\bar{F}_{T_q R}, A)$ and $\underline{Apr}_R(\mathfrak{T}_q) = (\underline{F_{T_q}}_R, A)$, respectively.
[Step 3.] Compute the weighting vector $W' = (\eta_1, \eta_2, \ldots, \eta_p)$ by Eq.(16).
[Step 4.] Compute the corresponding fuzzy sets $\mu_{\bar{\mathfrak{T}}'}$, $\mu_{\mathfrak{T}'}$ and $\mu_{\underline{\mathfrak{T}}'}$.
[Step 5.] Construct a fuzzy soft set $\mathfrak{F} = (G, C)$ using $\mu_{\bar{\mathfrak{T}}'}$, $\mu_{\mathfrak{T}'}$ and $\mu_{\underline{\mathfrak{T}}'}$.
[Step 6.] The optimal decision is to select e_j if $\nu(e_j) = \vee_{k \in \{1,2,\ldots,m\}} \nu(e_k)$.

Example 14. Suppose that we have an expert group $G = \{T_1, T_2, T_3, T_4\}$ consisting of 4 specialists and our goal is to choose an optimal candidate from a candidates set $A = \{e_1, e_2, \ldots, e_5\}$. For each candidate, every specialist will be asked to provide an evaluation as respect to all the professional skills in $U = \{o_1, o_2, \ldots, o_n\}$ and will be requested to give judgement if the candidate is good at these skill or not. In this case, the professional skills in U are divided into three classes/types: $(o_1, o_2) \in R$, $(o_3, o_4) \in R$ and $(o_5, o_6, o_7) \in R$ (R represent the type of some skills are equivalent). The evaluation result of all the candidates provided by expert T_q $(q = 1, 2, \ldots, 4)$ can be described as a soft set $\mathfrak{T}_q = (F_{T_q}, A)$ over U. Using rough approximations on soft sets, the tabular representations of soft sets $\bar{Apr}_R(\mathfrak{T}_q) = (\bar{F}_{T_q R}, A)(q = 1, 2, \ldots, 4)$ and $\underline{Apr}_R(\mathfrak{T}_q) = (\underline{F_{T_q}}_R, A)$ $(q = 1, 2, \ldots, 4)$ over U are obtained as (Table 18).

Finally, we can calculate the fuzzy soft set $\mathfrak{F} = (G, C)$. Assume that the weighting vector for confidence $W = (0.25, 0.5, 0.25)$ and to calculate the similarity between two soft sets (F, A) and (G, B) we use $S((F, A), (G, B)) = \frac{|A \cap B|}{|A \cup B|} \cdot \frac{\sum_{e \in A \cap B} |F(e) \cap G(e)|}{\sum_{e \in A \cap B} |F(e) \cup G(e)|}$ [41]. It is easy to obtain that

$$S(\bar{Apr}_R(\mathfrak{T}_1), \underline{Apr}_R(\mathfrak{T}_1)) = 0.5625,\ S(\bar{Apr}_R(\mathfrak{T}_2), \underline{Apr}_R(\mathfrak{T}_2)) = 0.625,$$

$$S(\bar{Apr}_R(\mathfrak{T}_3), \underline{Apr}_R(\mathfrak{T}_3)) = 0.32,\ S(\bar{Apr}_R(\mathfrak{T}_4), \underline{Apr}_R(\mathfrak{T}_4)) = 0.375.$$

Then, we can obtain $W' = (0.299, 0.332, 0.170, 0.199)$ by Eq. (16), the weighted evaluation value can be calculated by Eq. (17). Tabular representation of the fuzzy soft set $\mathfrak{F} = (G, C)$ with evaluation values is given by Table 19. Hence e_2 should be the most preferred candidate.

5. Conclusions

Fuzzy set theory, rough set theory and soft set theory are three relatively independent and closely related mathematical tools for dealing with uncertainty [42]. Based on the combination of these theories, various hybrid models, including fuzzy soft set theory and rough soft set theory, have been obtained to handle the vagueness in practical problems. In this paper, we focus on the application of fuzzy soft set theory and rough soft set theory in decision making. A classical fuzzy soft based decision making approach is improved to deal with decision making problems that contain updating information so that attributes need to be added/deleted in the fuzzy soft sets. We also present a new adjustable fuzzy soft sets based decision making approach by introducing comparison thresholds and corresponding level D-Score tables of fuzzy soft sets. This new approach has the potential to be extended to the intuitionistic fuzzy soft sets, interval-valued fuzzy soft sets situations, etc. Based on rough

Table 18
The tabular representations for soft sets in Example 14.

	o_1	o_2	o_3	o_4	o_5	o_6	o_7
Table for soft set $\bar{Apr}_R(\mathfrak{T}_1)$							
e_1	0	0	1	1	0	0	0
e_2	1	1	0	0	1	1	1
e_3	0	0	1	1	1	1	1
e_4	0	0	1	1	0	0	0
e_5	0	0	1	1	0	0	0
Table for soft set $\mathfrak{T}_1$							
e_1	0	0	0	1	0	0	0
e_2	1	1	0	0	1	1	1
e_3	0	0	1	0	1	1	0
e_4	0	0	1	1	0	0	0
e_5	0	0	1	1	0	0	0
Table for soft set $\underline{Apr}_R(\mathfrak{T}_1)$							
e_1	0	0	0	0	0	0	0
e_2	1	1	0	0	1	1	1
e_3	0	0	0	0	0	0	0
e_4	0	0	1	1	0	0	0
e_5	0	0	1	1	0	0	0
Table for soft set $\bar{Apr}_R(\mathfrak{T}_2)$							
e_1	1	1	1	1	0	0	0
e_2	1	1	0	0	1	1	1
e_3	0	0	1	1	1	1	1
e_4	0	0	1	1	1	1	1
e_5	0	0	1	1	1	1	1
Table for soft set $\mathfrak{T}_2$							
e_1	1	1	1	1	0	0	0
e_2	1	1	0	0	1	1	0
e_3	0	0	1	1	1	1	0
e_4	0	0	1	1	1	1	0
e_5	0	0	1	1	1	1	1
Table for soft set $\underline{Apr}_R(\mathfrak{T}_2)$							
e_1	1	1	1	1	0	0	0
e_2	1	1	0	0	0	0	0
e_3	0	0	1	1	0	0	0
e_4	0	0	1	1	0	0	0
e_5	0	0	1	1	1	1	1
Table for soft set $\bar{Apr}_R(\mathfrak{T}_3)$							
e_1	1	1	1	1	0	0	0
e_2	1	1	0	0	1	1	1
e_3	1	1	1	1	0	0	0
e_4	1	1	1	1	1	1	1
e_5	0	0	1	1	1	1	1
Table for soft set $\mathfrak{T}_3$							
e_1	1	1	1	1	0	0	0
e_2	0	1	0	0	1	1	0
e_3	1	0	1	1	0	0	0
e_4	1	0	1	0	1	1	0
e_5	0	0	1	1	0	0	1
Table for soft set $\underline{Apr}_R(\mathfrak{T}_3)$							
e_1	1	1	1	1	0	0	0
e_2	0	0	0	0	0	0	0
e_3	0	0	1	1	0	0	0
e_4	0	0	0	0	0	0	0
e_5	0	0	1	1	0	0	0
Table for soft set $\bar{Apr}_R(\mathfrak{T}_4)$							
e_1	1	1	1	1	0	0	0
e_2	1	1	0	0	1	1	1
e_3	0	0	1	1	1	1	1
e_4	0	0	0	0	1	1	1
e_5	1	1	1	1	1	1	1
Table for soft set $\mathfrak{T}_4$							
e_1	1	0	1	1	0	0	0
e_2	1	0	0	0	1	1	1
e_3	0	0	1	1	1	0	0
e_4	0	0	0	0	1	1	0
e_5	0	1	1	1	1	0	1
Table for soft set $\underline{Apr}_R(\mathfrak{T}_4)$							
e_1	0	0	1	1	0	0	0
e_2	0	0	0	0	1	1	1
e_3	0	0	1	1	0	0	0
e_4	0	0	0	0	0	0	0
e_5	0	0	1	1	0	0	0

Table 19
The tabular representation of $\mathfrak{F}=(G, C)$.

	e_1	e_2	e_3	e_4	e_5
L	0.486	0.714	0.690	0.578	0.643
M	0.415	0.590	0.476	0.429	0.538
H	0.344	0.394	0.200	0.180	0.428
$\nu(e_i)$	0.415	0.572	0.461	0.404	0.537

soft sets, some new algorithms are also provided to solve decision making and group decision making problems, different algorithms have different scopes of application. These original rough soft sets based approaches have the potential to be extended to the generation models of rough soft sets situations. In further research, the generation models of rough soft set theory and their corresponding application in decision making is an interesting issue to be addressed. The time complexity analysis of all the algorithms in the current work can be found in Appendix A.

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Appendix A. Complexity analysis of Algorithms 1–7

The complexity analysis of the algorithms in the current work are listed as follows:

- Algorithm 1: For calculating each entry of the comparison table from the fuzzy soft set, the complexity of running $|A|$ comparisons is $O(|A|)$, there are $|U|^2$ entries in the comparison table, hence the complexity of computing the comparison table is $O(|A||U|^2)$. The complexity of computing each score of each object by using the comparison score is $O(2|U|)=O(|U|)$, afterwards the complexity of selecting the max value is also $O(|U|)$. Thus, the complexity of Algorithm 1 is $O(|A||U|^2)+O(|U|)+O(|U|)=O(|A||U|^2)$.
- Algorithm 2: For calculating each entry of the D-Score table from the initial fuzzy soft set, the complexity of running $|A|$ comparisons is $O(|U|)$, there are $|U||A|$ entries in the comparison table, hence the complexity of computing the D-Score table is $O(|A||U|^2)$. The complexity of computing each D-Score of each object by using the D-Score table is $O(|A|)$, afterwards the complexity of selecting the max value is also $O(|U|)$. Thus, the complexity of Algorithm 2 is $O(|A||U|^2)+O(|A|)+O(|U|)=O(|A||U|^2)$.
- Algorithm 3: Compared to Algorithm 2, in Algorithm 3 we only introduce a threshold value when doing the comparisons to obtain the corresponding D-score Table, so the time complexity of Algorithm 3 is the same as Algorithm 2, that is, $O(|A||U|^2)$.
- Algorithm 4: The time complexity of Algorithm 4 is the same as Algorithm 3, that is, $O(|A||U|^2)$.
- Algorithm 5: For all $e_j \in A$, the time complexity of computing $\bar{F}_R(e_j)$ and $\underline{F}_R(e_j)$ from $F(e_j)$ is $O(|U|)$. There are $|A|$ parameters, therefore the complexity of computing the rough soft set from a given soft set $\mathfrak{S}=(F, A)$ is $O(|U||A|)$. The second step is to select a threshold λ manually, in which to compute the upper bound the time complexity is $O(|U|)$. The time complexity of the worst case to find the decision result is $O(c^1_{|A|}+c^2_{|A|}+\ldots+c^{|A|}_{|A|})=O(2^{|A|})$. It is easy to obtain the time complexity of Algorithm 5 is $O(|U||A|+2^{|A|})$.

  It is determined by the time complexity of Algorithm 5 that this algorithm is only suitable for decision making problems in which the number of attributes is relative small, which is a limitation of both Algorithm 5 in the current work and the Algorithm 9 in [34]. In the future it is worth paying attention to the further improvement of these algorithms to make them more feasible for large scale of data sets.
- Algorithm 6: For all $e_j \in A$, the time complexity of computing $\bar{F}_R(e_j)$ from $F(e_j)$ is $O(|U|)$. There are $|A|$ parameters, therefore the complexity of computing the upper approximation of a given soft set $\mathfrak{S}=(F, A)$ is $O(|U||A|)$. The complexity of selecting the max value of $\bar{F}_R(e_j)$, $e_j \in A$ is $O(|A|)$. It is easy to obtain the complexity of Algorithm 6 is $O(|U||A|+|A|)=O(|U||A|)$.
- Algorithm 7: For all $e_j \in A$, the time complexity of computing $\underline{F_{T_q}}_R(e_j)$ and $\bar{F}_{T_q R}(e_j)$ from $F_{T_q}(e_j)$ is $O(|U|)$. There are $|A|$ parameters, therefore the complexity of computing a rough soft set from a given soft set $\mathfrak{T}_q=(F_{T_q}, A)$ is $O(|U||A|)$. The complexity of obtaining all rough soft sets from all soft sets provided by experts $G=\{T_1, T_2, \ldots, T_p\}$ is $O(|U||A||G|)$. And the time complexity of computing each row of the fuzzy soft set from all rough soft sets is $O(|U||G||A|)$, three rows is $O(3|U||A||G|)=O(|U||A||G|)$, afterwards for computing $\nu(e_j)$, $e_j \in A$ from the fuzzy soft set is $O(3|A|)=O(|A|)$. The complexity of the last step to catch the largest value is obvious $O(|A|)$. The complexity of Step 3 has been ignored since it depends on the way for computing the similarity measures of soft sets. When the weights of experts are predefined, Step 3 should be skipped. Thus, the time complexity of Algorithm 7 is $O(|U||A||G|)+O(|U||A||G|)+O(|A|)+O(|A|)=O(|U||A||G|)$.

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4.2 A comparative study of some soft rough sets

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A Comparative Study of Some Soft Rough Sets

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Abstract: Through the combination of different types of sets such as fuzzy sets, soft sets and rough sets, abundant hybrid models have been presented in order to take advantage of each other and handle uncertainties. A comparative study of relationships and interconnections of some existing hybrid models has been carried out. Some foundational properties of modified soft rough sets (MSR sets) are analyzed. It is pointed out that MSR approximation operators are some kinds of Pawlak approximation operators, whereas approximation operators of Z-soft rough fuzzy sets are equivalent to approximation operators of rough fuzzy sets. The relationships among F-soft rough fuzzy sets, M-soft rough fuzzy sets and Z-soft rough fuzzy sets are surveyed. A new model called soft rough soft sets has been provided as the generalization of F-soft rough sets, and its application in group decision-making has been studied. Various soft rough sets models show great potential as a tool to solve decision-making problems, and a depth study of the connections among these models contributes to the flexible application of soft rough sets based decision-making approaches.

Keywords: rough set; soft set; soft rough set; soft rough fuzzy set

1. Introduction

Various types of uncertainties exist in real life situations, which calls for useful mathematic tools to meet various information process demands. Usually complicated problems take place with uncertainties, and most of these complex situations can not be handled by adopting classical mathematic methods, considering the fact that with classical mathematic tools all notions are requested to be strict. Up to now, abundant mathematic tools such as fuzzy set theory [1] and rough set theory [2,3] have already been developed and proved to be useful in handling several kinds of the problems that contain uncertainties, and all of these theories share a common inherent difficulty, which is mainly the inadequacy of the parametrization tool [4,5]. However, it is noticed that, without proper parametrization tools, sometimes a practical problem can not be described in a way as much as information collected from different aspects could be taken into account. To handle this issue and to enrich mathematical methodologies for coping with uncertainties, soft set theory was initially proposed by Molodtsov [4] in 1999, which considers every specific object from different attributes' aspects, in this way, this new model goes beyond all other existing mathematical tools to avoid the above-mentioned difficulties. After soft set theory comes out, in the past few years, there appears a continuous growth of interest in studying theoretical aspects of soft set theory, as well as the practical applications of soft sets.

Abundant mathematical models have already been designed in order to model and process vague concepts, among which it is noteworthy that fuzzy set theory and rough set theory have already drawn worldwide attention from researchers. The development of these two theories makes contributions to handle lots of complicated problems in engineering, economics, social science, et al. The main character of fuzzy set theory is that it describes a vague concept by using a membership function, and the allowance of partial memberships contributes to providing an appropriate framework to represent and process vague concepts. The character of rough set theory relies on handling vagueness and granularity in information systems by indirectly describing a vague concept through two exact concepts called its lower and upper approximations. In Pawlak's rough set model, the equivalence relation is a vital concept, by replacing the equivalence relation with a fuzzy similarity relation, fuzzy rough sets and rough fuzzy sets have been proposed [6].

The combinations of soft sets, rough sets and fuzzy sets have been extensively studied to benefit each other and to take the best advantage of them. Research on generalization models of soft sets is promising since usually the generalized models are not short of parameter tools, that is, all of the generalized soft set models usually keep the most important feature of soft set theory in considering issues from various aspects. The history of research on extending soft sets applying fuzzy set theory goes beyond fifteen years already since Maji et al. introduced fuzzy soft sets in [7]. Therefore far, the soft sets have been extended to intuitionistic fuzzy soft sets [8], interval-valued intuitionistic fuzzy soft sets [5,9], vague soft sets [10], soft interval sets [11] and many other hybrid soft sets models. The history of research on the generalization of soft sets by using rough set theory is relatively short. To introduce parametrization tools to rough set theory, Feng et al. [12,13] initially put forward the concept of soft rough sets and soft rough fuzzy sets, in which a soft set looks for the lower and upper approximations of a subset of the universe. Afterwards, Meng et al. [14] proposed soft fuzzy rough set, in which model the fuzzy soft set has been adopted into granulate the universe. Benefitting from similarity measures induced by soft sets and soft fuzzy sets, Qin et al. [15] provided several soft fuzzy rough set models through introducing confidence threshold values. Recently, Shabir et al. [16] noticed that Feng et al.'s soft rough sets [12] suffer from some unexpected properties such as the upper approximation of a non-empty set might be empty and a subset set X might not be contained in its upper approximation. To resolve this problem, Shabir et al. [16] modified their soft rough sets and introduced the modified soft rough set (MSR set), which has already been extended to fuzzy soft sets [17], and Z-soft rough fuzzy sets was proposed, and its application in decision-making problems was analyzed.

The exploitation of soft sets and hybrid soft sets models in decision-making shows a great development in the recent years [18–22]. The utilization of soft rough sets models in decision-making shows a promising prospect. Different decision-making approaches have been put forth based on MSR set [20], Z-soft rough fuzzy sets [17], Z-soft fuzzy rough set [21], and other soft rough sets models [23,24]. If the researchers could have a thorough knowledge of the connections among various soft rough sets, we believe that decision-making approaches under framework of soft rough sets could be applied in a more flexible and reliable way. However, the relationships among these hybrid sets have not been systematically studied so far. Furthermore, we notice that a soft set S can be looked upon an information system I_S. Based on this information system, we can establish Pawlak rough approximations and rough fuzzy approximations. What is the relationship between soft rough approximations (soft rough fuzzy approximations) in S and Pawlak rough approximations (rough fuzzy approximations) in I_S? Additionally, soft set and formal context are mathematically equivalent. The relationships among soft rough approximation operators and derivation operators used in formal concept analysis (FCA) are also interesting issues to be addressed. In this paper, we will concentrate on the discussion of these problems. The paper is structured as follows: Section 2 revises several basic concepts of soft sets, fuzzy sets and rough set. Section 3 studies relationships among several soft rough sets. The properties of MSR approximation operators and different connections between MSR approximation operators and F-soft rough approximation operators are analyzed. It is shown that MSR approximation operators and a kind of Pawlak approximation operators are equivalent, while Z-soft rough fuzzy approximation operators and a kind of rough fuzzy approximation operators are equivalent. The relationships among F-soft rough fuzzy sets, M-soft rough fuzzy sets and Z-soft rough fuzzy sets have also been investigated. Section 4 discusses the relationship between F-soft rough sets and modal-style operators in formal concept analysis. Section 5 proposes a new generalization of F-soft rough set, which is called a soft rough soft set, and a simple application of soft rough soft sets in group decision-making has been studied. Eventually, Section 6 concludes the paper by presenting some remarks and future works.

2. Preliminaries

Here, several concepts of fuzzy sets, soft sets and rough sets are briefly reviewed. Please refer to [1,2,4,7] for details.

An advantageous framework has been offered by fuzzy set theory [1] to handle vague concepts through the allowance for partial memberships. Let U be the universe set. Define a fuzzy set μ on U by its membership function $\mu : U \to [0,1]$. $\mu(x)$ indicates the degree to which x belongs to the fuzzy set μ for all $x \in U$. In what follows, we denote the family of all subsets of U by $P(U)$ and the family of all fuzzy sets on U by $F(U)$. The operations of fuzzy sets can be found in [1].

Molodtsov [4] introduced the concept of soft set. Let U be the universe set and E the set consisted of all parameters that is related to U. Hence, a soft set is defined as below:

Definition 1. *A pair (F, A) is called a soft set over U, where $A \subseteq E$ and F is a mapping given by $F : A \to P(U)$ [4].*

The soft set is characterized by a parameter set and a function defined on the parameter set. For every parameter $e \in A$, $F(e)$ is said to be the e-approximate elements and, correspondingly, the soft set can be viewed as a parameterized family of subsets of U.

A soft set (F, A) is called a full soft set if $\cup_{e \in A} F(e) = U$ [12]; $\tilde{N}_{(U,A)} = (N, A)$ is called a relative null soft set (with respect to the parameter set A), if $N(e) = \emptyset$ for all $e \in A$; $\tilde{W}_{(U,B)} = (W, B)$ is called a relative whole soft set (with respect to the parameter set B) if $W(e) = U$ for all $e \in B$ [25]. Maji et al. in [7] introduced the concept of fuzzy soft set.

Definition 2. *Let (U, E) be a soft space. A pair (F, A) is called a fuzzy soft set over U, where $A \subseteq E$ and F is a mapping defined as $F : A \to F(U)$ [7].*

The fuzzy soft set is also characterized by a parameter set and a function on the parameter set, whereas a fuzzy set on U takes place of a crisp subset of U corresponds to each parameter. It follows that, to a certain degree, a soft set can also be viewed as a special kind of fuzzy soft set.

Pawlak introduced rough set theory in [2], the application of which is based on a structure called information system.

Definition 3. *An information system is a pair $I = (U, A)$ of non-empty finite sets U and A, where U is a set of objects and A is a set of attributes; each attribute $a \in A$ is a function $a : U \to V_a$, where V_a is the set of all values (called domain) of attribute a [3].*

Soft sets and information systems are closely related [13,26,27]. $S = (F, A)$ is assumed to be a soft set over U and $I_S = (U, A)$ an information system induced by S. For any attribute $a \in A$, a function $a : U \to V_a = \{0, 1\}$ is defined by $a(x) = 1$ if $x \in F(a)$; or else $a(x) = 0$. In this way, every soft set could be viewed as an information system. In what follows, I_S is called the information system induced by soft set S.

By contrast, suppose the information system, $I = (U, A)$. It uses a parameter set as

$$B = \{(a, v_a); a \in A \wedge v_a \in V_a\},$$

and it follows that through setting $F(a, v_a) = \{x \in U; a(x) = v_a\}$ for each $a \in A$ and $v_a \in V_a$, a soft set (F, B) can be defined, which is the soft set induced by I.

Let U be the universe of discourse and R be an equivalence relation on U. (U, R) is called Pawlak approximation space. For each $X \subseteq U$, the upper approximation $\overline{R}(X)$ and lower approximation $\underline{R}(X)$ of X with respect to (U, R) are defined as [2]:

$$\overline{R}(X) = \{x \in U; [x]_R \cap X \neq \emptyset\}, \tag{1}$$

$$\underline{R}(X) = \{x \in U; [x]_R \subseteq X\}. \tag{2}$$

X is so-called definable in (U, R) if $\underline{R}(X) = \overline{R}(X)$, or else X is a rough set. Thus, in rough set theory, a rough concept is characterized by a couple of exact concepts, namely, its lower approximation and upper approximation. $Pos_R(X) = \underline{R}(X)$ and $Neg_R(X) = U - \overline{R}(X)$ are the R-positive region and R-negative region of X, respectively. Furthermore, $Bnd_R(X) = \overline{R}(X) - \underline{R}(X)$ is called the R-boundary region.

Up to now, various types of extension models of the Pawlak rough set have been proposed to enrich the theory and to meet different application demands [28,29]. In [12], by the combination of soft set, rough set and fuzzy set theory, soft rough sets and soft rough fuzzy sets were introduced. To make them easy to be distinguished from other models mentioned in the current work and also to facilitate the discussion, these two notions are called F-soft rough sets and F-soft rough fuzzy sets.

Definition 4. *Let $S = (f, A)$ be a soft set over U. $P = (U, S)$ is called a soft approximation space. Two operations can be defined based on P as follows [12]:*

$$\underline{apr}_P(X) = \{u \in U; \exists a \in A(u \in f(a) \subseteq X)\}, \tag{3}$$

$$\overline{apr}_P(X) = \{u \in U; \exists a \in A(u \in f(a), f(a) \cap X \neq \emptyset)\}. \tag{4}$$

For all $X \subseteq U$, $\underline{apr}_P(X)$ and $\overline{apr}_P(X)$ are respectively called the F-lower and F-upper soft rough approximations of X in S. X is F-soft definable in P if $\underline{apr}_P(X) = \overline{apr}_P(X)$, or else X is a F-soft rough set.

It is noted that we can present $\underline{apr}_P(X)$ and $\overline{apr}_P(X)$ in a more concise manner [13]:

$$\underline{apr}_P(X) = \cup\{f(a); a \in A \wedge f(a) \subseteq X\}, \tag{5}$$

$$\overline{apr}_P(X) = \cup\{f(a); a \in A \wedge f(a) \cap X \neq \emptyset\}. \tag{6}$$

In this definition, the soft set S is regarded as the elementary knowledge on the universe. F-lower and F-upper soft rough approximation operators are not dual to each other, that is, $\underline{apr}_P(X^c) = (\overline{apr}_P(X))^c$ usually does not hold, where the complement of set X is computed by $X^c = U - X$. If the condition $\cup_{a \in A} f(a) = U$ holds in a soft set $S = (f, A)$ over U, this soft set is a full soft set [12]. In this case, $\{f(a); a \in A\}$ comes into being a cover of the universe U. It is pointed out that $\underline{apr}_P$, $\overline{apr}_P$ and covering rough approximations [30] are closely related but fundamentally different [13]. Additionally, if $\{f(a); a \in A\}$ forms a partition of U, we will call $S = (f, A)$ a partition soft set [13,31].

It is pointed out by Shabir et al. [16] that $\exists\, x \in U$ s.t. $x \in Neg_P(X) = U - \overline{apr}_P(X)$ for all $X \subseteq U$, if $S = (f, A)$ is not a full soft set. In other words, $x \notin \overline{apr}_P(X)$ for all $X \subseteq U$. Thus, $X \subseteq \overline{apr}_P(X)$ and some basic properties of rough set do not hold in general. Based on these observations, modified soft rough sets (MSR sets) was defined as follows.

Definition 5. *Let (f, A) be a soft set over U and $\varphi : U \rightarrow P(A)$ be a map defined as $\varphi(x) = \{a \in A; x \in f(a)\}$. Then, (U, φ) is called MSR-approximation space and for any $X \subseteq U$, its lower MSR approximation $\underline{X}_\varphi$ and upper MSR approximation $\overline{X}_\varphi$ are defined as [16]:*

$$\underline{X}_\varphi = \{x \in U; \forall y \in X^c(\varphi(x) \neq \varphi(y))\}, \tag{7}$$

$$\overline{X}_{\varphi} = \{x \in U; \exists y \in X(\varphi(x) = \varphi(y)). \quad (8)$$

X is MSR definable if the condition $\underline{X}_{\varphi} = \overline{X}_{\varphi}$ holds, or else X is a MSR set.

Mathematically speaking, (U, φ) can be looked upon a soft set over A. In [32], (U, φ) was considered as a pseudo soft set that is induced by (f, A), afterwards a decision-making method related to pseudo soft set was provided.

3. Relationships among Several Soft Rough Sets

3.1. Relationships between F-Soft Rough Approximations and MSR Approximations

The notion of MSR set is the modification of a F-soft rough set, and some inherent connections between these two models should exist, which have not drawn enough attention from scholars yet. In this subsection, a theoretical analysis of F-soft rough sets and MSR sets will be provided, and some connections between F-soft rough approximations and MSR approximations will be pointed out.

It is noted that Ref. [16] $\underline{apr}_P(X) \subseteq \underline{X}_{\varphi}$ for any $X \subseteq U$ and the containment may be proper. Furthermore, in general, $\overline{X}_{\varphi} \subseteq \overline{apr}_P(X)$ or $\overline{apr}_P(X) \subseteq \overline{X}_{\varphi}$ does not hold. Now, we provide an example:

Example 1. *Let $A = \{a, b, c, d\}$ be a parameter set and $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ the universe. Suppose that $S = (f, A)$ is a soft set over U, in which $F(a) = \{x_1, x_6\}$, $F(b) = \{x_3\}$, $F(c) = \emptyset$, $F(d) = \{x_1, x_2, x_5\}$.*

(1) By the definition, $\overline{apr}_P(U) = \cup_{a \in A} f(a) = \{x_1, x_2, x_3, x_5, x_6\}$. It follows that $x_4 \notin \overline{apr}_P(U)$ and hence $x_4 \notin \overline{apr}_P(X)$ for any $X \subseteq U$.

(2) Let $X = \{x_3, x_4, x_5\}$. By direct computation, we know that $\overline{apr}_P(X) = \{x_1, x_2, x_3, x_5\}$, $\overline{X}_{\varphi} = \{x_2, x_3, x_4, x_5\}$. Thus, $\overline{apr}_P(X) \subseteq \overline{X}_{\varphi}$, or $\overline{X}_{\varphi} \subseteq \overline{apr}_P(X)$ does not hold.

However, only a shallow impression can be obtained noticing the above-mentioned conclusions in [16], and no details have been provided discussing the properties of and connections among $\overline{apr}_P(X)$, $\overline{X}_{\varphi}$, $\underline{apr}_P(X)$ and $\underline{X}_{\varphi}$. The questions still remain: is there any possibility $\overline{X}_{\varphi} \subseteq \overline{apr}_P(X)$ or $\overline{apr}_P(X) \subseteq \overline{X}_{\varphi}$ that holds? Which features will be requested if these conditions need to be established? Now, we will pay attention to these questions and provide answers.

A general assumption for Theorems 1–3 and Corollaries 1 and 2 is presented as below:

Let $S = (f, A)$ be a soft set over U and $P = (U, S)$ a soft approximation space.

Theorem 1. *S is a full soft set iff $\overline{X}_{\varphi} \subseteq \overline{apr}_P(X)$ for any $X \subseteq U$.*

Proof. ($\Rightarrow$). It is assumed that S is a full soft set and $X \subseteq U$. For all $x \in \overline{X}_{\varphi}$, $\exists\, y \in X$ s.t. $\varphi(x) = \varphi(y)$. By $y \in U = \cup_{a \in A} f(a)$, $\exists\, a \in A$ s.t. $y \in f(a)$. Then, $y \in X \cap f(a)$ and $X \cap f(a) \neq \emptyset$. By $y \in f(a)$ we obtain $a \in \varphi(y) = \varphi(x)$ and hence $x \in f(a)$. Consequently, $x \in \overline{apr}_P(X)$. Thus, $\overline{X}_{\varphi} \subseteq \overline{apr}_P(X)$.

($\Leftarrow$). Suppose that, for all $X \subseteq U$, the condition $\overline{X}_{\varphi} \subseteq \overline{apr}_P(X)$ holds. It can be observed that $x \in \overline{\{x\}}_{\varphi} \subseteq \overline{apr}_P(\{x\}) = \cup\{f(a); f(a) \cap \{x\} \neq \emptyset\} = \cup\{f(a); x \in f(a)\}$, for any x in U.

Thus, $\exists\, a \in A$ s.t. $x \in f(a)$. S is a full soft set by the arbitrary of x. □

Theorem 2. *$\overline{apr}_P(X) \subseteq \overline{X}_{\varphi}$ for any $X \subseteq U$ iff for any $a, b \in A$, $f(a) \cap f(b) = \emptyset$ whenever $f(a) \neq f(b)$.*

Proof. ($\Leftarrow$). Assume that for any $a, b \in A$, $f(a) \cap f(b) = \emptyset$ whenever $f(a) \neq f(b)$. Let $X \subseteq U$. For any $x \in \overline{apr}_P(X)$, attribute $a \in A$ exists s.t. $x \in f(a)$ and $f(a) \cap X \neq \emptyset$. Thus, we know that there exists $y \in U$ s.t. $y \in f(a) \cap X$. For any $b \in A$, if $f(a) \neq f(b)$, then $f(a) \cap f(b) = \emptyset$ and hence $x \notin f(b)$ by $x \in f(a)$. Thus, $\varphi(x) = \{b \in A; f(b) = f(a)\}$. Similarly, we have $\varphi(y) = \{b \in A; f(b) = f(a)\}$ and hence $\varphi(x) = \varphi(y)$. By $y \in X$, we know that $x \in \overline{X}_{\varphi}$ and consequently $\overline{apr}_P(X) \subseteq \overline{X}_{\varphi}$.

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($\Rightarrow$). Assume that $\overline{apr}_P(X) \subseteq \overline{X}_\varphi$ for any $X \subseteq U$. For any $a, b \in A$, if $f(a) \cap f(b) \neq \emptyset$, $\exists\, x \in U$ s.t. $x \in f(a) \cap f(b)$. By $x \in f(a)$, we conclude that

$$f(a) \subseteq \cup\{f(c); x \in f(c)\} = \cup\{f(c); \{x\} \cap f(c) \neq \emptyset\} = \overline{apr}_P(\{x\}) \subseteq \overline{\{x\}}_\varphi = \{y \in U; \varphi(y) = \varphi(x)\}.$$

Meanwhile, if $\varphi(y) = \varphi(x)$, then $a \in \varphi(x) = \varphi(y)$ and hence $y \in f(a)$. Therefore, $f(a) = \{y \in U; \varphi(y) = \varphi(x)\}$. Similarly, by $x \in f(b)$, we have $f(b) = \{y \in U; \varphi(y) = \varphi(x)\}$ and hence $f(a) = f(b)$. □

Theorems 1 and 2 shows that $\exists$ containment relationships between $\overline{X}_\varphi$ and $\overline{apr}_P(X)$ if some specific conditions hold. Based on these two theorems, we can have a clear idea about under which conditions the containment relationships can be held. Furthermore, by Theorems 1 and 2, we obtain

Corollary 1. *Let $f(e) \neq \emptyset$ for each $e \in A$. S is a partition soft set iff $\overline{apr}_P(X) = \overline{X}_\varphi$ for any $X \subseteq U$.*

Corollary 2. *S is a full soft set iff $X \subseteq \overline{apr}_P(X)$ for any $X \subseteq U$.*

Proof. It is assumed that S is a full soft set. For all $X \subseteq U$, it is obvious that $X \subseteq \overline{X}_\varphi \subseteq \overline{apr}_P(X)$ by Theorem 1. On the contrary, assume that $X \subseteq \overline{apr}_P(X)$ for any $X \subseteq U$. For each $x \in U$,

$$x \in \{x\} \subseteq \overline{apr}_P(\{x\}) = \cup\{f(a); f(a) \cap \{x\} \neq \emptyset\} = \cup\{f(a); x \in f(a)\}.$$

Thus, $\exists\, a \in A$ s.t. $x \in f(a)$. Consequently, S is a full soft set as required. □

Theorem 3. *$\underline{X}_\varphi \subseteq \underline{apr}_P(X)$ for any $X \subseteq U$ iff for any $x \in U$, $\exists\, a \in A$ s.t. $f(a) = \{y \in U; \varphi(y) = \varphi(x)\}$.*

Proof. ($\Rightarrow$). Suppose that $\underline{X}_\varphi \subseteq \underline{apr}_P(X)$ for all $X \subseteq U$. For any $x \in U$, let $X = \{y \in U; \varphi(y) = \varphi(x)\}$. It follows that

$$\underline{X}_\varphi = \{u \in U; \exists y \in X(\varphi(u) = \varphi(y))\} = \{u \in U; \varphi(u) = \varphi(x)\} = X.$$

By $x \in X$ and $\underline{X}_\varphi \subseteq \underline{apr}_P(X)$, then $x \in \underline{apr}_P(X)$ and hence $\exists\, a \in A$ s.t. $x \in f(a)$ and $f(a) \subseteq X$.

On the other hand, for any $y \in X$, we have $\varphi(y) = \varphi(x)$, therefore $a \in \varphi(x) = \varphi(y)$. Then, $y \in f(a)$ and hence $X \subseteq f(a)$. Thus, $f(a) = X = \{y \in U; \varphi(y) = \varphi(x)\}$.

($\Leftarrow$). Assume that $X \subseteq U$ and $x \in \underline{X}_\varphi$. For each $y \in U$, if $\varphi(x) = \varphi(y)$, we have $y \in X$ by $x \in \underline{X}_\varphi$. It follows that $\{y \in U; \varphi(y) = \varphi(x)\} \subseteq X$ and $\exists\, a \in A$ such that $f(a) = \{y \in U; \varphi(y) = \varphi(x)\}$. Thus, $x \in f(a)$ and $f(a) \subseteq X$. It follows that $x \in \underline{apr}_P(X)$ and consequently $\underline{X}_\varphi \subseteq \underline{apr}_P(X)$. □

By Theorem 3, we obtain a clear mind about the necessary conditions for $\underline{X}_\varphi \subseteq \underline{apr}_P(X)$ to be held, which has not been discussed in other literature yet. The connections between F-soft rough approximations and MSR approximations have been discussed in detail through the theorems presented above.

Keeping in mind that all of the theoretical research should serve practical applications. It is noted that F-soft rough sets and MSR sets group decision-making approaches have been put forward in [20,31], respectively. Based on the analysis about the connections of F-soft rough approximations and MSR approximations, the relationships between decision schemes by using these two different hybrid models could be further discussed in the future, and the decision results obtained by the two decision schemes may have some inherent relationship.

3.2. The Relationships between MSR Approximations and Pawlak's Rough Approximations

After the notion of MSR sets was put forward, it was applied to different circumstances to cope with practical problems. However, since there is systematic research on its relationship with Pawlak's rough sets up to now, the rationality of MSR sets may be questioned by scholars from a theoretical point of view.

Let $S = (f, A)$ be a soft set. S induces an information system $I_S = (U, A)$. According to Pawlak [2], A determines an indiscernibility relation R_S on U given by

$$R_S = \{(x, y) \in U \times U; \forall a \in A(a(x) = a(y))\}. \quad (9)$$

Clearly, (U, R_S) is a Pawlak approximation space. The equivalence class determined by the equivalence relation R_S that contains x is denoted by $[x]_{R_S}$. What is the relationship between Pawlak's rough approximations in (U, R_S) and F-soft rough approximations (MSR approximations) induced by soft set S? This section offers the discussion of this problem.

Theorem 4. *Let $S = (f, A)$ be a partition soft set over U and $P = (U, S)$ a soft approximation space. Define an equivalence relation R on U by*

$$R = \{(x, y) \in U \times U; \exists a \in A(\{x, y\} \subseteq f(a)\}. \quad (10)$$

Then, for all $X \subseteq U$, $\underline{apr}_P(X) = \underline{R}(X)$ and $\overline{apr}_P(X) = \overline{R}(X)$ [13,31].

Theorem 5. *Let $S = (f, A)$ be a partition soft set over U and $I_S = (U, A)$ the information system induced by soft set $S = (f, A)$. Then, $R_S = R$, where R is determined by Equation (10).*

Proof. Let $x, y \in U$ and $(x, y) \in R$. By the definition, $\exists\, a \in A$ s.t. $\{x, y\} \subseteq f(a)$. It follows that $a(x) = 1 = a(y)$. For any $b \in A - \{a\}$, if $f(b) = f(a)$, then $\{x, y\} \subseteq f(a) = f(b)$ and hence $b(x) = 1 = b(y)$; if $f(b) \neq f(a)$, then $f(b) \cap f(a) = \emptyset$ and hence $x \notin f(b)$, $y \notin f(b)$. Then, $b(x) = 0 = b(y)$. Thus, $c(x) = c(y)$ for each $c \in A$. Consequently, $(x, y) \in R_S$.

Conversely, let $x, y \in U$ and $(x, y) \in R_S$. By $x \in U = \cup_{a \in A} f(a)$, $\exists\, a \in A$ s.t. $x \in f(a)$. It follows that $a(y) = a(x) = 1$ and hence $y \in f(a)$. Consequently, $\{x, y\} \subseteq f(a)$ and thus $(x, y) \in R$. □

By Theorems 4 and 5, in cases when a partition soft set is used as the underlying soft set, F-soft rough sets in (U, S) could be identified with Pawlak's rough sets in (U, R_S). For MSR sets, we have the following results.

Theorem 6. *Let $S = (F, A)$ be a soft set over U and $I_S = (U, A)$ be the information system induced by soft set $S = (F, A)$.*

(1) For any $x \in U$, $[x]_{R_S} = \{y \in U; \varphi(x) = \varphi(y)\}$.
(2) For any $X \subseteq U$, $\underline{X}_\varphi = \underline{R_S}(X)$.
(3) For any $X \subseteq U$, $\overline{X}_\varphi = \overline{R_S}(X)$.

Proof. (1) Let $x, y \in U$ and $y \in [x]_{R_S}$. Then, $a(x) = a(y)$ for each $a \in A$. For any $b \in \varphi(x)$, we have $x \in f(b)$ and hence $b(x) = 1$. We can observe that $b(y) = b(x) = 1$ and $y \in f(b)$. Thus, $b \in \varphi(y)$ and hence $\varphi(x) \subseteq \varphi(y)$. Similarly, we have $\varphi(y) \subseteq \varphi(x)$ and consequently $\varphi(x) = \varphi(y)$.

On the contrary, suppose that $\varphi(x) = \varphi(y)$. For any $a \in A$, if $a(x) = 1$, then $x \in f(a)$ and hence $a \in \varphi(x) = \varphi(y)$. Thus, $y \in f(a)$ and $a(y) = 1$; if $a(x) = 0$, then $x \notin f(a)$ and hence $a \notin \varphi(x) = \varphi(y)$. Thus, $y \notin f(a)$ and $a(y) = 0$. Then, $a(x) = a(y)$ for any $a \in A$ and hence $y \in [x]_{R_S}$.

(2) Let $X \subseteq U$ and $x \in \underline{X}_\varphi$. For any $y \in [x]_{R_S}$, we have $\varphi(x) = \varphi(y)$ by (1). By $x \in \underline{X}_\varphi$, we have $\varphi(x) \neq \varphi(z)$ whenever $z \in X^c$. Thus, $y \in X$ by $\varphi(x) = \varphi(y)$. Then, $[x]_{R_S} \subseteq X$ and hence $x \in \underline{R_S}(X)$. We conclude that $\underline{X}_\varphi \subseteq \underline{R_S}(X)$.

On the contrary, assume that $x \in \underline{R_S}(X)$. It follows that $[x]_{R_S} \subseteq X$. For any $y \in X^c$, we have $y \notin X$ and hence $y \notin [x]_{R_S}$. Thus, $\varphi(x) \neq \varphi(y)$ by (1). Consequently, $x \in \underline{X}_\varphi$ and hence $\underline{R_S}(X) \subseteq \underline{X}_\varphi$.

(3) Let $X \subseteq U$ and $x \in \overline{X}_\varphi$. It follows that $\exists\, y \in X$ s.t. $\varphi(x) = \varphi(y)$. Thus, $y \in [x]_{R_S}$. Consequently, $[x]_{R_S} \cap X \neq \emptyset$ and hence $x \in \overline{R_S}(X)$.

Conversely, suppose that $x \in \overline{R_S}(X)$. Thus, $[x]_{R_S} \cap X \neq \emptyset$. It follows that there exists $y \in X$ s.t. $y \in [x]_{R_S}$. Consequently, $\varphi(x) = \varphi(y)$ and hence $x \in \overline{X}_\varphi$. □

Theorem 6 shows that MSR approximation operator is a kind of Pawlak rough approximation operator. The two mathematic models that correspond with these approximation operators have been interconnected by this theorem, which could be regarded as a theoretical proof for the rationality of MSR sets. Benefitting from the notion of MSR set, Zhan et al. provided the definition of Z-soft rough fuzzy set in a recent work [17] .

Definition 6. *Let (f, A) be a soft set over U and (U, φ) the MSR approximation space. For any fuzzy set $\mu \in F(U)$, the Z-lower and Z-upper soft rough approximations of μ are denoted by $\underline{\mu}_\varphi$ and $\overline{\mu}_\varphi$, respectively, which are fuzzy sets on U given by [17]:*

$$\underline{\mu}_\varphi(x) = \wedge\{\mu(y); y \in U \wedge \varphi(x) = \varphi(y)\}, \tag{11}$$

$$\overline{\mu}_\varphi(x) = \vee\{\mu(y); y \in U \wedge \varphi(x) = \varphi(y)\}, \tag{12}$$

for each $x \in U$, and the operators $\underline{\mu}_\varphi$ and $\overline{\mu}_\varphi$ are the Z-lower and Z-upper soft rough approximation operators on a fuzzy set, respectively. Specifically, if $\underline{\mu}_\varphi = \overline{\mu}_\varphi$, μ is a Z-soft definable; or else μ is a Z-soft rough fuzzy set.

By Theorem 6 (1), the following corollary could easily be achieved:

Corollary 3. *Let $S = (F, A)$ be a soft set over U and $I_S = (U, A)$ the information system induced by soft set $S = (F, A)$. Then,*

(1) $\underline{\mu}_\varphi(x) = \wedge\{\mu(y); y \in [x]_{R_S}\}$, and
(2) $\overline{\mu}_\varphi(x) = \vee\{\mu(y); y \in [x]_{R_S}\}$

for any $\mu \in F(U)$, $x \in U$.

By Corollary 3, Z-lower and Z-upper soft rough approximation operators are equivalent to Dubois and Prade's lower and upper rough fuzzy approximation operators in [6]. Benefitting from this corollary, the researchers may refer to both of the theories' aspects and the applications of rough fuzzy sets to better study the development of Z-soft rough sets. Furthermore, the utilization of rough set theory in decision system has been extensively studied during the past few decades. Through discussing the connections between F-soft rough set and and Pawlak rough set, as well as the connections between MSR approximation operators and Pawlak rough approximation operators, the exploitation of various soft rough sets models in decision-making may be studied in a more logic and systematic way in the future.

3.3. The Relationships among Several Soft Rough Fuzzy Sets

A soft rough fuzzy set can be viewed as an extension model of a soft rough set, where the approximations of a fuzzy set in a soft approximation space are characterized. There are several distinct soft rough fuzzy set models in the literature. In the current part, the connections between soft rough fuzzy set and rough fuzzy set will be discussed, as well as the relationships among several soft rough fuzzy sets.

Soft rough approximation operators on fuzzy sets were initially proposed by Feng et al. in [12].

Definition 7. *Let $S = (f, A)$ be a full soft set over U and $P = (U, S)$ a soft approximation space. The lower and upper soft rough approximations of a fuzzy set, $\mu \in F(U)$, with respect to P are noted as $\underline{sap}_P(\mu)$ and $\overline{sap}_P(\mu)$, respectively, which are defined by [12]:*

$$\underline{sap}_P(\mu)(x) = \wedge\{\mu(y); \exists a \in A(\{x, y\} \subseteq f(a))\}, \tag{13}$$

$$\overline{sap}_P(\mu)(x) = \vee\{\mu(y); \exists a \in A(\{x, y\} \subseteq f(a))\}, \tag{14}$$

for all $x \in U$. The operators $\underline{sap}_P$ and $\overline{sap}_P$ are the F-lower and F-upper soft rough approximation operators on fuzzy sets. If $\underline{sap}_P(\mu) = \overline{sap}_P(\mu)$, μ is said to be F-soft definable, or else μ is called a F-soft rough fuzzy set.

Note that $\underline{sap}_P$ and $\overline{sap}_P$ are dual to each other, i.e., $\overline{sap}_P(\mu^c) = (\underline{sap}_P(\mu))^c$ for every $\mu \in F(U)$. It has already been figured out that rough fuzzy sets in Pawlak approximation space (U, R) can be identified with F-soft rough fuzzy sets in soft approximation space (U, S) when the underlying soft set S is a partition soft set [13].

Meng et al. [14] noted that $\overline{sap}_P$ is a generalization of $\overline{apr}_P$, i.e., $\overline{sap}_P(X) = \overline{apr}_P(X)$ if $X \in P(U)$. On the contrary, $\underline{sap}_P$ is a not a generalization of $\underline{apr}_P$. Considering this issue, Meng et al. presented another soft rough fuzzy set model in [14].

Definition 8. *Let $S = (f, A)$ be a full soft set over U and $P = (U, S)$ a soft approximation space. The lower soft rough approximation $\underline{sap}'_P(\mu)$ and upper soft rough approximation $\overline{sap}'_P(\mu)$ of the fuzzy set $\mu \in F(U)$ are fuzzy sets in U defined as [14]:*

$$\underline{sap}'_P(\mu)(x) = \vee_{x \in f(a)} \wedge_{y \in f(a)} \mu(y), \tag{15}$$

$$\overline{sap}'_P(\mu)(x) = \wedge_{x \in f(a)} \vee_{y \in, f(a)} \mu(y) \tag{16}$$

for all $x \in U$. μ is called soft definable if the condition $\underline{sap}'_P(\mu) = \overline{sap}'_P(\mu)$ holds; or else μ is a soft rough fuzzy set. For avoiding confusion with other soft rough fuzzy set models, it will be called M-soft rough fuzzy set in the following parts.

It is proved that [14] $\underline{sap}'_P$ and $\overline{sap}'_P$ are dual to each other, and $\underline{sap}'_P$ is a generalization of $\underline{apr}_P$, i.e., $\underline{sap}'_P(X) = \underline{apr}_P(X)$ for any $X \subseteq U$.

Theorem 7. *Let $S = (f, A)$ be a partition soft set over U, $P = (U, S)$ a soft approximation space, and (U, R) a Pawlak approximation space, where R is given by Equation (10). For each $\mu \in F(U)$, $\underline{sap}'_P(\mu) = \underline{R}(\mu)$ and $\overline{sap}'_P(\mu) = \overline{R}(\mu)$.*

Proof. Assume that $\mu \in F(U)$ and $x \in U$. For each $y \in [x]_R$, $\exists\, a \in A$ s.t. $\{x, y\} \subseteq f(a)$. Suppose that $b \in A$ and $x \in f(b)$. We note that (f, A) is a partition soft set. By $x \in f(a) \cap f(b)$, it follows that $f(a) \cap f(b) \neq \emptyset$ and hence $f(a) = f(b)$. Hence,

$$\underline{sap}'_P(\mu)(x) = \vee_{x \in f(a)} \wedge_{z \in f(a)} \mu(z) = \wedge_{z \in f(a)} \mu(z) \leq \mu(y).$$

Consequently, $\underline{sap}'_P(\mu)(x) \leq \wedge\{\mu(y); y \in [x]_R\} = \underline{R}(\mu)(x)$.

Conversely, suppose that $x \in f(a)$. For each $y \in f(a)$, since $\{x, y\} \subseteq f(a)$, we get $y \in [x]_R$.

$$\mu(y) \geq \wedge\{\mu(z); z \in [x]_R\} = \underline{R}(\mu)(x),$$

hence $\wedge_{y \in f(a)} \mu(y) \geq \underline{R}(\mu)(x)$. Consequently,

$$\underline{sap}'_P(\mu)(x) = \vee_{x \in f(a)} \wedge_{z \in f(a)} \mu(z) \geq \underline{R}(\mu)(x),$$

and $\overline{sap}'_P(\mu) = \overline{R}(\mu)$ can be proved similarly. □

By this theorem, the (classical) rough fuzzy sets in Pawlak approximation space (U, R) and M-soft rough fuzzy sets in soft approximation space (U, S) are equivalent when the underlying soft set S is a partition soft set. It is shown by Corollary 3 that Z-soft rough fuzzy sets could be regarded as a kind of rough fuzzy set, which indicates that there also exist some fantastic relationships between these two distinct models. The following theorem demonstrates the correlation between Z-soft rough approximation operators and M-soft rough approximation operators.

Theorem 8. *Let $S = (f, A)$ be a full soft set over U, $P = (U, S)$ a soft approximation space and $\mu \in F(U)$:*

(1) $\underline{sap}'_P(\mu) \subseteq \underline{\mu}_\varphi$,
(2) $\overline{\mu}_\varphi \subseteq \overline{sap}'_P(\mu)$.

Proof. (1) Let $x \in U$, $a \in A$, $x \in f(a)$. For any $y \in U$, if $y \in [x]_R$, then $\varphi(x) = \varphi(y)$. It follows that $a \in \varphi(x) = \varphi(y)$ and hence $y \in f(a)$. Then, $[x]_R \subseteq f(a)$ and hence $\wedge_{y \in f(a)} \mu(y) \leq \wedge\{\mu(y); y \in [x]_R\} = \underline{\mu}_\varphi(x)$. Consequently, we conclude that

$$\underline{sap}'_P(\mu)(x) = \vee_{x \in f(a)} \wedge_{y \in f(a)} \mu(y) \leq \underline{\mu}_\varphi(x)$$

and hence $\underline{sap}'_P(\mu) \subseteq \underline{\mu}_\varphi$.

(2) Let $x \in U$, $a \in A$ and $x \in f(a)$. By (1), we have $[x]_R \subseteq f(a)$ and hence $\overline{\mu}_\varphi(x) = \vee\{\mu(y); y \in [x]_R\} \leq \vee_{y \in f(a)} \mu(y)$. It follows that

$$\overline{\mu}_\varphi(x) \leq \wedge_{x \in f(a)} \vee_{y \in f(a)} \mu(y) = \overline{sap}'_P(\mu)(x)$$

and hence $\overline{\mu}_\varphi \subseteq \overline{sap}'_P(\mu)$. □

It is noted that F-soft rough approximation operators $\underline{apr}_P(\mu)$, $\overline{apr}_P(\mu)$ can be expressed equivalently as [15]:

$$\underline{sap}_P(\mu)(x) = \wedge\{\mu(y); \exists a \in A(\{x, y\} \subseteq f(a))\} = \wedge_{x \in f(a)} \wedge_{y \in f(a)} \mu(y),$$
$$\overline{sap}_P(\mu)(x) = \vee\{\mu(y); \exists a \in A(\{x, y\} \subseteq f(a))\} = \vee_{x \in f(a)} \vee_{y \in f(a)} \mu(y).$$

Therefore, we have the following corollary:

Corollary 4. *Let $S = (f, A)$ be a full soft set over U and $P = (U, S)$ a soft approximation space. For any $\mu \in F(U)$,*

$$\underline{sap}_P(\mu) \subseteq \underline{sap}'_P(\mu) \subseteq \underline{\mu}_\varphi \subseteq \mu \subseteq \overline{\mu}_\varphi \subseteq \overline{sap}'_P(\mu) \subseteq \overline{sap}_P(\mu).$$

Meng et al. [14] presented a kind of soft fuzzy approximation space, where a fuzzy soft set is regarded as the elementary knowledge on the universe and used to granulate the universe.

Definition 9. *Let $\mathcal{F} = (f, A)$ be a fuzzy soft set over U. The pair $SF = (U, \mathcal{F})$ is called a soft fuzzy approximation space. For a fuzzy set $\mu \in F(U)$, the lower and upper soft fuzzy rough approximations of μ with respect to SF are denoted by $\underline{Apr}_{SF}(\mu)$ and $\overline{Apr}_{SF}(\mu)$, respectively, which are given by [14]:*

$$\underline{Apr}_{SF}(\mu)(x) = \wedge_{a \in A}((1 - f(a)(x)) \vee (\wedge_{y \in U}((1 - f(a)(y)) \vee \mu(y)))), \quad (17)$$

$$\overline{Apr}_{SF}(\mu)(x) = \vee_{a \in A}(f(a)(x) \wedge (\vee_{y \in U}(f(a)(y) \wedge \mu(y)))), \quad (18)$$

for all $x \in U$. The operators $\underline{Apr}_{SF}$ and $\overline{Apr}_{SF}$ are called the lower and upper soft fuzzy rough approximation operators on fuzzy sets.

It is proved that [14] $\underline{Apr}_{SF}$ and $\overline{Apr}_{SF}$ are extensions of $\underline{sap}_{SF}$ and $\overline{sap}_{SF}$, respectively, i.e., if $\mathcal{F} = (f, A)$ is a soft set, then $\underline{Apr}_{SF}(\mu) = \underline{sap}_{SF}(\mu)$ and $\overline{Apr}_{SF}(\mu) = \overline{sap}_{SF}(\mu)$ for any $\mu \in F(U)$.

Theorem 9. *Suppose that $\mathcal{F} = (f, A)$ is a fuzzy soft set over U and $SF = (U, \mathcal{F})$. Let $R_{\mathcal{F}}$ be the fuzzy relation on U given by $R_{\mathcal{F}}(x, y) = \vee_{a \in A}(f(a)(x) \wedge f(a)(y))$. For each $\mu \in F(U)$,*

(1) $\underline{Apr}_{SF}(\mu) = \underline{R_{\mathcal{F}}}(\mu)$,
(2) $\overline{Apr}_{SF}(\mu) = \overline{R_{\mathcal{F}}}(\mu)$.

By this theorem, the soft fuzzy rough approximation presented in Definition 9 is a kind of Dubois and Prade's fuzzy rough approximation in [6]. We note that $R_{\mathcal{F}}(x, y)$ describes a kind of similarity between x and y, and $R_{\mathcal{F}}$ is symmetric but $R_{\mathcal{F}}(x, x) \neq 1$ in general.

The utilization of Z-soft rough fuzzy set in decision-making has already been studied in [17]. Through discussing the connections among different soft rough fuzzy set models, we can further explore the applications of the other two kinds of soft rough fuzzy sets models in decision-making, enrich the decision mechanisms and pay attention to the selection of the most suitable mechanism according to environments. The soft fuzzy rough approximation operators on fuzzy sets proposed by Meng et al. [14] have the potential to be utilized to handle decision-making problems, discussion on the connections between which and fuzzy rough approximation operators confirm the rationality of this model from the theoretical perspective and lays the foundation for subsequent practical applications.

4. *F*-Soft Rough Sets and Modal-Style Operators in FCA

FCA [22,33,34] provides a methodology for knowledge description and summarization. In this section, several absorbing connections between F-soft rough sets and modal-style operators in FCA will be discussed. Formal concept analysis is carried out based on a formal context specifying which objects posses what properties or attributes. A formal concept is formulated as a pair of two sets, one is consists of objects and another consists of properties or attributes, and these two sets are connected by two set-theoretic operators. A complete lattice called concept lattice is constituted by the set of all formal concepts, which reflects the correlation of generalization and specialization for formal concepts.

Definition 10. *A formal context (G, M, I) consists of two sets G and M and a relation I between G and M. The elements of G are called the objects and the elements of M are called the attributes of the context. $(g, m) \in I$ indicate that the object g has the attribute m, or the attribute m is possessed by the object g [33].*

Let (G, M, I) be a formal context. For $A \subseteq G$, $B \subseteq M$, Duntsch and Gediga [6] defined a pair of modal-style operators $\triangle, \bigtriangledown$ as follows:

$$A^{\triangle} = \{m \in M; \exists g \in A((g, m) \in I)\}, \tag{19}$$

$$A^{\bigtriangledown} = \{m \in M; \forall g \in G((g, m) \in I \rightarrow g \in A)\}, \tag{20}$$

$$B^{\triangle} = \{g \in G; \exists m \in B((g, m) \in I)\}, \tag{21}$$

$$B^{\bigtriangledown} = \{g \in G; \forall m \in M((g, m) \in I \rightarrow m \in B)\}. \tag{22}$$

Recently, the granular computing based concept lattice theory has received much attention [35].

Rough set theory, soft set theory and concept lattices have similar basis data description. Mathematically speaking, the notions of soft set and formal context are equivalent. Furthermore, both a formal context and a soft set can be considered as a two-valued information system.

Theorem 10. *Let $S = (F, A)$ be a soft set over U. A formal context $C_S = (U, A, I_S)$ is induced by S, where I_S is provided as*

$$I_S = \{(x, a) \in U \times A; x \in F(a)\}.$$

Conversely, let $C = (U, A, I)$ be a formal context. A set-valued mapping $F_C : A \to P(U)$ is defined by

$$F_C(a) = \{x \in U; (x, a) \in I\}$$

for all $a \in A$, and $S_C = (F_C, A)$ is a soft set. Moreover, we have $S_{C_S} = S$ and $C_{S_C} = C$.

Proof. Only the proof for $S_{C_S} = S$ and $C_{S_C} = C$ will be provided here. Suppose that $S = (F, A)$ is a soft set over U and $a \in A$. For any $x \in U$, from the definition, we obtain that

$$x \in F_{C_S}(a) \Leftrightarrow (x, a) \in I_S \Leftrightarrow x \in F(a).$$

That is, $F_{C_S}(a) = F(a)$ for all $a \in A$. Thus, $F_{C_S} = F$, whence $S_{C_S} = S$.

Next, assume that $C = (U, A, I)$ is a formal context, $x \in U$ and $a \in A$. Then, by definition,

$$(x, a) \in I_{S_C} \Leftrightarrow x \in F_C(a) \Leftrightarrow (x, a) \in I.$$

Therefore, we conclude that $C_{S_C} = C$ as required. □

Theorem 11 shows the relationship among operators $\triangle$, $\triangledown$ and soft rough approximation operators.

Theorem 11. *Let $S = (F, A)$ be a soft set over U. For any $X \subseteq U$, $\underline{apr}_P(X) = X^{\triangledown\triangle}$, $\overline{apr}_P(X) = X^{\triangle\triangle}$.*

Proof. (1) For any $x \in \underline{apr}_P(X)$, $\exists\, a \in A$ s.t. $x \in f(a) \subseteq X$. Then, $x \in a^{\triangle}$ and $a^{\triangle} \subseteq X$. Therefore, $a \in X^{\triangledown}$ and consequently $x \in a^{\triangle} \subseteq X^{\triangledown\triangle}$. We conclude that $\underline{apr}_P(X) \subseteq X^{\triangledown\triangle}$.

Conversely, if $x \in X^{\triangledown\triangle}$, then $\exists\, a \in X^{\triangledown}$ s.t. $x \in a^{\triangle}$. Then, $x \in f(a)$ and $f(a) \subseteq X$. Thus, $x \in \{f(c); f(c) \subseteq X\} = \underline{apr}_P(X)$ and hence $X^{\triangledown\triangle} \subseteq \underline{apr}_P(X)$.

(2) For any $x \in \overline{apr}_P(X)$, $\exists\, a \in A$ satisfying $x \in f(a)$ and $f(a) \cap X \neq \emptyset$. It follows that $x \in a^{\triangle}$ and $a^{\triangle} \cap X \neq \emptyset$. Thus, $a \in X^{\triangle}$ and consequently $x \in a^{\triangle} \subseteq X^{\triangle\triangle}$.

Conversely, $\exists\, a \in X^{\triangle}$ s.t. $x \in a^{\triangle}$ if $x \in X^{\triangle\triangle}$. Then, $x \in f(a)$ and $f(a) \cap X \neq \emptyset$. Consequently, $x \in \{f(c); f(c) \cap X \neq \emptyset\} = \overline{apr}_P(X)$. □

FCA has become increasingly popular among various methods of conceptual data analysis, knowledge representation and decision-making. Depth study on the connections of soft rough sets theory and FCA contributes to the reference and fusion for decision-making approaches in these two different fields.

5. A New Generalization of F-Soft Rough Set: Soft Rough Soft Sets

In this section, by extending the notion of F-soft rough set, a new generalization model called soft rough soft set will be proposed. In this new model, we use a soft set is as the elementary knowledge to compute the approximations of soft set. In this way, parameterized tools can be used to the greatest extent. Some basic properties of the new proposed model are discussed. A multi-group decision-making approach based on soft rough soft sets has been provided.

Definition 11. *Let U be the universe set and A, A_1 be parameter sets. Let $\mathcal{S}_1 = (f_1, A_1)$ be a full soft set over U and $(U, \mathcal{S}_1)$ be a soft approximation space. Let $\mathcal{S} = (f, A)$ be a soft set over U. The lower and upper soft rough approximations of $\mathcal{S}$ in $(U, \mathcal{S}_1)$ are denoted by $\underline{sapr}_{\mathcal{S}_1}(\mathcal{S}) = (f_{\mathcal{S}_1}, A)$ and $\overline{sapr}_{\mathcal{S}_1}(\mathcal{S}) = (f^{\mathcal{S}_1}, A)$, which are soft sets over U defined by:*

$$f_{S_1}(e) = \{x \in U : \exists e' \in A_1[x \in f_1(e') \subseteq f(e)]\},$$
$$f^{S_1}(e) = \{x \in U : \exists e' \in A_1[x \in f_1(e'), f_1(e') \cap f(e) \neq \emptyset]\},$$

for all $e \in A$. $\underline{sapr}_{S_1}$, $\overline{sapr}_{S_1}$ *are the lower and the upper soft rough approximation operators on soft set* $\mathcal{S}$, *respectively. If* $\underline{sapr}_{S_1}(\mathcal{S}) = \overline{sapr}_{S_1}(\mathcal{S})$, *the soft set* $\mathcal{S}$ *is soft definable, or else* $\mathcal{S}$ *is so-called a soft rough soft set.*

Example 2. *Suppose that the universe set* $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ *and the parameters set* $E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$. *Let* $A = \{e_1, e_2, e_3, e_4\} \subseteq E$ *and* $A_1 = \{e_3, e_4, e_5, e_6, e_7\} \subseteq E$. *Let* $\mathcal{S}_1 = (f_1, A_1)$ *be a full soft set and* $\mathcal{S} = (f, A)$ *be a soft set over* U *as shown by Tables 1 and 2, respectively. In the soft approximation space* $(U, \mathcal{S}_1)$, *by Definition 11, we get the lower soft rough approximation* $\underline{sapr}_{S_1}(\mathcal{S}) = (f_{S_1}, A)$ *and the upper soft rough approximation* $\overline{sapr}_{S_1}(\mathcal{S}) = (f^{S_1}, A)$ *of soft set* $\mathcal{S} = (f, A)$, *as shown by Tables 3 and 4, respectively. In order to facilitate the readers to understand, Figure 1 is given to show the process of computing* $f_{S_1}(e_4)$ *and* $f^{S_1}(e_4)$ *from* $f(e_4)$.

Table 1. Soft set (f_1, A_1).

A \ U	x_1	x_2	x_3	x_4	x_5	x_6
e_3	1	0	0	0	0	1
e_4	0	1	1	0	0	0
e_5	0	0	0	0	0	0
e_6	0	0	0	0	1	0
e_7	0	0	0	1	1	1

Table 2. Soft set (f, A).

A \ U	x_1	x_2	x_3	x_4	x_5	x_6
e_1	1	1	0	1	0	1
e_2	0	1	1	0	0	0
e_3	0	0	0	1	1	1
e_4	1	1	1	1	0	1

Table 3. Soft set (f_{S_1}, A).

A \ U	x_1	x_2	x_3	x_4	x_5	x_6
e_1	1	0	0	0	0	1
e_2	0	1	1	0	0	0
e_3	0	0	0	1	1	1
e_4	1	1	1	0	0	1

Table 4. Soft set (f^{S_1}, A).

A \ U	x_1	x_2	x_3	x_4	x_5	x_6
e_1	1	1	1	1	1	1
e_2	0	1	1	0	0	1
e_3	1	0	0	1	1	1
e_4	1	1	1	1	1	1

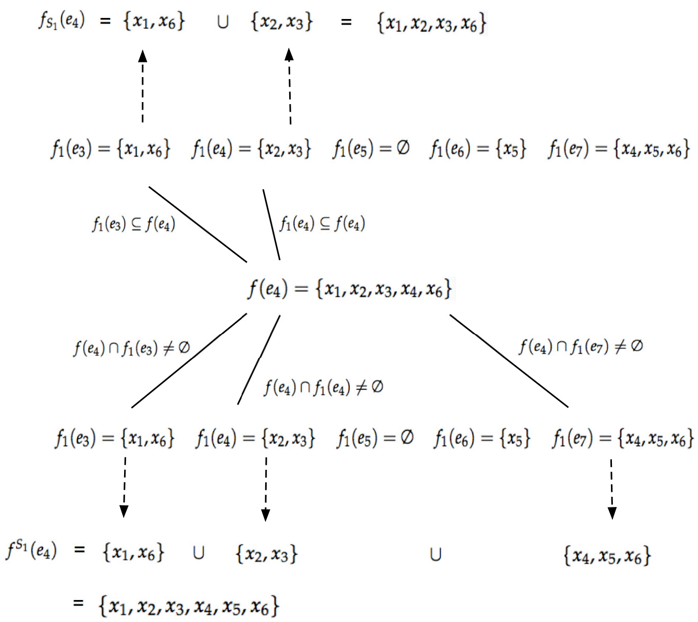

Figure 1. The process of computing $f_{S_1}(e_4)$ and $f^{S_1}(e_4)$ from $f(e_4)$ in Example 2.

Proposition 1. *Let $\mathcal{S}_1 = (f_1, A_1)$ be a full soft set over U and $(U, \mathcal{S}_1)$ be a soft approximation space. Let $\mathcal{S} = (f, A)$ be a soft set over U. The following properties hold:*

(1) $\underline{sapr}_{S_1}(\mathcal{S}) \subseteq \mathcal{S} \subseteq \overline{sapr}_{S_1}(\mathcal{S})$,
(2) $\underline{sapr}_{S_1}(\tilde{N}_{(U,A)}) = \tilde{N}_{(U,A)} = \overline{sapr}_{S_1}(\tilde{N}_{(U,A)})$,
(3) $\overline{sapr}_{S_1}(\tilde{W}_{(U,A)}) = \tilde{W}_{(U,A)} = \underline{sapr}_{S_1}(\tilde{W}_{(U,A)})$.

Proof. The lower and upper soft rough approximations of $\tilde{N}_{(U,A)} = (N, A)$ in $(U, \mathcal{S}_1)$ are denoted by $\underline{sapr}_{S_1}(\tilde{N}_{(U,A)}) = (N_{S_1}, A)$ and $\overline{sapr}_{S_1}(\tilde{N}_{(U,A)}) = (N^{S_1}, A)$; the lower and upper soft rough approximations of $\tilde{W}_{(U,A)} = (W, A)$ in $(U, \mathcal{S}_1)$ are denoted by $\underline{sapr}_{S_1}(\tilde{W}_{(U,A)}) = (W_{S_1}, A)$ and $\overline{sapr}_{S_1}(\tilde{W}_{(U,A)}) = (W^{S_1}, A)$.

(1a) For all $x \in U, e \in A$, if $x \in f_{S_1}(e) = \{x \in U : \exists e' \in A_1[x \in f_1(e') \subseteq f(e)]\}$, then we obtain $x \in f(e)$, so $f_{S_1}(e) \subseteq f(e)$;

(1b) For all $e \in A$, if $x \in f(e)$, since (f_1, A_1) is a full soft set, we obtain that $\exists e' \in A_1$, s.t. $x \in f_1(e')$, then $x \in f_1(e') \cap f(e) \neq \emptyset$, then $x \in \{x \in U : \exists e' \in A_1[x \in f_1(e'), f_1(e') \cap f(e) \neq \emptyset]\}$, that is, $x \in f^{S_1}(e)$ and $f(e) \subseteq f^{S_1}(e)$ for all $e \in A$.

Hence, we know that $f_{S_1}(e) \subseteq f(e) \subseteq f^{S_1}(e)$ for all $e \in A$, that is, $\underline{sapr}_{S_1}(\mathcal{S}) \subseteq \mathcal{S} \subseteq \overline{sapr}_{S_1}(\mathcal{S})$.

(2a) By the definition of relative null soft set, we know $N(e) = \emptyset$ for all $e \in A$. For all $e \in A$, we have $N_{S_1}(e) = \{x \in U : \exists e' \in A_1[x \in f_1(e') \subseteq N(e)]\} = \{x \in U : \exists e' \in A_1[x \in f_1(e') \subseteq \emptyset]\} = \emptyset = N(e)$, that is, $\underline{sapr}_{S_1}(\tilde{N}_{(U,A)}) = \tilde{N}_{(U,A)}$;

(2b) By the definition of relative null soft set, we know $N(e) = \emptyset$ for all $e \in A$. For all $e \in A$, we have $N^{S_1}(e) = \{x \in U : \exists e' \in A_1[x \in f_1(e'), f_1(e') \cap N(e) \neq \emptyset]\} = \emptyset = N(e)$, that is, $\overline{sapr}_{S_1}(\tilde{N}_{(U,A)}) = \tilde{N}_{(U,A)}$.

(3a) By the definition of relative whole soft set, we know $W(e) = U$ for all $e \in A$. For all $e \in A$, we have $W^{S_1}(e) = \{x \in U : \exists e' \in A_1[x \in f_1(e'), f_1(e') \cap W(e) \neq \emptyset]\} = U = W(e)$, that is, $\overline{sapr}_{S_1}(\tilde{W}_{(U,A)}) = \tilde{W}_{(U,A)}$.

(3b) By the definition of relative whole soft set, we know $W(e) = U$ for all $e \in A$. Since (f_1, A_1) is a full soft set over U, for all $e \in A$, we have $W_{S_1}(e) = \{x \in U : \exists e' \in A_1[x \in f_1(e') \subseteq W(e)]\} = \{x \in U : \exists e' \in A_1[x \in f_1(e') \subseteq U]\} = U = W(e)$, that is, $\underline{sapr}_{S_1}(\tilde{W}_{(U,A)}) = \tilde{W}_{(U,A)}$. □

Proposition 2. *Suppose that $\mathcal{S}_1 = (f_1, A_1)$ is a full soft set over U and $(U, \mathcal{S}_1)$ is a soft approximation space. Let $\mathcal{S} = (f, A)$, $\mathcal{T} = (g, A)$ be two soft sets over U. The following properties hold:*

(1) $\mathcal{S} \subseteq \mathcal{T} \Rightarrow \underline{sapr}_{S_1}(\mathcal{S}) \subseteq \underline{sapr}_{S_1}(\mathcal{T})$,
(2) $\mathcal{S} \subseteq \mathcal{T} \Rightarrow \overline{sapr}_{S_1}(\mathcal{S}) \subseteq \overline{sapr}_{S_1}(\mathcal{T})$,
(3) $\underline{sapr}_{S_1}(\mathcal{S} \cap \mathcal{T}) \subseteq \underline{sapr}_{S_1}(\mathcal{S}) \cap \underline{sapr}_{S_1}(\mathcal{T})$,
(4) $\underline{sapr}_{S_1}(\mathcal{S} \cup \mathcal{T}) \supseteq \underline{sapr}_{S_1}(\mathcal{S}) \cup \underline{sapr}_{S_1}(\mathcal{T})$,
(5) $\overline{sapr}_{S_1}(\mathcal{S} \cup \mathcal{T}) \supseteq \overline{sapr}_{S_1}(\mathcal{S}) \cup \overline{sapr}_{S_1}(\mathcal{T})$,
(6) $\overline{sapr}_{S_1}(\mathcal{S} \cap \mathcal{T}) \subseteq \overline{sapr}_{S_1}(\mathcal{S}) \cap \overline{sapr}_{S_1}(\mathcal{T})$.

Proof. The lower and upper soft rough approximations of $\mathcal{S}$ in $(U, \mathcal{S}_1)$ are denoted by $\underline{sapr}_{S_1}(\mathcal{S}) = (f_{S_1}, A)$ and $\overline{sapr}_{S_1}(\mathcal{S}) = (f^{S_1}, A)$; the lower and upper soft rough approximations of $\mathcal{T}$ in $(U, \mathcal{S}_1)$ are denoted by $\underline{sapr}_{S_1}(\mathcal{T}) = (g_{S_1}, A)$ and $\overline{sapr}_{S_1}(\mathcal{T}) = (g^{S_1}, A)$.

(1) If $\mathcal{S} \subseteq \mathcal{T}$, then for all $e \in A$, we have $f(e) \subseteq g(e)$. Assume that $x \in f_{S_1}(e) = \{x \in U : \exists e' \in A_1[x \in f_1(e') \subseteq f(e)]\}$. From $f(e) \subseteq g(e)$, we obtain $x \in \{x \in U : \exists e' \in A_1[x \in f_1(e') \subseteq g(e)]\} = g_{S_1}(e)$. Therefore, we get $f_{S_1}(e) \subseteq g_{S_1}(e)$ for all $e \in A$, i.e., $\underline{sapr}_{S_1}(\mathcal{S}) \subseteq \underline{sapr}_{S_1}(\mathcal{T})$;

(2) If $\mathcal{S} \subseteq \mathcal{T}$, then for all $e \in A$, we have $f(e) \subseteq g(e)$. Assume that $x \in f^{S_1}(e) = \{x \in U : \exists e' \in A_1[x \in f_1(e'), f_1(e') \cap f(e) \neq \emptyset]\}$, from $f(e) \subseteq g(e)$, we obtain $\exists e' \in A_1$, s.t. $x \in f_1(e'), f_1(e') \cap g(e) \neq \emptyset$, so $x \in g^{S_1}(e) = \{x \in U : \exists e' \in A_1[x \in f_1(e'), f_1(e') \cap g(e) \neq \emptyset]\}$, it follows that $f^{S_1}(e) \subseteq g^{S_1}(e)$ for all $e \in A$, i.e., $\overline{sapr}_{S_1}(\mathcal{S}) \subseteq \overline{sapr}_{S_1}(\mathcal{T})$;

(3) It is obvious that $\mathcal{S} \cap \mathcal{T} \subseteq \mathcal{S}$ and $\mathcal{S} \cap \mathcal{T} \subseteq \mathcal{T}$. From property (1), we obtain $\underline{sapr}_{S_1}(\mathcal{S} \cap \mathcal{T}) \subseteq \underline{sapr}_{S_1}(\mathcal{S})$ and $\underline{sapr}_{S_1}(\mathcal{S} \cap \mathcal{T}) \subseteq \underline{sapr}_{S_1}(\mathcal{T})$. Thus, $\underline{sapr}_{S_1}(\mathcal{S} \cap \mathcal{T}) \subseteq \underline{sapr}_{S_1}(\mathcal{S}) \cap \underline{sapr}_{S_1}(\mathcal{T})$.

(4) It is obvious that $\mathcal{S} \cup \mathcal{T} \supseteq \mathcal{S}$ and $\mathcal{S} \cup \mathcal{T} \supseteq \mathcal{T}$. From property (1), we obtain $\underline{sapr}_{S_1}(\mathcal{S} \cup \mathcal{T}) \supseteq \underline{sapr}_{S_1}(\mathcal{S})$ and $\underline{sapr}_{S_1}(\mathcal{S} \cup \mathcal{T}) \supseteq \underline{sapr}_{S_1}(\mathcal{T})$. Thus, $\underline{sapr}_{S_1}(\mathcal{S} \cup \mathcal{T}) \supseteq \underline{sapr}_{S_1}(\mathcal{S}) \cup \underline{sapr}_{S_1}(\mathcal{T})$.

(5) It is obvious that $\mathcal{S} \cup \mathcal{T} \supseteq \mathcal{S}$ and $\mathcal{S} \cup \mathcal{T} \supseteq \mathcal{T}$. From property (2), we obtain $\overline{sapr}_{S_1}(\mathcal{S} \cup \mathcal{T}) \supseteq \overline{sapr}_{S_1}(\mathcal{S})$ and $\overline{sapr}_{S_1}(\mathcal{S} \cup \mathcal{T}) \supseteq \overline{sapr}_{S_1}(\mathcal{T})$. Thus, $\overline{sapr}_{S_1}(\mathcal{S} \cup \mathcal{T}) \supseteq \overline{sapr}_{S_1}(\mathcal{S}) \cup \overline{sapr}_{S_1}(\mathcal{T})$.

(6) It is obvious that $\mathcal{S} \cap \mathcal{T} \subseteq \mathcal{S}$ and $\mathcal{S} \cap \mathcal{T} \subseteq \mathcal{T}$. From property (2), we obtain $\overline{sapr}_{S_1}(\mathcal{S} \cap \mathcal{T}) \subseteq \overline{sapr}_{S_1}(\mathcal{S})$ and $\overline{sapr}_{S_1}(\mathcal{S} \cap \mathcal{T}) \subseteq \overline{sapr}_{S_1}(\mathcal{T})$. Thus, $\overline{sapr}_{S_1}(\mathcal{S} \cap \mathcal{T}) \subseteq \overline{sapr}_{S_1}(\mathcal{S}) \cap \overline{sapr}_{S_1}(\mathcal{T})$. □

Proposition 3. *Let $\mathcal{S}_1 = (f_1, A_1)$ be a full soft set over U and $(U, \mathcal{S}_1)$ be a soft approximation space. Let $\mathcal{S} = (f, A)$ be a soft set over U. The following properties hold:*

(1) $\underline{sapr}_{S_1}(\mathcal{S}) \subseteq \underline{sapr}_{S_1}(\overline{sapr}_{S_1}(\mathcal{S}))$,
(2) $\overline{sapr}_{S_1}(\mathcal{S}) \supseteq \overline{sapr}_{S_1}(\underline{sapr}_{S_1}(\mathcal{S}))$.

Proof. From property (1) in Proposition 1, it is obvious that $\underline{sapr}_{S_1}(\mathcal{S}) \subseteq \mathcal{S} \subseteq \overline{sapr}_{S_1}(\mathcal{S})$. From property (1) and (2) in Proposition 2, we get $\underline{sapr}_{S_1}(\mathcal{S}) \subseteq \underline{sapr}_{S_1}(\overline{sapr}_{S_1}(\mathcal{S}))$ and $\overline{sapr}_{S_1}(\mathcal{S}) \supseteq \overline{sapr}_{S_1}(\underline{sapr}_{S_1}(\mathcal{S}))$, respectively. □

In [12], a group decision-making approach based on F-soft rough sets was proposed; however, if we carefully check their decision scheme, it is not hard to find that they actually use the tool of a soft rough soft set since the best alternatives provided by each specialist gather together to form a soft set and they compute the upper and lower soft rough approximations (soft sets) on the preliminary evaluation soft set during the decision process. That is, although the concept has not been formally proposed, the application of soft rough soft sets has already appeared in literature. From another perspective, the decision-making problem that can be solved by F-soft rough sets in [12] can also be solved by using soft rough soft sets. It is necessary to propose the concept for soft rough soft sets as well as its application to introduce parameter tools to the universe description, that is, make it feasible to describe objects in the universe from different aspects at the same time, information obtained from different aspects be able to be handled as a whole before the approximations of a soft set are computed, and allow the flexibility to make operations such as the restricted intersection "∩" [25] on soft sets whose soft rough approximations need to be computed; in this way, soft rough soft sets have the potential to be applied in more complex decision-making situations to meet demands of applications in real life cases. As follows, we provide a simple application of soft rough soft sets in decision-making.

Let $G = \{T_1, T_2, ..., T_p\}$ and $A_1 = \{e'_1, e'_2, ..., e'_q\}$ be two groups of specialists to evaluate all the candidates $U = \{x_1, x_2, ..., x_m\}$. In group G, each specialist is asked to point out if the candidates satisfy benefit properties in $A = \{e_1, e_2, ..., e_n\}$ or not. In this way, a serious of evaluations provided by specialists are obtained as (g_1, A), (g_2, A), (g_3, A), $\ldots$, (g_p, A). Afterwards, the evaluation made by group G could be obtained by $S = (f, A) = (g_1, A) \cap (g_2, A) \cap (g_3, A) \cap \ldots (g_p, A)$. Meanwhile, in another group $A_1 = \{e'_1, e'_2, ..., e'_q\}$, the specialists are under time pressure, and a lack of patience, or, because of some other issues, each specialist only points out the best alternatives; however, we have no clear idea about which properties are under their consideration. The best alternatives chosen by specialists in group A_1 form another soft set $S_1 = (f_1, A_1)$. We say the assessments provided by group G are more reliable since the assessments provided by them are more specific than group A_1. However, in order to make full use of information provided by the two independent groups, we can compute the lower soft rough approximation on (f, A) in soft approximation space (U, S_1). If $x_i \in f_{S_1}(e_j)$, from the axiomatic definition of soft rough soft sets, we know that the best alternatives of one or more specialists in A_1 are totally contained in $f(e_j)$, that is, the best alternatives chosen by some specialists in A_1 certainly occupy property e_j, which indicates that this benefit property e_j considered by group G may also be very important to group A_1. The final decision is to select the alternative that occupies the most number of beneficial properties that may be important for both groups.

The steps of this soft rough soft sets based multi-group decision-making approach can be listed as:

Step 1. Input the evaluations on alternatives $U = \{x_1, x_2, ..., x_m\}$ provided by specialists group $G = \{T_1, T_2, ..., T_p\}$ as $(g_1, A), (g_2, A), (g_3, A), \ldots, (g_p, A)$.

Step 2. Input the best alternatives selected by specialists group A_1 as $S_1 = (f_1, A_1)$.

Step 3. Compute the group evaluation made by the specialists in G as $S = (f, A) = (g_1, A) \cap (g_2, A) \cap (g_3, A) \cap \ldots (g_p, A)$.

Step 4. Compute the lower soft rough approximation of (f, A) in (U, S_1), i.e. (f_{S_1}, A).

Step 5. Compute the score of alternatives of each x_j $(j = 1, 2, \ldots, m)$ as $s(x_j) = \sum_{i=1}^{n} f_{S_1}(e_i)(x_j)$, and the decision result is x_k if it satisfies $s(x_k) = max_{j=1,2,...,m}\, s(x_j)$.

Example 3. *Suppose that a factory needs to purchase the best machine from $U = \{x_1, x_2, ..., x_6\}$ according to evaluations provided by two specialists groups G and A_1, which form a multi-group decision-making problem. $G = \{T_1, T_2, T_3, T_4\}$ consists of four specialists and each of them provides assessments on machines in U with respect to beneficial properties $A = \{e_1 =$ low price$, e_2 =$ high endurance$, e_3 =$ advanced technology$, e_4 =$ good compatibility$\}$. Each specialist in G points out if the machines satisfy properties in A or not. In this way, a serious of evaluation soft sets provided by specialists are obtained as (g_1, A), (g_2, A), (g_3, A), (g_4, A) (see Tables 5–8 as their tabular representations) and the group evaluation of G can be computed by $(f, A) = (g_1, A) \cap (g_2, A) \cap (g_3, A) \cap (g_4, A)$ (see also Table 2 as the tabular representation for (f, A)). Meanwhile, each specialist in another specialist group $A_1 = \{e'_3, e'_4, e'_5, e'_6, e'_7\}$ only points out the best machines according to his/her own cognition, which form soft set (f_1, A_1) (replace $e'_3 - e'_7$ by $e_3 - e_7$ and see also Table 1 for its tabular representation). The lower soft rough approximation of (f, A) in (U, S_1) can be easily computed as (f_{S_1}, A) (see also Table 3 for its tabular representation). It is easy to obtain that $s(x_1) = s(x_2) = s(x_3) = 2$,*

$s(x_4) = s(x_5) = 1$ *and* $s(x_6) = 3$, *hence* x_6 *should be the machine purchased by the factory since it satisfies largest number of beneficial properties that are important to two groups.*

Table 5. Soft set (g_1, A).

$A \backslash U$	x_1	x_2	x_3	x_4	x_5	x_6
e_1	1	1	1	1	0	1
e_2	0	1	1	0	0	0
e_3	0	0	0	1	1	1
e_4	1	1	1	1	0	1

Table 6. Soft set (g_2, A).

$A \backslash U$	x_1	x_2	x_3	x_4	x_5	x_6
e_1	1	1	0	1	0	1
e_2	1	1	1	0	0	0
e_3	0	0	0	1	1	1
e_4	1	1	1	1	0	1

Table 7. Soft set (g_3, A).

$A \backslash U$	x_1	x_2	x_3	x_4	x_5	x_6
e_1	1	1	0	1	0	1
e_2	0	1	1	0	0	0
e_3	1	0	0	1	1	1
e_4	1	1	1	1	0	1

Table 8. Soft set (g_4, A).

$A \backslash U$	x_1	x_2	x_3	x_4	x_5	x_6
e_1	1	1	0	1	0	1
e_2	0	1	1	0	0	0
e_3	0	0	1	1	1	1
e_4	1	1	1	1	0	1

As is mentioned at the beginning of this section, soft rough soft set is an extension model of F-soft rough set. Sometimes, in a practical situation, the universe set that needs to be granulated is presented from different attributes' aspects simultaneously. In other words, the parameter tools are necessary not only for the knowledge presentation, but also for the universe description. The new model provides a framework for dealing with these kinds of problems and the exploration of its potential use in decision-making is promising. Compared to F-soft rough sets, soft rough soft sets introduce parameter tools to the universe description and a soft set (instead of a subset of the universe) is approximated. Compared to rough soft set [12], a soft set instead of an equivalence relation has been adopted in soft rough soft sets to compute the approximations of soft sets [36,37]. In this section, only a small application attempt of soft rough soft sets in decision-making has been provided, which is far from enough to meet various demands in real life situations. More flexible and effective approaches need to be developed in the future.

6. Conclusions

This paper has presented a comparative study of some existing soft rough set models, and new discoveries on the relationships among various hybrid sets have been summarized in Table 9. It has been shown that the Z-soft rough fuzzy set is a kind of rough fuzzy set. Therefore, decision-making approaches based on rough fuzzy sets have the potential to be addicted to more specific situations in which Z-soft rough fuzzy sets should be applied to solve the problem. Various soft rough set models have shown great potential in coping with decision-making problems. Some potential applications of connections among various soft rough set models in decision-making have been briefly discussed in the current work. For instance, benefitting from the connections between F-soft rough approximations and MSR approximations that have been discussed, it is possible to further study the relationships between the decision results made by using soft rough sets and MSR sets. In future works, deeper and more specific research on the applications of these connections in decision-making will be conducted.

Table 9. Summary on relationships among various hybrid models.

Various Hybrid Models	Relationships
F-soft rough approximations and modified soft rough approximations (MSR approximations)	$\overline{X}_{\varphi} \subseteq \overline{apr}_P(X)$, $\overline{apr}_P(X) \subseteq \overline{X}_{\varphi}$, $\underline{X}_{\varphi} \subseteq \underline{apr}_P(X)$, if some specific conditions hold, respectively (see Theorems 1–3)
F-soft rough sets in (U, S) and Pawlak's rough sets in (U, R_S)	F-soft rough sets in (U, S) could be identified with Pawlak's rough sets in (U, R_S), when the underlying soft set is a partition soft set (see Theorems 4 and 5)
MSR approximations and Pawlak's rough approximations	MSR approximation operator is a kind of Pawlak rough approximation operator (see Theorem 6)
Z-lower, Z-upper soft rough approximation operators and Dubois and Prade's lower and upper rough fuzzy approximation operators in [6]	Z-lower and Z-upper soft rough approximation operators are equivalent to Dubois and Prade's lower and upper rough fuzzy approximation operators in [6] (see Corollary 3)
The (classical) rough fuzzy sets and M-soft rough fuzzy sets	The (classical) rough fuzzy sets in Pawlak approximation space (U, R) and M-soft rough fuzzy sets in soft approximation space (U, S) are equivalent when the underlying soft set S is a partition soft set (see Theorem 7)
Z-soft rough approximation operators and M-soft Rough approximation operators and F-soft rough approximation operators	$\underline{sap}_P(\mu) \subseteq \underline{sap}'_P(\mu) \subseteq \underline{\mu}_{\varphi} \subseteq \mu \subseteq \overline{\mu}_{\varphi} \subseteq \overline{sap}'_P(\mu)$ $\subseteq \overline{sap}_P(\mu)$ (see Theorem 8 and Corollary 4)
The soft fuzzy rough approximation in Definition 9 and Dubois and Prade's fuzzy rough approximation in [6]	The soft fuzzy rough approximation is a kind of Dubois and Prade's fuzzy rough approximation in [6] (see Theorem 9)
F-soft rough set and soft rough soft set	Soft rough soft set is an extension of F-soft rough set

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4.3 Hesitant linguistic expression soft sets: Application to group decision making

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Hesitant linguistic expression soft sets: Application to group decision making

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Abstract

The notion of linguistic value soft set was proposed to handle linguistic decision making problems under uncertainty. It is based on soft set theory and the fuzzy linguistic approach. However, in real-world situations the decision makers may not only need single linguistic terms to elicit their knowledge about alternatives but also more elaborate linguistic expressions such as, comparative linguistic expressions. Linguistic value soft sets cannot deal with these situations in a satisfactory manner. Hence, we propose one step further towards a complete combination of soft set theory and hesitant fuzzy linguistic term sets to put forward a new model called hesitant linguistic expression soft set that facilitates the elicitation of linguistic information with soft sets. Afterwards, a novel multi-criteria group decision making approach with a consensus reaching process, by using hesitant linguistic expression soft sets, is presented. Some examples show the feasibility and implementation of our novel proposals.

Keywords: Comparative linguistic expression, Hesitant fuzzy linguistic term set, Soft set, Consensus reaching process, Group decision making

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1. Introduction

Most of the real-world problems are under uncertainty and inadequacy contexts which can not be handled with classical crisp mathematical models. The demands for mathematic tools for dealing with uncertainty increases with the rapid development of socio-economic environment. During the past few decades, rough set theory (Pawlak (1982)), fuzzy set theory (Zadeh (1965)) and several other models contribute to meet the demands for coping with uncertain situations, however, all of these theories suffer from a common limitation, that is, incompatible with parameterization tools. To overcome this limitation, soft set theory was initiated by Molodtsov (1999), which makes it possible to consider objects in the universe from different parameter aspects.

To facilitate the applicability of soft set theory for dealing with uncertainty, one simple and feasible approach is the combination with other mathematic tools. For instance, fuzzy soft sets (Maji et al. (2001)), vague soft sets (Xu et al. (2010)) and interval-valued intuitionistic fuzzy soft sets (Jiang et al. (2010)) are some existing hybrid soft set models obtained in this way. Recently, the application of hybrid soft set models in decision making (DM) have drawn attention from researchers, such as, Alcantud & Santos-García (2017); Fatimah et al. (2017a,b); Ma et al. (2016). Taking advantage of the parameterization tools of soft set theory, during the process of decision making based on hybrid soft sets, the evaluations/assessments on alternatives provided by decision makers could be considered from different parameters aspects.

In real world situations, decision makers tend to provide linguistic information rather than quantitative forms considering the qualitative aspects of problems. The modeling of linguistic information has been facilitated by fuzzy linguistic approach (FLA) (Zadeh (1975a)) which represents qualitative terms by linguistic variables. To effectively model linguistic information in situations in which decision makers hesitate among different linguistic values for a linguistic variable, hesitant fuzzy linguistic term set ($HFLTS$) was introduced by Rodríguez et al. (2012). $HFLTSs$ could be flexibly applied in the computing with words (CW) process (Rodríguez et al. (2016); Rodríguez & Martínez (2013)) by means of their envelopes in form of linguistic intervals or fuzzy numbers (Liu & Rodríguez (2014); Rodríguez et al. (2012)).

Up to present, we are only aware of one paper that combines linguistic variables and soft set theory, Sun et al. (2017) introduced the concept of linguistic value soft set, in which the assessment on each alternative with respect to every parameter was presented as a linguistic term (LT). However, the application of single linguistic terms (LTs) constraints the elicitation of complex linguistic preferences in real life DM situations (Ma et al. (2007); Rodríguez et al. (2012); Tanga & Zhengb (2006)). Various proposals have been proposed by researchers for modeling complex linguistic preferences, such as, linguistic model based on fuzzy relation introduced by Tanga & Zhengb (2006), proportional 2-tuple linguistic model introduced by Wang & Hao (2006), and linguistic distribution assessment introduced by Dong et al. (2013). Compared with all other existing models, comparative linguistic expressions (CLEs) proposed by Rodríguez et al. (2012) are closer to cognition of human-being, easily generated by using a context free grammar in a formal way, and convenient to be transformed into HFLTSs for carrying out CW process. Subsequently, this research is focused on further combination of fuzzy linguistic approach and soft set theory to propose a novel model called hesitant linguistic expression soft set (HLE soft set), which overcomes previous limitation of linguistic value soft set and fulfills requirements in real world DM problems for elicitation of complex linguistic information in a human-being cognitive way by using CLEs modeled with $HFLTSs$.

The remainder of the paper is structured as follows: soft set, $HFLTS$, and related concepts are briefly reviewed in Section 2. In Section 3, it is proposed HLE soft set together with some operations. Section 4 introduces a multi-criteria DM approach based on HLE soft set. In Section 5, we present a novel group decision making (GDM) approach based on HLE soft set as well as a consensus model cooperates with this approach. Comparison between our GDM proposal and other existing approaches is provided in Section 6. Conclusions are given in Section 7.

2. preliminaries

In this section, we provide a brief review on soft set, FLA, $HFLTS$ and other related concepts which will be useful in the following sections.

2.1. Soft set theory

The concept of soft set theory was initiated by Molodtsov (1999) to overcome the inadequacy of the parametrization tools of many widely used mathematic tools for dealing with uncertainty.

Let U be the initial universe of objects and E be the set of parameters related to objects in U. Both U and E are assumed to be nonempty finite sets. Let $P(U)$ be the power set of U and $A \subseteq E$.

Definition 1. *(Molodtsov (1999)) A pair (F,A) is called a soft set over the universe U, where F is a mapping given by*

$$F : A \longrightarrow P(U).$$

A soft set can be viewed as a parameterized family of subsets of the universe U considering that $F(e)$ can be viewed as the set of $e-$approximate elements of the soft set (F,A) for parameter $e \in A$.

Null and absolute soft sets were defined by Maji et al. (2003): For (F,A), if $\forall e \in A, F(e) = \emptyset$, it is a null soft set denoted by $\emptyset$; if $\forall e \in A, F(e) = U$, it is an absolute soft set denoted by $\tilde{A}$.

Soft set and its generalization models show great applicability in different topics such as, data predicting (Liu et al. (2017)), rule mining (Feng et al. (2016)), medical diagnosis (Muthukumar & Krishnan (2016)) and decision making (Zhan et al. (2017a,b); Zhan & Zhu (2017)).

2.2. Fuzzy linguistic approach.

Fuzzy linguistic approach uses the fuzzy set theory to model linguistic information based on linguistic variable, that was described by Zadeh (1975a) as "a variable whose values are not numbers but words or sentences in a natural or artificial language" and formally defined as follows:

Definition 2. *(Zadeh (1975b)) A linguistic variable is characterized by a quintuple $(H, T(H), U, G, M)$ in which H is the name of the variable; $T(H)$ is the term set of H, i.e., the set of names of linguistic values of H, with each value being a fuzzy variable that is denoted by X and ranging across a universe of discourse U, which is associated with the base variable u, G is a syntactic rule (which usually takes the form of a grammar) for the generation of the names of values of H; and M is the semantic rule for associating its meaning with each H, $M(X)$, which is a fuzzy subset of U.*

Suitable descriptors for the terms as well as appropriate semantics are necessary for dealing with linguistic variables. Linguistic descriptors could be selected by using an ordered structure approach (Herrera et al. (2000); Yager (1995)), or a context-free grammar approach: the linguistic term set is defined by a context-free grammar G therefore the LTs are sentences generated by G (Bonissone (1980); Bordogna & Pasi (1993); Zadeh (1975b)).

Accordingly, the semantics for terms can be defined based on the ordered structure defined over the linguistic term set, or based on membership functions and a semantic rule (Rodríguez et al. (2012)). Mixed semantics are also allowed. Each linguistic term is assumed as a fuzzy number defined in $[0,1]$, and the use of parameters of the membership functions is effective to represent fuzzy numbers.

2.3. Hesitant fuzzy linguistic term set.

Recently, a new linguistic model was provided in Rodríguez et al. (2012) to improve the elicitation of CLEs by using $HFLTS$ and a context-free grammar.

Definition 3. *(Rodríguez et al. (2012)) Let S be a linguistic term set, and* $HFLTS$ H_S*, is an ordered finite subset of consecutive linguistic terms of S.*

The complement of $HFLTS$, H_S, is defined as $H_S^c = S - H_S = \{s_i/s_i \in S \text{ and } s_i \notin H_S\}$.

The following context-free grammar G_H was introduced by Rodríguez et al. (2013) to generate CLEs closer to the cognition of human-being.

Definition 4. *Let* G_H *be a context-free grammar and* $S = \{s_0, \ldots, s_g\}$ *be a linguistic term set. The elements of* $G_H = (V_N, V_T, I, P)$ *are defined as follows:*
$V_N = \{\langle primary\ term\rangle, \langle composite\ term\rangle, \langle unary\ relation\rangle, \langle binary\ relation\rangle, \langle conjunction\rangle\}$.
$V_T = \{at\ most, at\ least, between, and, s_0, \ldots, s_g\}$.
$I \in V_N$.
$P = \{I ::= \langle primary\ term\rangle | \langle composite\ term\rangle$
$\langle composite\ term\rangle ::= \langle unary\ relation\rangle\langle primary\ term\rangle | \langle binary\ relation\rangle\langle primary\ term\rangle\langle conjunction\rangle\langle primary\ term\rangle$
$\langle primary\ term\rangle ::= s_0|s_1|\ldots|s_g$
$\langle unary\ relation\rangle ::= at\ most|at\ least$
$\langle binary\ relation\rangle ::= between$
$\langle conjunction\rangle ::= and\}$.

Definition 5. *(Rodríguez et al. (2012)) A transformation function* E_{G_H} *from CLE, ll to HFLTS,* H_S*, where S is the linguistic term set used by* E_{G_H} *is defined as:*

$$E_{G_H} : ll \rightarrow H_S \tag{1}$$

Based on E_{G_H}, CLEs generated by G_H can be transformed into $HFLTSs$ in different ways according to their meaning in the processes of CW:
$E_{G_H}(s_i) = \{s_i | s_i \in S\}$,
$E_{G_H}(\text{at most } s_i) = \{s_j | s_j \leq s_i \text{ and } s_j \in S\}$,
$E_{G_H}(\text{at least } s_i) = \{s_j | s_j \geq s_i \text{ and } s_j \in S\}$,
$E_{G_H}(\text{between } s_i \text{ and } s_j) = \{s_k | s_i \leq s_k \leq s_j \text{ and } s_k \in S\}$.

Direct operations on CLEs are hard to be carried out, one feasible approach is to transform them into $HFLTSs$ by using E_{G_H}, afterwards perform the computations based on representation models for $HFLTSs$.

Envelope, as an important representation tool for $HFLTSs$, was firstly proposed in form of linguistic intervals by Rodríguez et al. (2012), after in form of trapezoidal fuzzy numbers (TFNs) by Liu & Rodríguez (2014). We will adopt the latter in this work since it keeps the fuzzy characters of linguistic information and follows the fuzzy linguistic approach. The fuzzy envelope F_{H_S} for a $HFLTS$ H_S could be computed using the proposal in Liu & Rodríguez (2014) and its parameterization could be $F_{H_S} = T(a,b,c,d)$.

3. Hesitant linguistic expression soft set

The concept of hesitant linguistic expression soft set (HLE soft set) will be introduced in this section by combining soft sets and CLEs. Operations on HLE soft sets will also be discussed.

3.1. Definition of HLE soft sets

HLE soft set will be defined under the background of decision making. In order to linguistically model the assessments provided by the decision makers, we choose linguistic descriptors by providing terms directly and the semantics for the LTs will be represented by fuzzy numbers defined in the interval $[0,1]$.

Definition 6. *Let $U = \{x_1, x_2, \ldots, x_m\}$ be the universe set and $E = \{e_1, e_2, \ldots, e_n\}$ be related parameters. Let $S = \{s_0, s_1, \ldots, s_g\}$ be a linguistic term set and $\mathscr{P}(U)$ be the power set of all CLEs built from S for the universe U. A pair (F^{cle}, E) is called a HLE soft set over U, where F^{cle} is a mapping from a parameter set E to the power set of all CLEs built from S for U, i.e., $F^{cle} : E \to \mathscr{P}(U)$.*

A HLE soft set (F^{cle}, E) is a parameterized family of CLEs built from S for the universe U. For any $e_j \in E$, $F^{cle}(e_j)$ is the set of e_j-approximation elements of HLE soft set (F^{cle}, E). $F^{cle}(e_j)(x_i)$ is a CLE that indicates to which degree object $x_i \in U$ satisfies parameter $e_j \in E$.

The HLE soft matrix is defined as

$$F^{cle} = (F^{cle}(e_j)(x_i))_{m \times n},$$

$(i = 1,2,\ldots,m,\ j = 1,2,\ldots,n)$. Each HLE soft set corresponds to a HLE soft matrix.

In the definition of HLE soft sets, CLEs are generated by using the context-free grammar G_H (Rodríguez et al. (2013)) based on S. The semantics for CLEs can be computed as parametric trapezoidal membership functions using the approach in Liu & Rodríguez (2014) considering that TFNs are good enough to capture the vagueness of linguistic assessments (Delgado et al. (1998a,b)). The descriptors for CLEs should be suitable to characterize the degree to which objects satisfy parameters. To achieve this goal, the first term in S should be a single term that indicates an object does not satisfy a parameter at all (label "none" will be used to represent this assessment in the current work) and the last term is a single term that means an object absolutely satisfies a parameter (label "absolute" will be adopted to represent this assessment in the current work). An example of the linguistic term set $S = \{s_0, s_1, \ldots, s_g\}$ $(g = 8)$ with its semantic is shown by Fig.1, where the TFNs representations for LTs "none" and "absolute" are the single terms $T(0,0,0,0)$ and $T(1,1,1,1)$, respectively.

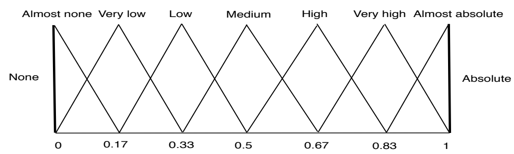

Figure 1: A linguistic term set of nine terms with its semantic.

LTs are special CLEs, hence the concept of HLE soft set generalizes the concept of linguistic value soft set proposed by Sun et al. (2017). If all CLEs in a HLE soft set degenerate to LTs, the HLE soft set will degenerate to a linguistic value soft set. In real world DM situations, decision makers might hesitate among different LTs when they are under time pressure, lack of confidence, knowledge or uncertain issues. The main limitation of linguistic value soft set is that it fails to deal with these hesitant situations. HLE soft set overcomes this limitation by allowing more flexible way in eliciting linguistic information, therefore we say that the use of HLE soft set will be more practical.

A simple example for the definition of HLE soft set could be the presented:

Example 1. *Suppose that* $U = \{x_1, x_2, x_3\}$ *is a set of houses,* $E = \{e_1 =$ *convenient traffic*$, e_2 =$ *low price*$, e_3 =$*good decoration*$, e_4 =$*nice environment*$, e_5 =$*large area*$\}$ *is a set of parameters expected by Mr. Jonhson when he chooses a house from* U*. An expert provides assessments on these houses by using CLEs built from a linguistic term set* $S = \{s_0 :$ *none*$(N), s_1 :$ *almost none* $(AN), s_2 :$ *very low* $(VL), s_3 :$ *low* $(L), s_4 :$*medium* $(M), s_5 :$*high* $(H), s_6 :$*very high* $(VH), s_7 :$*almost absolute* (AA), $s_8 :$ *absolute* $(A)\}$ *by using the context-free grammar* G_H*. The CLEs assessments for houses with respect to different parameters form a HLE soft set* (F^{cle}, E) *as is shown in Table 1.*

Table 1: Tabular representation of a HLE soft set

U	e_1	e_2	e_3	e_4	e_5
x_1	between VL and H	between VL *and* H	at most H	at least M	N
x_2	at most L	at most L	at least VL	at most H	at least M
x_3	between L *and* M	between L *and* VH	between L *and* VH	between VL *and* H	at most VL

In detail, by Def. 6, we obtain $F^{cle}(e_3) = \{x_1 : at\ most\ H, x_2 : at\ least\ VL, x_3 : between\ L\ and\ VH\}$*, which means that* $F^{cle}(e_3)(x_1) = at\ most\ H, F^{cle}(e_3)(x_2) = at\ least\ VL, F^{cle}(e_3)(x_3) = between\ L\ and\ VH$*. Here, "*$F^{cle}(e_3)(x_1) = at\ most\ H$*" indicates that the degree to which house* x_1 *satisfies parameter "good decoration"* (e_3) *is "at most high".*

Some relavant definitions will also be provided:

Definition 7. *Let* U *be a universe set,* E *be the set of parameters related to* U *and* $A \subseteq E$*. Let a linguistic term set* $S = \{s_0, s_1, \cdots, s_g\}$ *in which the term* s_0 *is "none" and the term* s_g *is "absolute". For a HLE soft set* (F^{cle}, A) *over* U*, if*

(i) for any $e \in A$*,* $x \in U$*,* $F^{cle}(e)(x) = none$*, we call* (F^{cle}, A) *a HLE null soft set (with respect to A), denoted as* $(\emptyset^{cle}, A)$*; and*

(ii) for any $e \in A$*,* $x \in U$*,* $F^{cle}(e)(x) = absolute$*, we call* (F^{cle}, A) *a HLE absolute soft set (with respect to A), denoted as* (I^{cle}, A)*.*

The definition of HLE absolute (null) soft set is consistent with the definition of absolute (null) soft set introduced by Maji et al. (2003), absolute (null) vague soft set introduced by Xu et al. (2010), and absolute (null) interval-valued intuitionistic fuzzy soft set introduced by Jiang et al. (2010) to indicate that all objects satisfy all parameters (any object does not satisfy any parameter at all).

3.2. Operations on HLE soft sets

A method to compute the fuzzy envelopes of $HFLTSs$ corresponds to $CLEs$ has been provided in Liu & Rodríguez (2014). However, LTs "none" and "absolute" were not taken into consideration when fuzzy envelopes are computed in Liu & Rodríguez (2014). To carry out operations on HLE soft set without ambiguity, the scheme for computing fuzzy envelopes of $HFLTSs$ based on a linguistic term set S has to be adjusted (A brief adjustment is given in Appendix A).

Operations on HLE soft sets are defined on the basis of ranking of CLEs in this work, therefore a ranking approach for CLEs based on ranking of the fuzzy envelopes for HFLTSs in form of TFNs will be introduced firstly. The ranking approaches for TFNs are various, in the current work the one in Abbasbandy & Hajjari (2009) will be adopted for ranking fuzzy envelopes of CLEs which is based on a measure called "magnitude", and the magnitude of a TFN u will be denoted as $Mag(u)$ (reviewed in Appendix B).

Definition 8. *Let S be a linguistic term set, ll_1, ll_2 be two CLEs built from S by using the context-free grammar G_H, and E_{G_H} be the transformation function from CLEs to HFLTSs, we say*
(1) $ll_1 < ll_2$ iff $F_{H_S^1} \prec F_{H_S^2}$; (2) $ll_1 > ll_2$ iff $F_{H_S^1} \succ F_{H_S^2}$; (3) $ll_1 = ll_2$ iff $F_{H_S^1} \sim F_{H_S^2}$, where $E_{G_H}(ll_1) = H_S^1$, $E_{G_H}(ll_2) = H_S^2$ are HFLTSs on S, and $F_{H_S^1}$, $F_{H_S^2}$ are their fuzzy envelopes, respectively. Sequently, the order $ll_1 \leq ll_2$, $ll_1 \geq ll_2$ can be formulated as:
(4) $ll_1 \leq ll_2$ iff $ll_1 < ll_2$ or $ll_1 = ll_2$; (5) $ll_1 \geq ll_2$ iff $ll_1 > ll_2$ or $ll_1 = ll_2$.
$\wedge$ and $\vee$ operators can be formulated as:
(6) $ll_1 \wedge ll_2 = ll_1$ iff $ll_1 \leq ll_2$; (7) $ll_1 \vee ll_2 = ll_1$ iff $ll_1 \geq ll_2$.

Afterwards, the linguistic complement of CLEs built from S will be defined considering their meaning.

Definition 9. *Let S be a linguistic term set and ll be a CLE built from $S = \{s_0, s_1, \cdots, s_g\}$ by using the context-free grammar G_H. The linguistic complement of ll is denoted by ll^c and defined by*

$$ll^c = \begin{cases} \text{at least } s_{g-i}, & \text{for } ll = \text{at most } s_i, \forall s_i \in S; \\ \text{at most } s_{g-i}, & \text{for } ll = \text{at least } s_i, \forall s_i \in S; \\ \text{between } s_{g-j} \text{ and } s_{g-i}, & \text{for } ll = \text{between } s_i \text{ and } s_j, \forall s_i, s_j \in S; \\ s_{g-i}, & for\ ll = s_i, \forall s_i \in S. \end{cases} \quad (2)$$

Afterwards, some operations on HLE soft sets will be defined.

Definition 10. *Let U be a universe set and E be the set of parameters related to U. Let $\mathscr{P}(U)$ be the power set of all CLEs built from S for the universe U. F, G are two mappings from parameter set E to $\mathscr{P}(U)$. For any $e \in E$, we note*
$F^{cle}(e) \subseteq G^{cle}(e)$ iff $F^{cle}(e)(x) \leq G^{cle}(e)(x)$, $\forall x \in U$.

Definition 11. *Let U be a universe set and E be the set of parameters related to U. Let (F^{cle}, A) and (G^{cle}, B) be two HLE soft sets over U, where $A, B \subseteq E$. We say that (F^{cle}, A) is a HLE soft subset of (G^{cle}, B), denoted as $(F^{cle}, A) \subseteq (G^{cle}, B)$, if*

(i) $A \subseteq B$, *and*
(ii) for any $e \in A$, $F^{cle}(e) \subseteq G^{cle}(e)$.

Example 2. *(Continued from Example 1) Suppose that* $A = \{e_1$ *=convenient traffic*, e_2 *=low price*$\}$, $B = \{e_1$ *=convenient traffic*, e_2 *=lowprice*, e_3 *=good decoration*$\}$ *are two parameter sets.* (F^{cle}, A)*and* (G^{cle}, B) *are two HLE soft sets.*
$F^{cle}(e_1) = \{x_1 : \text{between } VL \text{ and } H, x_2 : \text{at most } L, x_3 : \text{between } L \text{ and } M\}$;
$F^{cle}(e_2) = \{x_1 : \text{between } VL \text{ and } H, x_2 : \text{at most } L, x_3 : \text{between } L \text{ and } VH\}$;
$G^{cle}(e_1) = \{x_1 : \text{at least } H, x_2 : \text{at most } L, x_3 : \text{between } L \text{ and } VH\}$;
$G^{cle}(e_2) = \{x_1 : \text{between } M \text{ and } H, x_2 : \text{at least } H, x_3 : \text{at least } H\}$;
$G^{cle}(e_3) = \{x_1 : \text{at most } H, x_2 : \text{at least } L, x_3 : \text{between } M \text{ and } VH\}$;
By using the transformation function E_{G_H}, *we obtain* $HFLTSs$ *corresponds to* $F^{cle}(e_1)$ *and* $G^{cle}(e_1)$:
$E_{G_H}(F^{cle}(e_1)(x_1)) = E_{G_H}(\text{between } VL \text{ and } H) = \{VL, L, M, H\}$,
$E_{G_H}(F^{cle}(e_1)(x_2)) = E_{G_H}(\text{at most } L) = \{N, AN, VL, L\}$,
$E_{G_H}(F^{cle}(e_1)(x_3)) = E_{G_H}(\text{between } L \text{ and } M) = \{L, M\}$,
$E_{G_H}(G^{cle}(e_1)(x_1)) = E_{G_H}(\text{at least } H) = \{H, VH, AA, A\}$,
$E_{G_H}(G^{cle}(e_1)(x_2)) = E_{G_H}(\text{at most } L) = \{N, AN, VL, L\}$,
$E_{G_H}(G^{cle}(e_1)(x_3)) = E_{G_H}(\text{between } L \text{ and } VH) = \{L, M, H, VH\}$.
Denote the fuzzy envelope for $E_{G_H}(F^{cle}(e_j)(x_i))$ *as* F_{ij} *and* $E_{G_H}(G^{cle}(e_j)(x_i))$ *as* G_{ij}, *compute the fuzzy envelopes of the above* $HFLTSs$:
$F_{11} = T(0, 0.27, 0.57, 0.83)$, $F_{21} = T(0, 0, 0.15, 0.5)$, $F_{31} = T(0.17, 0.33, 0.50, 0.67)$;
$G_{11} = T(0.5, 0.85, 1, 1)$, $G_{21} = T(0, 0, 0.15, 0.5)$, $G_{31} = T(0.17, 0.43, 0.73, 1)$;
The magnitudes for these fuzzy envelopes are listed:
$Mag(F_{11}) \approx 0.42$, $Mag(F_{21}) \approx 0.10$, $Mag(F_{31}) \approx 0.42$,
$Mag(G_{11}) \approx 0.90$, $Mag(G_{21}) \approx 0.10$, $Mag(G_{31}) \approx 0.58$.
Since $Mag(F_{11}) \leq Mag(G_{11})$, $Mag(F_{21}) \leq Mag(G_{21})$, $Mag(F_{31}) \leq Mag(G_{31})$, *we know* $F_{11} \preceq G_{11}$, $F_{21} \preceq G_{21}$, $F_{31} \preceq G_{31}$.
From Def. 8, we know $F^{cle}(e_1)(x_1) \leq G^{cle}(e_1)(x_1)$, $F^{cle}(e_1)(x_2) \leq G^{cle}(e_1)(x_2)$, $F^{cle}(e_1)(x_3) \leq G^{cle}(e_1)(x_3)$, *then we obtain* $F^{cle}(e_1) \subseteq G^{cle}(e_1)$ *from Def. 10.*
In a similar way, we can obtain $F^{cle}(e_2) \subseteq G^{cle}(e_2)$.
Obviously, $A \subseteq B$, *then* $(F^{cle}, A) \subseteq (G^{cle}, B)$ *can be obtained from Def. 11.*

Definition 12. *(Maji et al. (2003)) Let* $E = \{e_1, e_2, \ldots, e_n\}$ *be a parameter set. The not set of* E *denoted by* $\neg E$ *is defined by* $\neg E = \{\neg e_1, \neg e_2, \ldots, \neg e_n\}$ *where* $\neg e_i = \text{not } e_i$.

Definition 13. *Let* U *be a universe set,* E *be the set of parameters related to* U, *and* $A \subseteq E$. *The complement of HLE soft set* (F^{cle}, A), *can be denoted by*

$(F^{cle},A)^c$ *and defined by:*

$$(F^{cle},A)^c = ((F^{cle})^c, \neg A)$$

where $\neg A \in \neg E$, *and* $(F^{cle})^c(\neg e)(x) = (F^{cle}(e)(x))^c, \neg e \in \neg A, x \in U$.

Another kind of complement operation called "relative complement" of HLE soft set is also provided:

Definition 14. *Let U be a universe set, E be the set of parameters related to U, and $A \subseteq E$. The relative complement of HLE soft set (F^{cle},A), can be denoted by $(F^{cle},A)'$ and defined by:*

$$(F^{cle},A)' = ((F^{cle})', A)$$

where $(F^{cle})'(e)(x) = (F^{cle}(e)(x))^c, e \in A, x \in U$.

An example is provided to show the difference between Defs. 13 and 14.

Example 3. *(Continued from Example 2) Let $U = \{x_1, x_2, x_3\}$ be the set of houses, $A = \{e_1 = \text{convenient traffic}, e_2 = \text{low price}\}$ be the considered parameters. Given a HLE soft set (F^{cle},A).*

The relative complement of (F^{cle},A) is $(F^{cle},A)'$ in which
$(F^{cle})'(e_1) = \{x_1 : \text{between L and } VH, x_2 : \text{at least } H, x_3 : \text{between M and } H\}$;
$(F^{cle})'(e_2) = \{x_1 : \text{between L and } VH, x_2 : \text{at least } H, x_3 : \text{between } VL \text{ and } H\}$,
*where "$(F^{cle})'(e_1)(x_2) = \text{at least } H$" means the satisfactive degree of house x_2 with respect to "**convenient traffic**" is "at least high".*

The complement of (F^{cle},A) is $(F^{cle},A)^c$ in which
$(F^{cle})^c(\neg e_1) = \{x_1 : \text{between L and } VH, x_2 : \text{at least } H, x_3 : \text{between M and } H\}$;
$(F^{cle})^c(\neg e_2) = \{x_1 : \text{between L and } VH, x_2 : \text{at least } H, x_3 : \text{between } VL \text{ and } H\}$,
*where "$(F^{cle})^c(\neg e_1)(x_2) = \text{at least } H$" means the satisfactive degree of house x_2 with respect to "**not convenient traffic**" is "at least high".*

Definition 15. *Let U be a universe set and E be the set of parameters related to U. For any two HLE soft sets (F^{cle},A) and (G^{cle},B) over U, where $A, B \subseteq E$,*

(i) The extended union of (F^{cle},A) and (G^{cle},B) is defined as

$$(H^{cle},C) = (F^{cle},A)\tilde{\cup}(G^{cle},B),$$

where $C = A \cup B$ *and*

$$H^{cle}(e)(x) = \begin{cases} F^{cle}(e)(x), & \text{for } e \in A, e \notin B; \\ F^{cle}(e)(x) \vee G^{cle}(e)(x), & \text{for } e \in A \cap B; \\ G^{cle}(e)(x), & \text{for } e \notin A, e \in B, \end{cases} \quad (3)$$

for all $x \in U$.

(ii) The restricted intersection of (F^{cle},A) *and* (G^{cle},B) *is defined as*

$$(H^{cle},C) = (F^{cle},A) \cap (G^{cle},B),$$

where $C = A \cap B$ *and*

$$H^{cle}(e)(x) = F^{cle}(e)(x) \wedge G^{cle}(e)(x) \tag{4}$$

for all $x \in U$, $e \in A \cap B$.

(iii) The restricted union of (F^{cle},A) *and* (G^{cle},B) *is defined as*

$$(H^{cle},C) = (F^{cle},A) \cup (G^{cle},B),$$

where $C = A \cap B$ *and*

$$H^{cle}(e)(x) = F^{cle}(e)(x) \vee G^{cle}(e)(x) \tag{5}$$

for all $x \in U$, $e \in A \cap B$.

(iv) The extended intersection of (F^{cle},A) *and* (G^{cle},B) *is defined as*

$$(H^{cle},C) = (F^{cle},A) \tilde{\cap} (G^{cle},B),$$

where $C = A \cup B$ *and*

$$H^{cle}(e)(x) = \begin{cases} F^{cle}(e)(x), & for\ e \in A, e \notin B; \\ F^{cle}(e)(x) \wedge G^{cle}(e)(x), & for\ e \in A \cap B; \\ G^{cle}(e)(x), & for\ e \notin A, e \in B. \end{cases} \tag{6}$$

for all $x \in U$.

In the following, some basic operation laws for HLE soft set are presented.

Proposition 1. *Let* U *be a universe set and* E *be the set of parameters related to* U. *Let* (F^{cle},A) *be a HLE soft set defined on* U, *where* $A \subseteq E$, *the following results hold.*
(1) $((F^{cle},A)^c)^c = (F^{cle},A)$; *(2)* $((F^{cle},A)')' = (F^{cle},A)$; *(3)* $(\emptyset^{cle},A)' = (I^{cle},A)$.

Proposition 2. *Let* U *be a universe set and* E *be the set of parameters related to* U. *Let* $(F^{cle},A),(G^{cle},B)$ *be HLE soft sets defined on* U, *where* $A,B \subseteq E$, *the following results hold.*
(1) $(F^{cle},A) \cup (\emptyset^{cle},A) = (F^{cle},A)$, $(F^{cle},A) \cap (\emptyset^{cle},A) = (\emptyset^{cle},A)$;
(2) $(F^{cle},A) \cup (I^{cle},A) = (I^{cle},A)$, $(F^{cle},A) \cap (I^{cle},A) = (F^{cle},A)$;
(3) $(F^{cle},A) \cup (F^{cle},A) = (F^{cle},A)$, $(F^{cle},A) \cap (F^{cle},A) = (F^{cle},A)$;
(4) $(F^{cle},A) \cap (G^{cle},B) = (G^{cle},B) \cap (F^{cle},A)$, $(F^{cle},A) \cup (G^{cle},B) = (G^{cle},B) \cup (F^{cle},A)$.

Proposition 3. *Let U be a universe set and E be the set of parameters related to U. Let $(F^{cle},A),(G^{cle},A)$ be HLE soft sets defined on U, where $A \subseteq E$, the De Morgan's laws hold as follows.*
(1) $((F^{cle},A)\cup(G^{cle},A))^c = (F^{cle},A)^c \cap (G^{cle},A)^c$,
(2) $((F^{cle},A)\cap(G^{cle},A))^c = (F^{cle},A)^c \cup (G^{cle},A)^c$.

Proof. We only provide the proof for (1), the proof for (2) can be obtained in a similar way.

(1) From Defs. 13 and 15, $(F^{cle}\cup G^{cle})^c(\neg e)(x) = ((F^{cle}\cup G^{cle})(e)(x))^c = (F^{cle}(e)(x)\vee G^{cle}(e)(x))^c$ for all $\neg e \in \neg A$, $x \in U$. $((F^{cle})^c \cap (G^{cle})^c)(\neg e)(x) = (F^{cle})^c(\neg e)(x) \wedge (G^{cle})^c(\neg e)(x) = (F^{cle}(e)(x))^c \wedge (G^{cle}(e)(x))^c$ for all $\neg e \in \neg A$, $x \in U$. Since $(F^{cle}(e)(x)\vee G^{cle}(e)(x))^c = (F^{cle}(e)(x))^c \wedge (G^{cle}(e)(x))^c$ from the definition of complement of CLE, we have $((F^{cle},A)\cup(G^{cle},A))^c = (F^{cle},A)^c \cap (G^{cle},A)^c$. □

Remark 1. *Properties $(1)^*$ and $(2)^*$ below do not hold unless $A = B$, considering that $\neg(A\cap B) \neq \neg A \cap \neg B$ unless $A = B$.*
$(1)^*$ $((F^{cle},A)\cup(G^{cle},B))^c = (F^{cle},A)^c \cap (G^{cle},B)^c$,
$(2)^*$ $((F^{cle},A)\cap(G^{cle},B))^c = (F^{cle},A)^c \cup (G^{cle},B)^c$.

Proposition 4. *Let U be a universe set and E be the set of parameters related to U. Let $(F^{cle},A),(G^{cle},B)$ be HLE soft sets defined on U, where $A,B \subseteq E$, the De Morgan's laws hold as follows.*
(1) $((F^{cle},A)\cup(G^{cle},B))' = (F^{cle},A)' \cap (G^{cle},B)'$,
(2) $((F^{cle},A)\cap(G^{cle},B))' = (F^{cle},A)' \cup (G^{cle},B)'$.

Proof. We only provide the proof for (1), the proof for (2) can be obtained in a similar way.

(1) From Defs. 14 and 15, it is easy to obtain that
$(F^{cle}\cup G^{cle})'(e)(x) = ((F^{cle}\cup G^{cle})(e)(x))^c = (F^{cle}(e)(x)\vee G^{cle}(e)(x))^c = (F^{cle}(e)(x))^c \wedge (G^{cle}(e)(x))^c = (F^{cle})'(e)(x) \wedge (G^{cle})'(e)(x) = ((F^{cle})' \cap (G^{cle})')(e)(x)$ for all $e \in A\cap B$, $x \in U$. □

Proposition 5. *Let U be a universe set and E be the set of parameters related to U. Let (F^{cle},A), (G^{cle},B), (H^{cle},C) be HLE soft sets defined on U, where $A,B,C \subseteq E$, the following results hold.*
(1) $((F^{cle},A)\cup(G^{cle},B))\cup(H^{cle},C) = (F^{cle},A)\cup((G^{cle},B)\cup(H^{cle},C))$;
(2) $((F^{cle},A)\cap(G^{cle},B))\cap(H^{cle},C) = (F^{cle},A)\cap((G^{cle},B)\cap(H^{cle},C))$;
(3) $((F^{cle},A)\cup(G^{cle},B))\cap(H^{cle},C) = ((F^{cle},A)\cap(H^{cle},C))\cup((G^{cle},B))\cap(H^{cle},C))$;
(4) $((F^{cle},A)\cap(G^{cle},B))\cup(H^{cle},C) = ((F^{cle},A)\cup(H^{cle},C))\cap((G^{cle},B))\cup(H^{cle},C))$.

Proof. (1) and (2) can be easily proved based on the definition of restricted union and restricted intersection of HLE soft sets. We only provide the proof for (3), the proof for (4) can be obtained in a similar way.

(3) From Def.15, it is easy to obtain that $((F^{cle} \cup G^{cle}) \cap H^{cle})(e)(x) = (F^{cle} \cup G^{cle})(e)(x) \wedge H^{cle}(e)(x) = (F^{cle}(e)(x)) \vee G^{cle}(e)(x)) \wedge H^{cle}(e)(x) = (F^{cle}(e)(x) \wedge H^{cle}(e)(x)) \vee (G^{cle}(e)(x) \wedge H^{cle}(e)(x)) = (F^{cle} \cap H^{cle})(e)(x) \vee (G^{cle} \cap H^{cle})(e)(x) = ((F^{cle} \cap H^{cle}) \cup (G^{cle} \cap H^{cle}))(e)(x)$ for all $e \in A \cap B \cap C$, $x \in U$. □

4. A multi-criteria decision making approach based on HLE soft set

Exploration on DM approaches based on generalized soft set models shows a huge development in recent years. However, most of these approaches fail to deal with linguistic information. In this section, we will provide a DM algorithm based on HLE soft set to deal with linguistic assessments on alternatives by extending an approach based on fuzzy soft set in Alcantud (2016).

4.1. Decision making scheme

Let $U = \{x_1, x_2, \dots, x_m\}$ be a universe of alternatives and $E = \{e_1, e_2, \dots, e_n\}$ be the set of parameters closely related to U. The evaluation on every alternative with respect to each parameter is presented by a CLE, in this way all assessments form a HLE soft set. The goal is to select the optimal alternative according to the linguistic information provided by a decision maker.

A multi-criteria DM algorithm based on HLE soft set can be presented as below:

Algorithm 1

Step 1 Input a HLE soft set (F^{cle}, E) on m alternatives $x_1, x_2, \dots, x_m$ as an input table whose cell (i, j) is $F^{cle}(e_j)(x_i)$.

Step 2 For each cost criteria/parameter e_j, replace column j by the result of applying linguistic complement (see Def. 9) of CLE in each cell. An uniformed HLE soft set could be presented as $(\overline{F}^{cle}, E)$ with corresponding cell (i, j) in tabular form as $\overline{F}^{cle}(e_j)(x_i)$.

Step 3 Denote the fuzzy envelope of $\overline{F}^{cle}(e_j)(x_i)$ by u_{ij}. For each parameter e_j, let M_j be the maximum magnitude value of fuzzy envelopes for $HFLTSs$ corresponds to CLEs on all alternatives, i.e., $M_j = max_{i=1,\dots,m} Mag(u_{ij})$ for each $j = 1, \dots, n$. Now we construct a comparison matrix $A = (a_{pq})_{m\times m}$ where for any p, q, and let a_{pq} be the sum of the non-negative values in the following finite sequence:
$\frac{Mag(u_{p1})-Mag(u_{q1})}{M_1}, \frac{Mag(u_{p2})-Mag(u_{q2})}{M_2}, \dots, \frac{Mag(u_{pn})-Mag(u_{qn})}{M_n}$.

Step 4 For each $i = 1, \ldots, m$, compute R_i as the sum of the elements in row i of comparison matrix A, and T_i as the sum of the elements in column i of A, then compute the score of each alternative x_i by $S_i = R_i - T_i$.

Step 5 The decision is the alternative x_k that maximizes the score, i.e., x_k s.t. $S_k = max_{i=1,\ldots,m} S_i$.

4.2. Illustrative example

An example is introduced to illustrate the application of Algorithm 1.

Suppose that a factory needs to purchase the best machine from $U = \{x_1, \ldots, x_6\}$ and parameters $E = \{e_1 =$high price$, e_2 =$long endurance$, e_3 =$advanced technology$\}$ are under their consideration. Let a linguistic term set $S = \{s_0$: none $(N), s_1$: almost none $(AN), s_2$: very low $(VL), s_3$: low $(L), s_4$: medium $(M), s_5$: high $(H), s_6$: very high $(VH), s_7$: almost absolute $(AA), s_8$: absolute$(A)\}$ (Fig. 1).

1) Input the initial HLE soft set.
The assessments form a HLE soft set (Table 2).

Table 2: Tabular representation of the initial HLE soft set

U	e_1	e_2	e_3
x_1	at most L	between L and M	at most L
x_2	at least H	at most L	M
x_3	between VL and H	between M and VH	between M and VH
x_4	H	between L and VH	at least H
x_5	at least H	between VL and H	between B and VH
x_6	between VL and H	at most L	between M and H

2) Normalize the decision information.
e_1 is a cost criteria, e_2 and e_3 are benefit criteria, the normalized HLE soft set is obtained (Table 3).

Table 3: Tabular representation of the normalized HLE soft set

U	e_1	e_2	e_3
x_1	at least H	between L and M	at most L
x_2	at most L	at most L	M
x_3	between L and VH	between M and VH	between M and VH
x_4	L	between L and VH	at least H
x_5	at most L	between VL and H	between L and VH
x_6	between L and VH	at most L	between M and H

3) Compute the comparison matrix.
Transform CLEs in the normalized HLE soft set into $HFLTSs$ (Table 4), the fuzzy envelopes of these $HFLTSs$ are computed (Table 5), as well as the magnitudes of these fuzzy envelopes (Table 6).

Table 4: $HFLTSs$ generated from the CLEs

U	e_1	e_2	e_3
x_1	$\{H,VH,AA,A\}$	$\{L,M\}$	$\{N,AN,VL,L\}$
x_2	$\{N,AN,VL,L\}$	$\{N,AN,VL,L\}$	$\{M\}$
x_3	$\{L,M,H,VH\}$	$\{M,H,VH\}$	$\{M,H,VH\}$
x_4	$\{L\}$	$\{L,M,H,VH\}$	$\{H,VH,AA,A\}$
x_5	$\{N,AN,VL,L\}$	$\{VL,L,M,H\}$	$\{L,M,H,VH\}$
x_6	$\{L,M,H,VH\}$	$\{N,AN,VL,L\}$	$\{M,H\}$

Table 5: Fuzzy envelopes for the $HFLTSs$

U	e_1	e_2	e_3
x_1	$T(0.5,0.85,1,1)$	$T(0.17,0.33,0.50,0.67)$	$T(0,0,0.15,0.5)$
x_2	$T(0,0,0.15,0.5)$	$T(0,0,0.15,0.5)$	$T(0.33,0.5,0.5,0.67)$
x_3	$T(0.17,0.43,0.73,1)$	$T(0.33,0.64,0.70,1)$	$T(0.33,0.64,0.70,1)$
x_4	$T(0.17,0.33,0.33,0.5)$	$T(0.17,0.43,0.73,1)$	$T(0.5,0.85,1,1)$
x_5	$T(0,0,0.15,0.5)$	$T(0,0.27,0.57,0.83)$	$T(0.17,0.43,0.73,1)$
x_6	$T(0.17,0.43,0.73,1)$	$T(0,0,0.15,0.5)$	$T(0.33,0.5,0.67,0.83)$

Table 6: Magnitudes of the fuzzy envelopes for the $HFLTSs$

U	e_1	e_2	e_3
x_1	0.896	0.416	0.104
x_2	0.104	0.104	0.5
x_3	0.581	0.669	0.669
x_4	0.331	0.581	0.896
x_5	0.104	0.419	0.581
x_6	0.581	0.104	0.584

From $M_j = max_{i=1,\ldots,6}Mag(u_{ij})$ for each $j = 1,2,3$, by Table 6, we obtain

$M_1 = max(0.896, 0.104, 0.581, 0.331, 0.104, 0.581) = 0.896$,
$M_2 = max(0.416, 0.104, 0.669, 0.581, 0.419, 0.104) = 0.669$,
$M_3 = max(0.104, 0.5, 0.669, 0.896, 0.581, 0.584) = 0.896$,
Comparison matrix A is obtained as Eq. (7).

$$A = \begin{pmatrix} 0.000 & 1.350 & 0.352 & 0.631 & 0.884 & 0.818 \\ 0.442 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 1.009 & 1.566 & 0.000 & 0.411 & 1.004 & 0.939 \\ 1.131 & 1.408 & 0.253 & 0.000 & 0.847 & 1.061 \\ 0.537 & 0.561 & 0.000 & 0.000 & 0.000 & 0.471 \\ 0.536 & 0.626 & 0.000 & 0.279 & 0.536 & 0.000 \end{pmatrix} \quad (7)$$

4) Compute the scores.
It is easy to obtain that
$R_1 = 4.035$, $R_2 = 0.442$, $R_3 = 4.929$, $R_4 = 4.7$, $R_5 = 1.569$, $R_6 = 1.977$.
$T_1 = 3.655$, $T_2 = 5.511$, $T_3 = 0.605$, $T_4 = 1.321$, $T_5 = 3.271$, $T_6 = 3.289$.
Then the scores of alternatives are
$S_1 = 0.38$, $S_2 = -5.069$, $S_3 = 4.324$, $S_4 = 3.379$, $S_5 = -1.702$, $S_6 = -1.312$.
5) Make the decision.
The decision is x_3, since $S_3 = max_{i=1,\ldots,6} S_i$.

5. Consensus group decision making based on HLE soft set

Multi-criteria group decision making problems are common in daily life, in these situations decision makers/experts may also use linguistic expressions rather than numerical values to express their evaluation over alternatives. To deal with these situations a novel GDM approach based on HLE soft set will be introduced in this section. Meanwhile, to pursue a reasonable decision result a novel consensus model will also be provided.

Before presenting the GDM scheme, we pay attention to the approach for aggregating HLE soft sets, so first the way to aggregate CLEs will be studied.

An operator for aggregating linguistic evaluations in DM under ignorance was defined in Yager (1995):

Definition 16. *(Yager (1995)) Let a set of linguistic valuables $L = \{L_1, L_2, \ldots, L_m\}$ such that $L_i > L_j$ if $i > j$. A mapping*

$$F_W : L^n \longrightarrow L$$

is called an ordinal OWA operator of dimension n if it has an associated weighting vector

$$W = \{w_1, w_2, \ldots, w_n\}^T$$

such that
1. $w_j \in L$,
2. $w_j \geq w_i$, *if* $j > i$,
3. $Max_j[w_j] = L_m$,
where for any set of values $a_1, \ldots, a_n$

$$F_W(a_1, \ldots, a_n) = Max_j[w_j \wedge b_j],$$

where b_j *is the jth largest of the* $a_1, \ldots, a_n$.

Several approaches to generate weighting vectors have been studied in Yager (1995) to implement the aggregation of linguistic values in a linguistic scale by using ordinal OWA operator. The idea in their work is to use linguistic values in the linguistic scale as the weights. However, if we simply follow their approach to generate weights for aggregating CLEs built from S, all CLEs built from S by using the context-free grammar G_H (denoted by $\mathscr{C}(S)$) should be listed a prior, which is hard to realize especially with the growth of numbers of terms in S. To simplify the calculation, we suggest to aggregate CLEs with weights as LTs in S, that is, use S instead of $\mathscr{C}(S)$ as the linguistic scale to generate the weighting vector. The simplification is feasible considering that:

(1) the LTs in S are special CLEs built from S, i.e., $S \subseteq \mathscr{C}(S)$;
(2) the largest CLE and smallest CLE in $\mathscr{C}(S)$ are all contained in S;
(3) the uniform and nondecreasing distribution of LTs in S.

In order to aggregate CLEs in $\mathscr{C}(S)$, now we define a CLE-OWA operator in which the weighting vector will be consisted of LTs in S:

Definition 17. *Let* $S = \{s_0, s_1, \ldots, s_g\}$ *be a linguistic term set and* $\mathscr{C}(S)$ *be all CLEs built from S by using the context-free grammar* G_H. *A mapping*

$$\theta_W : \mathscr{C}(S)^n \longrightarrow \mathscr{C}(S)$$

is called a CLE-OWA operator of dimension n if it has an associated weighting vector

$$W = \{w_1, w_2, \ldots, w_n\}^T$$

such that
1. $w_j \in S$,
2. $w_j \geq w_i$, *if* $j > i$,
3. $Max_j[w_j] = s_g$,
where for any set of CLEs in $\mathscr{C}(S)$, $a_1, \ldots, a_n$

$$\theta_W(a_1, \ldots, a_n) = Max_j[w_j \wedge b_j],$$

where b_j *is the jth largest of the* $a_1, \ldots, a_n$.

Remark 2. *The "largest" of CLEs $a_1,\dots,a_n$ in Def. 17 refers to a comparison based on Def. 8.*

Remark 3. *CLE-OWA operators are special ordinary OWA operators Yager (1995) in which the weights are LTs while the aggregation objects are CLEs.*

Example 4. *Assume that a linguistic term set $S = \{s_0,s_1,s_2,s_3,s_4,s_5,s_6,s_7,s_8\}$, and four CLEs built from S by G_H: "at least s_4", "at most s_3", "between s_3 and s_6", "at least s_5" need to be aggregated using the following weighting vector:*

$$W = \{s_1,s_2,s_6,s_8\}^T$$

Ordering the elements in "at least s_4, at most s_3, between s_3 and s_6, at least s_5" by using Def. 8, we obtain
$b_1 =$ at least s_5, $b_2 =$ at least s_4, $b_3 =$ between s_3 and s_6, $b_4 =$ at most s_3,
hence,
$\theta_W(\textit{at least } s_4, \textit{at most } s_3, \textit{between } s_3 \textit{ and } s_6, \textit{at least } s_5) = Max[s_1 \wedge \textit{at least } s_5, s_2 \wedge \textit{at least } s_4, s_6 \wedge \textit{between } s_3 \textit{ and } s_6, s_8 \wedge \textit{at most } s_3] = Max[s_1, s_2, \textit{between } s_3 \textit{ and } s_6, \textit{at most } s_3] = \textit{between } s_3 \textit{ and } s_6$.

By using the CLE-OWA operator, HLE soft sets can be aggregated in the following way:

Definition 18. *Let $U = \{x_1,x_2,\dots,x_m\}$ be the universe, $E = \{e_1,e_2,\dots,e_n\}$ be parameters related to U and $S = \{s_0,s_1,\dots,s_g\}$ be a linguistic term set. Let (F_1^{cle},E), (F_2^{cle},E), ..., (F_f^{cle},E) be HLE soft sets over U, θ_W-related collective HLE soft set comes from (F_1^{cle},E), (F_2^{cle},E), ..., (F_f^{cle},E) is defined by*

$$\tilde{F}^{cle}(e_j)(x_i) = \theta_W(F_1^{cle}(e_j)(x_i),\dots,F_f^{cle}(e_j)(x_i)) \tag{8}$$

for all $e_j \in E$ and $x_i \in U$, where θ_W is a CLE-OWA operator associated with a weighting vector W generated from S.

A consensus GDM algorithm based on HLE soft sets is given as follows:
Algorithm 2.

Step 1 Determine the GDM problem. This phase consists of defining experts $G = \{t_1,t_2,\dots t_f\}$, alternatives $U = \{x_1,x_2,\dots x_m\}$, parameters $E = \{e_1,e_2,\dots e_n\}$, and a linguistic term set $S = \{s_0,s_1,\dots,s_g\}$.

Step 2 Each expert $t_k \in G$ provides evaluations on alternatives in U with respect to all parameters in E by using CLEs built from S. The CLEs provided by each expert t_k $(k = 1,2,\dots,f)$ form a HLE soft set (F_k^{cle},E).

Step 3 Select a CLE-OWA operator θ_W of dimension f associated with a weighting vector W generated from S for aggregating CLEs.

Step 4 Carry out the consensus reaching process and denote the HLE soft set provided by expert t_k after several rounds of adjustments which finally reaches consensus by (F'^{cle}_k, E) $(k \in \{1,2,\ldots,f\})$.

Step 5 Compute the θ_W-related collective HLE soft set $(\tilde{F}'^{cle}, E)$ comes from (F'^{cle}_1, E), (F'^{cle}_2, E), $\ldots$, (F'^{cle}_f, E).

Step 6 Apply Algorithm 2 on $(\tilde{F}'^{cle}, E)$ and select the optimal alternative x_k which satisfies $S_k = maxS_i$ $(i = 1,2,\ldots,m)$, where S_i is the score of x_i.

The optimal alternative can be one or several, and the CLE-OWA operators that could be adopted in Step 3 are various.

When the GDM is performed with HLE soft sets, the assessments from different experts are collected from various parameter aspects, which makes more comprehensive the use of information, however increase the amount of data and the necessary for the consensus reaching process (CRP) (Palomares et al. (2014)). As follows a consensus model is presented for HLE soft set based linguistic GDM (see Fig. 2).

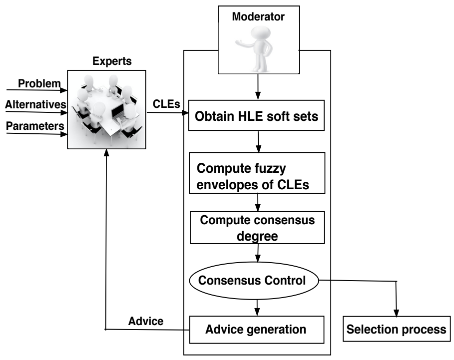

Figure 2: Consensus model

1) Compute the fuzzy envelopes for CLEs in all HLE soft sets provided by experts.
   Transform CLE $F^{cle}_k(e_j)(x_i)$ into a $HFLTS$ by using the transformation function E_{G_H}. Denote the fuzzy envelope of $E_{G_H}(F^{cle}_k(e_j)(x_i))$ by a TFN u^k_{ij}.

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2) Compute the consensus degree based on the magnitudes of fuzzy envelopes for CLEs.

   (a) For each pair of experts t_l and t_k, compute a similarity matrix $SM_{lk} = (sm_{ij}^{lk})_{m\times n}$ in which $sm_{ij}^{lk} \in [0,1]$ represents the agreement level between t_l and t_k on alternative x_i with respect to parameter e_j, computed by

$$sm_{ij}^{lk} = 1 - |Mag(u_{ij}^{l}) - Mag(u_{ij}^{k})| \quad (9)$$

   (b) The consensus degree of all experts on alternative x_i with respect to parameter e_j is defined as

$$cp_{ij} = 1 - \frac{2}{f(f-1)} \sum_{k,l\in\{1,2,\dots,f\},k\neq l} |Mag(u_{ij}^{l}) - Mag(u_{ij}^{k})| \quad (10)$$

   (c) The consensus level between experts t_l and t_k ($t_l, t_k \in G$) should be computed by

$$cl(t_l,t_k) = \frac{1}{mn}\sum_{j=1}^{n}\sum_{i=1}^{m} sm_{ij}^{lk} \quad (11)$$

   (d) The group consensus level among the group $G = \{t_1, t_2, \dots t_f\}$ should be defined by

$$CL = \frac{2}{f(f-1)} \sum_{k,l\in\{1,2,\dots,f\},k\neq l} cl(t_l,t_k) \quad (12)$$

3) Consensus control.
   In this phrase, the consensus threshold μ established at the beginning is compared with the group consensus level, if the consensus is not enough, the assessments of some experts should be adjusted:
   if $CL > \mu$, the CRP ends and the selection process (steps 5-6 in Algorithm 2) is carried out;
   if $CL < \mu$, some experts are suggested to adjust their assessments.
   A number $Maxround \in N$ could be set a prior to determine the maximum number of adjusting rounds.

4) Advise Generation.

   (a) Compute θ_W-related collective HLE soft set:
   In this phase, θ_W-related collective HLE soft set $(\tilde{F}^{cle}, E)$ will be computed from (F_1^{cle}, E), (F_2^{cle}, E), $\dots$, (F_f^{cle}, E) (see Def. 18).
   The fuzzy envelope of $E_{G_H}(\tilde{F}^{cle}(e_j)(x_i))$ is denoted by $\tilde{u}_{ij}$, the HLE soft matrix corresponds to $(\tilde{F}^{cle}, E)$ is denoted by $\tilde{F}^{cle}$, then a proximity

matrix $P_k = (p^k_{pq})_{m\times n}$ between each expert t_k ($k \in \{1,2,\ldots,f\}$) and $\tilde{F}^{cle}$ could be obtained:

$$p^k_{ij} = 1 - |Mag(u^k_{ij}) - Mag(\tilde{u}_{ij})| \tag{13}$$

where $i \in \{1,2,\ldots,m\}$, $j \in \{1,2,\ldots,n\}$.
An average proximity corresponds to x_i and e_j should be computed by using an aggregation operator γ:

$$\overline{p}_{ij} = \gamma(p^1_{ij}, p^2_{ij}, \ldots, p^f_{ij}) \tag{14}$$

(b) Identify the assessments of experts to be changed:
If $CL \leq \mu$, there must exist (i, j), s.t. $cp_{ij} \leq \mu$, we should determine that the experts need to adjust their assessments, as well as the position (p,q) corresponds to x_p and e_q they should change. To do so,
   i. Determine position (p,q) by $cp_{pq} = min(cp_{ij})$ where $i \in \{1,2,\ldots,m\}$ and $j \in \{1,2,\ldots,n\}$.
   ii. Determine the experts to adjust assessments in position (p,q): Expert t_k who satisfies $p^k_{pq} < \overline{p}_{pq}$ should be suggested to modify their assessments on x_p with respect to e_q. Here, the experts who should change assessments can be not unique.

(c) Determine the change direction.
In this phase, a positive value close to zero denoted by ε will be adopted to define a margin of acceptability, and some direction rules are presented as follows:
   - If $(Mag(u^k_{pq}) - Mag(\tilde{u}_{pq})) < -\varepsilon$, then expert t_k should increase assessment on alternative x_p with respect to parameter e_q.
   - If $(Mag(u^k_{pq}) - Mag(\tilde{u}_{pq})) > \varepsilon$, then expert t_k should decrease assessment on alternative x_p with respect to parameter e_q.
   - If $-\varepsilon \leq (Mag(u^k_{pq}) - Mag(\tilde{u}_{pq})) \leq \varepsilon$, then expert t_k should not change assessment on alternative x_p with respect to parameter e_q.

An example to illustrative the GDM scheme cooperates with the proposed consensus model:

Example 5. *Suppose that there are four experts $G = \{t_1,t_2,t_3,t_4\}$ who provides evaluations on alternatives in $U = \{x_1,x_2,x_3,x_4\}$ to determine the best one. Parameters $E = \{e_1,e_2,\ldots,e_5\}$ are considered and the evaluations form four HLE soft sets, in which CLEs are built from a linguistic term set $S = \{s_0,s_1,\ldots,s_8\}$ (Fig. 1).*

In the CRP, consensus threshold $\mu = 0.9$ and margin of acceptability $\varepsilon = 0.1$, maximum adjusting rounds $Maxround = 10$. We use average aggregation as the operator γ.

Step 1. *Collect the CLEs provided by experts.*
Assessments of each expert $t_k \in G$ form a HLE soft set (F_k^{cle}, E). Corresponding HLE soft matrices are denoted by $F_1^{cle}, F_2^{cle}, \ldots, F_4^{cle}$ (see Eqs. (15)-(18)).

$$F_1^{cle} = \begin{pmatrix} \text{between } s_2 \text{ and } s_4 & \text{at least } s_5 & \text{at most } s_3 & \text{at most } s_4 & \text{between } s_2 \text{ and } s_4 \\ \text{at least } s_4 & \text{between } s_3 \text{ and } s_6 & s_5 & \text{at least } s_4 & \text{at least } s_5 \\ \text{between } s_2 \text{ and } s_5 & \text{at most } s_3 & s_3 & \text{at least } s_5 & s_0 \\ \text{at most } s_3 & \text{at least } s_5 & \text{at most } s_3 & \text{between } s_2 \text{ and } s_4 & \text{at least } s_4 \end{pmatrix} \quad (15)$$

$$F_2^{cle} = \begin{pmatrix} \text{between } s_4 \text{ and } s_6 & \text{at least } s_5 & s_2 & \text{at most } s_4 & \text{between } s_2 \text{ and } s_4 \\ \text{between } s_4 \text{ and } s_5 & \text{between } s_3 \text{ and } s_6 & s_5 & \text{between } s_4 \text{ and } s_5 & \text{at least } s_5 \\ s_2 & \text{at most } s_3 & s_0 & \text{at least } s_5 & s_0 \\ \text{between } s_2 \text{ and } s_3 & \text{at least } s_5 & \text{at most } s_3 & \text{between } s_2 \text{ and } s_4 & s_8 \end{pmatrix} \quad (16)$$

$$F_3^{cle} = \begin{pmatrix} \text{between } s_4 \text{ and } s_5 & \text{at least } s_5 & s_2 & \text{at most } s_4 & \text{between } s_2 \text{ and } s_4 \\ \text{between } s_4 \text{ and } s_5 & \text{between } s_3 \text{ and } s_6 & s_0 & \text{at least } s_5 & \text{at least } s_5 \\ s_8 & \text{at most } s_3 & \text{at most } s_3 & \text{at least } s_5 & s_0 \\ \text{at least } s_5 & \text{at least } s_5 & \text{at most } s_3 & \text{between } s_2 \text{ and } s_4 & \text{at least } s_4 \end{pmatrix} \quad (17)$$

$$F_4^{cle} = \begin{pmatrix} \text{between } s_4 \text{ and } s_6 & \text{at least } s_5 & s_2 & \text{at most } s_4 & \text{between } s_2 \text{ and } s_4 \\ \text{between } s_4 \text{ and } s_5 & \text{between } s_3 \text{ and } s_6 & s_8 & \text{at least } s_5 & \text{between } s_2 \text{ and } s_3 \\ \text{between } s_3 \text{ and } s_4 & \text{between } s_2 \text{ and } s_3 & \text{at most } s_3 & \text{at least } s_5 & \text{between } s_2 \text{ and } s_3 \\ \text{at most } s_3 & \text{at least } s_5 & \text{at most } s_3 & \text{between } s_2 \text{ and } s_4 & \text{at least } s_5 \end{pmatrix} \quad (18)$$

Step 2. *Choose a CLE-OWA operator θ_W of dimension f.*
Here we carry out the unitor function[1] from $[0,1]$ to S to construct the weighting vector W for aggregating CLEs built from S, and obtain $W = \{s_0, s_3, s_6, s_8\}$.

[1]Assume $L = \{L_1, \ldots, L_m\}$, an ordinal unitor function is a mapping defined in Yager (1995) as $H : [0,1] \longrightarrow L$ s.t. $H(r) = L_i$, $\frac{i-1}{m} \le r \le \frac{i}{m}$, $i = 1, \ldots, m$ and $H(1) = L_m$.

The reason why we choose this weighting vector is that it seems to be the analog of the normative aggregation, $w_i = 1/n$, in the numeric case. The weights in W of dimension n was computed by Yager (1995):

$$w_j = H(\frac{j-1}{n-1}), \qquad j = 1, \ldots, n.$$

In this case, the number of linguistic values in S is 9, and the dimension of W is $n = 4$. Here the unit interval is divided into 9 pieces and maps onto unitor function H. Then we get
$w_1 = H(\frac{0}{3}) = s_0$; $w_2 = H(\frac{1}{3}) = s_3$ *(since* $\frac{3}{9} \leq r < \frac{4}{9}$*);* $w_3 = H(\frac{2}{3}) = s_6$ *(since* $\frac{6}{9} \leq r < \frac{7}{9}$*);* $w_4 = H(\frac{3}{3}) = s_8$.

Step 3. *Carry out the CRP.*

1) *Compute the consensus degree based on the magnitudes of fuzzy envelopes for CLEs.*

   (a) *Transform the CLEs into $HFLTSs$ (see Eqs. (19)-(22)).*

$$F_1^{cle} = \begin{pmatrix} \{s_2,s_3,s_4\} & \{s_5,s_6,s_7,s_8\} & \{s_0,s_1,s_2,s_3\} & \{s_0,s_1,s_2,s_3,s_4\} & \{s_2,s_3,s_4\} \\ \{s_4,s_5,s_6,s_7,s_8\} & \{s_3,s_4,s_5,s_6\} & s_5 & \{s_4,s_5,s_6,s_7,s_8\} & \{s_5,s_6,s_7,s_8\} \\ \{s_2,s_3,s_4,s_5\} & \{s_0,s_1,s_2,s_3\} & s_3 & \{s_5,s_6,s_7,s_8\} & s_0 \\ \{s_0,s_1,s_2,s_3\} & \{s_5,s_6,s_7,s_8\} & \{s_0,s_1,s_2,s_3\} & \{s_2,s_3,s_4\} & \{s_4,s_5,s_6,s_7,s_8\} \end{pmatrix} \quad (19)$$

$$F_2^{cle} = \begin{pmatrix} \{s_4,s_5,s_6\} & \{s_5,s_6,s_7,s_8\} & s_2 & \{s_0,s_1,s_2,s_3,s_4\} & \{s_2,s_3,s_4\} \\ \{s_4,s_5\} & \{s_3,s_4,s_5,s_6\} & s_5 & \{s_4,s_5\} & \{s_5,s_6,s_7,s_8\} \\ s_2 & \{s_0,s_1,s_2,s_3\} & s_0 & \{s_5,s_6,s_7,s_8\} & s_0 \\ \{s_2,s_3\} & \{s_5,s_6,s_7,s_8\} & \{s_0,s_1,s_2,s_3\} & \{s_2,s_3,s_4\} & s_8 \end{pmatrix} \quad (20)$$

$$F_3^{cle} = \begin{pmatrix} \{s_4,s_5\} & \{s_5,s_6,s_7,s_8\} & s_2 & \{s_0,s_1,s_2,s_3,s_4\} & \{s_2,s_3,s_4\} \\ \{s_4,s_5\} & \{s_3,s_4,s_5,s_6\} & s_0 & \{s_5,s_6,s_7,s_8\} & \{s_5,s_6,s_7,s_8\} \\ s_8 & \{s_0,s_1,s_2,s_3\} & \{s_0,s_1,s_2,s_3\} & \{s_5,s_6,s_7,s_8\} & s_0 \\ \{s_5,s_6,s_7,s_8\} & \{s_5,s_6,s_7,s_8\} & \{s_0,s_1,s_2,s_3\} & \{s_2,s_3,s_4\} & \{s_4,s_5,s_6,s_7,s_8\} \end{pmatrix} \quad (21)$$

$$F_4^{cle} = \begin{pmatrix} \{s_4,s_5,s_6\} & \{s_5,s_6,s_7,s_8\} & s_2 & \{s_0,s_1,s_2,s_3,s_4\} & \{s_2,s_3,s_4\} \\ \{s_4,s_5\} & \{s_3,s_4,s_5,s_6\} & s_8 & \{s_5,s_6,s_7,s_8\} & \{s_2,s_3\} \\ \{s_3,s_4\} & \{s_2,s_3\} & \{s_0,s_1,s_2,s_3\} & \{s_5,s_6,s_7,s_8\} & \{s_2,s_3\} \\ \{s_0,s_1,s_2,s_3\} & \{s_5,s_6,s_7,s_8\} & \{s_0,s_1,s_2,s_3\} & \{s_2,s_3,s_4\} & \{s_5,s_6,s_7,s_8\} \end{pmatrix} \quad (22)$$

   (b) *Compute the fuzzy envelopes of $HFLTSs$ (see Eqs. (23)-(26)).*

$$F_1^{cle} = \begin{pmatrix} T(0,0.30,0.36,0.67) & T(0.5,0.85,1,1) & T(0,0,0.15,0.5) & T(0,0,0.35,0.67) & T(0,0.30,0.36,0.67) \\ T(0.33,0.65,1,1) & T(0.17,0.43,0.73,1) & T(0.5,0.67,0.67,0.83) & T(0.33,0.65,1,1) & T(0.5,0.85,1,1) \\ T(0,0.27,0.57,0.83) & T(0,0,0.15,0.5) & T(0.17,0.33,0.33,0.5) & T(0.5,0.85,1,1) & T(0,0,0,0) \\ T(0,0,0.15,0.5) & T(0.5,0.85,1,1) & T(0,0,0.15,0.5) & T(0,0.30,0.36,0.67) & T(0.33,0.65,1,1) \end{pmatrix} \quad (23)$$

$$F_2^{cle} = \begin{pmatrix} T(0.33,0.64,0.7,1) & T(0.5,0.85,1,1) & T(0,0.17,0.17,0.33) & T(0,0,0.35,0.67) & T(0,0.30,0.36,0.67) \\ T(0.33,0.5,0.67,0.83) & T(0.17,0.43,0.73,1) & T(0.5,0.67,0.67,0.83) & T(0.33,0.5,0.67,0.83) & T(0.5,0.85,1,1) \\ T(0,0.17,0.17,0.33) & T(0,0,0.15,0.5) & T(0,0,0,0) & T(0.5,0.85,1,1) & T(0,0,0,0) \\ T(0,0.17,0.33,0.5) & T(0.5,0.85,1,1) & T(0,0,0.15,0.5) & T(0,0.30,0.36,0.67) & T(1,1,1,1) \end{pmatrix} \quad (24)$$

$$F_3^{cle} = \begin{pmatrix} T(0.33,0.5,0.67,0.83) & T(0.5,0.85,1,1) & T(0,0.17,0.17,0.33) & T(0,0,0.35,0.67) & T(0,0.30,0.36,0.67) \\ T(0.33,0.5,0.67,0.83) & T(0.17,0.43,0.73,1) & T(0,0,0,0) & T(0.5,0.85,1,1) & T(0.5,0.85,1,1) \\ T(1,1,1,1) & T(0,0,0.15,0.5) & T(0,0,0.15,0.5) & T(0.5,0.85,1,1) & T(0,0,0,0) \\ T(0.5,0.85,1,1) & T(0.5,0.85,1,1) & T(0,0,0.15,0.5) & T(0,0.30,0.36,0.67) & T(0.33,0.65,1,1) \end{pmatrix} \quad (25)$$

$$F_4^{cle} = \begin{pmatrix} T(0.33,0.64,0.7,1) & T(0.5,0.85,1,1) & T(0,0.17,0.17,0.33) & T(0,0,0.35,0.67) & T(0,0.30,0.36,0.67) \\ T(0.33,0.5,0.67,0.83) & T(0.17,0.43,0.73,1) & T(1,1,1,1) & T(0.5,0.85,1,1) & T(0,0.17,0.33,0.5) \\ T(0.17,0.33,0.5,0.67) & T(0,0.17,0.33,0.5) & T(0,0,0.15,0.5) & T(0.5,0.85,1,1) & T(0,0.17,0.33,0.5) \\ T(0,0,0.15,0.5) & T(0.5,0.85,1,1) & T(0,0,0.15,0.5) & T(0,0.30,0.36,0.67) & T(0.5,0.85,1,1) \end{pmatrix} \quad (26)$$

2) *By using Eqs. (9), (11) and (12), the consensus degree of the group is* $CL = 0.868$.
3) *Consensus control.*
   Since $CL = 0.868 < \mu$*, we go to next step.*
4) *Advise Generation.*
   (a) *Compute* θ_W-*related collective HLE soft set* $(\tilde{F}^{cle}, E)$ *comes from HLE soft sets* (F_1^{cle}, E), (F_2^{cle}, E), $\ldots$, (F_f^{cle}, E). *Corresponding HLE soft matrix is denoted by* $\tilde{F}^{cle}$ *(see Eq. (27)).*

$$\tilde{F}^{cle} = \begin{pmatrix} \text{between } s_4 \text{ and } s_5 & \text{at least } s_5 & s_2 & \text{at most } s_4 & \text{between } s_2 \text{ and } s_4 \\ \text{between } s_4 \text{ and } s_5 & \text{between } s_3 \text{ and } s_6 & s_5 & \text{at least } s_4 & s_6 \\ \text{between } s_3 \text{ and } s_4 & \text{at most } s_3 & \text{at most } s_3 & \text{at least } s_5 & s_0 \\ \text{between } s_2 \text{ and } s_3 & \text{at least } s_5 & \text{at most } s_3 & \text{between } s_2 \text{ and } s_4 & \text{at least } s_4 \end{pmatrix} \quad (27)$$

By using Eq. (13), proximity matrix $P_k = (p^k_{pq})_{i\times j}$ $(k = 1,2,3,4)$ *between each expert* t_k *and* $\tilde{F}^{cle}$ *could be obtained as:*

$$P_1 = \begin{pmatrix} 0.752 & 1 & 0.935 & 1 & 1 \\ 0.786 & 1 & 1 & 1 & 0.935 \\ 0.997 & 1 & 0.773 & 1 & 1 \\ 0.854 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$P_2 = \begin{pmatrix} 0.915 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0.786 & 0.935 \\ 0.753 & 1 & 0.896 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0.798 \end{pmatrix}$$

$$P_3 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0.331 & 0.902 & 0.935 \\ 0.416 & 1 & 1 & 1 & 1 \\ 0.354 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$P_4 = \begin{pmatrix} 0.915 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0.669 & 0.902 & 0.419 \\ 1 & 0.854 & 1 & 1 & 0.75 \\ 0.854 & 1 & 1 & 1 & 0.902 \end{pmatrix}$$

(b) *Identify the assessments of experts to be changed:*

  A. *Determine the position* (p,q)*: since* $cp_{23} = min(cp_{ij}) = 0.5$ *where* $i \in \{1,2,\ldots,m\}$, $j \in \{1,2,\ldots,n\}$*, we know the assessment on* x_2 *with respect to* e_3 *should be changed.*

  B. *Determine the expert to adjust ssessments:*
  By Eq.(14), we obtain $\overline{p}_{23} = 0.75$*. Since* $p^3_{23} = 0.331 < \overline{p}_{23}$, $p^4_{23} = 0.669 < \overline{p}_{23}$, t_3 *and* t_4 *should modify assessments on* x_2 *with respect to* e_3*.*

(c) *Determine the change direction.*
In this phase, $\varepsilon = 0.1$ *is the margin of acceptability, and:*

  - $Mag(u^3_{23}) - Mag(\tilde{u}_{23}) = 0 - 0.669 < -\varepsilon$*, then* t_3 *should increase assessment on* x_2 *with respect to* e_3*.*
  - $Mag(u^4_{23}) - Mag(\tilde{u}_{23}) = 1 - 0.669 > \varepsilon$*, then* t_4 *should decrease assessment on* x_2 *with respect to* e_3*.*

(d) *Suppose that expert* t_3 *increase assessment to "between* s_3 *and* s_6*", and expert* t_4 *decrease assessment to "*s_4*", the new consensus degree is computed as 0.888 which has still not reach the threshold* 0.90*, turn to second round.*

5) Consensus reaching.
In the second round, expert t_3 is suggested to decrease his/her assessment on x_4 with respective to e_1. Suppose that t_3 decreases this assessment to "between s_3 and s_4", then the new group consensus degree becomes $0.903 > \mu$, the CRP ends.

Step 4. *The adjusted assessments provided by experts after CRP form HLE soft sets (F'^{cle}_1, E), (F'^{cle}_2, E), ..., (F'^{cle}_f, E). Now the new θ_W-related collective HLE soft set $(\tilde{F}'^{cle}, E)$ is computed from them, and its corresponding HLE soft matrix is denoted by $\tilde{F'}^{cle}$ (see Eq. (28)).*

$$\tilde{F'}^{cle} = \begin{pmatrix} \text{between } s_4 \text{ and } s_5 & \text{at least } s_5 & s_2 & \text{at most } s_4 & \text{between } s_2 \text{ and } s_4 \\ \text{between } s_4 \text{ and } s_5 & \text{between } s_3 \text{ and } s_6 & \text{between } s_3 \text{ and } s_6 & \text{at least } s_4 & s_6 \\ \text{between } s_3 \text{ and } s_4 & \text{at most } s_3 & \text{at most } s_3 & \text{at least } s_5 & s_0 \\ \text{between } s_2 \text{ and } s_3 & \text{at least } s_5 & \text{at most } s_3 & \text{between } s_2 \text{ and } s_4 & \text{at least } s_4 \end{pmatrix} \tag{28}$$

Step 5. *Based on $(\tilde{F}'^{cle}, E)$, we obtain the scores of alternatives as $S_1 = -0.726$, $S_2 = 5.746$, $S_3 = -4.380$, $S_4 = -0.635$.*

Step 6. *The final decision is alternative x_2.*

6. Comparative study

From our knowledge, there is only one exiting algorithm (Algorithm 1 in Sun et al. (2017), denoted by Sun et al.'s algorithm) proposed for dealing with linguistic GDM problems under the framework of soft set theory, that is based on the introduction of a model called linguistic value soft set. A HLE soft set will degenerate to a linguistic value soft set when all CLEs in it degenerate to LTs. Therefore, Algorithm 2 based on HLE soft set is also able to deal with situations in which experts' assessments are LTs. To carry out a comparison study between Algorithm 2 and Sun et al.'s algorithm, in this section, we will present a linguistic GDM problem in which experts' assessments are LTs, and afterwards deal with the problem by using Algorithm 2 and Sun et al.'s algorithm separately. Based on different decision results obtained from these two approaches, we will show the advantage of our proposal.

6.1. Decision making problem

Suppose that experts $G = \{t_1, t_2, t_3, t_4\}$ provide assessments on alternatives $U = \{x_1, x_2, x_3, x_4\}$ by using LTs in a linguistic term set $S = \{s_0, s_1, \ldots, s_8\}$ in order to determine the best one. The parameters considered are $E = \{e_1, e_2, \ldots, e_5\}$

and the information provided by experts form four linguistic value soft sets (see Eqs. (29)-(32) as corresponding linguistic value soft matrices [2]).

$$F_1^L = \begin{pmatrix} s_6 & s_5 & s_4 & s_0 & s_8 \\ s_8 & s_1 & s_2 & s_5 & s_2 \\ s_3 & s_8 & s_2 & s_0 & s_2 \\ s_2 & s_8 & s_3 & s_1 & s_3 \end{pmatrix} \tag{29}$$

$$F_2^L = \begin{pmatrix} s_5 & s_5 & s_4 & s_0 & s_8 \\ s_8 & s_1 & s_2 & s_0 & s_1 \\ s_5 & s_8 & s_2 & s_0 & s_2 \\ s_2 & s_8 & s_3 & s_1 & s_4 \end{pmatrix} \tag{30}$$

$$F_3^L = \begin{pmatrix} s_6 & s_8 & s_4 & s_0 & s_1 \\ s_8 & s_1 & s_2 & s_0 & s_1 \\ s_8 & s_8 & s_2 & s_0 & s_2 \\ s_2 & s_8 & s_3 & s_1 & s_4 \end{pmatrix} \tag{31}$$

$$F_4^L = \begin{pmatrix} s_8 & s_6 & s_4 & s_0 & s_8 \\ s_8 & s_6 & s_2 & s_0 & s_1 \\ s_3 & s_8 & s_2 & s_0 & s_2 \\ s_2 & s_8 & s_3 & s_1 & s_4 \end{pmatrix} \tag{32}$$

6.2. Deal with the problem by using Sun et al.'s algorithm in Sun et al. (2017).

The linguistic term set S used in Sun et al. (2017) is symmetric about a middle term s_0, however to facilitate comparison here we make tiny adjustments by setting the middle term as s_4 and smallest term as s_0, the adjustments will not cause perturbation to the application.

Denote the parameter set considered by expert t_j by t_j^E, since the parameters considered by all experts are the same, the choice value matrices (see Def. 11 in Sun et al. (2017)), $C_{(t_k^E, \cap_{j=1,j\neq k}^{|G|} t_j^E)}(k = 1,2,3,4)$, are always

[2] The definition of linguistic value soft metric is provided in Sun et al. (2017).

$$C_{(t_k^E, \cap_{j=1, j\neq k}^{|G|} t_j^E)} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Transformations $-1 \triangleq s_0$ and $1 \triangleq s_8$ will be adopted to handle the choice value matrix to perform the product operation (see Def. 12 in Sun et al. (2017)) with linguistic value soft sets. For each expert t_k $(k = 1, 2, 3, 4)$, the product operation result could be achieved as

$$P_k^L = F_k^L \otimes C_{(t_k^E, \cap_{j=1, j\neq k}^{|G|} t_j^E)} = \begin{pmatrix} s_8 & s_8 & s_8 & s_8 & s_8 \\ s_8 & s_8 & s_8 & s_8 & s_8 \\ s_8 & s_8 & s_8 & s_8 & s_8 \\ s_8 & s_8 & s_8 & s_8 & s_8 \end{pmatrix}.$$

Meanwhile, the weights of the four experts can be easily computed as the same, that is, $W = (w_1, w_2, w_3, w_4) = (0.25, 0.25, 0.25, 0.25)$ (see Def. 13 in Sun et al. (2017)), then the result of the weighted sum is

$$P^L = \sum_{k=1}^{|G|} w_k P_k^L = \begin{pmatrix} s_8 & s_8 & s_8 & s_8 & s_8 \\ s_8 & s_8 & s_8 & s_8 & s_8 \\ experts_8 & s_8 & s_8 & s_8 & s_8 \\ s_8 & s_8 & s_8 & s_8 & s_8 \end{pmatrix}.$$

According to Sun et al's algorithm the decision will be made based on P^L. However, it can be observed that the aggregated result P^L is far from the evaluations of each expert. For instance, assessment on x_2 with respect to e_3 provided by every expert is always s_2, whereas the aggregation result in P^L is s_8, which is far from assessment of the majority. Here, we can not make any decision from P^L since the ranking function (see Def. 14 in Sun et al. (2017)) for all alternatives are the same, that is, $R_{F^L}(x_i) = s_8$, $i = 1, 2, 3, 4$.

6.3. Deal with the problem by using Algorithm 2.

Here we adopt the same weighting vector used in Example 5 to carry out the aggregation of HLE soft sets. The GDM process is similar to Example 5, so only a brief description will be provided here:

The initial consensus degree among experts is 0.891. During the consensus process, in the first round t_4 will be suggested to increase the assessment on alternative x_1 with respect to parameter e_5. At this moment, the group assessment on x_1 with respect to e_5 is s_6, whereas the assessment of t_4 is s_1. Suppose that t_4 increases it from s_1 to s_6, the new consensus degree reaches 0.912.

After the CRP, the HLE soft matrix $\tilde{F}'^{cle}$ corresponds to θ_W-collective HLE soft set $(\tilde{F}'^{cle}, E)$ is obtained as Eq. (33), from which the scores of alternatives are $S_1 = 3.826$, $S_2 = -4.699$, $S_3 = -2.684$, $S_4 = 3.557$. Obviously the optimal alternative is x_1.

$$\tilde{F}'^{cle} = \begin{pmatrix} s_6 & s_5 & s_4 & s_0 & s_6 \\ s_8 & s_1 & s_2 & s_0 & s_1 \\ s_3 & s_8 & s_2 & s_0 & s_2 \\ s_2 & s_8 & s_3 & s_1 & s_4 \end{pmatrix} \tag{33}$$

6.4. Comparison analysis

Based on the decision result obtained from Algorithm 2 and Sun et al.'s algorithm, now we present a comparison analysis between them.

1. It is shown by the above example that there exist GDM problems which can not be handled by Sun et al' algorithm, however can be solved by Algorithm 2.
2. Sun et al's algorithm is proposed based on linguistic value soft set, while Algorithm 2 is based on HLE soft set. It is determined by the models that Sun et al's algorithm can only be applied when experts' assessments are LTs, whereas Algorithm 2 can be applied when assessments could be both CLEs and LTs.
3. Although the concept of consensus has been mentioned in Sun et al. (2017), experts' assessments have not been adjusted to get closer to the majority to ensure a group consensus in Sun et al' algorithm. In Algorithm 2, by introducing a consensus model, the assessments provided by experts farthest from the majority has been adjusted, the goal of consensus has been reached.
4. In the computation process of Sun et al's algorithm, virtual terms have been applied (see example in Section 4.3 in Sun et al. (2017)), which are actually not linguistic values (no syntax) (Rodríguez & Martínez (2013)), and don't follow the fuzzy linguistic approach. In the computation process of Algorithm 2, only linguistic values have been applied with the help of their fuzzy representations.

Table 7 is provided to summarize the comparison. From comparison analysis we conclude that Algorithm 2 goes beyond Sun et al's algorithm.

Table 7: Comparison between Algorithm 2 and Sun et al's algorithm

Method	Model	Assessments	Consensus	Decision result
Sun et al' algorithm	Linguistic value soft set	LTs	No consensus	No result in some situations
Algorithm 2	HLE soft set	CLEs (contain LTs)	Consensus	Result obtained

7. Conclusion

This paper introduces a new generalization of soft set called HLE soft set. It is an extension of linguistic value soft set which overcomes its limitations in eliciting complex linguistic information in hesitant DM settings. Based on HLE soft sets, we provide a multi-criteria DM algorithm and afterwards a GDM algorithm. Remarkably, a novel consensus model based on HLE soft set supports the GDM scheme. Through numerical examples the effectiveness and feasibility of the proposed algorithms are shown.

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Appendix A: Adjustment for the approach for computing fuzzy envelops of $HFLTSs$ corresponds to CLEs in HLE soft sets

In HLE soft sets, suppose that CLEs are built from a linguistic term set $S=\{s_0,s_1,\ldots,s_g\}$, where $s_0 =$ “$none''$ and $s_g =$ “$absolute''$ and $g+1$ is the granularity. LTs $s_k \in S$ are defined by trapezoidal (triangular) membership functions $A^k = T(a_L^k, a_M^k, a_M^k, a_R^k)$, $k = 0,1,\ldots,g$. The semantic for LT *none* is defined as $T(0,0,0,0)$ and LT $absolute$ is defined as $T(1,1,1,1)$.

When LTs “$none''$ and “$absolute''$ are considered, for CLEs built from S, scheme for computing their fuzzy envelopes in Liu & Rodríguez (2014) could be adjusted as follows:

1. Fuzzy envelope for CLE “at least s_i”.
   (1) If $s_1 \le s_i \le s_{g-1}$, the fuzzy envelope for “at least s_i” could be computed as following.
      i. Obtain the elements to aggregate.
         Assume that LTs in S are defined as a triangular membership functions, the set of elements to aggregate is
         $T = \{a_L^i, a_M^i, a_L^{i+1}, a_R^i, a_M^{i+1}, a_L^{i+2}, a_R^{i+1}, \ldots, a_L^g, a_R^{g-1}, a_M^g, a_R^g\}$,
         From fuzzy partitions in Ruspini (1969) it can be simplified as
         $T = \{a_L^i, a_M^i, a_M^{i+1}, \ldots, a_M^{g-1}, a_R^{g-1}\}$.
      ii. Compute the parameters of the trapezoidal fuzzy membership function.
         A trapezoidal fuzzy membership function $F_{H_S} = T(a,b,c,d)$ is used as the envelope of $HFLTS$, H_S, transformed from “at least s_i”, where a and d can be easily computed by the *min* and *max* operators, i.e.,
         $a = min\{a_L^i, a_M^i, \ldots, a_M^{g-1}, a_R^{g-1}\} = a_L^i$,
         $d = max\{a_L^i, a_M^i, \ldots, a_M^{g-1}, a_R^{g-1}\} = a_R^{g-1}$,

at the same time, b and c are obtained by aggregating the remaining elements $a_M^i, a_M^{i+1}, \ldots, a_M^{g-1}$ with *OWA* operators, i.e.,

$$b = OWA_{W^2}(a_M^i, a_M^{i+1}, \ldots, a_M^{g-1}), \tag{15}$$

$$c = OWA_{W^2}(a_M^i, a_M^{i+1}, \ldots, a_M^{g-1}) \tag{16}$$

where W^2 will be given in iii.

iii. Obtain the *OWA* weights.
The *OWA* weights in Fileva & Yagerb (1998) will be adopted in the current research to reflect importance of different LTs.
b and c are computed by W^2 with $n = g - i$, i.e. $W^2 = (w_1^2, w_2^2, \ldots, w_{g-i}^2)$, where
$w_1^2 = \alpha^{g-i-1}$, $w_2^2 = (1-\alpha)\alpha^{g-i-2}$, $w_3^2 = (1-\alpha)\alpha^{g-i-3}$, $\ldots$, $w_{g-i-1}^2 = (1-\alpha)\alpha$, $w_{g-i}^2 = 1-\alpha$.
c is computed by W^2 with $\alpha = 1$, so $c = a_M^{g-1}$.

iv. Obtain the fuzzy envelope.
For the $HFLTS$, H_S, from the CLE "at least s_i", its fuzzy envelope F_{H_S} is defined as a TFN $T(a_L^i, b, a_M^{g-1}, a_R^{g-1})$, where b is computed by Eq. (15).

Remark 1 An approach to determine α for computing b:
Let us consider the value of α to compute b, it should support the properties:
(a) $0 = a_M^1 \leq a_M^i \leq b \leq a_M^{g-1} = 1$;
(b) For a fixed s_i in "at least s_i", if $\alpha \to 0$, then $b \to a_M^i$, if $\alpha > 0$, then $b > a_M^i$, if $\alpha \to 1$, then $b \to a_M^{g-1}$;
(c) If $s_i \to s_1$, then $\alpha \to 0$ and $b \to a_M^1 = 0$;
(d) If $s_i \to s_{g-1}$, then $\alpha \to 1$ and $b \to a_M^{g-1} = 1$.
The value α increases from 0 to 1 as s_i increases from s_1 to s_{g-1}. That is, α depends on the index of s_i. To compute α, a linear function is defined as
$f_1(i) = \beta i + r$, s.t. $\alpha = f_1(i)$,
which satisfies the boundary conditions
$f_1(1) = 0$, $f_1(g-1) = 1$
the form of f_1 can be obtained as:
$f_1(i) = \frac{i-1}{g-2}$, i.e., $\alpha = \frac{i-1}{g-2}$.

(2) If $s_i = s_0$, consider that s_0 can be regarded as an inside term of s_1, the fuzzy envelop for the CLE "at least s_0" is the same as the fuzzy envelope for the CLE "at least s_1" which can be computed by steps in (1) with the values of $\alpha = \frac{i-1}{g-2}$ by Remark 1.

(3) If $s_i = s_g$, the fuzzy envelop for the CLE "at least s_g" will be $T(1,1,1,1)$.

2. Fuzzy envelope for the CLE "at most s_i".
When "$none''$" and "$absolute''$" are taken into consideration, the steps to achieve the fuzzy envelope for the CLE "at most s_i" in Liu & Rodríguez (2014) should be adjusted as below:

(1) If $s_1 \leq s_i \leq s_{g-1}$, the steps for computing the fuzzy envelope of CLE "at most s_i" in Liu & Rodríguez (2014) should be adjusted as follows.

i. Obtain the elements to aggregate.
The set of elements to aggregate is
$T = \{a_L^0, a_M^0, a_L^1, a_R^0, a_M^1, a_L^2, a_R^1, \ldots, a_L^i, a_R^{i-1}, a_M^i, a_R^i\}$
which can be simplified as
$T = \{a_L^1, a_M^1, a_M^2, \ldots, a_M^i, a_R^i\}$

ii. Compute the parameters of the trapezoidal fuzzy membership function.
A TFN $F_{H_S} = T(a,b,c,d)$ is used as the envelope of $HFLTS$, H_S, transformed from the CLE "at most s_i", where a and d can be computed as
$a = min\{a_L^1, a_M^1, a_M^2, \ldots, a_M^i, a_R^i\} = a_L^1$,
$d = max\{a_L^1, a_M^1, a_M^2, \ldots, a_M^i, a_R^i\} = a_R^i$,
b and c are obtained by aggregating the remaining elements $a_M^1, a_M^2, \ldots, a_M^i$ with OWA operators, i.e.,

$$b = OWA_{W^1}(a_M^1, a_M^2, \ldots, a_M^i), \tag{17}$$

$$c = OWA_{W^1}(a_M^1, a_M^2, \ldots, a_M^i) \tag{18}$$

where W^1 will be defined in iii.

iii. Obtain the OWA weights.
The weights used to compute b and c are in form of W^1 with $n = i$, i.e.
$W^1 = (w_1^1, w_2^1, \ldots, w_i^1)$, where
$w_1^1 = \alpha$, $w_2^1 = \alpha(1-\alpha)$, $w_3^1 = \alpha(1-\alpha)^2$, $\ldots$, $w_{i-1}^1 = \alpha(1-\alpha)^{i-2}$, $w_i^1 = (1-\alpha)^{i-1}$.
The weights W^1 used to compute b is with $\alpha = 0$, so $b = a_M^1$.

iv. Obtain the fuzzy envelope.
For the $HFLTS$, H_S, from the CLE "at least s_i", its fuzzy envelope F_{H_S} is defined as a TFN $T(a_L^1, a_M^1, c, a_R^i)$, where c is computed using Eq. (18).

Remark 2 An approach to determine α for computing c:
Let us consider the value of α to compute c, it should support the properties:
(a) $0 = a_M^1 \leq c \leq a_M^i \leq a_M^{g-1} = 1$;
(b) For a fixed s_i, if $\alpha \to 0$, then $c \to a_M^1$, if $\alpha > 0$, then $c > a_M^1$, if $\alpha \to 1$, then $c \to a_M^i$.
(c) If $s_i \to s_1$, then $\alpha \to 0$ and $c \to a_M^1 = 0$.
(d) If $s_i \to s_{g-1}$, then $\alpha \to 1$ and $c \to a_M^{g-1} = 1$.
The value of α increases from 0 to 1 as s_i increases from s_1 to s_{g-1}, it can be computed in a similar way as "at least s_i", i.e., $\alpha = \frac{i-1}{g-2}$.

(2) If $s_i = s_0$, the fuzzy envelope for the CLE "at most s_0" will be $T(0,0,0,0)$.

(3) If $s_i = s_g$, consider that s_g can be regarded as an inside term of s_{g-1}, the fuzzy envelope for "at most s_g" is the same as the fuzzy envelope for "at most s_{g-1}" which can be computed by steps in (1) with $\alpha = \frac{i-1}{g-2}$ by Remark 2.

3. Fuzzy envelope for the CLE "between s_i and s_j".
   (1) If $s_1 \leq s_i \leq s_j \leq s_{g-1}$, the approach of computing the fuzzy envelope for the CLE "between s_i and s_j" should follow steps in Liu & Rodríguez (2014).
   A TFN $F_{H_S} = T(a,b,c,d)$ is used as the envelope of $HFLTS$, H_S, transformed from the CLE "between s_i and s_j", where a and d can be easily computed by the *min* and *max* operators, b and c are obtained by seperatly aggregating some of the remaining elements $a_M^i, a_M^{i+1}, \ldots, a_M^j$ with *OWA* operators. b is computed with weight W^2 which is related to a parameter α_1 and c is computed with weight W^1 which is related to a parameter α_2. Please see more details in Liu & Rodríguez (2014).
   However, the approach for computing α_1 and α_2 should be adjusted as follows:
   Remark 3 Let us consider the value of α_1 and α_2 to compute c and b. The method for computing values α_1 and α_2 in the weights W^2 and W^1 has to be adjusted consider the following two extreme cases:
   (a) if $j-i=1$, in this case there is no necessary to aggregate, α_1 should be set as 1 so that this assumption will not affect the result, since $b = \alpha_1 \times a_M^i = a_M^i$.
   (b) if $s_i \to s_1$ and $s_j \to s_{g-1}$, we have $j-i \to g-2$ and $\alpha_1 \to 0$.
   Thus, there exists a function $f_2 : [1, g-2) \to (0,1]$, so that $\alpha_1 = f_2(j-i)$, which satisfies boundary conditions $f_2(1) = 1$ and $f_2(g-2) = 0$.
   Here f_2 is also assumed as a linear function, i.e. $f_2(j-i) = \beta(j-i) + \gamma$, where β, γ are unknown parameters.
   f_2 can be obtained as: $f_2(j-i) = \frac{g-2-(j-i)}{g-3}$, where $i = \text{index}(s_i)$, $j = \text{index}(s_j)$, and $g+1$ is the granularity of $S = \{s_0, \ldots, s_g\}$.
   Therefore, α_1 is defined by $\alpha_1 = \frac{g-2-(j-i)}{g-3}$, and $\alpha_2 = 1 - \alpha_1 = \frac{(j-i)-1}{g-3}$.
   (2) If $s_i = s_0$, consider that s_0 can be regarded as an inside term of s_1, the fuzzy envelop for the CLE "between s_0 and s_j" is the same as the fuzzy envelope for the CLE "between s_1 and s_j" which can be computed by the approach in Liu & Rodríguez (2014) with the values of α_1 and α_2 computed as in Remark 3.
   (3) If $s_j = s_g$, consider that s_g can be regarded as an inside term of s_{g-1}, the fuzzy envelop for the CLE "between s_i and s_g" is the same as the fuzzy envelope for the CLE "between s_i and s_{g-1}" which can be computed by the approach in Liu & Rodríguez (2014) with the values of α_1 and α_2 chosen as in Remark 3.

Appendix B: An approach for ranking trapezoidal fuzzy numbers.

A ranking approach of TFNs based on magnitude in Abbasbandy & Hajjari (2009) will be briefly recalled here.

Firstly, we recall the parametric form of fuzzy numbers presented in Ma et al. (1999) which was considered in Abbasbandy & Hajjari (2009):

Definition 19. *(Ma et al. (1999)) A fuzzy number u in parametric form is a pair* $(\underline{u}, \overline{u})$ *of functions* $\underline{u}(r)$, $\overline{u}(r)$, $0 \leq r \leq 1$, *which satisfies the following requirements:*
1. $\underline{u}(r)$ *is a bounded monotonic increasing left continuous function,*
2. $\overline{u}(r)$ *is a bounded monotonic decreasing right continuous function,*
3. $\underline{u}(r) \leq \overline{u}(r), 0 \leq r \leq 1.$

The trapezoidal fuzzy number $u = (x_0, y_0, \alpha, \beta)$ (Fig. 3), with two defuzzifiers x_0, y_0, and left fuzziness $\alpha > 0$ and right fuzziness $\beta > 0$ is a fuzzy set where the membership function is

$$u(x) = \begin{cases} \frac{1}{\alpha}(x - x_0 + \alpha), & x_0 - \alpha \leq x \leq x_0 \\ 1, & x \in [x_0, y_0] \\ \frac{1}{\alpha}(y_0 - x + \beta), & y_0 \leq x \leq y_0 + \beta \\ 0, & \text{otherwise,} \end{cases} \tag{19}$$

and its parametric form is $\underline{u}(r) = x_0 - \alpha + \alpha r$, $\overline{u}(r) = y_0 + \beta - \beta r$. u is a triangular fuzzy number which can be written as $u = (x_0, \alpha, \beta)$ if it is provided that $x_0 = y_0$.

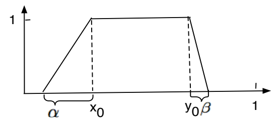

Figure 3: Trapezoidal fuzzy number $u = (x_0, y_0, \alpha, \beta)$.

Based on the parametric form of FNs, a measure called magnitudes of TFNs was introduced in Abbasbandy & Hajjari (2009) for the purpose of ranking TFNs:

For an arbitrary trapezoidal fuzzy number $u = (x_0, y_0, \alpha, \beta)$, with parametric form $u = (\underline{u}(r), \overline{u}(r))$, the magnitude of the trapezoidal fuzzy number is defined as

$$Mag(u) = \frac{1}{2}\Big(\int_0^1 (\underline{u}(r) + \overline{u}(r) + x_0 + y_0) f(r) dr\Big) \tag{20}$$

where the function $f(r)$ is a non-negative and increasing function on $[0,1]$ with $f(0) = 0$, $f(1) = 1$ and $\int_0^1 f(r)dr = \frac{1}{2}$. Function $f(r)$ can be chosen according to the actual situation. In this paper we use $f(r) = r$, following the way in Abbasbandy & Hajjari (2009).

The rule for ranking TFNs is the larger $Mag(u)$, the larger the fuzzy number, formalized as: for any two trapezoidal fuzzy numbers u and v, their ranking is determined by:

- $u \prec v$ iff $Mag(u) < Mag(v)$,

- $u \succ v$ iff $Mag(u) > Mag(v)$,
- $u \sim v$ iff $Mag(u) = Mag(v)$,

then the order $u \preceq v$, $u \succeq v$ can be formulated as

- $u \preceq v$ iff $u \prec v$ or $u \sim v$,
- $u \succeq v$ iff $u \succ v$ or $u \sim v$.

4.4 Type-2 fuzzy envelope for HFLTSs and its application to multi-criteria decision making

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Type-2 fuzzy envelope of hesitant fuzzy linguistic term set: a new representation model of comparative linguistic expressions

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***Abstract*—The use of hesitant fuzzy linguistic term sets contributes to the elicitation of comparative linguistic expressions in decision contexts when experts hesitate among different linguistic terms to provide their assessments. Since the existing representation models for linguistic expressions based on hesitant fuzzy linguistic term sets do not consider properly the uncertainty caused by the inherent vagueness of such linguistic expressions, it is necessary to improve their modeling to cope with such vagueness. In this paper, we propose a new fuzzy envelope for the hesitant fuzzy linguistic term sets in form of type-2 fuzzy sets for representing comparative linguistic expressions. Such an envelope overcomes the limitation of existing representations in coping with inherent uncertainties and facilitates the processes of computing with words for linguistic decision making problems dealing with comparative linguistic expressions.**

***Index Terms*—Hesitant fuzzy linguistic term sets, Type-2 fuzzy sets, Envelope, Comparative linguistic expressions.**

I. Introduction

In many real world decision making situations the use of linguistic information is appropriate due to the qualitative aspects of the problem [26]. The application of linguistic information usually implies to carry out computing with words (CW) processes, which is defined as a methodology for reasoning, computing and decision making using linguistic information [24]. CW in decision making will enhance the reliability and flexibility of classical decision models, since it does not only makes the reasoning processes related to the decision making closer to human cognition, but also improves the resolution of decision making under uncertainty with linguistic information [15].

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One common approach that has provided successful and reliable results in linguistic decision making is the fuzzy linguistic approach [39], taking advantage of the fact that it represents the qualitative terms by means of linguistic variables rather than numerical values. The fuzzy linguistic approach [18], [39] facilitates the modeling of linguistic information to capture inherent language uncertainties. However, usually the use of the fuzzy linguistic approach is restricted to the elicitation of single and simple terms to express the information provided by the experts, which may not reflect exactly the expert's real assessment in a linguistic context with a high level of uncertainty, in which experts hesitate among multiple terms.

To overcome this limitation, several linguistic approaches have been introduced to elicit more elaborated linguistic expressions than single linguistic terms [25], including the proportional 2-tuple model that adopts the proportion of two consecutive linguistic terms [35], the linguistic model that merges different single linguistic terms into a new synthesized term [13], and the linguistic model built by logical connectives and fuzzy relations that measure the similarity between any two linguistic terms [33]. Although these proposals provide greater flexibility to elicit linguistic expressions in hesitant decision situations, it is noticed that the expressions generated by them are either far from common language used by experts in decision problems or lack of systematic formalization.

Recently, Rodríguez et al. introduced the concept of Hesitant Fuzzy Linguistic Term Set ($HFLTS$) [27], its application to CW processes improves previous approaches by eliciting comparative linguistic expressions (CLEs) that are closer to human beings' cognition based on context-free grammars, which formalize the generation of flexible linguistic expressions.

In CW the statement *"words mean different things for the different people"* has been studied and managed from different views such as the use of multi-granularity linguistic term sets in order to deal with multiple sources of linguistic information [7], [10], and the linguistic model based on type-2 fuzzy sets representation that represents the semantics of linguistic terms as type-2 membership functions [17], [34], [40].

Similarly, CLEs also mean different things to different people, fuzzy models can be used to capture the uncertainties of such expressions. For CW dealing with CLEs represented as $HFLTSs$, it is necessary to explore suitable fuzzy representations for $HFLTSs$. So far, two different representation models of $HFLTS$ have been developed to facilitate CW processes. In [27], [28] the computational linguistic model cooperates with the envelope of $HFLTS$ which is represented as linguistic intervals and the process of CW is finally accomplished with the help of a symbolic model, losing information. However, in [12] a fuzzy envelope for $HFLTS$ has been introduced whose representation is a type-1 fuzzy membership function obtained through the aggregation of the fuzzy membership functions of the linguistic terms contained in the $HFLTS$. Both envelopes for $HFLTS$ fail to reflect and deal with the fact that linguistic expressions mean different things for different people, therefore the semantics of CLEs generated by the context-free grammar and based on $HFLTS$ should be improved to overcome such a limitation.

Following the idea of type-2 fuzzy sets in CW with linguistic terms, in this paper we adopt interval type-2 fuzzy sets for representing the meaning of CLEs based on $HFLTSs$, by using the entropy of $HFLTS$ [38] to compute the uncertainties contained in such expressions. Such a proposal offers a new way of providing a wide and adaptive vocabulary for decision making problems dealing with linguistic information.

The remainder of the paper is structured as follows: Section 2 reviews the type-2 fuzzy set theory, the CLEs based on $HFLTSs$ and several related concepts. Section 3 proposes a type-2 fuzzy envelope for CLEs based on $HFLTSs$. Section 4 presents examples of the type-2 fuzzy envelope. In Section 5, a comparison between type-1 and type-2 fuzzy envelopes of $HFLTSs$ in decision making is provided. And finally, Section 6 provides several conclusions and future work.

II. Preliminaries

This section reviews some basis concepts of type-2 fuzzy sets (T2 FS), interval type-2 fuzzy sets (IT2 FS), the elicitation of CLEs based on $HFLTS$ with its type-1 fuzzy envelope and the entropy measure of $HFLTS$ that will be used to obtain the type-2 fuzzy envelope.

A. Type-2 fuzzy sets and interval type-2 fuzzy sets

Some basic concepts on T2 FS and IT2 FS are reviewed in this subsection. The background materials are mainly taken from [16], [19], [21], [22], adapted to the recommendations in [23].

A T2 FS, initiated by Zadeh [39] as an extension of an ordinary fuzzy set (type-1 fuzzy set), is denoted by $\tilde{A}$ and characterized by a type-2 membership function $\mu_{\tilde{A}}(x,u)$ [20], [21], where $x \in X$ and

$$\tilde{A} = \{((x,u), \mu_{\tilde{A}}(x,u)) | x \in X, u \in [0,1]\} \quad (1)$$

in which $0 \leq \mu_{\tilde{A}}(x,u)) \leq 1$.

If all $\mu_{\tilde{A}}(x,u) = 1$, the T2 FS $\tilde{A}$ turns to an IT2 FS (see Fig. 1), it is characterized as

$$\tilde{A} = \int_{x \in X} \int_{u \in J_x} 1/(x,u) = \int_{x \in X} [\int_{u \in J_x} 1/u]/x, \quad J_x \subseteq [0,1] \quad (2)$$

where x, the primary variable, has domain X; $u \in U$, the secondary variable, has domain J_x at each $x \in X$; J_x is called the primary membership of x and is defined in Eq. (6).

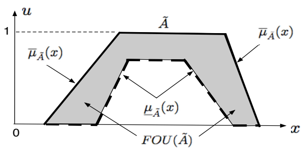

Fig. 1. FOU (shaded), LMF (dashed) and UMF (solid) for IT2 FS $\tilde{A}$

Note that Eq. (2) means: $\tilde{A} : X \rightarrow \{[a,b] : 0 \leq a \leq b \leq 1\}$. Uncertainty about IT2 FS $\tilde{A}$ is conveyed by a bounded region called the footprint of uncertainty (FOU), that is the aggregation of all primary memberships, i.e.,

$$FOU(\tilde{A}) = \{(x,u) : u \in J_x \subseteq [0,1]\} \quad (3)$$

The upper membership function (UMF) of $\tilde{A}$, denoted by $\overline{\mu}_{\tilde{A}}(x)$, $\forall x \in X$, and lower membership function (LMF) of $\tilde{A}$, denoted by $\underline{\mu}_{\tilde{A}}(x)$, $\forall x \in X$, are two type-1 membership functions that bound the FOU, i.e.

$$\overline{\mu}_{\tilde{A}}(x) = \ sup\{u | u \in [0,1], \mu_{\tilde{A}}(x,u) > 0\}, \quad \forall x \in X \quad (4)$$

$$\underline{\mu}_{\tilde{A}}(x) = inf\{u | u \in [0,1], \mu_{\tilde{A}}(x,u) > 0\}, \quad \forall x \in X \quad (5)$$

Note that J_x is an interval set, i.e.

$$J_x = \{(x,u) : u \in [\underline{\mu}_{\tilde{A}}(x), \overline{\mu}_{\tilde{A}}(x)]\} \quad (6)$$

$FOU(\tilde{A})$ in Eq. (3) can also be expressed as

$$FOU(\tilde{A}) = \{(x,u) : x \in X, u \in [\underline{\mu}_{\tilde{A}}(x), \overline{\mu}_{\tilde{A}}(x)]\} \quad (7)$$

An IT2 FS $\tilde{A}$ can also be represented as

$$\tilde{A} = 1/FOU(\tilde{A}) \quad (8)$$

with the understanding that this means putting a secondary grade of 1 at all points of $FOU(\tilde{A})$.

Recently, the relationship between interval-valued fuzzy sets (IVFS) [30] and IT2 FS have been discussed in [20], [32]. It is pointed out that the phrase "IT2 FS" is a more general term than the phrase "IVFS" and includes IVFS as a special case [32]. The operations, methods, and systems that have been developed and published about IT2 FSs are, so far, only valid in the special case when IT2 FS = IVFS [20]. Actually when J_x is defined as Eq. (6), the IT2 FS should be called a CIT2 FS [1]. Since every CIT2 FS is an IVFS [23], the proposed envelope in form of IT2 FS calculated by Eqs. (7) and (8) will not lead to ambiguous operations in future applications. Besides, in this paper when phrase "IT2 FS" is used it means CIT2 FS. At this moment, the use of $FOU(\tilde{A})$ is unambiguous, as well as Eqs. (6) and (7) [23].

[1] An IT2 FS should be called a closed IT2 FS (CIT2 FS) if $\{u \in [0,1] | \mu_{\tilde{A}}(x,u) = 1\}$ is a closed interval for every $x \in X$ [23].

B. Type-1 fuzzy envelope of $HFLTS$

Recently, several proposals have been provided using linguistic expressions richer than a single linguistic term [13], [33], [35], they are not close to common language of human being or lack of defined formalization to generate the linguistic expressions. A new linguistic model was provided in [27] that overcomes this limitation by using $HFLTS$ and the context-free grammars which provide a formal way to generate CLEs.

Definition 1. *[27] Let $S = \{s_0, ..., s_g\}$ be a linguistic term set. A $HFLTS$, H_S , is an ordered finite subset of the consecutive linguistic terms of S.*

To generate simple but rich linguistic expressions suitable to provide preferences in decision making problems modeled by means of $HFLTS$, a context-free grammar [2] G_H was defined in [28]. Usually the CLEs generated by the context-free grammar G_H are hard to be directly used in the processes of CW, to tackle this issue a transformation function E_{G_H} was defined in [27], which can transform CLEs into $HFLTSs$.

The concept of envelope for $HFLTS$ was firstly provided in form of linguistic intervals [27]. Nevertheless, the models based on linguistic intervals loss the initial fuzzy representation for linguistic information. A type-1 fuzzy envelope of $HFLTS$ has been proposed to overcome this limitation [12]. Here we make a brief review on the steps to achieve the type-1 fuzzy envelope.

1. Obtain the elements to aggregate. Assume that every linguistic term can be defined by a triangular membership function, the set of elements to aggregate is $T = \{a^i_L, a^i_M, a^{i+1}_L, a^i_R, a^{i+1}_M, a^{i+2}_L \ldots, a^g_L, a^{g-1}_R, a^g_M, a^g_R\}$ It follows the fuzzy partitions [29] that $a^{k-1}_R = a^k_M = a^{k+1}_L$ $(k = 1, 2, \ldots, g-1)$, hence the elements to aggregate can be simplified as $T = \{a^i_L, a^i_M, a^{i+1}_M, \ldots, a^g_M, a^g_R\}$
2. Compute the parameters of the trapezoidal fuzzy membership function. A trapezoidal fuzzy membership function $F_{H_S} = T(a, b, c, d)$ is used as the type-1 fuzzy set representation of a CLE using $HFLTS$ H_S as a media, where a and d can be easily computed by the min and max operators, i.e.,
   $a = min\{a^i_L, a^i_M, a^{i+1}_M, \ldots, a^j_M, a^j_R\} = a^i_L$,
   $d = max\{a^i_L, a^i_M, a^{i+1}_M, \ldots, a^j_M, a^j_R\} = a^j_R$,
   b and c are obtained by aggregating the remaining elements $a^i_M, a^{i+1}_M, \ldots, a^j_M$ with OWA operators, i.e.,
   $b = OWA_{W^s}(a^i_M, a^{i+1}_M, \ldots, a^j_M)$,
   $d = OWA_{W^t}(a^i_M, a^{i+1}_M, \ldots, a^j_M)$,
   with $s, t = 1, 2, s \neq t$ or $s = t$.
3. Obtain the OWA weights. Different importance degrees of linguistic terms are reflected by means of the OWA weights, which can be computed as follows:

   Definition 2. *[8] Let $\alpha \in [0, 1]$, the first type of OWA weights $W^1 = (w^1_1, w^1_2, \ldots, w^1_n)$ is defined as*
   $w^1_1 = \alpha$, $w^1_2 = \alpha(1-\alpha)$, $w^1_3 = \alpha(1-\alpha)^2$, $\ldots$, $w^1_{n-1} = \alpha(1-\alpha)^{n-2}$, $w^1_n = (1-\alpha)^{n-1}$;
   the second type of OWA weights $W^2 = (w^2_1, w^2_2, \ldots, w^2_n)$ is defined as
   $w^2_1 = \alpha^{n-1}$, $w^2_2 = (1-\alpha)\alpha^{n-2}$, $w^2_3 = (1-\alpha)\alpha^{n-3}$, $\ldots$, $w^2_{n-1} = (1-\alpha)\alpha$, $w^2_n = 1-\alpha$.
4. Obtain the type-1 fuzzy envelope. Let H_S be a $HFLTS$, its type-1 fuzzy envelope F_{H_S} can be defined as the trapezoidal fuzzy membership function $T(a, b, c, d)$, i.e., $F_{H_S} = T(a, b, c, d)$, where parameters a, b, c, d are computed using the previous steps. The type-1 fuzzy membership function in accordance with F_{H_S} is denoted by $F_{H_S}(x)$, $x \in X$.

C. Entropy for $HFLTS$

Recently, Wei et al. [38] studied the entropy measures for extended hesitant fuzzy linguistic term set ($EHFLTS$) considering not only the fuzziness, but also the hesitation of the $EHFLTS$. $HFLTS$ is a special case of $EHFLTS$. Taking into account that $HFLTS$ is the only tool that will be applied in the current work, we will deduce the entropy measures of $EHFLTS$ in [38] into $HFLTS$ cases in the following review.

Definition 3. *[38] Let $S = \{s_0, s_1, \ldots, s_g\}$ be a linguistic term set, $H_S = \{s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_l}\}$ be a $HFLTS$ on S. The deviation function of a $HFLTS$ H_S is defined as:*

$$\eta(H_S) = \frac{2}{l(l-1)} \sum_{i=1}^{l-1} \sum_{j=i+1}^{l} (I(s_{\alpha_j}) - I(s_{\alpha_i})) \qquad (9)$$

where $I(s_{\alpha_k})$ is the index of the linguistic term s_{α_k}.

Definition 4. *[38] Let $H_S = \{s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_l}\}$ be a $HFLTS$ on the linguistic term set $S = \{s_0, s_1, \ldots, s_g\}$, and $\mathbb{H}(S)$ be the set of all the $HFLTSs$ on S. Let $E_f, E_h, E_c : \mathbb{H}(S) \longrightarrow [0, 1]$ be three mappings, if they satisfy the following axiomatic requirements:*
$(F1)$ $E_f(H_S) = 0$ if and only if $H_S = \{s_0\}$, $H_S = \{s_g\}$, $H_S = \{s_0, s_g\}$;
$(F2)$ $E_f(H_S) = 1$ if and only if $H_S = \{s_{\frac{g}{2}}\}$;
$(F3)$ Let $H^1_S = \{s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_l}\}$ be a $HFLTS$, and H^2_S be another $HFLTS$ given by changing any element $s_{\alpha_i}(i = 1, 2, \ldots, l)$ in H^1_S to $s_{\alpha'_i}$. If $|I(s_{\alpha_i}) - \frac{g}{2}| \geq |I(s_{\alpha'_i}) - \frac{g}{2}|$, then $E_f(H^1_S) \leq E_f(H^2_S)$;
$(F4)$ $E_f(H_S) = E_f(Neg(H_S))$, where $Neg(H_S)$ is the negation operator of H_S,
and
$(H1)$ $E_h(H_S) = 0$, if and only if $H_S = \{s_{\alpha_1}\}$ (no hesitancy);
$(H2)$ $E_h(H_S) = 1$, if and only if $H_S = \{s_0, s_1, \ldots, s_g\}$ (whole hesitancy);
$(H3)$ $E_h(H^1_S) \leq E_h(H^2_S)$, if $\eta(H^1_S) \leq \eta(H^2_S)$;
$(H4)$ $E_h(H_S) = E_h(Neg(H_S))$, where $Neg(H_S)$ is the negation operator of H_S,
and
$(E1)$ $E_c(H_S) = 0$ if and only if $H_S = \{s_0\}$, $H_S = \{s_g\}$;
$(E2)$ $E_c(H_S) = 1$ if and only if $H_S = \{s_{\frac{g}{2}}\}$;
$(E3)$ Let $H^1_S = \{s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_l}\}$ be a $HFLTS$, and H^2_S be another $HFLTS$ given by changing any element $s_{\alpha_i}(i = 1, 2, \ldots, l)$ in H^1_S to $s_{\alpha'_i}$. If $|I(s_{\alpha_i}) - \frac{g}{2}| \geq |I(s_{\alpha'_i}) - \frac{g}{2}|$ and $\eta(H^1_S) \leq \eta(H^2_S)$, then $E_c(H^1_S) \leq E_c(H^2_S)$;
$(E4)$ $E_c(H_S) = E_c(Neg(H_S))$.

Then E_f, E_h and E_c are the fuzzy, hesitant and comprehensive entropies of a $HFLTS$, respectively.

The axiomatic definition of comprehensive entropies is actually based on the combination of fuzzy and hesitant entropies, therefore it arises naturally that the specific calculation formulas for comprehensive entropies can be achieved through the combination of calculation formulas of fuzzy and hesitant entropies. It is noteworthy that a general formula to construct the comprehensive entropies of $HFLTS$ through fuzzy entropies and the hesitant entropies have been provided in [38], as well as a class of comprehensive entropies with a parameter to control the importance degree of hesitancy when the overall uncertainty need to be computed.

Theorem 1. *[38] If a real valued function $E_c : \mathbb{H}(S) \longrightarrow [0,1]$ is defined by*

$$E_c(H_S) = f(E_f(H_S), E_h(H_S)) \tag{10}$$

where the function $f : [0,1] \times [0,1] \longrightarrow [0,1]$ satisfies the following conditions

1)$f(0,0) = 0$, $f(1,0) = 1$;

2)$f(x,y)$ is strictly monotone increasing with respect to x and y, respectively,

then E_c is a comprehensive entropy measure of the $HFLTS$ H_S.

Define $f : [0,1] \longrightarrow [0,1]$ as $f(x,y) = \frac{x+\beta y}{1+\beta y}$, $\beta \in [0,1]$. Since $f(x,y)$ satisfies the two conditions in Theorem 1, the comprehensive entropy measure of a $HFLTS$, H_S, can be:

$$E_c(H_S) = \frac{E_f(H_S) + \beta E_h(H_S)}{1 + \beta E_h(H_S)} \tag{11}$$

with $\beta \in [0,1]$ that can be fixed according to the importance of the hesitation of H_S. The smaller value of β indicates that the less hesitancy will be taken into consideration if the overall uncertainty of H_S is evaluated by using Eq. (11), while $\beta = 0$ indicates that only the fuzzy uncertainty will be considered.

III. TYPE-2 FUZZY ENVELOPE OF $HFLTS$

Linguistic expressions are inherently vague and uncertain, i.e. linguistic expressions mean different things to different people, so a qualified representation model of linguistic expressions must be able to incorporate these uncertainties. However, previous representations for CLEs based on $HFLTSs$ in form of linguistic intervals [12] and type-1 fuzzy set [27] do not consider these uncertainties. To overcome this limitation, we propose the use of a type-2 fuzzy membership function as the representation, that is similar to the way in which linguistic terms may be represented by type-2 fuzzy membership functions.

One precondition to deal with the uncertainty contained in $HFLTS$ is the proper estimation of the uncertainty. The current proposal is proposed based on noticing that the hesitancy among linguistic terms more fuzzy will result in more uncertainty. Here is an example to illustrate:

Example 1: Suppose that a company need to make a decision on purchase of a machine or not based on the assessment of an expert on this machine, the linguistic term set S = {super bad, very bad, bad, a little bad, a little good, good, very good, super good}. If the expert provides the evaluation result as H_{S_1} = {super bad, very bad}, then the company intends to avoid this machine; if the estimation is H_{S_2} = {very good, super good}, then the company intends to buy this machine. However, if the assessment is H_{S_3} = {a little bad, a little good}, the company may be much more harder to make a decision on "buy it" or "not buy it" considering the uncertainty on the general condition of the machine.

Based on the above consideration, we obtain:

Lemma 1. *The hesitancy among linguistic terms more fuzzy will lead to more uncertainty compared with hesitancy among linguistic terms less fuzzy.*

Therefore, the hesitancy of different $HFLTSs$ should be treated differently when dealing with the overall uncertainty. In this proposal, we will use the comprehensive entropies [38] as Eq. (11) to consider two different types of uncertainty contained in $HFLTS$, the hesitancy and the fuzzy uncertainty for constructing the type-2 fuzzy envelope. Furthermore a mechanism to determine the importance of hesitancy according to the linguistic terms contained in $HFLTS$ themselves will be carried out to achieve the goal of different treatments of hesitancy of different $HFLTSs$.

In the following subsections a general process to construct the type-2 fuzzy envelope for $HFLTS$ will be introduced and afterwards its applications in different CLEs generated from the context-free grammar will be discussed separately.

A. Type-2 fuzzy envelope of HFLTS: general process

Let $H_S = \{s_i, s_{i+1}, \ldots, s_j\}$ be a $HFLTS$, where $s_k \in S = \{s_0, \ldots, s_g\}, k \in \{i, \ldots, j\}$. A three-step process is carried out to compute the type-2 fuzzy envelope of $HFLTS$ (see Fig. 2).

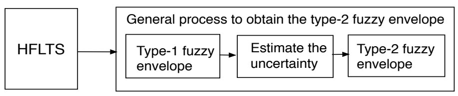

Fig. 2. General process to obtain the type-2 fuzzy envelope

1. Calculate the type-1 fuzzy envelope of $HFLTS$. According to [12], reviewed in Section II.
2. Evaluate the uncertainty contained in $HFLTS$. Taking into account both types of uncertainty contained in a $HFLTS$. The fuzzy uncertainty determined by the deviation of the linguistic terms contained in the $HFLTS$ from the fuzziest element, and the hesitancy related to the number of terms in $HFLTS$ and to the fuzziness of terms (see lemma 1). Therefore, comprehensive entropies introduced in [38] can be constructed by using fuzzy and hesitancy entropies, which provide the advantage of controlling the importance degree of hesitancy when evaluating the overall uncertainty. We prefer to use this class of comprehensive entropies to measure the uncertainty contained in the $HFLTSs$ in order

to perform different treatments for hesitations contained in different $HFLTSs$ by determining their importance degrees of hesitancy according to their specific characteristics . Thus, the comprehensive entropy of a $HFLTS$, H_S, will be calculated by

$$E_c(H_S) = \frac{E_f(H_S) + \beta(H_S)E_h(H_S)}{1 + \beta(H_S)E_h(H_S)}, \quad (12)$$

where $E_f(H_S)$ and $E_h(H_S)$ are the fuzzy and hesitant entropy of H_S, respectively. The function $\beta(H_S)$ represents the importance/emphasis degree of hesitancy when evaluating the overall uncertainty contained in H_S. From Eq. (12), the larger $\beta(H_S)$, the greater the value of overall uncertainty $E_c(H_S)$, because the more hesitancy is considered for computing the overall uncertainty in H_S. Two main factors will be considered for determining the value of $\beta(H_S)$ according to H_S:

- The number of linguistic terms contained in the $HFLTS$ H_S;
- The positions of the terms in the $HFLTS$ H_S.

Several principles to determine $\beta(H_S)$ are listed below:

P1. $\beta(H_S) = 1$ if $H_S = \{s_0, s_1, \ldots, s_g\}$.
If all the linguistic terms are contained in H_S, the hesitancy reaches the max, which calls for a highest level of attention being focused on hesitancy.

P2. $\beta(H_S) = 0$ if $H_S = \{s_i\}$, $s_i \in S$.
If only one single linguistic term is contained in H_S, the hesitancy reaches the min, which calls for a lowest level of attention being focused on hesitancy. It is noteworthy that if the value of $\beta(H_S)$ reaches 0, by Eq. (12) the comprehensive entropy degenerates to the fuzzy entropy of H_S.

P3. $|H'_S| > |H_S| \Rightarrow \beta(H'_S) > \beta(H_S)$.
It reflects that the larger number of linguistic terms contained in H_S the higher importance of hesitancy. Keeping in mind that our goal is to evaluate the uncertainty contained in a CLE represented by the $HFLTS$ H_S, the larger number of terms in H_S the higher level of hesitancy of the CLE, as well as the higher impact of hesitancy on the overall uncertainty, which is implemented by a larger value of $\beta(H_S)$ when the overall uncertainty is estimated as $E_c(H_S)$ with Eq. (12).

P4. The change quantity of $\beta(H_S)$ should be positively correlated to the fuzzy degree of the linguistic term added in/deleted from H_S, i.e.,

- Add a linguistic term s_p into H_S and then it turns to H'_S, add another linguistic term s_q into H'_S and therefore it turns to H''_S. If $E_f(s_p) \leq E_f(s_q)$, then $|\beta(H'_S) - \beta(H_S)| \leq |\beta(H''_S) - \beta(H'_S)|$;
- Delete a linguistic term s_p from H_S and then it turns to H'_S, delete another linguistic term s_q from H'_S and therefore it turns to H''_S. If $E_f(s_p) \leq E_f(s_q)$, then $|\beta(H'_S) - \beta(H_S)| \leq |\beta(H''_S) - \beta(H'_S)|$.

Lemma 1 can be implemented by controlling the change quantity of $\beta(H_S)$ to be positively correlated to the fuzzy degree of the linguistic term added in/deleted from a given $HFLTS$, H_S. If a term is added into a H_S, the value of $\beta(H_S)$ should increase, positively correlated to the fuzzy degree of the term added, i.e., the larger fuzzy degree of the added term, the more importance of hesitancy; on the other hand, if a term is deleted from H_S, the decrease quantity of $\beta(H_S)$ should be positively correlated to the fuzzy degree of the terms deleted from the $HFLTS$, i.e., the more fuzzy the deleted term, the less importance of hesitancy.

P5. $\beta(H_S) = \beta(Neg(H_S))$.
According to the axiomatic definitions, $(F4)$ and $(H4)$ the fuzzy uncertainty and hesitancy contained in the $HFLTSs$, H_S and $Neg(H_S)$ are always the same. Hence the importance of hesitancy should be the same for $E_c(H_S)$ and $E_c(Neg(H_S))$.

Our ultimate goal is to construct a suitable representation for CLEs based on $HFLTSs$. For different CLEs generated from the context-free grammar which can be transformed to H_S, $\beta(H_S)$ will be defined as different functions with respect to variables closely related to the number and positions of the terms in H_S. The functions for calculating $\beta(H_S)$ provided in this proposal will satisfy principles P1-P5, and they will always be twice differentiable with respect to the corresponding variables in order to be easily proved satisfying principles P3 and P4.

3. Construct the type-2 fuzzy envelope for $HFLTS$.
For a $HFLTS$ H_S, its type-2 fuzzy envelope, denoted as $\tilde{F}_{H_S}$, will be built based on its type-1 fuzzy envelope [12], denoted as F_{H_S}. As an initial exploration of type-2 fuzzy envelope for $HFLTS$, it will be defined as an IT2 FS for simplification of calculation.
When the type-2 fuzzy envelope $\tilde{F}_{H_S}$ of H_S is constructed, the type-1 fuzzy envelope F_{H_S} will be used as the UMF, and then the LMF of $\tilde{F}_{H_S}$ determined by its UMF considering the uncertainty contained in H_S, which is measured by the comprehensive entropy $E_c(H_S)$, i.e., the LMF of $\tilde{F}_{H_S}$ can be presented as
$\underline{\mu}_{\tilde{F}_{H_S}}(x) = max\{0, F_{H_S}(x) - E_c(H_S)\},\ \forall x \in X$.
An IT2 FS can be uniquely determined by its FOU, when the LMF and UMF are determined, the FOU is uniquely determined, as well as the IT2 FS. By constructing LMF and UMF using the above approach, the uncertainty contained in H_S can be approximately reflected by the width of the FOU, and envelope can be presented as an IT2 FS $\tilde{F}_{H_S} = 1/FOU(\tilde{F}_{H_S})$ with

$$FOU(\tilde{F}_{H_S}) = \{(x, u) : x \in X, u \in [max\{0, F_{H_S}(x) - E_c(H_S)\}, F_{H_S}(x)]\}.$$

B. Type-2 fuzzy envelope for $HFLTS$

Now the specific type-2 fuzzy envelopes for different CLEs represented by $HFLTSs$ will be discussed. A mechanism to make different treatments for hesitancy of different CLEs will be provided by means of calculating the importance degrees of hesitancy for different expressions using different functions.

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*1) Type-2 fuzzy envelope for the CLE "**at least** s_i":* This expression is used by an expert when he/she hesitates among different linguistic terms however is clear about the worst assessment. By using the transformation function E_{G_H}, it is easy to obtain a $HFLTS$ as

E_{G_H}(at least s_i)= $\{s_i, s_{i+1}, \ldots, s_g\}$

The type-2 fuzzy envelope is computed as:

1. Compute the type-1 fuzzy envelope for the $HFLTS$ E_{G_H}(at least s_i).
   a) Obtain the elements to aggregate: $T = \{a^i_L, a^i_M, a^{i+1}_M, \ldots, a^g_M, a^g_R\}$
   b) Remaining steps refer to [12], and the constructed type-1 fuzzy envelope $F_{E_{G_H}(\text{at least } s_i)}(x)$, $x \in X$ is represented as $T(a^i_L, b, a^g_M, a^g_R)$.
2. Compute the uncertainty contained in E_{G_H}(at least s_i).
   The uncertainty of $HFLTS$ E_{G_H}(at least s_i) is evaluated by $E_c(E_{G_H}$(at least s_i)) that is computed by Eq. (15). In this case, both the number and the positions of linguistic terms are closely related to the label of s_i, i.e., i. To define $\beta(E_{G_H}$(at least s_i)) as a unary function related to independent variable i, it cannot be determined only by the number of linguistic terms in E_{G_H}(at least s_i), but also by the fuzzy degree of linguistic terms added into/deleted from E_{G_H}(at least s_i), associated with the positions of linguistic terms contained in E_{G_H}(at least s_i). $\beta(E_{G_H}$(at least s_i)) is twice differentiable with respect to i to facilitate the discussion, while the first-order derivative and second-order derivative are denoted by $\frac{d(\beta(E_{G_H}(\text{at least } s_i)))}{di}$ and $\frac{d^2(\beta(E_{G_H}(\text{at least } s_i)))}{di^2}$, respectively.
   In order to ensure principles P1-P4, $\beta(E_{G_H}$(at least s_i)) should fulfill the following properties:
   i. $\beta(E_{G_H}(\text{at least } s_i)) = 1$ if $i = 0$.
      If $i \to 0$, i.e., $s_i \to s_0$, we have $|E_{G_H}(\text{at least } s_i)| \to |E_{G_H}(\text{at least } s_0)| = |\{s_0, s_1, \ldots, s_g\}| = g + 1$. This property ensures that if all the linguistic terms in S are contained in E_{G_H}(at least s_i), the importance degree of hesitancy reaches the max, i.e., $\beta(E_{G_H}(\text{at least } s_i)) = 1$.
   ii. $\beta(E_{G_H}(\text{at least } s_i)) = 0$ if $i = g$.
      If $i \to g$, i.e., $s_i \to s_g$, we have $|E_{G_H}(\text{at least } s_i)| \to |E_{G_H}(\text{at least } s_g)| = |\{s_g\}| = 1$. This property ensures that if there is only one single term in S contained in E_{G_H}(at least s_i), the importance degree of hesitancy reaches the min, i.e., $\beta(E_{G_H}(\text{at least } s_i)) = 0$.
   iii. $\frac{d(\beta(E_{G_H}(\text{at least } s_i)))}{di} \leq 0$ when $i \in [0, g]$.
      Since the number of linguistic terms contained in E_{G_H}(at least s_i) decreases when i increases, the monotone decreasing property of $\beta(E_{G_H}$(at least s_i)) with respect to i ensures that the value $\beta(E_{G_H}$(at least s_i)) decreases when the number of terms in E_{G_H}(at least s_i) decreases.
   iv. $\frac{d^2(\beta(E_{G_H}(\text{at least } s_i)))}{di^2} \leq 0$ when $i \in [0, \frac{g}{2}]$, and $\frac{d^2(\beta(E_{G_H}(\text{at least } s_i)))}{di^2} \geq 0$ when $i \in [\frac{g}{2}, g]$.
      From $\frac{d^2(\beta(E_{G_H}(\text{at least } s_i)))}{di^2} \leq 0$ when $i \in [0, \frac{g}{2}]$ it is easy to obtain $\frac{d(\beta(E_{G_H}(\text{at least } s_i)))}{di}$ is a monotone decreasing function with respect to $i \in [0, \frac{g}{2}]$, which indicates that if the left-most linguistic terms in E_{G_H}(at least s_i) are deleted one by one, the decrease quantity of $\beta(E_{G_H}$(at least s_i)) increases when the left-most linguistic term s_i changes from s_0 to $s_{\frac{g}{2}}$ (if g is even), or changes from s_0 to $s_{\frac{g-1}{2}}$ (if g is odd); meanwhile, from $\frac{d^2(\beta(E_{G_H}(\text{at least } s_i)))}{di^2} \geq 0$ when $i \in [\frac{g}{2}, g]$ it is easy to obtain $\frac{d(\beta(E_{G_H}(\text{at least } s_i)))}{di}$ is a monotone increasing function with respect to $i \in [\frac{g}{2}, g]$, which indicates that if the left-most linguistic terms are deleted one by one from E_{G_H}(at least s_i), the decrease quantity of $\beta(E_{G_H}$(at least s_i)) decreases when the left-most linguistic term s_i changes from $s_{\frac{g}{2}}$ to s_g (if g is even), or from $s_{\frac{g+1}{2}}$ to s_g (if g is odd). Considering the axiomatic definition of fuzzy entropy for linguistic terms, the above properties of $\frac{d^2(\beta(E_{G_H}(\text{at least } s_i)))}{di^2}$ ensure that the change quantity of $\beta(E_{G_H}$(at least s_i)) is positively correlated to the fuzzy degree of the linguistic terms being deleted from E_{G_H}(at least s_i), in other words, the more fuzzy the deleted term, the more $\beta(E_{G_H}$(at least s_i)) decreases.

   Based on the above analysis, now we give an example definition of $\beta(E_{G_H}$(at least s_i)) as Eq. (13) (see Fig. 4), which can be easily proved satisfying above properties i-iv. The definition method is not unique, as long as it satisfies principles P1-P5.

$$\beta(E_{G_H}(\text{at least } s_i)) = \frac{1}{2}cos\frac{\pi}{g}i + \frac{1}{2}, i \in [0, g]. \quad (13)$$

3. Compute the type-2 fuzzy envelope for E_{G_H}(at least s_i).
   For the $HFLTS$, E_{G_H}(at least s_i) obtained from the CLE "at least s_i", its type-2 fuzzy envelope can be defined as an IT2 FS $\tilde{F}_{E_{G_H}(\text{at least } s_i)} = 1/FOU(\tilde{F}_{E_{G_H}(\text{at least } s_i)})$ with its footprint (see Fig. 3):

$$FOU(\tilde{F}_{E_{G_H}(\text{at least } s_i)}) = \{(x, u) : x \in X, u \in [max\{0, F_{E_{G_H}(\text{at least } s_i)}(x) - E_c(E_{G_H}(\text{at least } s_i))\}, F_{E_{G_H}(\text{at least } s_i)}(x)]\}.$$

*2) Type-2 fuzzy envelope for the comparative linguistic expression " **at most** s_i":* This expression is used by an expert when he/she is clear about the best assessment however still hesitates among different linguistic terms. By using the transformation function E_{G_H}, the $HFLTS$ is obtained as

E_{G_H}(at most s_i)= $\{s_0, s_1, \ldots, s_i\}$

The type-2 fuzzy envelope is computed as:

1. Compute the type-1 fuzzy envelope for the $HFLTS$ E_{G_H}(at most s_i).
   a) Obtain the elements to aggregate: $T = \{a^0_L, a^0_M, a^1_M, \ldots, a^i_M, a^i_R\}$
   b) Remaining steps refer to [12], and the constructed type-1 fuzzy envelope $F_{E_{G_H}(\text{at most } s_i)}(x)$, $x \in X$ is represented as $T(a^0_L, a^0_M, c, a^i_R)$.
2. Compute the uncertainty contained in E_{G_H}(at most s_i).
   The uncertainty of $HFLTS$ E_{G_H}(at most s_i) is evaluated by $E_c(E_{G_H}$(at most s_i)) and computed by Eq.(16). In

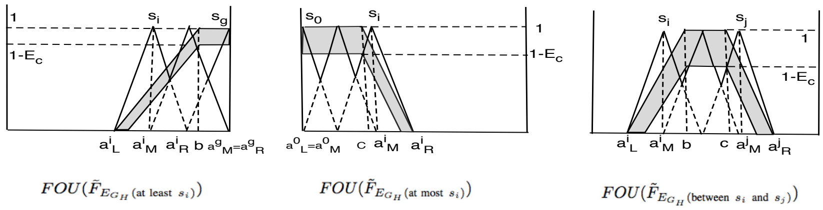

Fig. 3. FOU for type-2 fuzzy envelopes of E_{G_H}(at least s_i), E_{G_H}(at most s_i) and E_{G_H}(between s_i and s_j)

this case, both the number and the positions of linguistic terms contained in E_{G_H}(at most s_i) are closely related to the label of s_i, i.e., i, we define $\beta(E_{G_H}(\text{at most } s_i))$ as a unary function related to independent variable i. $\beta(E_{G_H}(\text{at most } s_i))$ is twice differentiable with respect to i to facilitate the discussion, while the first-order derivative and second-order derivative are denoted by $\frac{d(\beta(E_{G_H}(\text{at most } s_i)))}{di}$ and $\frac{d^2(\beta(E_{G_H}(\text{at most } s_i)))}{di^2}$, respectively. In order to ensure principles P1-P4, $\beta(E_{G_H}(\text{at most } s_i))$ should fulfill the following properties:

i. $\beta(E_{G_H}(\text{at most } s_i)) = 1$ if $i = g$.
If $i \to g$, i.e., $s_i \to s_g$, then $|E_{G_H}(\text{at most } s_i)| \to |E_{G_H}(\text{at most } s_g)| = |\{s_0, s_1, \ldots, s_g\}| = g + 1$. It ensures that if all the terms in S are contained in E_{G_H}(at most s_i), the importance degree of hesitancy reaches the max, i.e., $\beta(E_{G_H}(\text{at most } s_i)) = 1$.

ii. $\beta(E_{G_H}(\text{at most } s_i)) = 0$ if $i = 0$.
If $i \to 0$, i.e., $s_i \to s_0$, then $|E_{G_H}(\text{at most } s_i)| \to |E_{G_H}(\text{at most } s_0)| = |\{s_0\}| = 1$. This property ensures that if there is only one single term in S contained in E_{G_H}(at most s_i), the importance degree of hesitancy reaches the min, i.e., $\beta(E_{G_H}(\text{at most } s_i)) = 0$.

iii. $\frac{d(\beta(E_{G_H}(\text{at most } s_i)))}{di} \geq 0$ when $i \in [0, g]$.
As i increases, the number of terms in the $HFLTS$ increases. From the monotone of the first-order derivative, it is easy to obtain that the value of $\beta(E_{G_H}(\text{at most } s_i))$ increases when i increases, which indicates that the value of $\beta(E_{G_H}(\text{at most } s_i))$ increases when the number of terms increases.

iv. $\frac{d^2(\beta(E_{G_H}(\text{at most } s_i)))}{di^2} \geq 0$ when $i \in [0, \frac{g}{2}]$, and $\frac{d^2(\beta(E_{G_H}(\text{at most } s_i)))}{di^2} \leq 0$ when $i \in [\frac{g}{2}, g]$.
From the negative and positive of the second-order derivative, we obtain that the first-order derivative $\frac{d(\beta(E_{G_H}(\text{at most } s_i)))}{di}$ is a monotone increasing function with respect to $i \in [0, \frac{g}{2}]$, which ensures that if new linguistic terms are added to E_{G_H}(at most s_i) from the right side, the increase quantity of $\beta(E_{G_H}(\text{at most } s_i))$ increases when the right-most linguistic term changes from s_0 to $s_{\frac{g}{2}}$ (if g is even), or changes from s_0 to $s_{\frac{g-1}{2}}$ (if g is odd); meanwhile, $\frac{d(\beta(E_{G_H}(\text{at most } s_i)))}{di}$ is a monotone decreasing function with respect to i on domain $[\frac{g}{2}, g]$, which ensures that if new linguistic terms are added to E_{G_H}(at most s_i) from the right side, the increase quantity of $\beta(E_{G_H}(\text{at most } s_i))$ decreases when the right-most linguistic term changes from $s_{\frac{g}{2}}$ to s_g (if g is even), or changes from $s_{\frac{g-1}{2}}$ to s_g (if g is odd).
The above properties of $\frac{d^2(\beta(E_{G_H}(\text{at most } s_i)))}{di^2}$ ensure that the increase quantity of $\beta(E_{G_H}(\text{at most } s_i))$ is positively correlated to the fuzzy degree of the linguistic term being added in E_{G_H}(at most s_i), that is, the more fuzzy the added linguistic term is, the more $\beta(E_{G_H}(\text{at most } s_i))$ increases.

Based on the above analysis, now we offer an example definition of $\beta(E_{G_H}(\text{at most } s_i))$ as Eq. (14) (see Fig. 4), which can be easily proved satisfying above properties i-iv. The definition method is not unique, as long as it satisfies principles P1-P5.

$$\beta(E_{G_H}(\text{at most } s_i)) = \frac{1}{2}sin(\frac{\pi}{g}i - \frac{\pi}{2}) + \frac{1}{2}, i \in [0, g]. \quad (14)$$

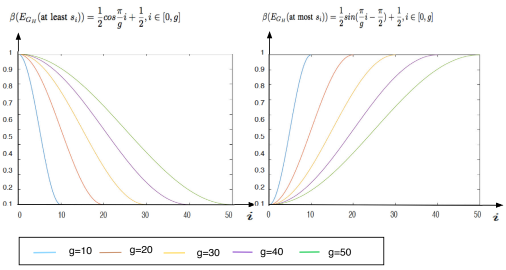

Fig. 4. $\beta(E_{G_H}(\text{at least } s_i))$ defined by Eq.(13) and $\beta(E_{G_H}(\text{at most } s_i))$ defined by Eq. (14), when $g = 10, 20, \ldots, 50$.

3. Compute the type-2 fuzzy envelope for E_{G_H}(at most s_i). For the $HFLTS$, E_{G_H}(at most s_i) obtained from the CLE "at most s_i", its type-2 fuzzy envelope can be defined as an IT2 FS $\tilde{F}_{E_{G_H}(\text{at most } s_i)} = 1/FOU(\tilde{F}_{E_{G_H}(\text{at most } s_i)})$ with its footprint (see Fig. 3):

$$FOU(\tilde{F}_{E_{G_H}(\text{at most } s_i)}) = \{(x, u) : x \in X, u \in [max\{0, F_{E_{G_H}(\text{at most } s_i)}(x) - E_c(E_{G_H}(\text{at most } s_i))\}, F_{E_{G_H}(\text{at most } s_i)}(x)]\}.$$

3) Type-2 fuzzy envelope for the CLE " **between** s_i **and** s_j*":* This expression hesitates among different linguistic terms

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$$E_c(E_{G_H}(\text{at least } s_i)) = \frac{E_f(E_{G_H}(\text{at least } s_i)) + \beta(E_{G_H}(\text{at least } s_i)) \cdot E_h(E_{G_H}(\text{at least } s_i))}{1 + \beta(E_{G_H}(\text{at least } s_i)) \cdot E_h(E_{G_H}(\text{at least } s_i))} \quad (15)$$

$$E_c(E_{G_H}(\text{at most } s_i)) = \frac{E_f(E_{G_H}(\text{at most } s_i)) + \beta(E_{G_H}(\text{at most } s_i)) \cdot E_h(E_{G_H}(\text{at most } s_i))}{1 + \beta(E_{G_H}(\text{at most } s_i)) \cdot E_h(E_{G_H}(\text{at most } s_i))} \quad (16)$$

$$E_c(E_{G_H}(\text{between } s_i \text{ and } s_j)) = \frac{E_f(E_{G_H}(\text{between } s_i \text{ and } s_j)) + \beta(E_{G_H}(\text{between } s_i \text{ and } s_j)) \cdot E_h(E_{G_H}(\text{between } s_i \text{ and } s_j))}{1+\beta(E_{G_H}(\text{between } s_i \text{ and } s_j))\cdot E_h(E_{G_H}(\text{between } s_i \text{ and } s_j))} \quad (17)$$

but the best and worst assessments are clear. By using the transformation function, we obtain the $HFLTS$ as $E_{G_H}(\text{between } s_i \text{ and } s_j)=\{s_i, s_{i+1}, \dots, s_j\}$.
The type-2 fuzzy envelope is computed as:

1. Compute the type-1 fuzzy envelope for the $HFLTS$ $E_{G_H}(\text{between } s_i \text{ and } s_j)$.
   a) Obtain the elements to aggregate: $T = \{a_L^i, a_M^i, a_M^{i+1}, \dots, a_M^j, a_R^j\}$
   b) Remaining steps refer to [12], and the constructed type-1 fuzzy envelope, $F_{E_{G_H}(\text{between } s_i \text{ and } s_j)}(x)$, $x \in X$ is represented as $T(a_L^i, b, c, a_R^j)$.
2. Compute the uncertainty in $E_{G_H}(\text{between } s_i \text{ and } s_j)$.
   The uncertainty of $HFLTS$ $E_{G_H}(\text{between } s_i \text{ and } s_j)$ is evaluated as $E_c(E_{G_H}(\text{between } s_i \text{ and } s_j))$ and computed by Eq. (17). In this case, both the number and the positions of terms in the $HFLTS$ $E_{G_H}(\text{between } s_i \text{ and } s_j)$ are actually determined by the positions of s_i and s_j, in order to construct a link between $\beta(E_{G_H}(\text{between } s_i \text{ and } s_j))$ and the number of linguistic terms contained in $E_{G_H}(\text{between } s_i \text{ and } s_j)$, as well as the positions of linguistic terms contained in $E_{G_H}(\text{between } s_i \text{ and } s_j)$, we define $\beta(E_{G_H}(\text{between } s_i \text{ and } s_j))$ as a function of two variables, i and j. $\beta(E_{G_H}(\text{between } s_i \text{ and } s_j))$ is twice differentiable with respect to i and j respectively. The first-order partial derivative and second-order partial derivative with respect to i are denoted by $\frac{\partial(\beta(E_{G_H}(\text{between } s_i \text{ and } s_j)))}{\partial i}$ and $\frac{\partial^2(\beta(E_{G_H}(\text{between } s_i \text{ and } s_j)))}{\partial i^2}$, respectively, while the first-order partial derivative and second-order partial derivative with respect to j are denoted by $\frac{\partial(\beta(E_{G_H}(\text{between } s_i \text{ and } s_j)))}{\partial j}$ and $\frac{\partial^2(\beta(E_{G_H}(\text{between } s_i \text{ and } s_j)))}{\partial j^2}$, respectively.
   To ensure principles P1-P4, $\beta(E_{G_H}(\text{between } s_i \text{ and } s_j))$ should fulfill some properties which will be shown in three different cases as follows:
   a) if
   $$0 \leq i \leq \frac{g}{2} \leq j \leq g.$$
   $\beta(E_{G_H}(\text{between } s_i \text{ and } s_j))$ should satisfy the following properties:
      i. if $i = 0$, $j = g$, $\beta(E_{G_H}(\text{between } s_i \text{ and } s_j)) = 1$;
      ii. if $i = j = \frac{g}{2}$, $\beta(E_{G_H}(\text{between } s_i \text{ and } s_j)) = 0$;
      iii. (1) $\frac{\partial(\beta(E_{G_H}(\text{between } s_i \text{ and } s_j)))}{\partial i} \leq 0$ $(i \in [0, \frac{g}{2}])$, (2) $\frac{\partial(\beta(E_{G_H}(\text{between } s_i \text{ and } s_j)))}{\partial j} \geq 0$ $(j \in [\frac{g}{2}, g])$;
      If j is already determined, the larger i, the less number of terms in the $HFLTS$. From iii. (1), $\beta(E_{G_H}(\text{between } s_i \text{ and } s_j))$ is monotonically decreasing on the independent variable i, which ensures that $\beta(E_{G_H}(\text{between } s_i \text{ and } s_j))$ decreases when the number of terms in a $HFLTS$ decreases; If i is determined, the larger j, the more number of terms in the $HFLTS$. From iii. (2), $\beta(E_{G_H}(\text{between } s_i \text{ and } s_j))$ is monotonically increasing on the independent variable j, which ensures that the importance of hesitancy $\beta(E_{G_H}(\text{between } s_i \text{ and } s_j))$ increases with the growth of terms in a $HFLTS$;
      iv. (1) $\frac{\partial^2(\beta(E_{G_H}(\text{between } s_i \text{ and } s_j)))}{\partial i^2} \leq 0$ $(i \in [0, \frac{g}{2}])$, (2) $\frac{\partial^2(\beta(E_{G_H}(\text{between } s_i \text{ and } s_j)))}{\partial j^2} \leq 0$ $(j \in [\frac{g}{2}, g])$.
      From iv. (1), suppose that j is a fixed value, $\frac{\partial(\beta(E_{G_H}(\text{between } s_i \text{ and } s_j)))}{\partial i}$ is monotonically decreasing with respect to $i \in [0, \frac{g}{2}]$, which ensures that the decrease quantity of $\beta(E_{G_H}(\text{between } s_i \text{ and } s_j))$ increases when the left-most linguistic term changes from s_0 to $s_{\frac{g}{2}}$ (if g is even), or changes from s_0 to $s_{\frac{g-1}{2}}$ (if g is odd). From the axiomatic definition of fuzzy entropy for linguistic terms, that means if the left-most linguistic terms are deleted one by one from a $HFLTS$, the more fuzzy the deleted term, the more $\beta(E_{G_H}(\text{between } s_i \text{ and } s_j))$ decreases.
      From iv. (2), suppose that i is a fixed value, $\frac{\partial(\beta(E_{G_H}(\text{between } s_i \text{ and } s_j)))}{\partial j}$ is monotonically decreasing with respect to $j \in [\frac{g}{2}, g]$, which ensures that the increase quantity of $\beta(E_{G_H}(\text{between } s_i \text{ and } s_j))$ decreases when the right-most linguistic term changes from $s_{\frac{g}{2}}$ to s_g (if g is even), or changes from $s_{\frac{g+1}{2}}$ to s_g (if g is odd), that is, if linguistic terms are added one by one into $E_{G_H}(\text{between } s_i \text{ and } s_j)$ from the right hand end, the less fuzzy the added linguistic term is, the less $\beta(E_{G_H}(\text{between } s_i \text{ and } s_j))$ increases.
   b) if
   $$0 \leq i \leq j \leq \frac{g}{2}.$$
   $\beta(E_{G_H}(\text{between } s_i \text{ and } s_j))$ should satisfy the following properties:
      i. if $i = 0$, $j = \frac{g}{2}$, $\beta(E_{G_H}(\text{between } s_i \text{ and } s_j)) = \frac{1}{2}$;
      ii. if $i = j$, $\beta(E_{G_H}(\text{between } s_i \text{ and } s_j)) = 0$;

iii. (1) $\frac{\partial(\beta(E_{G_H}(\text{between } s_i \text{ and } s_j)))}{\partial i} \leq 0$ $(i \in [0, \frac{g}{2}])$,
(2) $\frac{\partial(\beta(E_{G_H}(\text{between } s_i \text{ and } s_j)))}{\partial j} \geq 0$ $(j \in [0, \frac{g}{2}])$;

iv. (1) $\frac{\partial^2(\beta(E_{G_H}(\text{between } s_i \text{ and } s_j)))}{\partial i^2} \leq 0$ $(i \in [0, \frac{g}{2}])$,
(2) $\frac{\partial^2(\beta(E_{G_H}(\text{between } s_i \text{ and } s_j)))}{\partial j^2} \geq 0$ $(j \in [0, \frac{g}{2}])$.
Here we only illustrate property iv. (2): From iv. (2), $\frac{\partial(\beta(E_{G_H}(\text{between } s_i \text{ and } s_j)))}{\partial j}$ is monotonically increasing with respect to $j \in [0, \frac{g}{2}]$, which ensures that the increase quantity of $\beta(E_{G_H}(\text{between } s_i \text{ and } s_j))$ increases when right-most linguistic term s_j changes from s_0 to $s_{\frac{g}{2}}$ (if g is even), or changes from s_0 to $s_{\frac{g-1}{2}}$ (if g is odd), that is, if linguistic terms are one by one added into $E_{G_H}(\text{between } s_i \text{ and } s_j)$ from the right hand end, the more fuzzy the added linguistic term is, the more $\beta(E_{G_H}(\text{between } s_i \text{ and } s_j))$ increases.

c) if

$$\frac{g}{2} \leq i \leq j \leq g.$$

$\beta(E_{G_H}(\text{between } s_i \text{ and } s_j))$ should satisfy the following properties:

i. if $i = \frac{g}{2}$, $j = g$, $\beta(E_{G_H}(\text{between } s_i \text{ and } s_j)) = \frac{1}{2}$;
ii. if $i = j$, $\beta(E_{G_H}(\text{between } s_i \text{ and } s_j)) = 0$;
iii. (1) $\frac{\partial(\beta(E_{G_H}(\text{between } s_i \text{ and } s_j)))}{\partial i} \leq 0$ $(i \in [\frac{g}{2}, g])$,
(2) $\frac{\partial(\beta(E_{G_H}(\text{between } s_i \text{ and } s_j)))}{\partial j} \geq 0$ $(j \in [\frac{g}{2}, g])$;
iv. (1) $\frac{\partial^2(\beta(E_{G_H}(\text{between } s_i \text{ and } s_j)))}{\partial i^2} \geq 0$ $(i \in [\frac{g}{2}, g])$,
(2) $\frac{\partial^2(\beta(E_{G_H}(\text{between } s_i \text{ and } s_j)))}{\partial j^2} \leq 0$ $(j \in [\frac{g}{2}, g])$.
Here we only illustrate property iv. (1): From iv. (1), $\frac{\partial(\beta(E_{G_H}(\text{between } s_i \text{ and } s_j)))}{\partial i}$ is monotonically increasing with respect to $i \in [\frac{g}{2}, g]$, which ensures that the decrease quantity of $\beta(E_{G_H}(\text{between } s_i \text{ and } s_j))$ decreases when the left-most linguistic term changes from $s_{\frac{g}{2}}$ to s_g (if g is even), or changes from $s_{\frac{g+1}{2}}$ to s_g (if g is odd), that is, if the left-most linguistic terms are one by one deleted from $E_{G_H}(\text{between } s_i \text{ and } s_j)$, the less fuzzy the deleted linguistic term is, the less $\beta(E_{G_H}(\text{between } s_i \text{ and } s_j))$ decreases.

Remark 1. The property *iv* in cases (a)-(c) can only ensure that the change quantity of $\beta(E_{G_H}(\text{between } s_i \text{ and } s_j))$ is positively related to the fuzzy degree of linguistic terms added to/deleted from $E_{G_H}(\text{between } s_i \text{ and } s_j)$ from one single side. A supplementary condition to ensure that $\beta(E_{G_H}(\text{between } s_i \text{ and } s_j))$ satisfies principle P4 is needed, that is, if linguistic terms are respectively added into $E_{G_H}(\text{between } s_i \text{ and } s_j)$ from the right side and from the left side, as long as the fuzzy degree of added linguistic term is larger, the increase quantity of $\beta(E_{G_H}(\text{between } s_i \text{ and } s_j))$ will be larger. This supplementary condition can be formalized as follows:

(1) $\beta(E_{G_H}(\text{between } s_{i-1} \text{ and } s_j))$-$\beta(E_{G_H}(\text{between } s_i \text{ and } s_j))$ = $\beta(E_{G_H}(\text{between } s_i \text{ and } s_{j+1}))$-$\beta(E_{G_H}(\text{between } s_i \text{ and } s_j))$ if $E_f(s_{i-1}) = E_f(s_{j+1})$, $i, j \in \{1, 2, \ldots, g-1\}$ and $i \leq j$.
(2) $\beta(E_{G_H}(\text{between } s_{i-1} \text{ and } s_j))$-$\beta(E_{G_H}(\text{between } s_i \text{ and } s_j))$<$\beta(E_{G_H}(\text{between } s_i \text{ and } s_{j+1}))$-$\beta(E_{G_H}(\text{between } s_i \text{ and } s_j))$ if $E_f(s_{i-1}) < E_f(s_{j+1})$, $i, j \in \{1, 2, \ldots, g-1\}$ and $i \leq j$.
(3) $\beta(E_{G_H}(\text{between } s_{i-1} \text{ and } s_j))$-$\beta(E_{G_H}(\text{between } s_i \text{ and } s_j))$>$\beta(E_{G_H}(\text{between } s_i \text{ and } s_{j+1}))$-$\beta(E_{G_H}(\text{between } s_i \text{ and } s_j))$ if $E_f(s_{i-1}) > E_f(s_{j+1})$, $i, j \in \{1, 2, \ldots, g-1\}$ and $i \leq j$.

Based on the above analysis, an example of $\beta(E_{G_H}(\text{between } s_i \text{ and } s_j))$ is presented as Eq. (18), which can be easily proved satisfying properties i-iv in cases (a)-(c) and satisfying the supplementary condition in Remark 1 as will be shown in Theorem 2. The formula is not unique, as long as it satisfies principles P1-P5.

$$\beta(E_{G_H}(\text{between } s_i \text{ and } s_j)) = \frac{1}{2}cos\frac{\pi}{g}i + \frac{1}{2}sin(\frac{\pi}{g}j - \frac{\pi}{2}), \quad i, j \in [0, g]. \tag{18}$$

Theorem 2. *$\beta(E_{G_H}(\text{between } s_i \text{ and } s_j))$ calculated by Eq. (18) satisfies the condition in Remark 1.*

Proof. The proof can be found in Appendix A. □

3. Compute the type-2 fuzzy envelope for $E_{G_H}(\text{between } s_i \text{ and } s_j)$. For the $HFLTS$, $E_{G_H}(\text{between } s_i \text{ and } s_j)$ obtained from the CLE "between s_i and s_j", its type-2 fuzzy envelope can be defined as an IT2 FS

$$\tilde{F}_{E_{G_H}(\text{between } s_i \text{ and } s_j)} = 1/FOU(\tilde{F}_{E_{G_H}(\text{between } s_i \text{ and } s_j)})$$

whose footprint is (see Fig. 3):

$$FOU(\tilde{F}_{E_{G_H}(\text{between } s_i \text{ and } s_j)}) = \{(x, u) : x \in X, u \in [max\{0, F_{E_{G_H}(\text{between } s_i \text{ and } s_j)}(x) - E_c(E_{G_H}(\text{between } s_i \text{ and } s_j))\}, F_{E_{G_H}(\text{between } s_i \text{ and } s_j)}(x)]\}.$$

Several theorems to support the mechanism to determine the importance degree of hesitancy by means of Eqs. (13), (14) and (18) are provided.

Theorem 3. *If $\beta(E_{G_H}$(at most s_i)) is calculated by Eq. (14) and $\beta(E_{G_H}$(between s_0 and s_i)) is calculated by Eq. (18), then $\beta(E_{G_H}$(at most s_i))= $\beta(E_{G_H}$(between s_0 and s_i)), $i \in \{0, 1, \ldots, g\}$.*

Proof. $\beta(E_{G_H}(\text{between } s_0 \text{ and } s_i)) = \frac{1}{2}cos0 + \frac{1}{2}sin(\frac{\pi}{g}i - \frac{\pi}{2}) = \frac{1}{2} + \frac{1}{2}sin(\frac{\pi}{g}i - \frac{\pi}{2}) = \beta(E_{G_H}(\text{at most } s_i))$. □

Theorem 4. *If $\beta(E_{G_H}$(at least s_i)) is calculated by Eq. (13) and $\beta(E_{G_H}$(between s_i and s_g)) is calculated by Eq. (18), then $\beta(E_{G_H}$(at least s_i))= $\beta(E_{G_H}$(between s_i and s_g)), $i \in \{0, 1, \ldots, g\}$.*

Proof. $\beta(E_{G_H}(\text{between } s_i \text{ and } s_g)) = \frac{1}{2}cos\frac{\pi}{g}i + \frac{1}{2}sin(\frac{\pi}{g}g - \frac{\pi}{2}) = \frac{1}{2}cos\frac{\pi}{g}i + \frac{1}{2}sin\frac{\pi}{2} = \frac{1}{2}cos\frac{\pi}{g}i + \frac{1}{2} = \beta(E_{G_H}(\text{at least } s_i))$. □

Remark 2. Noticing that E_{G_H}(at most s_i)=E_{G_H}(between s_0 and s_i) and E_{G_H}(at least s_i) = E_{G_H}(between s_i and s_g), from theorems 3 and 4, we conclude that Eqs. (13), (14) and (18) are consistent.

Theorem 5. *If $\beta(E_{G_H}$(at most s_i)) is calculated by Eq. (14) and $\beta(E_{G_H}$(at least s_{g-i})) is calculated by Eq. (13), then $\beta(E_{G_H}$(at most s_i))= $\beta(E_{G_H}$(at least s_{g-i})), $i \in \{0, 1, \ldots, g\}$.*

Proof. β (E_{G_H}(at least s_{g-i}))= $\frac{1}{2}cos(\frac{\pi}{g}(g-i)) + \frac{1}{2} = \frac{1}{2}cos(\pi - \frac{\pi}{g}i) + \frac{1}{2} = -\frac{1}{2}cos\frac{\pi}{g}i + \frac{1}{2} = -\frac{1}{2}sin(\frac{\pi}{2} + \frac{\pi}{g}i) + \frac{1}{2} = -\frac{1}{2}sin(\pi + (\frac{\pi}{g}i - \frac{\pi}{2})) + \frac{1}{2} = \frac{1}{2}sin(\frac{\pi}{g}i - \frac{\pi}{2}) + \frac{1}{2}$=$\beta(E_{G_H}$(at most s_i)). □

Theorem 6. *If $\beta(E_{G_H}$(between s_i and s_j)) and $\beta(E_{G_H}$(between s_{g-j} and s_{g-i})) are calculated by Eq. (18), then $\beta(E_{G_H}$(between s_i and s_j))=$\beta(E_{G_H}$(between s_{g-j} and s_{g-i})), $i, j \in \{0, 1, \ldots, g\}$ and $i \leq j$.*

Proof. $\beta(E_{G_H}$(between s_{g-j} and s_{g-i})) =$\frac{1}{2}cos(\frac{\pi}{g}(g-j)) + \frac{1}{2}sin(\frac{\pi}{g}(g-i) - \frac{\pi}{2}) = \frac{1}{2}cos(\pi - \frac{\pi}{g}j) + \frac{1}{2}sin(\pi - \frac{\pi}{g}i - \frac{\pi}{2}) = \frac{1}{2}cos(\frac{\pi}{2} - (\frac{\pi}{g}j - \frac{\pi}{2})) + \frac{1}{2}sin(\frac{\pi}{g}i + \frac{\pi}{2}) = \frac{1}{2}sin(\frac{\pi}{g}j - \frac{\pi}{2}) + \frac{1}{2}cos\frac{\pi}{g}i$=$\beta(E_{G_H}$(between s_i and s_j)). □

Remark 3. Noticing that E_{G_H}(at most s_i)= $Neg(E_{G_H}$(at least s_{g-i})) and E_{G_H}(between s_i and s_j) = $Neg(E_{G_H}$(between s_{g-j} and s_{g-i})), from theorems 5 and 6 it is obvious that the importance degrees of hesitancy obtained by Eqs. (13), (14) and (18) satisfy principle P5. For future formulas computing the importance degrees of hesitancy that will replace Eqs. (13), (14) and (18), they should satisfy $\beta(E_{G_H}$(at most s_i))= $\beta(E_{G_H}$(at least s_{g-i})), $i \in \{0, 1, \ldots, g\}$ and $\beta(E_{G_H}$(between s_i and s_j))=$\beta(E_{G_H}$(between s_{g-j} and s_{g-i})), $i, j \in \{0, 1, \ldots, g\}$ and $i \leq j$ in order to satisfy principle P5.

IV. Computing the type-2 fuzzy envelopes: Illustrative examples

Let us reconsider the example illustrated in [12] that presented type-1 fuzzy envelopes for CLEs, the type-2 fuzzy envelopes for such expressions generated by the context-free grammar G_H will be introduced as below.

Example 2 Let $S = \{s_0 : \text{nothing}, s_1 : \text{very bad}, s_2 : \text{bad}, s_3 : \text{medium}, s_4 : \text{good}, s_5 : \text{very good}, s_6 : \text{perfect}\}$ be a linguistic term set shown in Fig. 5.

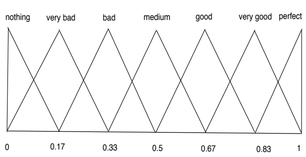

Fig. 5. The linguistic term set $S = \{s_0, s_1, \ldots, s_6\}$

In this example, we will adopt the measures proposed in [38] to estimate the fuzzy entropy of $HFLTS$ $H_S = \{s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_l}\}$ by $E_f(H_S) = \frac{1}{l}\sum_{i=1}^{l} 4\frac{I(s_{\alpha_i})}{g}(1 - \frac{I(s_{\alpha_i})}{g})$, and the hesitant entropy of H_S by $E_h(H_S) = \frac{1}{g}\eta(H_S)$, where $\eta(H_S)$ is calculated by Eq. (9).

Several type-2 fuzzy envelopes for $HFLTSs$ transformed from different CLEs are computed as follows.

- Type-2 fuzzy envelope for the $HFLTS$ E_{G_H}(at least s_4) $= \{s_4, s_5, s_6\}$ corresponds to ll_1 = at least s_4.
  1. The type-1 fuzzy envelope for E_{G_H}(at least s_4) is $F_{E_{G_H}(\text{at least } s_4)} = T(0.5, 0.85, 1, 1)$.
  2. It is easy to obtain that $E_f(E_{G_H}(\text{at least } s_4)) = \frac{1}{3} \times \sum_{i=1}^{3} 4\frac{I(s_{\alpha_i})}{6}(1 - \frac{I(s_{\alpha_i})}{6}) \approx 0.48$ ($s_{\alpha_1} = s_4, s_{\alpha_2} = s_5, s_{\alpha_3} = s_6$); $E_h(E_{G_H}(\text{at least } s_4)) = \frac{1}{6} \times \frac{2}{3(3-1)} \times \sum_{i=1}^{2}\sum_{j=i+1}^{3}(I(s_{\alpha_j}) - I(s_{\alpha_i})) \approx 0.22$ ($s_{\alpha_1} = s_4, s_{\alpha_2} = s_5, s_{\alpha_3} = s_6$); $\beta(E_{G_H}(\text{at least } s_4)) = \frac{1}{2}cos(\frac{\pi}{6} \times 4) + \frac{1}{2} = 0.25$ by Eq. (13), therefore $E_c(E_{G_H}(\text{at least } s_4)) \approx 0.51$ by Eq. (15).
  3. The type-2 fuzzy envelope is given as an IT2 FS $\tilde{F}_{E_{G_H}(\text{at least } s_4)} = 1/FOU(\tilde{F}_{E_{G_H}(\text{at least } s_4)})$ with its footprint (see Fig. 6)

$$FOU(\tilde{F}_{E_{G_H}(\text{at least } s_4)}) = \{(x, u) : x \in [0, 1], u \in [max\{0, F_{E_{G_H}(\text{at least } s_4)}(x) - 0.51\}, F_{E_{G_H}(\text{at least } s_4)}(x)]\}$$

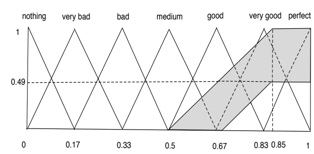

Fig. 6. FOU for the type-2 fuzzy envelope of the $HFLTS$ corresponds to ll_1

- Type-2 fuzzy envelope for the $HFLTS$ E_{G_H}(at most s_2) $= \{s_0, s_1, s_2\}$ corresponds to ll_2 = at most s_2, E_{G_H}(between s_3 and s_5) $= \{s_3, s_4, s_5\}$ corresponds to ll_3 = between s_3 and s_5 and E_{G_H}(between s_4 and s_6) $= \{s_4, s_5, s_6\}$ corresponds to ll_4 = between s_4 and s_6 can be found in the supplementary material Appendix B.

V. Comparisons on applications of type-1 and type-2 fuzzy envelopes in decision making.

In this section, we will make a comparison on the applications of type-1 fuzzy envelopes and type-2 fuzzy envelopes in decision making and show the advantages of the use of type-2 fuzzy envelopes through reconsidering the multi-criteria decision making problem in [12].

Example 3 Suppose that the manager of a company wants to select a material supplier to purchase some key components of a new product. After preliminary screening, four alternatives $X = \{x_1, x_2, x_3, x_4\}$ remained in the candidate list. The considered criteria are $C = \{c_1 = \text{quality}, c_2 = \text{delivery speed}\}$. Because of the lack of information and knowledge about problem, the manager hesitates among several linguistic terms therefore use CLEs close to the natural language used by human beings in decision making problems. To do so, the context-free grammar G_H in [28] and the linguistic term set $S = \{s_0 : \text{nothing (N)}, s_1 : \text{very bad (VB)}, s_2 : \text{bad (B)}, s_3 : \text{medium (M)}, s_4 : \text{good (G)}, s_5 : \text{very good (VG)}, s_6 : \text{perfect (P)}\}$ are used. The assessments provided for this problem are shown in Table I.

To avoid the interference induced by different treatment methodologies for cost criteria in different fuzzy TOPSIS approaches, only benefit criteria have been adopted. Meanwhile, the weights of all criteria are supposed to be equal in order to make the comparison feasible. It is based on the consideration that type-1 fuzzy TOPSIS schemes are usually carried out with setting the weights of criteria type-1 fuzzy numbers [1], [3], [14], [37], [36], type-2 fuzzy TOPSIS schemes are usually carried out with setting the weights of criteria type-2 fuzzy numbers [4], [5], [6], [11], [31], however it will be unreasonable to make the comparison of the applications of type-1 and type-2 envelopes in a decision making problem if the weights of criteria are in different forms.

The CLEs (rating) for alternative x_i $(i \in \{1, 2, 3\})$ with respect to criteria c_j $(j \in \{1, 2\})$ are denoted by ll_{ij} $(i \in \{1, 2, 3\}, j \in \{1, 2\})$ and shown in Table I. The corresponding $HFLTS$ $\tilde{F}_{E_{G_H}(ll_{ij})}$ of these CLEs are shown in Table II.

TABLE I
ASSESSMENTS (I) OF THE PROBLEM.

-	c_1	c_2
x_1	between B and M	between B and M
x_2	at most B	at least G
x_3	between M and G	between B and M
x_4	between B and VG	between VB and G

TABLE II
$HFLTSs$ GENERATED FROM THE CLEs.

-	c_1	c_2
x_1	$\{B, M\}$	$\{B, M\}$
x_2	$\{N, VB, B\}$	$\{G, VG, P\}$
x_3	$\{M, G\}$	$\{B, M\}$
x_4	$\{B, M, G, VG\}$	$\{VB, B, M, G\}$

Now we handle this problem with type-1 fuzzy envelopes of linguistic expressions by using the approach in [12], which actually follows the fuzzy TOPSIS model in [3].

The calculation process can be found in the supplementary material Appendix C, the alternatives are ranked according to their closeness coefficients:

$x_1 \prec x_2 = x_3 = x_4$.

Therefore, the best alternative is x_2 or x_3 or x_4.

In [6], an interval type-2 trapezoidal fuzzy TOPSIS method has been proposed for dealing with fuzzy decision making problems. By conducting several comparisons with other interval type-2 fuzzy multiple criteria decision analysis approaches, they proved that their method is easy to implement and produces effective and valid results for solving multiple criteria decision-making problems. Since the type-2 fuzzy envelopes proposed in the current work are also interval type-2 trapezoidal fuzzy numbers (IT2TrFN), we can handle the problem in this example by using their method.

The calculation process can be found in the supplementary material Appendix D, the alternatives are ranked according to the likelihood based closeness coefficients:

$x_1 \prec x_4 \prec x_3 \prec x_2$.

Therefore, the best alternative is x_2.

Some other examples can be found in the supplementary material Appendix E.

Analysis:

The use of type-2 fuzzy envelope is consistent with type-1 fuzzy envelope in obtaining best result. However, through the above example it is shown that in situations when two alternatives cannot be distinguished by using type-1 fuzzy envelope, the use of type-2 envelope provides a more precise result. It is determined by the fact that, compared with type-1 fuzzy envelope, the construction of type-2 fuzzy envelope considers more comprehensive information contained in linguistic expressions, taking better use of the linguistic uncertainties which have been reflected by the fuzzy uncertainty and hesitancy of the $HFLTSs$. That is, compared with type-1 TOPSIS cooperates with type-1 fuzzy envelope of CLEs, type-2 TOPSIS cooperates with type-2 fuzzy envelope of such expressions (with closeness coefficient as crisp values) perform better in reducing information loss in the decision process, that is the reason why it achieves more accurate decision result.

VI. CONCLUSIONS AND FUTURE WORKS

The use of CLEs based on context-free grammars and $HFLTSs$ has already provided some successful applications in linguistic decision making. The context-free grammar provides the generative mechanism for comparative linguistic preferences, meanwhile the use of $HFLTSs$ provides a manner to present CLEs by using several consecutive linguistic terms. In order to facilitate the CW process using CLEs presented by $HFLTSs$, it is necessary to figure out a suitable representation model which is capable to deal with the uncertainty contained in such expressions. To reach this goal, in the current work,

- A reasonable and effective way to estimate the uncertainties contained in $HFLTSs$ has been provided, taking into account both the fuzzy uncertainty and the hesitancy.
- A new fuzzy envelope of $HFLTSs$ in form of IT2 FSs has been constructed based on its type-1 fuzzy envelope, which can be successfully used to deal with the uncertainty contained in a $HFLTS$.
- By means of transforming CLEs to $HFLTSs$ and then construct the type-2 fuzzy envelopes of $HFLTSs$, a representation method for CLEs in form of IT2 FSs has been achieved.

In the future, we plan to work on the following issues.

- Considering that linguistic decision making problems require CLEs in real life cases, it would be promising to study the application of the proposed representation for CLEs in the information representation process and the CW process of such problems.

- New group decision making models will be developed that manage complex linguistic preference information presented as CLEs by means of type-2 fuzzy envelopes.
- Since IT2 FSs are special cases of general T2 FSs, it would be interesting to see how the envelope can be extended to generate general T2 FS models for CLEs.

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1

APPENDIX A
THE PROOF FOR THEOREM 2.

(1) If $E_f(s_{i-1}) = E_f(s_{j+1})$, considering $i \leq j$, by the axiomatic definition of fuzzy entropy for linguistic terms in [38], we obtain $(i-1)+(j+1) = g$, i.e. $i + j = g$. It is easy to prove that $\frac{1}{2}cos\frac{\pi}{g}(i-1)+\frac{1}{2}sin(\frac{\pi}{g}j-\frac{\pi}{2}) = \frac{1}{2}cos\frac{\pi}{g}i+\frac{1}{2}sin(\frac{\pi}{g}(j+1)-\frac{\pi}{2})$, that is, $\beta(E_{G_H}$(between s_{i-1} and s_j)) $= \beta(E_{G_H}$(between s_i and s_{j+1})), then $\beta(E_{G_H}$(between s_{i-1} and s_j))-$\beta(E_{G_H}$(between s_i and s_j))=$\beta(E_{G_H}$(between s_i and s_{j+1}))-$\beta(E_{G_H}$(between s_i and s_j)).

(2) From the axiomatic definition of fuzzy entropy of linguistic terms, there are only two possibilities for $E_f(s_{i-1}) < E_f(s_{j+1})$:

(i) $0 \leq i-1 < i \leq j < j+1 \leq \frac{g}{2}$.
We only need to prove $\beta(E_{G_H}$(between s_{i-1} and s_j))<$\beta(E_{G_H}$(between s_i and s_{j+1})),
i.e., $\frac{1}{2}cos\frac{\pi}{g}(i-1)+\frac{1}{2}sin(\frac{\pi}{g}j-\frac{\pi}{2}) < \frac{1}{2}cos\frac{\pi}{g}i+\frac{1}{2}sin(\frac{\pi}{g}(j+1)-\frac{\pi}{2})$,
i.e., $\frac{1}{2}cos\frac{\pi}{g}(i-1)-\frac{1}{2}cos\frac{\pi}{g}i < \frac{1}{2}sin(\frac{\pi}{g}(j+1)-\frac{\pi}{2})-\frac{1}{2}sin(\frac{\pi}{g}j-\frac{\pi}{2})$,
i.e., $\frac{1}{2}sin(\frac{\pi}{g}i-\frac{\pi}{2})-\frac{1}{2}sin(\frac{\pi}{g}(i-1)-\frac{\pi}{2}) < \frac{1}{2}sin(\frac{\pi}{g}(j+1)-\frac{\pi}{2})-\frac{1}{2}sin(\frac{\pi}{g}j-\frac{\pi}{2})$.
Construct a function $f(x) = \frac{1}{2}sin(\frac{\pi}{g}x-\frac{\pi}{2})$, it is easy to obtain $f'(x) = \frac{1}{2}\cdot\frac{\pi}{g}\cdot cos(\frac{\pi}{g}x-\frac{\pi}{2}) \geq 0$, $x \in [0, \frac{g}{2}]$ and $f''(x) = -\frac{1}{2}\cdot\frac{\pi^2}{g^2}\cdot sin(\frac{\pi}{g}x-\frac{\pi}{2}) \geq 0$, $x \in [0, \frac{g}{2}]$, that indicates $f(x)$ is an increase function and its increase quantity increases when x changes from 0 to $\frac{g}{2}$, since $i-1 < j$, it is obvious that $f(x)$ changes more when x changes from j to $j+1$ than it changes from $i-1$ to i, that is, $\frac{1}{2}sin(\frac{\pi}{g}i-\frac{\pi}{2})-\frac{1}{2}sin(\frac{\pi}{g}(i-1)-\frac{\pi}{2}) < \frac{1}{2}sin(\frac{\pi}{g}(j+1)-\frac{\pi}{2})-\frac{1}{2}sin(\frac{\pi}{g}j-\frac{\pi}{2})$.

(ii) $0 \leq i-1 < i \leq \frac{g}{2} \leq j < j+1 \leq g$ and $\frac{g}{2}-(i-1) > (j+1)-\frac{g}{2}$, i.e., $j < g-i$.
From (1) we obtain that $\beta(E_{G_H}$(between s_{i-1} and s_{g-i}))-$\beta(E_{G_H}$(between s_i and s_{g-i}))= $\beta(E_{G_H}$(between s_i and s_{g-i+1}))-$\beta(E_{G_H}$(between s_i and s_{g-i})) since $E_f(s_{i-1}) = E_f(s_{g-i+1})$ by the axiomatic definition of fuzzy entropy for linguistic terms and the definition of negation operator of a linguistic term in [38].
Considering $1 \leq i \leq \frac{g}{2} \leq j < g-i$, it is easy to obtain $\frac{g}{2} < j+1 < g-i+1$. Refer to the analysis for property iv. (2) in case (a), when s_i is fixed, $\beta(E_{G_H}$(between s_i and s_{g-i+1}))-$\beta(E_{G_H}$(between s_i and s_{g-i})) < $\beta(E_{G_H}$(between s_i and s_{j+1}))-$\beta(E_{G_H}$(between s_i and s_j)) since $E_f(s_{g-i+1}) < E_f(s_{j+1})$.
By Eq. (18), it is easy to prove $\beta(E_{G_H}$(between s_{i-1} and s_j))-$\beta(E_{G_H}$(between s_i and s_j)) $= \beta(E_{G_H}$(between s_{i-1} and s_{g-i}))-$\beta(E_{G_H}$(between s_i and s_{g-i})).
Based on the above considerations, we obtain that $\beta(E_{G_H}$(between s_{i-1} and s_j))-$\beta(E_{G_H}$(between s_i and s_j)) $=$ $\beta(E_{G_H}$(between s_{i-1} and s_{g-i}))-$\beta(E_{G_H}$(between s_i and s_{g-i}))= $\beta(E_{G_H}$(between s_i and s_{g-i+1}))-$\beta(E_{G_H}$(between s_i and s_{g-i})) < $\beta(E_{G_H}$(between s_i and s_{j+1}))-$\beta(E_{G_H}$(between s_i and s_j)).

(3) It can be proved in a similar way of (2).

SUPPLEMENTARY MATERIAL

APPENDIX B
COMPUTING THE TYPE-2 FUZZY ENVELOPES: ILLUSTRATIVE EXAMPLES

- Type-2 fuzzy envelope for the $HFLTS$ $E_{G_H}(\text{at most } s_2) = \{s_0, s_1, s_2\}$ corresponds to $ll_2 = \text{at most } s_2$.

1. The type-1 fuzzy envelope for $E_{G_H}(\text{at most } s_2)$ is $F_{E_{G_H}(\text{at most } s_2)} = T(0, 0, 0.15, 0.5)$.
2. It is easy to obtain that $E_f(E_{G_H}(\text{at most } s_2)) = \frac{1}{3} \times \sum_{i=1}^{3} 4\frac{I(s_{\alpha_i})}{6}(1 - \frac{I(s_{\alpha_i})}{6}) \approx 0.48$ ($s_{\alpha_1} = s_0, s_{\alpha_2} = s_1, s_{\alpha_3} = s_2$);
$E_h(E_{G_H}(\text{at most } s_2)) = \frac{1}{6} \times \frac{2}{3(3-1)} \times \sum_{i=1}^{2}\sum_{j=i+1}^{3}(I(s_{\alpha_j}) - I(s_{\alpha_i})) \approx 0.22$ ($s_{\alpha_1} = s_0, s_{\alpha_2} = s_1, s_{\alpha_3} = s_2$);
$\beta(E_{G_H}(\text{at most } s_2)) = \frac{1}{2}sin(\frac{\pi}{6} \times 2 - \frac{\pi}{2}) + \frac{1}{2} = 0.25$ by Eq. (14),
therefore $E_c(E_{G_H}(\text{at most } s_2)) \approx 0.51$ by Eq. (16). If the result of theorem 5 is used, the computation can be significantly simplified.
3. The type-2 fuzzy envelope is given as an IT2 FS $\tilde{F}_{E_{G_H}(\text{at most } s_2)} = 1/FOU(\tilde{F}_{E_{G_H}(\text{at most } s_2)})$ with its footprint (see Fig. 7)

$$FOU(\tilde{F}_{E_{G_H}(\text{at most } s_2)}) = \{(x, u) : x \in [0, 1], u \in [max\{0, F_{E_{G_H}(\text{at most } s_2)}(x) - 0.51\}, F_{E_{G_H}(\text{at most } s_2)}(x)]\}.$$

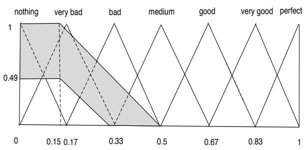

Fig. 7. FOU for the type-2 fuzzy envelope of the $HFLTS$ corresponds to ll_2

- Type-2 fuzzy envelope for the $HFLTS$ $E_{G_H}(\text{between } s_3 \text{ and } s_5) = \{s_3, s_4, s_5\}$ corresponds to $ll_3 = \text{between } s_3 \text{ and } s_5$.

1. The type-1 fuzzy envelope for $E_{G_H}(\text{between } s_3 \text{ and } s_5)$ is $F_{E_{G_H}(\text{between } s_3 \text{ and } s_5)} = T(0.33, 0.64, 0.70, 1)$.
2. It is easy to obtain that $E_f(E_{G_H}(\text{between } s_3 \text{ and } s_5)) = \frac{1}{3}\times\sum_{i=1}^{3} 4\frac{I(s_{\alpha_i})}{6}(1-\frac{I(s_{\alpha_i})}{6}) \approx 0.82$ ($s_{\alpha_1} = s_3, s_{\alpha_2} = s_4, s_{\alpha_3} = s_5$);
$E_h(E_{G_H}(\text{between } s_3 \text{ and } s_5)) = \frac{1}{6} \times \frac{2}{3(3-1)} \times \sum_{i=1}^{2}\sum_{j=i+1}^{3}(I(s_{\alpha_j}) - I(s_{\alpha_i})) \approx 0.22$ ($s_{\alpha_1} = s_3, s_{\alpha_2} = s_4, s_{\alpha_3} = s_5$);
$\beta(E_{G_H}(\text{between } s_3 \text{ and } s_5)) = \frac{1}{2}cos(\frac{\pi}{6} \times 3) + \frac{1}{2}sin(\frac{\pi}{6} \times 5 - \frac{\pi}{2}) \approx 0.43$ by Eq. (18),
therefore $E_c(E_{G_H}(\text{between } s_3 \text{ and } s_5)) \approx 0.84$ by Eq. (17).
3. The type-2 fuzzy envelope is given as an IT2 FS

$$\tilde{F}_{E_{G_H}(\text{between } s_3 \text{ and } s_5)} = 1/FOU(\tilde{F}_{E_{G_H}(\text{between } s_3 \text{ and } s_5)})$$

with its footprint (see Fig. 8)

$$FOU(\tilde{F}_{E_{G_H}(\text{between } s_3 \text{ and } s_5)}) = \{(x, u) : x \in [0, 1], u \in [max\{0, F_{E_{G_H}(\text{between } s_3 \text{ and } s_5)}(x) - 0.84\}, F_{E_{G_H}(\text{between } s_3 \text{ and } s_5)}(x)]\}.$$

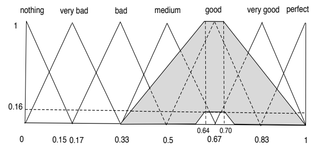

Fig. 8. FOU for the type-2 fuzzy envelope of the $HFLTS$ corresponds to ll_3

- Type-2 fuzzy envelope for the $HFLTS$ $E_{G_H}(\text{between } s_4 \text{ and } s_6) = \{s_4, s_5, s_6\}$ corresponds to $ll_4 = \text{between } s_4 \text{ and } s_6$.

1. The type-1 fuzzy envelope for $E_{G_H}(\text{between } s_4 \text{ and } s_6)$ is $F_{E_{G_H}(\text{between } s_4 \text{ and } s_6)} = T(0.5, 0.80, 0.86, 1)$.
2. It can be easily obtained that $E_f(E_{G_H}(\text{between } s_4 \text{ and } s_6)) \approx 0.48$, $E_h(E_{G_H}(\text{between } s_4 \text{ and } s_6)) \approx 0.22$, and $\beta(E_{G_H}(\text{between } s_4 \text{ and } s_6)) = \frac{1}{2}cos(\frac{\pi}{6} \times 4) + \frac{1}{2}sin(\frac{\pi}{6} \times 6 - \frac{\pi}{2}) = 0.25$ by Eq. (18),
therefore $E_c(E_{G_H}(\text{between } s_4 \text{ and } s_6)) \approx 0.51$ by Eq. (17).
3. The type-2 fuzzy envelope is given as an IT2 FS

$$\tilde{F}_{E_{G_H}(\text{between } s_4 \text{ and } s_6)} = 1/FOU(\tilde{F}_{E_{G_H}(\text{between } s_4 \text{ and } s_6)}).$$

with its footprint (see Fig. 9)

$$FOU(\tilde{F}_{E_{G_H}(\text{between } s_4 \text{ and } s_6)}) = \{(x, u) : x \in [0, 1], u \in [max\{0, F_{E_{G_H}(\text{between } s_4 \text{ and } s_6)}(x) - 0.51\}, F_{E_{G_H}(\text{between } s_4 \text{ and } s_6)}(x)]\}.$$

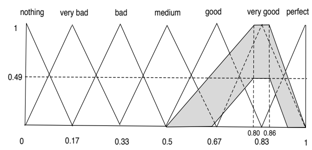

Fig. 9. FOU for the type-2 fuzzy envelope of the $HFLTS$ corresponds to ll_4

In the above example, the $HFLTS$, $\{s_4, s_5, s_6\}$ occupies two different type-1 fuzzy envelope $F_{E_{G_H}(\text{at least } s_4)} = T(0.5, 0.85, 1, 1)$ and $F_{E_{G_H}(\text{between } s_4 \text{ and } s_6)} = T(0.5, 0.80, 0.86, 1)$, meanwhile it occupies different type-2 fuzzy envelope $\tilde{F}_{E_{G_H}(\text{at least } s_4)}$ and $\tilde{F}_{E_{G_H}(\text{between } s_4 \text{ and } s_6)}$, according to different CLEs it represents. In real life cases, we believe that the representations of "at least s_4" and "between s_4 and s_6" should be different, since "at least s_4" and "between s_4 and s_6" means different things according to human-being's cognition/intuition. However, these two different linguistic expressions are translated into the same $HFLTS$ by using the transform function E_{G_H}. It is caused by the information losing during the transforming process from linguistic expressions to $HFLTS$. Constructing different envelopes for $HFLTS$ according to corresponding CLEs is a good way to reduce this kind of information losing.

APPENDIX C
THE CALCULATION PROCESS FOR DECISION MAKING WITH TYPE-1 FUZZY ENVELOPES OF $HFLTSs$.

(1) Transform the CLEs into $HFLTSs$ and their type-1 fuzzy envelopes.
The type-1 fuzzy envelopes for $HFLTSs$ transformed from CLEs can be listed as follows:
$F_{E_{G_H}(ll_{11})} = (0.17, 0.33, 0.5, 0.67)$;
$F_{E_{G_H}(ll_{12})} = (0.17, 0.33, 0.5, 0.67)$;
$F_{E_{G_H}(ll_{21})} = (0, 0, 0.15, 0.5)$;
$F_{E_{G_H}(ll_{22})} = (0.5, 0.85, 1, 1)$;
$F_{E_{G_H}(ll_{31})} = (0.33, 0.5, 0.67, 0.83)$;
$F_{E_{G_H}(ll_{32})} = (0.17, 0.33, 0.5, 0.67)$;
$F_{E_{G_H}(ll_{41})} = (0.17, 0.43, 0.73, 1)$;
$F_{E_{G_H}(ll_{42})} = (0, 0.27, 0.57, 0.83)$.
The above type-1 fuzzy envelopes remain unchanged after normalization and therefore the normalization process is omitted here.

(2) Calculate the distance from the type-1 fuzzy envelope of each $HFLTS$ to the fuzzy positive ideal solution $A^+ = (T(1,1,1,1), T(1,1,1,1))$, and the fuzzy negative ideal solution $A^- = (T(0,0,0,0), T(0,0,0,0))$.
The same geometrical distance [9] as in [12] is adopted here to calculate the distances, and the distances are listed as follows:
$D_1^+ = 4.66$, $D_1^- = 3.34$, $D_2^+ = 4$, $D_2^- = 4$, $D_3^+ = 4$, $D_3^- = 4$, $D_4^+ = 4$, $D_4^- = 4$.

(3) Calculate the closeness coefficient of each alternative.
$CC_1 \approx 0.418$, $CC_2 = 0.5$, $CC_3 = 0.5$, $CC_4 = 0.5$.

(4) Selection process.
In this phase, the alternatives are ranked according to their closeness coefficients:
$x_1 \prec x_2 = x_3 = x_4$.
Therefore, the best alternative is x_2 or x_3 or x_4.

APPENDIX D
THE CALCULATION PROCESS FOR DECISION MAKING WITH TYPE-2 FUZZY ENVELOPES OF $HFLTSs$

For the convenience of discussion, as follows all type-2 fuzzy numbers will be adjusted to the form as presented in [6]. That is, let an IT2TrFN $\tilde{A}_{ij}$ denote the assessment of the alternative $x_i \in X$ with respect to criteria $c_j \in C$. $\tilde{A}_{ij}$ is expressed as

$$\tilde{A}_{ij} = [\tilde{A}_{ij}^-, \tilde{A}_{ij}^+] = [(a_{1ij}^-, a_{2ij}^-, a_{3ij}^-, a_{4ij}^-; h_{\tilde{A}_{ij}}^-), (a_{1ij}^+, a_{2ij}^+, a_{3ij}^+, a_{4ij}^+; h_{\tilde{A}_{ij}}^+)],$$

where $\tilde{A}_{ij}^+$ and $\tilde{A}_{ij}^-$ denote the respective lower and upper membership functions of $\tilde{A}_{ij}$.

(1) Transform the CLEs into $HFLTSs$ and their type-2 fuzzy envelopes.
The type-2 fuzzy envelopes for $HFLTSs$ transformed from CLEs can be listed as follows:

$$\tilde{A}_{11} = [\tilde{A}_{11}^-, \tilde{A}_{11}^+] = [(0.32, 0.33, 0.5, 0.51; 0.05), (0.17, 0.33, 0.5, 0.67; 1)],$$

$$\tilde{A}_{12} = [\tilde{A}_{12}^-, \tilde{A}_{12}^+] = [(0.32, 0.33, 0.5, 0.51; 0.05), (0.17, 0.33, 0.5, 0.67; 1)],$$

$$\tilde{A}_{21} = [\tilde{A}_{21}^-, \tilde{A}_{21}^+] = [(0, 0, 0.15, 0.32; 0.49), (0, 0, 0.15, 0.5; 1)],$$

$$\tilde{A}_{22} = [\tilde{A}_{22}^-, \tilde{A}_{22}^+] = [(0.68, 0.85, 1, 1; 0.49), (0.5, 0.85, 1, 1; 1)],$$

$$\tilde{A}_{31} = [\tilde{A}_{31}^-, \tilde{A}_{31}^+] = [(0.49, 0.5, 0.67, 0.68; 0.05), (0.33, 0.5, 0.67, 0.83; 1)],$$

$$\tilde{A}_{32} = [\tilde{A}_{32}^-, \tilde{A}_{32}^+] = [(0.32, 0.33, 0.5, 0.51; 0.05), (0.17, 0.33, 0.5, 0.67; 1)],$$

$$\tilde{A}_{41} = [\tilde{A}_{41}^-, \tilde{A}_{41}^+] = [(0.39, 0.43, 0.73, 0.77; 0.14), (0.17, 0.43, 0.73, 1; 1)],$$

$$\tilde{A}_{42} = [\tilde{A}_{42}^-, \tilde{A}_{42}^+] = [(0.23, 0.27, 0.57, 0.61; 0.14), (0, 0.27, 0.57, 0.83; 1)].$$

(2) Calculate the negative-ideal solution and positive-ideal solution.
Since the criteria considered in this example are all benefit criteria, according to [6], the negative-ideal solution $\tilde{A}_{\eta j}$ and positive-ideal solution $\tilde{A}_{\rho j}$ with respect to each criteria $c_j (j = 1, 2)$ are defined as following:

$$\tilde{A}_{\eta j} = [\tilde{A}_{\eta j}^-, \tilde{A}_{\eta j}^+] = [(a_{1\eta j}^-, a_{2\eta j}^-, a_{3\eta j}^-, a_{4\eta j}^-; h_{\tilde{A}_{\eta j}}^-), (a_{1\eta j}^+, a_{2\eta j}^+, a_{3\eta j}^+, a_{4\eta j}^+; h_{\tilde{A}_{\eta j}}^+)]$$

where

$$\tilde{A}_{\eta j}^- = (\wedge_{i=1}^m a_{1ij}^-, \wedge_{i=1}^m a_{2ij}^-, \wedge_{i=1}^m a_{3ij}^-, \wedge_{i=1}^m a_{4ij}^-; \wedge_{i=1}^m h_{\tilde{A}_{ij}}^-)$$

and

$$\tilde{A}_{\eta j}^+ = (\wedge_{i=1}^m a_{1ij}^+, \wedge_{i=1}^m a_{2ij}^+, \wedge_{i=1}^m a_{3ij}^+, \wedge_{i=1}^m a_{4ij}^+; \wedge_{i=1}^m h_{\tilde{A}_{ij}}^+);$$

meanwhile,

$$\tilde{A}_{\rho j} = [\tilde{A}_{\rho j}^-, \tilde{A}_{\rho j}^+] = [(a_{1\rho j}^-, a_{2\rho j}^-, a_{3\rho j}^-, a_{4\rho j}^-; h_{\tilde{A}_{\rho j}}^-), (a_{1\rho j}^+, a_{2\rho j}^+, a_{3\rho j}^+, a_{4\rho j}^+; h_{\tilde{A}_{\rho j}}^+)]$$

where

$$\tilde{A}_{\rho j}^- = (\vee_{i=1}^m a_{1ij}^-, \vee_{i=1}^m a_{2ij}^-, \vee_{i=1}^m a_{3ij}^-, \vee_{i=1}^m a_{4ij}^-; \wedge_{i=1}^m h_{\tilde{A}_{ij}}^-)$$

and

$$\tilde{A}^{+}_{\rho j} = (\vee_{i=1}^{m} a^{+}_{1ij}, \vee_{i=1}^{m} a^{+}_{2ij}, \vee_{i=1}^{m} a^{+}_{3ij}, \vee_{i=1}^{m} a^{+}_{4ij}; \wedge_{i=1}^{m} h^{+}_{\tilde{A}_{ij}}).$$

In this case, it is easy to obtain that

$\tilde{A}_{\eta 1} = [\tilde{A}^{-}_{\eta 1}, \tilde{A}^{+}_{\eta 1}] = [(0, 0, 0.15, 0.32; 0.05), (0, 0, 0.15, 0.5; 1)]$,

$\tilde{A}_{\eta 2} = [\tilde{A}^{-}_{\eta 2}, \tilde{A}^{+}_{\eta 2}] = [(0.23, 0.27, 0.5, 0.51; 0.05), (0, 0.27, 0.5, 0.67; 1)]$,

meanwhile,

$\tilde{A}_{\rho 1} = [\tilde{A}^{-}_{\rho 1}, \tilde{A}^{+}_{\rho 1}] = [(0.49, 0.5, 0.73, 0.77; 0.05), (0.33, 0.5, 0.73, 1; 1)]$,

$\tilde{A}_{\rho 2} = [\tilde{A}^{-}_{\rho 2}, \tilde{A}^{+}_{\rho 2}] = [(0.68, 0.85, 1, 1; 0.05), (0.5, 0.85, 1, 1; 1)]$.

(3) Calculate the likelihood of IT2TrF binary relation.
Let

$\tilde{A}_{ij} (= [\tilde{A}^{-}_{ij}, \tilde{A}^{+}_{ij}] = [(a^{-}_{1ij}, a^{-}_{2ij}, a^{-}_{3ij}, a^{-}_{4ij}; h^{-}_{\tilde{A}_{ij}}), (a^{+}_{1ij}, a^{+}_{2ij}, a^{+}_{3ij}, a^{+}_{4ij}; h^{+}_{\tilde{A}_{ij}})]),$

$\tilde{B}_{ij} (= [\tilde{B}^{-}_{ij}, \tilde{B}^{+}_{ij}] = [(b^{-}_{1ij}, b^{-}_{2ij}, b^{-}_{3ij}, b^{-}_{4ij}; h^{-}_{\tilde{B}_{ij}}), (b^{+}_{1ij}, b^{+}_{2ij}, b^{+}_{3ij}, b^{+}_{4ij}; h^{+}_{\tilde{B}_{ij}})])$

be any two $IT2TrF$ numbers in X. Let ς be a positive integer. Assume that at least one of $h^{-}_{A} \neq h^{+}_{B}$, $a^{-}_{4} \neq a^{-}_{1}$, $b^{+}_{4} \neq b^{+}_{1}$, and $a^{-}_{\varsigma} \neq b^{+}_{\varsigma}$ holds, and at least one of $h^{+}_{A} \neq h^{-}_{B}$, $a^{+}_{4} \neq a^{+}_{1}$, $b^{-}_{4} \neq b^{-}_{1}$, and $a^{+}_{\varsigma} \neq b^{-}_{\varsigma}$ holds, where $\varsigma = 1, 2, 3, 4$.
The lower likelihood $L^{-}(\tilde{A} \geq \tilde{B})$ of an IT2TrF binary relation $\tilde{A} \geq \tilde{B}$ is defined by Eq. (19).

$$L^{-}(\tilde{A} \geq \tilde{B}) = max\{1 - max[\frac{\sum_{\varsigma=1}^{4} max(b^{+}_{\varsigma} - a^{-}_{\varsigma}, 0) + (b^{+}_{4} - a^{-}_{1}) + 2max(h^{+}_{B} - h^{-}_{A}, 0)}{\sum_{\varsigma=1}^{4} |b^{+}_{\varsigma} - a^{-}_{\varsigma}| + (a^{-}_{4} - a^{-}_{1}) + (b^{+}_{4} - b^{+}_{1}) + 2|h^{+}_{B} - h^{-}_{A}|}, 0], 0\} \quad (19)$$

The upper likelihood $L^{-}(\tilde{A} \geq \tilde{B})$ of an IT2TrF binary relation $\tilde{A} \geq \tilde{B}$ is defined by Eq. (20).

$$L^{+}(\tilde{A} \geq \tilde{B}) = max\{1 - max[\frac{\sum_{\varsigma=1}^{4} max(b^{-}_{\varsigma} - a^{+}_{\varsigma}, 0) + (b^{-}_{4} - a^{+}_{1}) + 2max(h^{-}_{B} - h^{+}_{A}, 0)}{\sum_{\varsigma=1}^{4} |b^{-}_{\varsigma} - a^{+}_{\varsigma}| + (a^{+}_{4} - a^{+}_{1}) + (b^{-}_{4} - b^{-}_{1}) + 2|h^{-}_{B} - h^{+}_{A}|}, 0], 0\} \quad (20)$$

The likelihood $L(\tilde{A} \geq \tilde{B})$ of an IT2TrF binary relation $\tilde{A} \geq \tilde{B}$ is defined by:

$$L(\tilde{A} \geq \tilde{B}) = \frac{L^{-}(\tilde{A} \geq \tilde{B}) + L^{+}(\tilde{A} \geq \tilde{B})}{2} \quad (21)$$

In this case, the necessary likelihoods of IT2TrF binary relations are listed as follows, which will be adopted in Eq. (22):
$L(\tilde{A}_{11} \geq \tilde{A}_{\eta 1}) \approx 0.692$, $L(\tilde{A}_{21} \geq \tilde{A}_{\eta 1}) \approx 0.524$, $L(\tilde{A}_{31} \geq \tilde{A}_{\eta 1}) \approx 0.777$, $L(\tilde{A}_{41} \geq \tilde{A}_{\eta 1}) \approx 0.770$,
$L(\tilde{A}_{12} \geq \tilde{A}_{\eta 2}) \approx 0.568$, $L(\tilde{A}_{22} \geq \tilde{A}_{\eta 2}) \approx 0.876$, $L(\tilde{A}_{32} \geq \tilde{A}_{\eta 2}) \approx 0.568$, $L(\tilde{A}_{42} \geq \tilde{A}_{\eta 2}) \approx 0.544$;
$L(\tilde{A}_{\rho 1} \geq \tilde{A}_{11}) \approx 0.679$, $L(\tilde{A}_{\rho 1} \geq \tilde{A}_{21}) \approx 0.789$, $L(\tilde{A}_{\rho 1} \geq \tilde{A}_{31}) \approx 0.534$, $L(\tilde{A}_{\rho 1} \geq \tilde{A}_{41}) \approx 0.556$,
$L(\tilde{A}_{\rho 2} \geq \tilde{A}_{12}) \approx 0.793$, $L(\tilde{A}_{\rho 2} \geq \tilde{A}_{22}) \approx 0.449$, $L(\tilde{A}_{\rho 2} \geq \tilde{A}_{32}) \approx 0.793$, $L(\tilde{A}_{\rho 2} \geq \tilde{A}_{42}) \approx 0.778$.

(4) Calculate the likelihood based closeness coefficient of each alternative.
Since the criteria considered in this example are all benefit criteria, according to [6], the likelihood based closeness coefficient can be calculated by

$$LC_i = \frac{\sum_{j=1}^{n} L(\tilde{A}_{ij} \geq \tilde{A}_{\eta j})}{\sum_{j=1}^{n} (L(\tilde{A}_{\rho j} \geq \tilde{A}_{ij}) + L(\tilde{A}_{ij} \geq \tilde{A}_{\eta j}))} \quad (22)$$

It is easy to obtain that $LC_1 \approx 0.461$, $LC_2 \approx 0.531$, $LC_3 \approx 0.503$, $LC_4 \approx 0.496$. (The process for determining likelihood-based comparison indexes has been omitted here, considering that all criteria are of benefit type.)

(5) Selection process.
In this phase, the alternatives are ranked according to the likelihood based closeness coefficients:
$x_1 \prec x_4 \prec x_3 \prec x_2$.
Therefore, the best alternative is x_2.

APPENDIX E
MORE EXAMPLES

If the problem assessment I change to assessment II as is shown by Table III, the ranking obtained by using type-1 fuzzy envelopes is $x_2 \prec x_1 \prec x_4 = x_3$, whereas the ranking obtained by using type-2 fuzzy envelopes is $x_2 \prec x_1 \prec x_4 \prec x_3$. If the problem assessment I change to assessment III as is shown by Table IV, the ranking obtained by using type-1 fuzzy envelopes is $x_2 \prec x_4 = x_3 \prec x_1$, whereas the ranking obtained by using type-2 fuzzy envelopes is $x_2 \prec x_4 \prec x_3 \prec x_1$.

TABLE III
ASSESSMENTS (II) OF THE PROBLEM.

-	c_1	c_2
x_1	between M and VG	between VB and B
x_2	at most B	between M and VG
x_3	between M and G	between B and M
x_4	between B and VG	between VB and G

TABLE IV
ASSESSMENTS (III) OF THE PROBLEM.

-	c_1	c_2
x_1	at most B	at least G
x_2	at most B	between M and VG
x_3	between M and G	between B and M
x_4	between B and VG	between VB and G

Chapter 5

Conclusions and Future Works

Finally, this section concludes the current research memory, reviews the main proposals and results, and points out some future works.

5.1 Conclusions

Linguistic DM problems under uncertainty are common in our daily life. The diversity and complexity of uncertainties calls for useful mathematic models to deal with these problems. The generalized models of soft sets obtained by combing them with fuzzy sets, rough sets and linguistic models show great potential dealing with uncertain DM situations, since the hybrid models take advantages of different models and therefore enhance the ability for dealing with diverse uncertainties.

For DM approaches based on fuzzy soft sets and rough soft sets, there are still arguments that need to be settled and limitations need to be overcome. By providing the following proposals, we successfully reach the first objective (see Section 1.2):

1. An analysis on limitations for fuzzy soft sets based DM approaches has been carried out. One of the popular DM approaches based on fuzzy soft sets, called the score based approach, has been improved by introducing the concepts of D-Score and D-Score table. The improved approach decreases successfully the time consumption when parameters need to be deleted/added during the process of DM.

2. An adjustable approach based on fuzzy soft sets has been proposed by introducing threshold values or threshold fuzzy sets when the scores for alternatives are computed. The proposed approach can be used to solve problems which cannot be handled by existing ones.

3. An analysis on limitations for rough soft sets based DM approaches has been carried out. Two DM approaches and one GDM approach based on rough soft sets have been introduced. In the GDM approach, the weights of decision makers are obtained by using similarity measures between soft sets.

Various soft rough set models have been proposed in the literature by constructing different approximation operators on crisp sets [15, 63] or fuzzy sets [15, 49, 83]. However, no systematic research on them has been carried out about the connections among them. By carrying out the following researches, the second objective (see Section 1.2) has been reached:

1. The relationships among various soft rough approximations have been discussed in a systematic way.

2. A novel model called *soft rough soft sets* has been proposed by using a soft set as the knowledge to compute the approximations of another soft set. The application of soft rough soft sets in DM has been illustrated by using an example.

Although a model called linguistic value soft sets was already introduced in the literature, which combines soft set theory with linguistic information, this model only allows decision makers to provide initial assessments by using single linguistic terms, which limits the elicitation of linguistic information because sometimes decision makers need to use more complex expressions to express their knowledge. To facilitate the elicitation of more complex linguistic expressions with soft set models, it is necessary to construct new hybrid soft set models that combines soft set theory and CLEs. By providing the following proposals we successfully reach the third objective (see Section 1.2):

1. A novel model called HLE soft sets has been introduced, in which assessments of decision makers could be both linguistic terms or CLEs. A decision making approach based on CLEs soft sets has been presented.

2. A GDM approach based on HLE soft sets has been introduced. A consensus model to cooperate with the GDM process is proposed. Comparisons of our proposed GDM approach and existing approaches based on linguistic value soft set have been done to show some advantages of the proposed model.

New fuzzy representation models for CLEs need to be constructed to deal with linguistic uncertainties. These representation models are expected to facilitate the CW processes when handling with DM problems in which experts provide evaluations on alternatives using CLEs. By constructing the following representation model or CLEs and comparing it with some existing models, the fourth objective (see Section 1.2) has been reached:

1. A new representation model for CLEs called type-2 fuzzy envelope has been introduced. This new representation model follows the fuzzy linguistic approach and can successfully reflect the uncertainties contained in CLEs.

2. The comparison between type-2 fuzzy envelope and type-1 fuzzy envelope has been carried out by using an illustration example in DM. Finally it has been shown that the decision results are consistent however the result is more precise when type-2 fuzzy envelope is used.

5.2 Future works

Despite several proposals have been made in this research, there are still some challenges to deal with DM and GDM problems under uncertainties by using hybrid soft set models. In the near future, we will concentrate on the extension of the proposals presented:

1. To research in deep the inner relationships among different hybrid soft set models.

2. By using different hybrid soft set models and corresponds decision making methodologies, different decision results could be obtained. To explore the relationships among different decision results by using the relationships among different models.

3. To carry out the CW processes in GDM under the framework of HLE soft sets by using type-2 fuzzy TOPSIS method [10] and type-2 fuzzy envelopes for HFLTSs. Deal with a GDM problem based on a linguistic value soft set, and then solve the problem by using type-1 fuzzy TOPSIS [9] (with type-1 fuzzy envelopes for HFLTSs) and type-2 fuzzy TOPSIS (with type-2 fuzzy envelopes for HFLTSs), respectively. To make a comparison on different decision results when two decision schemes are applied, and analyze the advantages of the application of type-2 fuzzy envelope.

4. To research in further detail the construction of representation models for linguistic expressions and apply them to hybrid soft sets based decision making. IT2 FSs are special cases of general type-2 FSs, it would be interesting to see how general T2 FSs could be constructed to serve as representation models for CLEs.

Additional Publications

Regarding the diffusion of our scientific results, besides the publications included in this memory, we highlight the following contributions:

- International Journals
  - Y Liu, K Qin. Object-parameter approaches to predicting unknown data in incomplete fuzzy soft sets. *International Journal of Applied Mathematics and Computer Science*, vol. 27, issue 1, pp. 157-167, 2017.
  - Y Liu, J Luo, B Wang, K Qin. A theoretical development on the entropy of interval-valued intuitionistic fuzzy soft sets based on the distance measure. *International Journal of Computational Intelligence Systems*, vol. 10, issue 1, pp. 569, 2017.
  - A Labella, Y Liu, R M Rodríguez, L Martínez. Analyzing the Performance of Classical Consensus Models in Large Scale Group Decision Making: A comparative Study. *Applied Soft Computing*, vol. 67, issue C, pp. 677-690, 2018.
- International Conferences
  - Y Liu, R M Rodríguez, K Qin, L Martínez. Improved score based decision making method by using fuzzy soft sets. *The 13th International FLINS Conference on Data Science an knowledge Engineering for Sensing Decision Support (FLINS 2018) held on Belfast (UK) in August 21-24, 2018.*

Appendix A

Resumen escrito en Español

Título de la tesis: *Toma de decisiones lingüística basada en modelos matemáticos híbridos.*

Este apéndice incluye el título, índice, introducción, resumen y conclusiones escritas en español como parte de los requisitos necesarios para obtener el doctorado según el artículo 23.2 del Reglamento de Estudios de Doctorado de la Universidad de Jaén.

En primer lugar se muestra el índice de esta memoria de investigación. A continuación se introduce brevemente la investigación llevada acabo, indicando la motivación, objetivos planteados y la estructura en capítulos que componen esta tesis. Se presenta también un resumen de la misma, y finalmente se describen las conclusiones obtenidas y trabajos futuros.

Contenidos

A.2 Motivación

Los modelos matemáticos clásicos no son capaces de resolver problemas reales de toma de decisiones que presentan información vaga e incierta. Existen algunos enfoques matemáticos tales como, la teoría de conjuntos difusos, la teoría *soft sets* y la teoría *rough sets*, que han sido ampliamente utilizados en problemas de toma de decisiones bajo incertidumbre, sin embargo, presentan una limitación ya que no tienen herramientas de parametrización. Esta limitación indica que individualmente estos modelos no pueden considerar alternativas con diferentes aspectos en los parámetros. Molodstov [50] propuso un modelo llamado *soft set* (conjuntos suaves), que evita esta limitación satisfactoriamente. La combinación de este modelo con otros ha generado la aparición de nuevos modelos generalizados para modelar diferentes tipos de incertidumbre.

Los modelos *soft sets híbridos* pueden ser clasificados en dos categorías: (i) modelos híbridos obtenidos mediante la combinación de conjuntos difusos (y modelos generalizados de conjuntos difusos) con *soft sets*; y (ii) modelos híbridos obtenidos de la combinación de *rough sets* (y modelos generalizados de rough sets) con *soft sets*. Los *soft sets* pertenecen a la primera categoría, mientras *rough soft sets* y *soft rough sets* pertenecen a la segunda categoría. Estos modelos *soft sets híbridos* son bastante simples, por tanto, se han propuesto modelos híbridos más complejos para generalizarlos. Por ejemplo, *intuitionistic fuzzy soft sets* [36] e *interval valued intuitionistic fuzzy soft sets* [28] podrían ser vistos como extensiones de *fuzzy soft sets*. Jiang et al. [27] y Zhang et al. [85] extendieron el modelo de toma de decisiones basado en *fuzzy soft sets* introducido por Feng et al. [14] definiendo dos modelos de toma de decisiones, uno basado en *intucionistic fuzzy soft sets* y el otro basado en *interval-valued intuitionistic fuzzy soft sets*.

Los dos enfoques más populares de toma de decisiones basados en *fuzzy soft sets* son: i) el enfoque basado en *choice value* [30] y ii) el enfoque basado en *score* [61]. Diferentes investigadores han discutido sobre cual de ellos es el más razonable [14, 30], aunque ambos presentan algunas limitaciones. Por ejemplo, el enfoque basado en *score* introducido por Roy y Maje en [61] requiere una gran cantidad de cálculos cuando se añaden o se eliminan parámetros durante el proceso de toma de decisiones, ésto causa algunas limitaciones cuando los problemas tienen información dinámica. Los estudios realizados sobre toma de decisiones y toma de decisiones en grupo basados en *rough soft sets* están aún en una fase inicial. No hay métodos que permitan a los expertos proporcionar sus valoraciones sobre las alternativas utilizando la combinación de *rough sets* y *soft sets*. Los modelos mencionados anteriormente requieren que cada experto proporcione una decisión óptima antes de aplicar un proceso de toma de decisiones en grupo. Teniendo en cuenta estas limitaciones, vemos necesario realizar un estudio sistemático para mejorar los enfoques de toma de decisiones basados en *fuzzy soft sets* y *rough soft sets*.

Aunque varios algoritmos basados en los modelos *soft rough sets* y sus extensiones difusas han sido propuestos para resolver problemas de toma de decisiones [81, 82, 83], no se han realizado aún estudios de la relación entre ellos. Para hacer más flexible la aplicación de estos modelos, *soft rough sets*, en toma de decisiones y poder aplicar el más adecuado dependiendo de cada problema, es importante tener en cuenta estas relaciones.

En algunos problemas de toma de decisiones del mundo real, los expertos pueden usar información lingüística en lugar de valores *crisp* para proporcionar sus valoraciones sobre las alternativas. Entre los distintos modelos *soft sets híbridos*, el modelo *linguistic value soft set* es el único que podría usarse para modelar la información lingüística en el marco de trabajo de *soft sets*. Sin embargo, si aplicamos este modelo, los expertos que participan en el problema de toma de decisiones siempre tienen que proporcionar sus valoraciones mediante términos lingüísticos simples, lo que puede ser difícil en algunas situaciones donde los expertos duden entre varios términos lingüísticos y el uso de un único término lingüístico no es suficiente para reflejar su conocimiento de forma adecuada. Por tanto, es conveniente definir nuevos modelos *soft sets híbridos* capaces de utilizar no sólo términos lingüísticos simples sino también expresiones lingüísticas más complejas.

Existen diferentes enfoques para modelar la información lingüística, uno de ellos es el enfoque lingüístico difuso [78] que proporciona un método directo para modelar la incertidumbre mediante variables lingüísticas. Este enfoque ha sido ampliamente utilizado en problemas de toma de decisiones en los que se han obtenido muy buenos resultados. En el enfoque lingüístico difuso las palabras significan cosas diferentes para diferentes personas, por tanto, un conjunto difuso es utilizado para capturar la incertidumbre contenida en una palabra. Sin embargo, la mayoría de los modelos lingüísticos [39, 40] están limitados, ya que los expertos proporcionan sus preferencias mediante términos lingüísticos simples definidos a priori y en algunas situaciones, debido a la falta de información, o presión del tiempo, los expertos pueden dudar entre varios términos lingüísticos por lo que el uso de un único término lingüístico no es suficiente para expresar sus opiniones. Para evitar esta limitación, recientemente se ha introducido un modelo llamado conjunto de términos lingüísticos difuso dudoso (CTLDD) [58]. También se definió una gramática libre de contexto para generar expresiones lingüísticas comparativas cercanas al modelo cognitivo de los seres humanos y una función de transformación que transforma las expresiones lingüísticas comparativas en CTLDD [58]. El uso de las expresiones lingüísticas comparativas basadas en CTLDD permite a los expertos expresar sus valoraciones de una forma más flexible y elaborada que los términos lingüísticos simples. Por tanto, parece interesante investigar el uso de expresiones lingüísticas comparativas y definir nuevos modelos soft sets capaces de utilizar dichas expresiones.

El uso de expresiones lingüísticas comparativas en estos modelos híbridos implica realizar procesos computacionales, y al igual que sucede con los términos lingüísticos simples en los procesos de computación con palabras, las expresiones lingüísticas comparativas significan diferentes cosas para diferentes personas. De ahí, que para modelar la incertidumbre contenida en una expresión lingüística comparativa, sea necesario construir un modelo de representación adecuado para CTLDD. Hasta ahora, los modelos de representación para CTLDD están basados en intervalos lingüisticos [58] o conjuntos difusos tipo-1 [33], pero ninguno de estos modelos tiene en cuenta la duda y la incertidumbre contenida en los CTLDD, lo que puede causar pérdida de información cuando las expresiones lingüísticas comparativas se usan en toma de decisiones. Por tanto, es necesario definir nuevos modelos de representación para CTLDD que puedan reflejar y modelar la incertidumbre lingüística de una forma más adecuada.

A.3 Objetivos

Teniendo en cuenta la motivación y consideraciones mencionadas en la sección anterior, el propósito de esta investigación se centra en mejorar las metodologías de modelos matemáticos híbridos en toma de decisiones, concretamente en toma de decisiones con información lingüística.

Los objetivos que perseguimos para alcanzar este propósito son los siguientes:

1. Realizar un estudio comparativo de los enfoques de toma de decisiones basados en *fuzzy soft sets* y *rough soft sets* existentes, destacando y analizando sus limitaciones. Proponer nuevas metodologías para evitar estas limitaciones, así como explorar nuevos enfoques basados en modelos híbridos que satisfagan diferentes demandas en aplicaciones reales.

2. Realizar un estudio comparativo de los modelos *soft rough sets* existentes, así como de sus extensiones difusas, analizando la relación entre los diferentes modelos y destacando su uso en toma de decisiones. Investigar nuevas formas de combinar la teoría soft set con la teoría rough set y proponer nuevos modelos *soft rough set*. Estudiar la aplicación de los modelos propuestos en toma de decisiones.

3. Definir un nuevo modelo *soft set híbrido* capaz de utilizar expresiones lingüísticas

comparativas para mejorar la elicitación de información lingüística y construir nuevos algorítmos basados en el modelo propuesto para resolver problemas de toma de decisiones en grupo. Una vez definidos estos algoritmos, es necesario examinar su funcionalidad comparándolos con los algoritmos existentes basados en otros modelos *soft sets híbridos*.

4. Por último, construir un nuevo modelo difuso para representar expresiones lingüísticas comparativas. Este modelo puede utilizarse para reflejar y modelar la incertidumbre lingüística contenida en tales expresiones. Dado que las expresiones lingüísticas comparativas pueden transformarse en CTLDD, el nuevo modelo de representación basado en conjuntos difusos tipo-2 debería reflejar y modelar ambos tipos de incertidumbre: difusa y dudosa. Además, examinaremos la funcionalidad del nuevo modelo de representación para expresiones lingüísticas comparativas y lo compararemos con otro modelo de representación basado en conjuntos difusos tipo-1.

A.4 Estructura

Para alcanzar los objetivos planteados y según lo establecido en el artículo 23, punto 3, de la normativa vigente para los Estudios de Doctorado en la Universidad de Jaén, correspondiente al programa establecido en el RD 99/2011, esta investigación será presentada como un conjunto de artículos publicados por el estudiante de doctorado.

Dichas publicaciones constituyen el núcleo de la tesis y corresponden a dos artículos científicos publicados en revistas internacionales indexadas por la base de datos JCR (Journal Citation Reports), producida por ISI (Institute for Scientfic Information), junto con otros dos artículos que se encuentran sometidos bajo revisión en dos revistas internationales también indexadas por JCR al finalizar esta memoria. Por tanto, la memoria se compone de un total de cuatro publicaciones, dos de ellas publicadas en revistas de reconocido prestigio.

A continuación hacemos una breve descripción de la estructura de esta memoria:

- Capítulo 2: Este capítulo revisa los conceptos teóricos que son utilizados en nuestras propuestas para alcanzar los objetivos planteados: definición de soft sets, fuzzy soft sets, rough soft sets y otros modelos *soft sets híbridos*; el concepto de enfoque lingüístico difuso, CTLDD y expresiones lingüísticas comparativas generadas mediante una gramática libre de contexto.

- Capítulo 3: Este capítulo introduce brevemente las propuestas publicadas y sometidas que forman parte de esta memoria de investigación. Para cada artículo, se realiza una breve discusión de los resultados obtenidos.

- Capítulo 4: Constituye el núcleo de la tesis doctoral, incluyendo un compendio de las publicaciones obtenidas como resultado de la investigación realizada. Para cada publicación se indican los índices de calidad donde la propuesta ha sido publicada.

- Capítulo 5: Expone las conclusiones finales extraidas de esta investigación y propuestas para trabajos futuros.

A.5 Resumen

Existen diferentes modelos matemáticos híbridos, obtenidos de la combinación de *soft sets* con otros modelos, tales como conjuntos difusos y *rough sets*, llamados *soft sets híbridos*. Estos modelos han sido recientemente aplicados a problemas de toma de decisiones. Sin embargo, presentan algunas limitaciones cuando los problemas se definen en contextos cualitativos y es necesario usar información lingüística. Por tanto, esta tesis se centra en mejorar las metodologías de los modelos existentes, así como proponer nuevos modelos *soft sets híbridos* capaces de modelar distintos tipos de incertidumbre. También se definen modelos *soft sets híbridos* para resolver problemas de toma de decisiones con información lingüística. Para ello, se presentan las siguientes propuestas:

1. Se introducen metodologías para mejorar algunos enfoques de toma de decisiones basados en *fuzzy soft sets* y *rough soft sets*. Se proponen nuevos modelos de toma de decisiones basados en estos dos modelos matemáticos híbridos para hacer frente a algunas limitaciones existentes.

2. Se realiza un estudio sistemático de la relación entre diferentes modelos *soft rough set*. Se propone un modelo matemático híbrido llamado *soft rough soft sets* el cual es aplicado a toma de decisiones.

3. Se define un modelo matemático híbrido llamado, *hesitant linguistic expression soft set*, que combina la teoría *soft set* con el enfoque lingüístico difuso. Este modelo es capaz de modelar la duda que pueden tener los expertos cuando expresan sus valoraciones y el uso de un término lingüístico no es suficiente para reflejar su conocimiento. Este modelo se aplica a problemas de toma de decisones y toma de decisiones en grupo. Además, se introduce un modelo de consenso basado en *hesitant linguistic expression soft set* para obtener soluciones consensuadas y aceptadas por todos los participantes.

4. Se construye un nuevo modelo de representación para expresiones lingüisticas comparativas basado en conjuntos difusos tipo-2 para modelar la incertidum-

bre contenida en dichas expresiones. Este nuevo modelo facilita los procesos de computación con palabras en problemas de toma de decisiones lingüísticos. Se realiza también un estudio comparativo entre el modelo tipo-2 propuesto y otro modelo de representación existente basado en tipo-1, en el que se muestra que el modelo propuesto obtiene resultados más precisos.

A.6 Conclusiones y Trabajos Futuros

Esta sección cierra la memoría de investigación revisando las diferentes conclusiones obtenidas de las propuestas que se han realizado en la misma y exponiendo líneas de investigación sobre trabajos futuros que podrían realizarse partiendo de los resultados presentados en ella. Finalmente, se indican las publicaciones adicionales derivadas de la investigación realizada.

A.6.1 Conclusiones

Los problemas de toma de decisiones bajo incertidumbre son comunes en nuestra vida diaria. La diversidad y complejidad de la incertidumbre hace necesario el uso de modelos matemáticos que sean capaces de resolver este tipo de problemas. La generalización de modelos *soft sets* obtenidos de la combinación de éstos con conjuntos difusos, *rough sets* y modelos lingüísticos, muestran gran potencial para modelar la incertidumbre que aparece en problemas de toma de decisiones, ya que los modelos híbridos presentan ventajas de diferentes modelos y por tanto, fortalecen su capacidad para modelar diversos tipos de incertidumbre.

Los enfoques de toma de decisiones basados en *fuzzy soft sets* y *rough soft sets* existentes en la literatura, presentan algunas limitaciones que necesitan ser evitadas. Por tanto, el primer objetivo planteado en la sección 1.2 se ha alcanzado mediante las siguientes propuestas.

1. Hemos realizado un análisis sobre las limitaciones de los enfoques de toma de decisiones basados en *fuzzy soft sets*. Uno de los enfoques de toma de decisiones basados en *fuzzy soft sets* más populares, llamado enfoque basado en *score* ha sido mejorado introduciendo los conceptos de *D-Score* y *D-Score table*. Este nuevo enfoque decrementa el coste temporal cuando añadimos o eliminamos parámetros durante el proceso de toma de decisiones.

2. Se ha propuesto un enfoque adaptativo de *fuzzy soft sets* con umbrales o umbrales difusos al calcular las valoraciones de las alternativas. El enfoque propuesto puede resolver problemas que no puedían ser resueltos con los modelos previos.

3. Hemos estudiado las limitaciones de los enfoques de toma de decisiones basados en *rough soft sets* y hemos definido dos enfoques de toma de decisiones y otro de toma de decisiones en grupo basados en *rough soft sets*. En este último enfoque, los pesos de los expertos que participan en el problema se obtienen mediante medidas de similitud entre *soft sets*.

En la literatura se han propuesto varios modelos de *soft rough sets* mediante la construcción de diferentes operadores de aproximación sobre conjuntos clásicos [15, 63] o conjuntos difusos [15, 49, 83]. Sin embargo, no se ha realizado un estudio sistemático sobre las relaciones que existen entre ellos. Para alcanzar el segundo objetivo indicado en la sección 1.2, hemos realizado los siguientes estudios.

1. Hemos analizado y discutido de forma sistemática las relaciones entre varios modelos de *soft rough*.

2. Hemos propuesto un nuevo modelo llamado *soft rough soft sets* utilizando un soft set para representar el conocimiento y calcular las aproximaciones a otro soft set. Un ejemplo ilustrativo del nuevo modelo aplicado a toma de decisiones ha sido mostrado.

A pesar de que existe un modelo llamado *linguistic value soft sets* que combina la teoría *soft set* con información lingüística, este modelo sólo permite que los expertos expresen sus valoraciones mediante términos lingüísticos simples, ésto limita la elicitación de información lingüística ya que en algunas ocasiones los expertos necesitan usar expresiones más complejas que términos lingüísticos simples para expresar su conocimiento. Para facilitar la elicitación de expresiones lingüísticas más complejas mediante el uso de modelos *soft sets*, es necesario construir nuevos modelos *soft set* híbridos que combinen la teoría *soft sets* con expresiones lingüísticas comparativas. Este objetivo (sección 1.2) se ha alcanzado con las siguientes propuestas.

1. Hemos definido un nuevo modelo llamado *hesitant linguistic expression soft sets*, que permite que los expertos proporcionen sus valoraciones mediante términos lingüísticos simples o expresiones lingüísticas comparativas. Un enfoque de toma de decisiones basado en dicho modelo ha sido también propuesto.

2. Teniendo en cuenta el modelo anterior, hemos introducido un enfoque de toma de decisiones en grupo basado en *hesitant linguistic expression soft sets* y un modelo de consenso para obtener soluciones consensuadas. Además se ha realizado un estudio comparativo para mostrar las ventajas y mejoras que presenta nuestra propuesta respecto a los modelos existentes basados en *linguistic value soft sets*.

Otro de los objetivos que nos planteamos al inicio de esta investigación fue, definir modelos de representación difusos para expresiones lingüísticas comparativas que sean capaces de reflejar y modelar la incertidumbre lingüística que aparecen en dichas expresiones. Estos modelos deben facilitar los procesos computacionales, cuando son aplicados a problemas de toma de decisiones en el que los expertos proporcionan sus valoraciones mediante expresiones lingüísticas comparativas. Para ello, se han presentado las siguientes propuestas:

1. Se ha construido un nuevo modelo de representación para expresiones lingüísticas comparativas basado en conjuntos tipo-2, *type-2 fuzzy envelope*. Este modelo sigue la base del enfoque lingüístico difuso y es capaz de reflejar de forma satisfactoria la incertidumbre contenida en las expresiones lingüísticas comparativas.

2. Se ha realizado una comparación entre el modelo propuesto basado en tipo-2 y el modelo existente en la literatura basado en tipo-1 mediante un ejemplo ilustrativo aplicado a toma de decisiones. Los resultados obtenidos de este ejemplo muestran que cuando se aplica el modelo propuesto basado en tipo-2 se obtienen resultados más precisos.

A.6.2 Trabajos futuros

A pesar de las propuestas presentadas en esta investigación, aún existen algunos retos por alcanzar y que aquí presentamos como trabajos futuros.

1. Profundizar en el estudio de las relaciones existentes entre los distintos modelos *soft sets híbridos*.

2. Estudiar la relación entre modelos *soft sets híbridos* y analizar los resultados obtenidos cuando estos modelos son aplicados a problemas de toma de decisiones.

3. Definir modelos computacionales para realizar procesos de computación con palabras en problemas de toma de decisiones en grupo dentro del marco de trabajo de *hesitant linguistic expression soft sets* utilizando el método TOPSIS difuso para tipo-2 [10] y el método de representación propuesto para CTLDD basado también en tipo-2. Buscar un problema de toma de decisiones en grupo basado en *linguistic value soft set* y resolverlo mediante el método TOPSIS difuso tipo-1 [9] (with type-1 fuzzy envelopes for HFLTSs) y el método TOPSIS difuso tipo-2 (with type-2 fuzzy envelopes for HFLTSs), para posteriormente realizar un análisis comparativo de los dos métodos mostrando las ventajas de usar el método basado en tipo-2.

4. Desarrollar otros modelos de representación para expresiones lingüíticas comparativas y aplicarlos a problemas de toma de decisiones que utilicen *soft sets híbridos*. Los conjuntos difusos intucionistas tipo-2 son un caso especial de conjuntos difusos tipo-2, por tanto, sería interesante estudiar la generalización de los conjuntos difusos tipo-2, para construir modelos de representación de expresiones lingüísticas comparativas.

Publicaciones adicionales

En relación a la difusión y publicación de los resultados presentados, además de las publicaciones presentadas en esta memoria, destacamos las siguientes aportaciones:

- Revistas Internationales
  - Y Liu, K Qin. Object-parameter approaches to predicting unknown data in incomplete fuzzy soft sets. *International Journal of Applied Mathematics and Computer Science*, vol. 27, issue 1, pp. 157-167, 2017.
  - Y Liu, J Luo, B Wang, K Qin. A theoretical development on the entropy of interval-valued intuitionistic fuzzy soft sets based on the distance measure. *International Journal of Computational Intelligence Systems*, vol. 10, issue 1, pp. 569, 2017.
  - A Labella, Y Liu, R M Rodríguez, L Martínez. Analyzing the Performance of Classical Consensus Models in Large Scale Group Decision Making: A comparative Study. *Applied Soft Computing*, vol. 67, issue C, pp. 677-690, 2018.
- Congresos Internationales
  - Y Liu, R M Rodríguez, K Qin, L Martínez. Improved score based decision making method by using fuzzy soft sets. *The 13th International FLINS Conference on Data Science an knowledge Engineering for Sensing Decision Support (FLINS 2018) held on Belfast (UK) in August 21-24, 2018.*

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